Amplitude Relations

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arXiv:1702.08158, 1703.01269, 1708.04514,1808.06835, 1904.05997

USTC, September 07, 2019

Contents

I: Motivation

Overview

- We have witnessed the amazing progress for scattering amplitudes. One of the key concepts behind these achievements is the on-shell.
- Using on-shell properties, many deep results can be reached, such as on-shell recursion relations, the unitarity cut method for loop calculation, the uniqueness of gauge theories in 4D, etc.

- Among these deep results, one interesting us is the relations of amplitudes among different theories. A famous example is the so called **KLT relation** (by Kawai, Lewellen and Tye), which states that: the tree-level scattering amplitude $\mathcal{M}$ can be written as

$$\mathcal{M}_n = \sum_{\alpha,\beta} A_n(\alpha)\mathcal{S}[\alpha|\beta]\widetilde{A}_n(\beta)$$

where $A_n, \widetilde{A}_n$ are color ordered scattering amplitudes of Yang-Mills theory, and the $\mathcal{S}$ is the momentum kernel. For example

$$\begin{aligned}\mathcal{M}_3(1,2,3) &= A_3(1,2,3)\widetilde{A}_3(1,2,3),\\ \mathcal{M}_4(1,2,3,4) &= A_4(1,2,3,4)s_{12}\widetilde{A}_4(3,4,2,1)\end{aligned}$$

[Kawai, Lewellen, Tye; 1985] [Bern, Dixon, Perelstein, Rozowsky; 1999] [Bjerrum-Bohr, Damgaard, Feng, Sondergaard; 2010]

Such relation is found when we try to calculate the scattering amplitude of close string theory. Although in string theory closed string is roughly decomposed to the left-moving and right-moving open string, from the point of view of field theory (especially by Lagrangian formulism), it is big surprising:

- Gauge symmetry is symmetry for inner quantities while gravity theory is based on the space-time symmetry, the general equivalence principal for the choice of coordinate.
- More importantly, the Lagrangian of gauge theory is polynomial with finite number of interaction terms ($\int d^4x F_{\mu\nu} F^{\mu\nu}$), while the Einstein Lagrangian is highly non-linear and infinite number of interaction terms after perturbative expansion ($\frac{1}{2\kappa^2} \int d^4x \sqrt{-g}\, R$)

Then why there is the KLT relation?

- The key is the distinction between particles and fields
- The field carries the tensor representation of Lorentz group, while particle is categorized under the little group.
- Or in another word, the Lagrangian description uses the field, so is a off-shell description, while the scattering amplitude uses the particle, so is on-shell quantity.
- The existence of KLT relation is a on-shell property, so it is hard to see by off-shell description.

- Recent study shows that KLT relation between gauge theory and gravity theory is just a particular example of wild zoo. However, we need to have a proper framework to view these things.
- Such a framework is provided by Cachazo, He and Yuan, the so called CHY-frame, which is fully on-shell

[Cachazo, He, and Yuan, 2014]

II: General Relation in CHY-frame

In 2013, new formula for tree amplitudes of massless theories has been proposed by Cachazo, He and Yuan:

$$\mathcal{A}_n = \boxed{\int \frac{\left(\prod_{i=1}^{n} dz_i\right)}{d\omega} \Omega(\mathcal{E})} \mathcal{I},$$

[Freddy Cachazo, Song He, Ellis Ye Yuan , 2013, 2014]

In this frame:

- Each particle is represented by a puncture in Riemann sphere,i.e., a complex number z_i
- The expression holds for general D-dimension
- The box part is universal for all theories
- The CHY-integrand $\mathcal{I}$ determines the particular theory

For the universal part,

$$\Omega(\mathcal{E}) \equiv \prod_a{}' \delta(\mathcal{E}_a) = z_{ij} z_{jk} z_{ki} \prod_{a \neq i,j,k} \delta(\mathcal{E}_a)$$

provides the constraints:

- Scattering equations are defined

$$\mathcal{E}_a \equiv \sum_{b \neq a} \frac{2k_a \cdot k_b}{z_a - z_b} = 0, \qquad a = 1, 2, ..., n$$

- Only $(n-3)$ of them are independent by $SL(2, C)$ symmetry

$$\sum_a \mathcal{E}_a = 0, \quad \sum_a \mathcal{E}_a z_a = 0, \quad \sum_a \mathcal{E}_a z_a^2 = 0,$$

