Generalized Positivity Bounds

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EFT & Amplitude Workshop, Sep 7, 2019, Hefei

de Rham, Melville, Tolley & **SYZ**,
1702.06134, 1702.08577, 1706.02712, 1804.10624

Zhang & **SYZ**, 1808.00010; Bi, Zhang & **SYZ**, 1902.08977

Effective field theories (EFTs)

- EFTs are widely used in physics

particle physics, GR, inflation, dark energy, …

- Separation of physics at different scales

write down all local operators consistent
with symmetries, suppressed by cut-off scale

$$\mathscr{L} = \sum_i \Lambda^4 \; f_i \; \mathcal{O}_i\left(\frac{\text{boson}}{\Lambda}, \frac{\text{fermion}}{\Lambda^{3/2}}, \frac{\partial}{\Lambda}\right)$$

f_i : Wilson coefficients

Are all EFTs allowed?

Answer: No!

Not every set of Wilson coefficients are allowed!

$$e^{\frac{i}{\hbar}S_W[\text{light}]} = \int D[\text{heavy}]\ e^{\frac{i}{\hbar}S_{\text{UV}}[\text{light,heavy}]}$$

UV completion satisfies: Lorentz invariance, unitarity, locality, causality, crossing symmetry, analyticity,…

Positivity bounds on Wilson coefficients

Simplest example: P(X)

$$\mathscr{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{\lambda}{\Lambda^4}(\partial_\mu\phi\partial^\mu\phi)^2 + \cdots$$

$$A(s,t) = \frac{\lambda}{\Lambda^4}\left[s^2 + t^2 + u^2\right] + \cdots$$

Positivity bound: $\lambda > 0$

Theories with $\lambda < 0$ do not have a local and Lorentz invariant UV completion

Outline

- Positivity Bounds for Spin 0: Main idea
- Positivity Bounds for Spin >0: Subtleties
- Applications of Positivity Bounds
    - Massive gravity
- Summary