Universal part: $(n-3)$ integrations with $(n-3)$ delta-functions, so the integration becomes the sum over all solutions of scattering equations

$$\sum_{z\in\mathrm{Sol}} \frac{1}{\det'(\Phi)}\mathcal{I}(z)$$

where $\det'(\Phi)$ is the Jacobi coming from solving $\mathcal{E}_a$

$$\Phi_{ab} = \frac{\partial \mathcal{E}_a}{\partial z_b} = \begin{cases} \frac{s_{ab}}{z_{ab}^2} & a \neq b \\ -\sum_{c\neq a} \frac{s_{ac}}{z_{ac}^2} & a = b \end{cases},$$

Building elements I: four $n \times n$ matrices

$$A_{ab} = \begin{cases} \frac{k_a \cdot k_b}{z_a - z_b} \\ 0 \end{cases}, \quad \text{for} \quad \begin{array}{c} a \neq b \\ a = b \end{array},$$

$$B_{ab} = \begin{cases} \frac{\epsilon_a \cdot \epsilon_b}{z_a - z_b} \\ 0 \end{cases}, \quad \text{for} \quad \begin{array}{c} a \neq b \\ a = b \end{array},$$

$$C_{ab} = \begin{cases} \frac{\epsilon_a \cdot k_b}{z_a - z_b} \\ -\sum_{c \neq a} \frac{\epsilon_a \cdot k_c}{z_a - z_c} \end{cases} \quad \text{for} \quad \begin{array}{c} a \neq b \\ a = b \end{array},$$

$$X_{ab} = \begin{cases} \frac{1}{z_a - z_b} \\ 0 \end{cases}, \quad \text{for} \quad \begin{array}{c} a \neq b \\ a = b \end{array},$$

Building elements II: open chain and closed cycle

$$
\begin{aligned}
[a_1 a_2 ... a_n] &= (z_{a_1} - z_{a_2})(z_{a_2} - z_{a_3})...(z_{a_{n-1}} - z_{a_n}) \\
(a_1 a_2 ... a_n) &= (z_{a_1} - z_{a_2})(z_{a_2} - z_{a_3})...(z_{a_{n-1}} - z_{a_n})(z_{a_n} - z_{a_1})
\end{aligned}
$$

Building elements III: Ψ **matrix:** A $2n \times 2n$ antisymmetric matrix Ψ:

$$
\Psi(\{k_i, \epsilon_i\} = \begin{pmatrix} A & -C^t \\ C & B \end{pmatrix}
$$

One nice property of matrix Ψ is that it makes the gauge invariance manifest!

Two most important building blocks of weight two:

(I) reduced Pfaffian of Ψ

$$\mathrm{Pf}'\Psi = 2\frac{(-1)^{i+j}}{z_i - z_j}\mathrm{Pf}\Psi^{ij}_{ij} ,$$

where $1 \leq i, j \leq n$ and Ψ^{ij}_{ij} is the matrix Ψ removing rows i, j and columns i, j. It is independent of the choice (i, j).

(II) color ordered Parker-Taylor factor

$$\mathcal{C}(\alpha) = \mathrm{PT}(\alpha) = \frac{1}{(\alpha(1)...\alpha(n))}$$

One key fact: for all theories described in this frame, CHY-integrand is factorized to two parts:

	$\mathcal{I}_L$	$\mathcal{I}_R$
bi-adjoint scalar	$\mathcal{C}_n(\alpha)$	$\mathcal{C}_n(\alpha)$
Yang-Mills	$\mathcal{C}_n(\alpha)$	$\mathrm{Pf}'\Psi_n$
Einstein gravity	$\mathrm{Pf}'\Psi_n$	$\mathrm{Pf}'\Psi_n$
Born-Infeld	$(\mathrm{Pf}'A_n)^2$	$\mathrm{Pf}'\Psi_n$
Non-linear sigma model	$\mathcal{C}_n(\alpha)$	$(\mathrm{Pf}'A_n)^2$
Yang-Mills-scalar	$\mathcal{C}_n(\alpha)$	$\mathrm{Pf}X_n\mathrm{Pf}'A_n$
Einstein-Maxwell-scalar	$\mathrm{Pf}X_n\mathrm{Pf}'A_n$	$\mathrm{Pf}X_n\mathrm{Pf}'A_n$
Dirac-Born-Infeld (scalar)	$(\mathrm{Pf}'A_n)^2$	$\mathrm{Pf}X_n\mathrm{Pf}'A_n$
Special Galileon	$(\mathrm{Pf}'A_n)^2$	$(\mathrm{Pf}'A_n)^2$

[Freddy Cachazo, Song He, Ellis Ye Yuan , 2014] [Freddy Cachazo, Peter Cha, Sebastian Mizera, 2016]

Solving polynomial equations with multiple variables is not easy, but for the CHY case, systematically method has been developed, so we can read out analytic expression straightforward:

- For the case with only simple pole: [Freddy Cachazo, Song He, Ellis Ye Yuan , 2013,2014] [Baadsgaard, Bjerrum-Bohr, Bourjaily and Damgaard, 2015] [Cachazo, Gomez, 2015] [Lam, Yao, 2016]
- For general case with higher order pole [Baadsgaard, Bjerrum-Bohr, Bourjaily and Damgaard, 2015] [Huang, Feng, Luo, Zhu, 2016] [Cardona, Feng, Gomez, Huang, 2016]
- Final conclusion: Any weight two integrand can be decomposed as the sum of PT-factors with kinematic rational coefficients [Bjerrum-Bohr, Bourjaily, Damgaard, Feng, 2016]

The generalized KLT relation is the natural consequece of the factorized form of the CHY-integrand:

- First observation:

$$
\begin{aligned}
m[\alpha|\beta] &= \sum_{z\in \mathrm{Sol}} \mathrm{PT}(\alpha)\, \frac{1}{\det'(\Phi)}\, \mathrm{PT}(\beta) \\
&= (\mathcal{S}[\alpha|\beta])^{-1}
\end{aligned}
$$

where the $\mathcal{S}$ is KLT kernel and $m[\alpha|\beta]$ the amplitudes of bi-color-ordered ϕ^3 theory

[Freddy Cachazo, Song He, Ellis Ye Yuan , 2013] [Bern, Dixon, Perelstein, Rozowsky, 1998] [Bjerrum-Bohr, Damgaard, Feng, Sondergaard, 2010]

- For general theory with

$$\mathcal{I} = \mathcal{I}_L \times \mathcal{I}_R$$

we derive

$$\begin{aligned} \mathrm{A} &= \sum_{z \in \mathrm{Sol}} \mathcal{I}_L \frac{1}{\det'(\Phi)} \mathcal{I}_R \\ &= \frac{\mathrm{PT}(\alpha)\, \mathcal{I}_L}{\det'(\Phi)} \times \frac{\det'(\Phi)}{\mathrm{PT}(\alpha)\, \mathrm{PT}(\beta)} \times \frac{\mathrm{PT}(\beta)\, \mathcal{I}_R}{\det'(\Phi)} \\ &= \mathrm{A}_L(\alpha) \times (\phi^3)^{-1} \times \mathrm{A}_R(\beta) \\ &= \mathrm{A}_L(\alpha) \times \mathcal{S}[\alpha|\beta] \times \mathrm{A}_R(\beta) \end{aligned}$$

- With both $\mathcal{I}_L = \mathcal{I}_R = \mathrm{Pf}'\Psi$ we get the original KLT relation.

Expansion:

- If we define coefficient $c(\alpha) = \sum_\beta \mathcal{S}[\alpha|\beta] \times \mathrm{A}_R(\beta)$ we have

$$\begin{aligned} \mathrm{A} &= \sum_{\alpha,\beta} \mathrm{A}_L(\alpha) \times \mathcal{S}[\alpha|\beta] \times \mathrm{A}_R(\beta) \\ &= \sum_{\alpha} c(\alpha)\mathrm{A}_L(\alpha) \end{aligned}$$

- Similar we have

$$\mathrm{A} = \sum_{\beta} \widetilde{c}(\beta)\mathrm{A}_R(\beta)$$

with $\widetilde{c}(\beta) = \sum_\alpha \mathrm{A}_L(\alpha) \times \mathcal{S}[\alpha|\beta]$

- A systematic way has been developed to expand any weight two CHY-integrands on the sum of Parker-Taylor factor.

  [Bjerrum-Bohr, Bourjaily, Damgaard, Feng, 2016][Huang, Du, Feng, 2017]

- We must emphasize that the expansion of amplitudes of one theory by another theory is hard to guess from the Lagrangian, but is very natural from CHY frame!

Some examples of expansions I:

	$\mathcal{I}_L$	$\mathcal{I}_R$
Yang-Mills	$\mathcal{C}_n(\alpha)$	$\mathrm{Pf}'\Psi_n$
Einstein gravity	$\mathrm{Pf}'\Psi_n$	$\mathrm{Pf}'\Psi_n$
Einstein-YM	$\mathrm{PT}_r(\alpha)\ \mathrm{Pf}\Psi_{\mathcal{S}}$	$\mathrm{Pf}'\Psi_n$
Born-Infeld	$(\mathrm{Pf}'A_n)^2$	$\mathrm{Pf}'\Psi_n$

Thus we see that amplitudes of Einstein, Einstein-YM, Born-Infeld can be expanded by the color-ordered YM amplitudes.

$$L_{BI} = \ell^2\left(\sqrt{-\det(\eta_{\mu\nu} - \ell^2 F_{\mu\nu})} - 1\right)$$

[Stieberger, Taylor, 2016] [Nandan, Plefka, Schlotterer, Wen, 2016] [de la Cruz, Kniss, Weinzierl , 2016]