Crossing symmetry

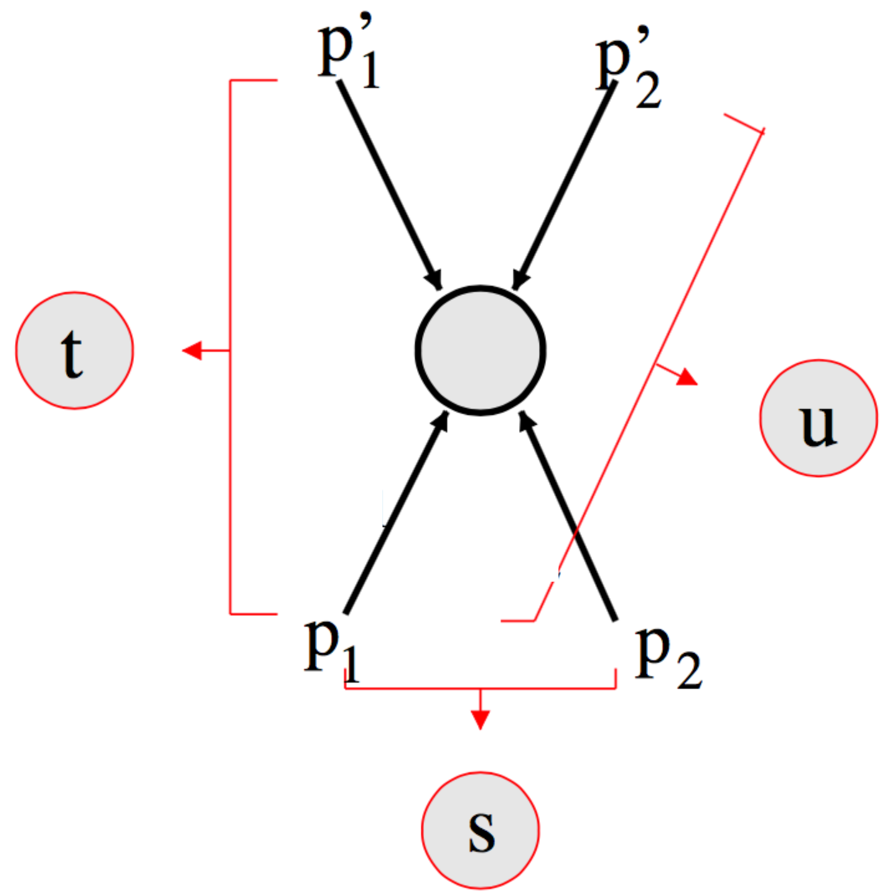
To start with one scalar with mass

Mandelstam variables

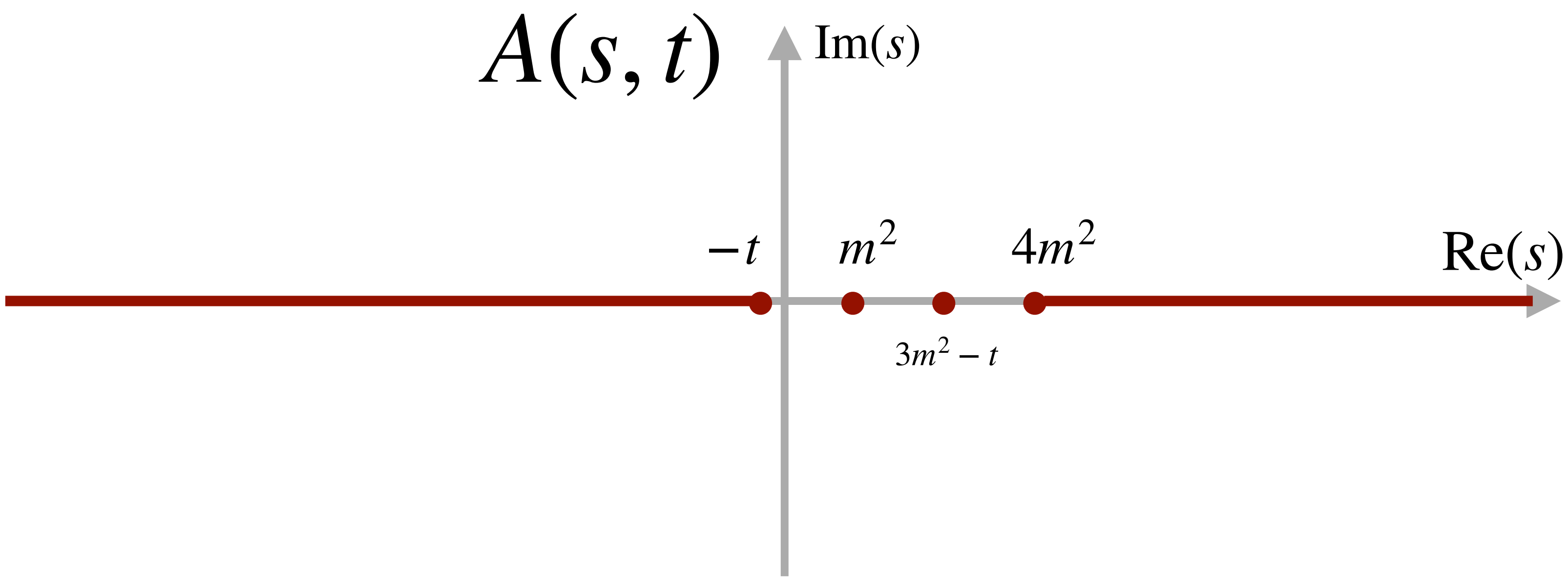
$$s = -(p_1 + p_2')^2 = E_{\rm cm}^2$$

$$t = -(p_1 + p_1')^2 = -\frac{s - 4m^2}{2}(1 - \cos\theta)$$

$$u = -(p_1 + p_2')^2 = 4m^2 - s - t$$



$A(s,t)$



Crossing symmetry

$$A(s,t) = A(u,t) = A(t,s)$$

Optical theorem

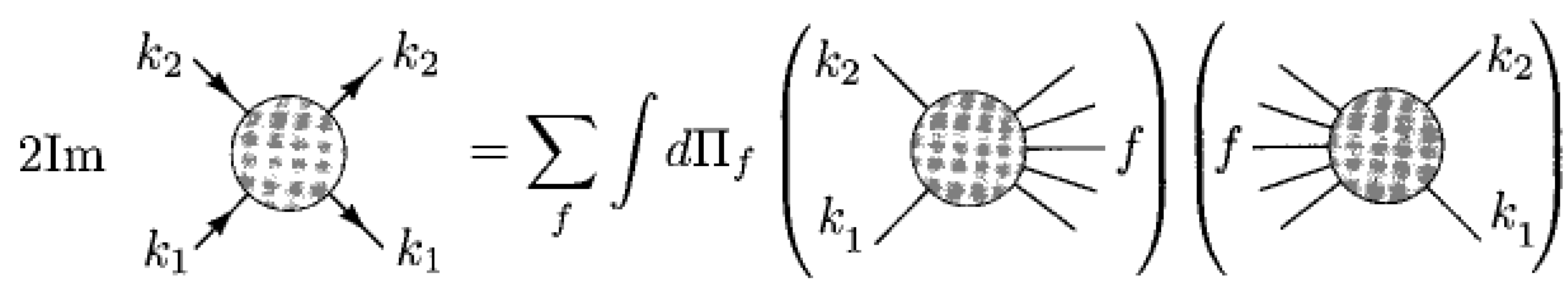
Unitarity:

$$S^\dagger S = 1, \qquad S = 1 + iT \quad \longrightarrow \quad (T - T^\dagger) = iT^\dagger T$$

Acting initial and final states:

$$\langle F|T|I\rangle - \langle I|T|F\rangle^* = \sum_f \int \mathrm{d}\Pi_f \langle f|T|F\rangle^* \langle f|T|I\rangle$$