More examples of expansions II:

	$\mathcal{I}_L$	$\mathcal{I}_R$
Yang-Mills-scalar	$\mathcal{C}_n(\alpha)$	$\text{Pf}X_n\text{Pf}'A_n$
Einstein-Maxwell-scalar	$\text{Pf}X_n\text{Pf}'A_n$	$\text{Pf}X_n\text{Pf}'A_n$
Dirac-Born-Infeld (scalar)	$(\text{Pf}'A_n)^2$	$\text{Pf}X_n\text{Pf}'A_n$

we see that amplitudes of Einstein-Maxwell-scalar and Dirac-Born-Infeld can be expanded by color ordered Yang-Mills-scalar amplitudes.

$$
\begin{aligned}
L_{YM-S} &= -\text{Tr}\left(\frac{1}{4}F^2 + \frac{1}{2}(D\phi)^2 - \frac{g^2}{4}\sum[\phi^I, \phi^J]^2\right) \\
L_{DBIs} &= \ell^2\left(\sqrt{-\det(\eta_{\mu\nu} - \ell^2\partial_\mu\phi\partial_\nu\phi)} - 1\right)
\end{aligned}
$$

More examples of expansions III:

	$\mathcal{I}_L$	$\mathcal{I}_R$
Non-linear sigma model	$\mathcal{C}_n(\alpha)$	$(\mathrm{Pf}' A_n)^2$
Special Galileon	$(\mathrm{Pf}' A_n)^2$	$(\mathrm{Pf}' A_n)^2$

Amplitudes of Special Galileon theory can be expanded by amplitudes of Non-linear sigma model

$$L_{NLSM} = \frac{1}{8\lambda^2}\mathrm{Tr}(\partial_\mu U^\dagger \partial^\mu U), \quad U = (I+\lambda\Phi)(I-\lambda\Phi)^{-1}$$

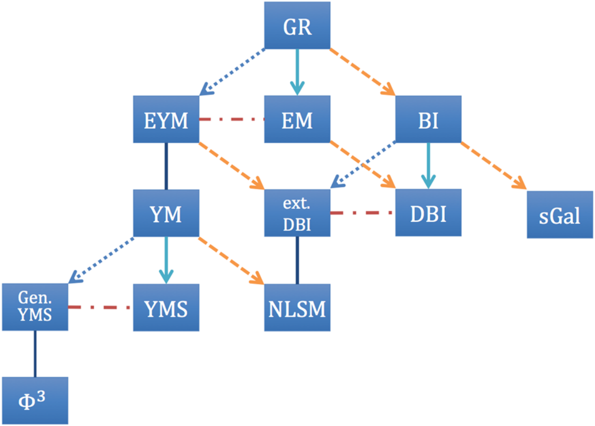
$$LGal = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \sum_{m=3} g_m L_m, \quad L_m = \phi \det(\partial^{\mu_i}\partial_{\nu_j}\phi)_{i,j=1,\ldots,m-1}$$

III: Web of theories

In previous part, we have seen the KLT relation from CHY-framework. In this part, we will see another connection among different theories, i.e., the web of theories, from two different angles: (1) from CHY integrand; (2) using some differential operators

[Cachazo, He, Yuan, 2014] [Cheung, Shen, Wen, 2017]

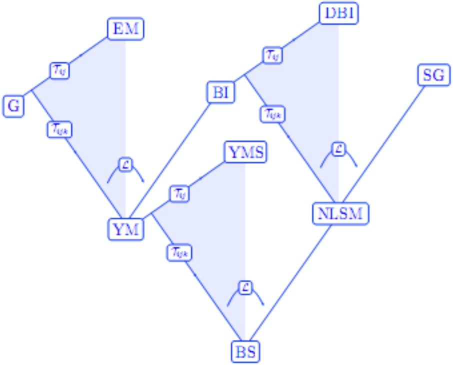
In the CHY-frame, CHY-integrand of different theories are related by a few operations (Dimension reduction, Squeezing, Generalized Dimension reduction):



[Yuan's talk, 2017]

- **Compactifying**: Components of gauge field in higher space-time dimension become scalar field in lower space-time dimension.
- **Generalizing/Squeezing**: Producing the multi-trace structure.
- **Generalized Dimension Reduction**: While removing polarization vector (so turning a gluon to scalar), it adds derivative interaction to vertex.

Similar web has been given by Cheung, Shen and Wen using differential operators:



Now we present some detail of CSW constructions. There are four basic differential operators:

- Trace operator: $\mathcal{T}_{ij} \equiv \partial_{\epsilon_i \epsilon_j}$
- Insertion operator: $\mathcal{T}_{ikj} \equiv \partial_{k_i \epsilon_k} - \partial_{k_j \epsilon_k}$
- Longitudinal operators: $\mathcal{L}_i \equiv \sum_{j \neq i} k_i k_j \partial_{k_j \epsilon_i}$
- $\mathcal{L}_{ij} \equiv -k_i k_j \partial_{\epsilon_i \epsilon_j}$

All of them remove polarization vectors, but only $\mathcal{L}_i$ and $\mathcal{L}_{ij}$ add derivative interactions.

Combination of differential operators:

- The operator $\mathcal{T}[\alpha]$ for a length-m set $\alpha = \{\alpha_1, \alpha_2, \cdots, \alpha_m\}$ is defined as

$$\mathcal{T}[\alpha] \equiv \mathcal{T}_{\alpha_1\alpha_m} \cdot \prod_{i=2}^{m-1} \mathcal{T}_{\alpha_{i-1}\alpha_i\alpha_m} \,.$$

  Its role is to turn gravitons to a trace of gluons.

- The operator $\mathcal{L}$ is defined through longitudinal operators as

$$\mathcal{L} \equiv \prod_i \mathcal{L}_i = \widetilde{\mathcal{L}} + \cdots \,, \quad \text{with } \widetilde{\mathcal{L}} \equiv \sum_{\rho\in\text{pair}} \prod_{i,j\in\rho} \mathcal{L}_{ij}.$$

  Its role is to remove polarization vector while adding derivative interaction to the vertex.

- For a given length-$2m$ set I, we define a new operator as

$$\mathcal{T}_{X_{2m}} \equiv \sum_{\rho\in\text{pair}} \prod_{i,j\in\rho} \mathcal{T}_{i_k j_k} .$$

- Similarly we can also define the operator $\mathcal{T}_{\mathcal{X}_{2m}}$ as

$$\mathcal{T}_{\mathcal{X}_{2m}} \equiv \sum_{\rho\in\text{pair}} \prod_{i,j\in\rho} \delta^{I_{i_k}, I_{j_k}} \mathcal{T}_{i_k j_k} .$$

  The new feature is that we have distinguished different kind of matters.

Unified web:

- $$\begin{aligned} \mathcal{A}^{\text{EYM}} &= \mathcal{T}[\text{Tr}_1]\cdots\mathcal{T}[\text{Tr}_m]\,\mathcal{A}^{\text{G}}\,, \\ \mathcal{A}^{\text{YM}} &= \mathcal{T}[i_1\cdots i_n]\,\mathcal{A}^{\text{G}}\,, \\ \mathcal{A}^{\text{EM}} &= \mathcal{T}_{X_{2m}}\,\mathcal{A}^{\text{G}}\,, \\ \mathcal{A}^{\text{EM}}_{\text{flavor}} &= \mathcal{T}_{\mathcal{X}_{2m}}\,\mathcal{A}^{\text{G}}\,, \\ \mathcal{A}^{\text{BI}} &= \mathcal{L}\cdot\mathcal{T}[ab]\,\mathcal{A}^{\text{G}}\,, \end{aligned}$$

- $$\begin{aligned} \mathcal{A}^{\rm YMS} &= \mathcal{T}[{\rm Tr}_1]\cdots\mathcal{T}[{\rm Tr}_m]\,\mathcal{A}^{\rm YM}\,, \\ \mathcal{A}^{\rm YMS}_{\rm special} &= \mathcal{T}_{\mathcal{X}_{2m}}\,\mathcal{A}^{\rm YM}\,, \\ \mathcal{A}^{\rm BS} &= \mathcal{T}[i_1\cdots i_n]\,\mathcal{A}^{\rm YM}\,, \\ \mathcal{A}^{\rm NLSM} &= \mathcal{L}\cdot\mathcal{T}[ab]\,\mathcal{A}^{\rm YM}\,, \\ \mathcal{A}^{\phi^4} &= \mathcal{T}_{X_n}\,\mathcal{A}^{\rm YM}\,, \end{aligned}$$

- $$\begin{aligned} \mathcal{A}_{\text{ex}}^{\text{DBI}} &= \mathcal{T}[\text{Tr}_1]\cdots\mathcal{T}[\text{Tr}_m]\,\mathcal{A}^{\text{BI}}\,, \\ \mathcal{A}^{\text{DBI}} &= \mathcal{T}_{\mathcal{X}_{2m}}\,\mathcal{A}^{\text{BI}}\,, \\ \mathcal{A}^{\text{NLSM}} &= \mathcal{T}[i_1\cdots i_n]\,\mathcal{A}^{\text{BI}}\,, \\ \mathcal{A}^{\text{SG}} &= \mathcal{L}\cdot\mathcal{T}[ab]\,\mathcal{A}^{\text{BI}}\,, \end{aligned}$$

Now we have same web coming from different angles, thus there must be relation among these two methods. It is easy to show the equivalence by acting differential operators to CHY-integrands directly and get the matched the expressions.