Optical theorem:



forward limit

$t = 0,\ \theta = 0$

$$\mathrm{Im}[A(s,0)] = \sqrt{s(s-4m^2)}\sigma(s) > 0$$

Partial wave unitarity

Partial wave expansion:

$$A(s,t) = 16\pi\sqrt{\frac{s}{s-4m^2}}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)a_l(s)$$

Partial wave unitary bounds:

$$0 \leq |a_l(s)|^2 \leq \operatorname{Im} a_l(s) \leq 1$$

Also $\partial_t^n P_l(1+t)|_{t=0} \geq 0$, we get

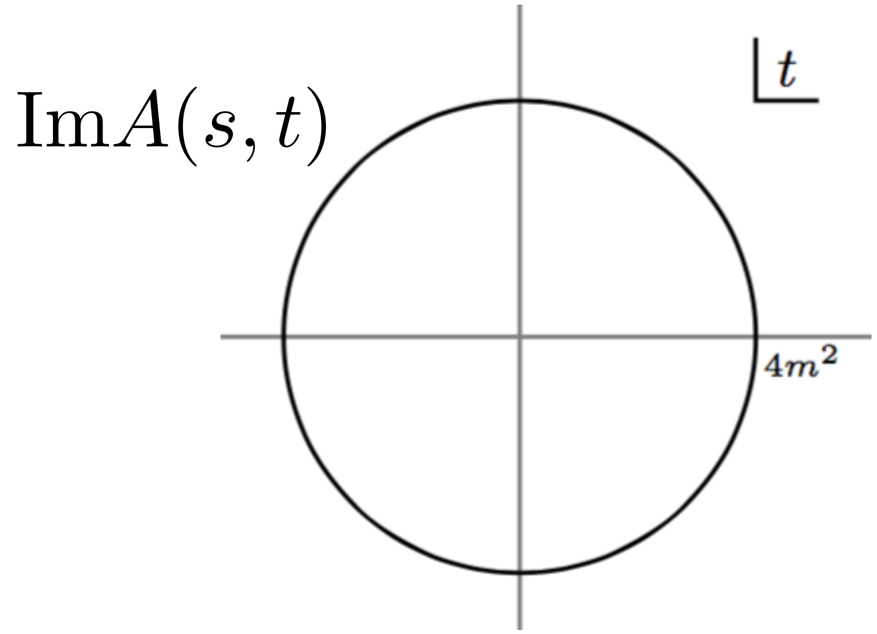
$$\frac{\partial^n}{\partial t^n}\operatorname{Im}[A(s,t)]\Big|_{t=0} > 0 \quad \forall\, n \geq 0 \quad \text{and} \quad s \geq 4m^2$$

What about $\operatorname{Im}[A(s, t \neq 0)]$?

$$\operatorname{Im}[A(s,t)] = \operatorname{Im}[A(s,0)] + \frac{\partial}{\partial t}\operatorname{Im}[A(s,0)]\ t + \frac{1}{2!}\frac{\partial^2}{\partial t^2}\operatorname{Im}[A(s,0)]\ t^2 + \cdots$$

Analyticity

Fixed *s*:

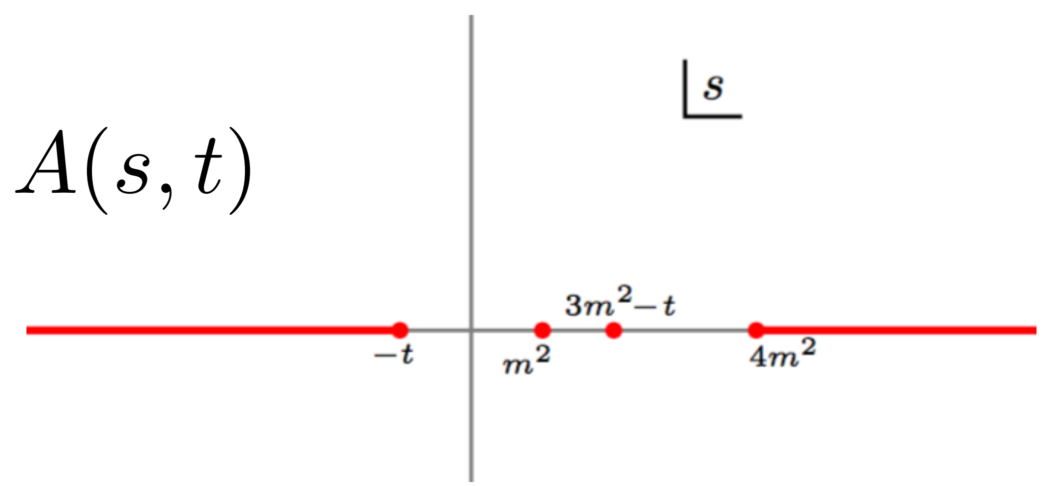


Martin, 1965

Key ingredient 1:

$$\frac{\partial^n}{\partial t^n}\mathrm{Im}[A(s,t)] > 0 \qquad s \geq 4m^2\,,\ 0 \leq t < 4m^2.$$

Fixed *t*:



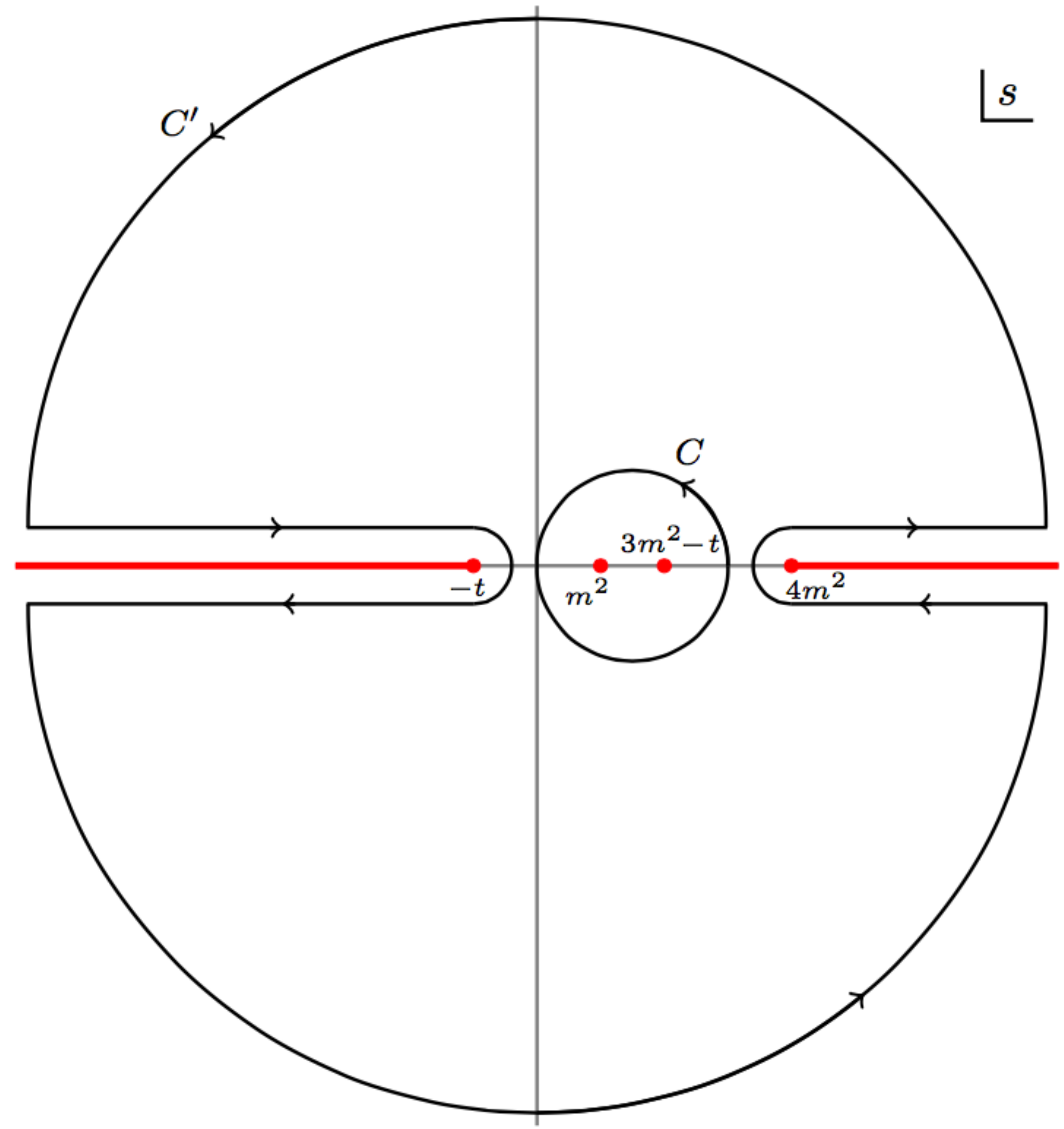
Dispersion relation (1)

$$A(s,t) = \frac{1}{2\pi i}\oint_{\mathcal{C}} ds' \, \frac{A(s',t)}{s'-s},$$

$$= \frac{\lambda}{m^2-s} + \frac{\lambda}{m^2-u} + \int_{\mathcal{C}^{\pm}_{\infty}} ds' \frac{A(s',t)}{s'-s}$$