[Zhou, Feng, 2018] [Bollmann, Ferro, 2018]

Expansion to KK-basis:

- The key is the construction of building blocks, by noticing the multi-linear property for polarization vectors.
- For sEYM amplitude with one graviton $A^{\rm EYM}_{n,1}(1,\cdots,n;h_1)$, the $(\epsilon_{h_1}\cdot k_i)$, $i=1,...,n-1$ give a basis for Lorentz contracted linear function of ϵ_{h_1}. Thus we can write down

$$A^{\rm EYM}_{n,1}(1,\cdots,n;h_1)=\sum_{i=1}^{n-1}(\epsilon_{h_1}\cdot K_i)B_i,$$

with $K_i=\sum_{j=1}^{i}k_i$, $i=1,...,n-1$.

- We apply insertion operator $\mathcal{T}_{ah_1(a+1)}$ with $a = 1, \cdots, n-1$ to the expansion:

$$\begin{aligned} \mathcal{L} &= \mathcal{T}_{ah_1(a+1)} A^{\text{EYM}}_{n,1}(1, \cdots, n; h_1) \\ &= A^{\text{YM}}_{n+1}(1, \cdots, a, h_1, a+1, \cdots, n-1) \\ \mathcal{R} &= = \sum_{i=1}^{n-1} \left\{ \mathcal{T}_{ah_1(a+1)}(\epsilon_{h_1} \cdot K_i) \right\} B_i = \sum_{i=1}^{n-1} \delta_{a,i} B_i = B_a. \end{aligned}$$

- Putting together, we get

$$\begin{aligned} & A^{\text{EYM}}_{n,1}(1, \cdots, n; h_1) \\ = & \sum_{i=1}^{n-1} (\epsilon_{h_1} \cdot K_i) A^{\text{YM}}_{n+1}(1, \cdots, i, h_1, i+1, \cdots, n-1) \\ = & \sum_{\shuffle} (\epsilon_{h_1} \cdot Y_{h_1}) A^{\text{YM}}_{n+1}(1, \{2, \cdots, n-1\} \shuffle \{h_1\}, n) \end{aligned}$$

With two gravitons:

- First we write expansion regarding to the first graviton:

$$A^{\text{EYM}}_{n,2}(1,\cdots,n;h_1,h_2) = \sum_{i=1}^{n-1}(\epsilon_{h_1}\cdot K_i)B_i$$
$$+(\epsilon_{h_1}\cdot k_{h_2})(\epsilon_{h_2}\cdot D_{h_2}) + (\epsilon_{h_1}\cdot\epsilon_{h_2})E_{h_2}$$

- Using insertion operators $\mathcal{T}_{ah_1(a+1)}$ with $a=1,\cdots,n-1$ like in previous example, we can solve

$$B_a = A^{\text{EYM}}_{n+1,1}(1,\cdots,a,h_1,a+1,\cdots,n;h_2) =$$

with iterative structure.

- Using the gauge invariance, i.e., the corresponding operator relation

$$[\mathcal{T}_{ijk}, \mathcal{W}_l] = \delta_{il}\mathcal{T}_{ij} - \delta_{kl}\mathcal{T}_{jk} ,$$

we arrive

$$(k_{h_2} \cdot D_{h_2}) + E_{h_2} = 0$$

Thus we have

$$\begin{aligned} & A^{\text{EYM}}_{n,2}(1, \cdots, n; h_1, h_2) \\ = \; & \sum_{\shuffle} (\epsilon_{h_1} \cdot Y_{h_1}) A^{\text{EYM}}_{n+1,1}(1, \{2, \cdots, n-1\} \shuffle \{h_1\}, n; h_2) \\ & +(\epsilon_{h_1} \cdot f_{h_2} \cdot D_{h_2}) \; , \end{aligned}$$

- To determine D_{h_2}, we expand it to the building blocks as

$$\sum_{i=1}^{n-1}(\epsilon_{h_1}\cdot f_{h_2}\cdot K_i)H_i+(\epsilon_{h_1}\cdot f_{h_2}\cdot k_{h_1})H_{h_1}\ .$$

- Using $\mathcal{T}_{ah_2(a+1)}\mathcal{T}_{h_2h_1(a+1)}$ we solve

$$H_a=\sum_{\sqcup\sqcup}A^{\rm YM}_{n+2}(1,\cdots,a,h_2,\{a+1,\cdots,n-1\}\sqcup\sqcup\{h_1\},n)$$

- For remaining $(\epsilon_{h_1}\cdot f_{h_2}\cdot k_{h_1})H_{h_1}=\frac{1}{2}{\rm tr}(f_{h_1}f_{h_2})H_{h_1}$, one can prove for general building blocks with "index circle" structure, the corresponding coefficient is zero.