$$+ \int_{4m^2}^{\infty} \frac{d\mu}{\pi}\left(\frac{\mathrm{Im}A(\mu,t)}{\mu-s} + \frac{\mathrm{Im}A(\mu,t)}{\mu-u}\right)$$

$$A(s,t) = A(u,t)$$



Froissart-Martin bound:

$$\lim_{s\to\infty} |A(s,t)| < C s^{1+\epsilon(t)}, \qquad 0 \le t < 4m^2,$$

$$\epsilon(t) \;<\; 1$$

Froissart, 1961
Martin, 1962

Dispersion relation (2)

Identity: $$\frac{1}{\mu-s}=\frac{(s-\mu_p)^2}{(\mu-\mu_p)^2}\frac{1}{\mu-s}+2\frac{(s-\mu_p)}{(\mu-\mu_p)^2}+\frac{(\mu-s)}{(\mu-\mu_p)^2}$$

Twice subtracted dispersion relation:

$$A(s,t)=a(t)+\frac{\lambda}{m^2-s}+\frac{\lambda}{m^2-u}$$
$$+\int_{4m^2}^{\infty}\frac{d\mu}{\pi}\left(\frac{(s-\mu_p)^2\mathrm{Im}A(\mu,t)}{(\mu-\mu_p)^2(\mu-s)}+\frac{(u-\mu_p)^2\mathrm{Im}A(\mu,t)}{(\mu-\mu_p)^2(\mu-u)}\right)$$

Define: $\tilde{B}(v,t)=A(s,t)-\frac{\lambda}{m^2-s}-\frac{\lambda}{m^2-u}$ $\quad v=s+\frac{t}{2}-2m^2$

$$\bar{s}=s-\frac{4m^2}{3}$$

Key ingredient 2: $$\tilde{B}(v,t)=a(t)+\int_{4m^2}^{\infty}\frac{d\mu}{\pi(\bar{\mu}+\bar{t}/2)}\frac{2v^2\,\mathrm{Im}A(\mu,t)}{(\bar{\mu}+\bar{t}/2)^2-v^2}$$

Key ingredients: Recap

Key ingredient 1: $$\frac{\partial^n}{\partial t^n}\mathrm{Im}[A(s,t)] > 0 \qquad s \geq 4m^2\,,\, 0 \leq t < 4m^2.$$

Key ingredient 2: $$\tilde{B}(v,t) = a(t) + \int_{4m^2}^{\infty} \frac{d\mu}{\pi(\bar{\mu}+\bar{t}/2)} \frac{2v^2\,\mathrm{Im}A(\mu,t)}{(\bar{\mu}+\bar{t}/2)^2 - v^2}$$

$$B^{(2N,M)}(t) = \frac{1}{M!}\partial_v^{2N}\partial_t^M \tilde{B}(v,t)\Big|_{v=0} = \sum_{k=0}^{M} \frac{(-1)^k}{k!2^k} I^{(2N+k,M-k)}$$

$$I^{(q,p)}(t) = \frac{q!}{p!}\frac{2}{\pi}\int_{4m^2}^{\infty} \frac{d\mu\;\; \partial_t^p \mathrm{Im}\,A(\mu,t)}{(\bar{\mu}+\bar{t}/2)^{q+1}} > 0$$

$$I^{(q,p)} < \frac{q}{\mathcal{M}^2} I^{(q-1,p)}$$

$$\mathcal{M}^2 = (t+4m^2)/2$$

First few positivity bounds

0-th *t* derivative bound: $Y^{(2N,0)} = B^{(2N,0)}(t) > 0$

Making use of $I^{(q,p)} < \frac{q}{\mathcal{M}^2} I^{(q-1,p)}$

1-st *t* derivative bound:

$$Y^{(2N,1)} = B^{(2N,1)} + \frac{2N+1}{2\mathcal{M}^2} B^{(2N,0)} > I^{(2N,1)} > 0$$

2-nd *t* derivative bound:

$$Y^{(2N,2)} = B^{(2N,2)} + \frac{2N+1}{2\mathcal{M}^2} Y^{(2N,1)} - \frac{1}{8} Y^{(2(N+1),0)} > I^{(2N,2)} > 0$$

An infinite number of positivity bounds

Recurrence relation:

de Rham, Melville, Tolley & **SYZ**, arXiv:1702.06134

$$Y^{(2N,M)} = \sum_{r=0}^{M/2} c_r B^{(2N+2r,M-2r)} + \frac{1}{\mathcal{M}^2} \sum_{k \text{ even}}^{(M-1)/2} (2(N+k)+1)\beta_k Y^{(2(N+k),M-2k-1)} > 0$$

$$\operatorname{sech}(x/2) = \sum_{k=0}^{\infty} c_k x^{2k} \quad \text{and} \quad \tan(x/2) = \sum_{k=0}^{\infty} \beta_k x^{2k+1}$$

$$\mathcal{M}^2 = (t+4m^2)/2$$

At low energies, EFT amplitude $\simeq$ full amplitude

Forward limit positivity bound

Adams, Arkani-Hamed, Dubovsky, Nicolis, Rattazzi, 2006
(also, a few earlier works)

Forward limit positivity bound

$$f(s_p) = \frac{1}{2\pi i}\oint_{C'} ds' \frac{A(s',0)}{(s'-s_p)^3} = \frac{\partial^2 B(s,t=0)}{2\,\partial s^2}$$

$$= \sum_{N=1}^{\infty} \frac{s_p^{2N-2}}{2(2N-2)!} Y^{(2N,0)}(0) > 0$$

$$0 \le s_p < 4m^2$$

$$\mathrm{Im}[A(s,0)] = \sqrt{s(s-4m^2)}\sigma(s) > 0$$

Our generalizations:

1. an infinite number of *t* derivative bounds
2. away from the forward limit $0 \le t < 4m^2$
3. applicable to general spins

How effective are the higher order bounds?

Amplitude $B^{(2N,M)}$

"s" derivative → $2N$; t derivative → M

Structure of bounds

$$Y^{(2,0)} = B^{(2,0)} > 0$$

$$Y^{(2,1)} = \frac{1}{2\mathscr{M}^2} Y^{(2,0)} + B^{(2,1)} > 0$$

$$Y^{(2,2)} = \frac{3}{2\mathscr{M}^2} Y^{(2,1)} + B^{(2,2)} - \frac{1}{8} B^{(4,0)} > 0$$

$$\vdots$$

$$\mathscr{M}^2 = \frac{1}{2}\left(t + 4m^2\right) \sim m^2$$

Improved positivity bounds

$A(s,t)$ calculable within EFT: $E < \epsilon\Lambda$ $\epsilon\Lambda \gg m$

Low energy subtracted amplitude:

$$B^{(2N,M)}_{\epsilon\Lambda}(t) = B^{(2N,M)}(t) - \sum_{k=0}^{M} \frac{2(-1)^k}{\pi k! 2^k} \frac{(2N+k)!}{(M-k)!} \int_{4m^2}^{\epsilon^2\Lambda^2} \mathrm{d}\mu \frac{\partial_t^{2N+k} \mathrm{Im}\, A(\mu,t)}{(\mu + t/2 - 2m^2)^{M-k+1}}$$

Raised $\mathcal{M}^2 = \epsilon^2\Lambda^2 + t/2 - 2m^2 \approx \epsilon^2\Lambda^2$.