Above iterative construction can be generalized to arbitrary number of gravitons, thus we arrive the structure, i.e., turning one graviton at each step:

$$\begin{aligned}
& A^{EYM}_{n,3}(1,\ldots,n;p,q,r) \\
= & \sum_{\shuffle}(\epsilon_p\cdot Y_p)A^{EYM}_{n+1,2}(1,\{2,\ldots,n-1\}\shuffle\{p\},n;q,r) \\
& +\sum_{\shuffle}(\epsilon_p\cdot F_q\cdot Y_q)A^{EYM}_{n+2,1}(1,\{2,\ldots,n-1\}\shuffle\{q,p\},n;r) \\
& +\sum_{\shuffle}(\epsilon_p\cdot F_r\cdot Y_r)A^{EYM}_{n+2,1}(1,\{2,\ldots,n-1\}\shuffle\{r,p\},n;q) \\
& +\sum_{\shuffle}(\epsilon_p\cdot F_q\cdot F_r\cdot Y_r)A^{YM}_{n+3}(1,\{2,\ldots,n-1\}\shuffle\{r,q,p\},n) \\
& +\sum_{\shuffle}(\epsilon_p\cdot F_r\cdot F_q\cdot Y_q)A^{YM}_{n+3}(1,\{2,\ldots,n-1\}\shuffle\{q,r,p\},n)\ .
\end{aligned}$$

Above expansion is to KK-basis. However, since the BCJ-relation, KK-basis is not fully independent (so the coefficients are not manifest gauge invariant). Now we consider the expansion to BCJ-basis.

- First observation is that since BCJ-basis is independent to each other, its expansion coefficient must be gauge invariant.
- Thus to building blocks for expansion must be gauge invariant. Let us see how it works.
- The sEYM amplitude with one graviton can be expanded as

$$A_{n,1}^{\rm EYM}(1,2,\cdots,n;p) \;=\; \sum_{a=2}^{n-1}\frac{(k_1 f_p K_a)}{(k_1 k_p)}B_a\,.$$

- Applying $\mathcal{T}_{jp(j+1)}$ (with $2 \leq j \leq n-1$) we solve

$$B_j = A^{\rm YM}_{n+1}(1,2,\cdots,j,p,j+1,\cdots,n)\ .$$

and get

$$A^{\rm EYM}_{n,1}(1,\cdots,n;p) = \frac{(k_1 f_p Y_p)}{(k_1 k_p)} A^{\rm YM}_{n+1}(1,2,\{3,\cdots,n-1\}\sqcup\{p\},n)$$

- The action of left $\mathcal{T}_{jp(j+1)}$ with $j=1$ gives

$$\begin{aligned} & A^{\rm YM}_{n+1}(1,p,2,\cdots,n) \\ = & \sum_{a=2}^{n-1} \frac{(-k_p\cdot K_a)}{k_1\cdot k_p} A^{\rm YM}_{n+1}(1,2,\cdots,a,p,a+1,\cdots,n-1,n)\ . \end{aligned}$$

which is nothing, but fundamental BCJ relation

$$\sum_{\sqcup}(k_p\cdot X_p)A^{\rm YM}_{n+1}(1,\{2,\cdots,n-1\}\sqcup\{p\},n) = 0\ ,$$

For two gravitons:

- The expansion of building blocks are

$$A^{\rm EYM}_{n,2}(1,2,\cdots,n;p,q)=\sum_{a=2}^{n-1}\sum_{b=2}^{n-1}\frac{(k_1 f_p K_a)}{(k_1 k_p)}\frac{(k_1 f_q K_b)}{(k_1 k_q)}B_{ab}$$
$$+\sum_{b=2}^{n-1}\frac{(k_1\cdot f_p\cdot k_q)}{(k_1\cdot k_p)}\frac{(k_1 f_q K_b)}{(k_1 k_q)}B_{qb}+\sum_{a=2}^{n-1}\frac{(k_1\cdot f_q\cdot k_p)}{(k_1\cdot k_q)}\frac{(k_1 f_p K_a)}{(k_1 k_p)}B_{ap}$$
$$+\frac{(k_1\cdot f_p\cdot f_q\cdot k_1)}{(k_1\cdot k_p)(k_1\cdot k_q)}D+\frac{(k_1\cdot f_p\cdot k_q)}{(k_1\cdot k_p)}\frac{(k_1\cdot f_q\cdot k_p)}{(k_1\cdot k_q)}E\ .$$

- item **For** E**:** Since the building block contains the index cycle structure $(k_1 \cdot f_q \cdot f_p \cdot k_q)$, i.e.,

$$(k_1 \cdot f_p \cdot k_q)(k_1 \cdot f_q \cdot k_p) = (k_1 \cdot \boxed{f_q \cdot f_p \cdot k_q})(k_1 \cdot k_p) - (k_1 \cdot f_p \cdot f_q \cdot k_1$$

it is zero.

- **For** B_{ab}**:** For this one, we consider the action of $\mathcal{T}_{jp(j+1)}\mathcal{T}_{mq(m+1)}$ with $j, m = 2, ..., n-1$. The first term is

$$\frac{(k_1 f_p Y_p)}{(k_1 k_p)}\frac{(k_1 f_q Y_q)}{(k_1 k_q)} A^{\text{YM}}_{n+2}(1, 2, \{3, \cdots, n-1\} \sqcup \{p\} \sqcup \{q\}, n)$$

- **For B_{qb}:** For this one, let us act with operator $\mathcal{T}_{mq(m+1)}$ with $m = 2, ..., n-1$ first, and then $\mathcal{T}_{mpq}$ to get

$$B_{qm} = \sum_{\sqcup\!\sqcup} A^{\rm YM}_{n+2}(1, 2, \cdots, m, q, \{m+1, \cdots, n-1\} \sqcup\!\sqcup \{p\}, n)$$

- **For B_{ap}:** Similarly as B_{qb}, we get

$$B_{mp} = \sum_{\sqcup\!\sqcup} A^{\rm YM}_{n+2}(1, 2, \cdots, m, p, \{m+1, \cdots, n-1\} \sqcup\!\sqcup \{q\}, n)$$

- **For the D:** For the last one, we use $\mathcal{T}_{1qp}\mathcal{T}_{1p2}$ and solve

$$D = \frac{(K_q \cdot X_q)(k_p \cdot (Y_p - k_1))}{\mathcal{K}_{1pq}} A^{\text{YM}}_{n+1}(1, 2, \{3, \ldots, n-1\} \sqcup \{p, q\}, n) + \frac{(K_p \cdot X_p)(k_q \cdot (Y_q - k_1))}{\mathcal{K}_{1pq}} A^{\text{YM}}_{n+2}(1, 2, \{3, ..., n-1\} \sqcup \{q, p\}, n)$$

Finally we reach

$$
\begin{aligned}
&A^{\text{EYM}}_{n,2}(1,2,\cdots,n;p,q) = \sum_{\shuffle} \frac{(k_1 f_p Y_p)}{(k_1 k_p)} \frac{(k_1 f_q Y_q)}{(k_1 k_q)} A^{\text{YM}}_{n+2}(1,2,\{3,\cdots,n-1\} \shuffle \{p\} \shuffle \{q\}, n) \\
&+ \sum_{\shuffle} \frac{(k_1 \cdot f_p \cdot k_q)}{(k_1 \cdot k_p)} \frac{(k_1 f_q Y_q)}{(k_1 k_q)} A^{\text{YM}}_{n+2}(1,2,\{3,\cdots,n-1\} \shuffle \{q,p\}, n) \\
&+ \sum_{\shuffle} \frac{(k_1 \cdot f_q \cdot k_p)}{(k_1 \cdot k_q)} \frac{(k_1 f_p Y_p)}{(k_1 k_p)} A^{\text{YM}}_{n+2}(1,2,\{3,\cdots,n-1\} \shuffle \{p,q\}, n) \\
&+ \frac{(k_1 \cdot f_p \cdot f_q \cdot k_1)}{(k_1 \cdot k_p)(k_1 \cdot k_q)} \left\{ \frac{(K_q \cdot X_q)(k_p \cdot (Y_p - k_1))}{\mathcal{K}_{1pq}} A^{\text{YM}}_{n+1}(1,2,\{3,\ldots,n-1\} \shuffle \{p,q\}, n) \right. \\
&\left. + \frac{(K_p \cdot X_p)(k_q \cdot (Y_q - k_1))}{\mathcal{K}_{1pq}} A^{\text{YM}}_{n+2}(1,2,\{3,\ldots,n-1\} \shuffle \{q,p\}, n) \right\} ,
\end{aligned}
$$

V: Final Remark

- The revolution in scattering amplitudes in recent years has pushed forward some important concepts, i.e., such as on-shell, proper description variables etc.
- Various relations discussed in this talk is hard to understand from off-shell Lagrangian formula, but is easy to see from some on-shell frames
- These things hint that there maybe different descriptions other than familiar Lagrangian formula, although we have not completely figured out
- Although there are huge progresses we have made, there are still more waiting us to discover and to understand!

Thanks a lot for the attension !!!