→ Improved bounds $Y^{(2N,M)}_{\epsilon\Lambda}(t) > 0$

de Rham, Melville, Tolley & **SYZ**, arXiv:1702.08577

Tree level positivity bounds

$$I^{(q,p)}(t) = \frac{q!}{p!}\frac{2}{\pi}\int_{4m^2}^{\infty} \frac{d\mu \;\; \partial_t^p \mathrm{Im}\, A(\mu,t)}{(\bar{\mu}+\bar{t}/2)^{q+1}} > 0 \quad \longrightarrow \quad \mathcal{M}^2 = (t+4m^2)/2$$

weakly coupled
tree level UV completion

$$I^{(q,p)}(t) = \frac{q!}{p!}\frac{2}{\pi}\int_{\Lambda_{\mathrm{th}}^2}^{\infty} \frac{d\mu \;\; \partial_t^p \mathrm{Im}\, A(\mu,t)}{(\bar{\mu}+\bar{t}/2)^{q+1}} > 0 \quad \longrightarrow \quad \mathcal{M}^2 \simeq \Lambda_{\mathrm{th}}^2$$

Λ_{th}: mass of first state that lies outside the EFT

$$Y_{\mathrm{tree}}^{(2N,M)}(t) = Y^{(2N,M)}(t)\Big|_{\mathcal{M}\to\Lambda_{\mathrm{th}}} > 0$$

Positivity Bounds for Spin >0: Subtleties

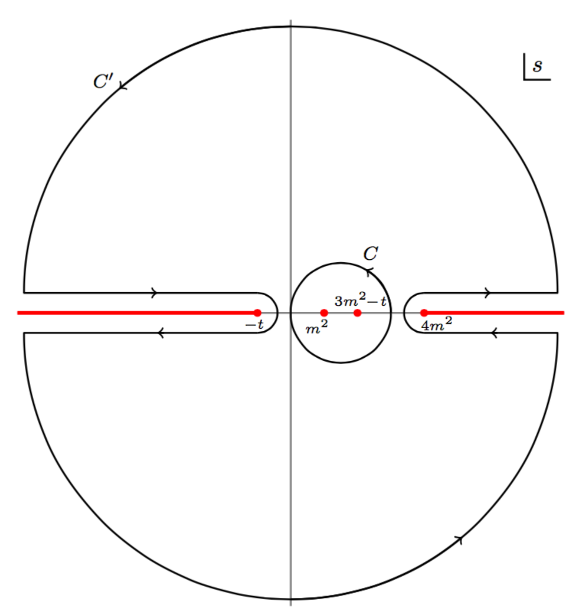
Subtleties with nonzero spins (1)

crossing for scalar fields $A(s,t) = A(u,t)$

Crossing is nontrivial in helicity formalism:

$$\mathcal{H}^{s}_{\lambda_1\lambda_2\lambda_3\lambda_4}(s,t,u) = (-1)^{2S_2} \sum_{\lambda_i'} e^{i\pi(\lambda_1'-\lambda_3')} d^{S_1}_{\lambda_1'\lambda_1}(\chi_u) d^{S_2}_{\lambda_2'\lambda_2}(-\pi+\chi_u) \cdot d^{S_1}_{\lambda_3'\lambda_3}(-\chi_u) d^{S_2}_{\lambda_4'\lambda_4}(\pi-\chi_u) \mathcal{H}^{u}_{\lambda_1'\lambda_4'\lambda_3'\lambda_2'}(u,t,s),$$

Positivity of LH cut can not be established!



$$\frac{\partial^n}{\partial t^n} \mathrm{Im}\mathcal{H}^{s}_{\lambda_1\lambda_2\lambda_3\lambda_4} > 0$$

Subtleties with nonzero spins (2)

Extra kinematical singularities:

Pole at threshold $s = 4m^2$

Branch point at $stu = 0$

Branch point at $u = 0$ for boson-fermion scattering

$$\cos\theta = 1 + \frac{2t}{s - 4m^2}, \quad \sin\theta = \frac{2\sqrt{tu}}{s - 4m^2}.$$

$$\cos\frac{\theta}{2} = \sqrt{\frac{-u}{s - 4m^2}}$$

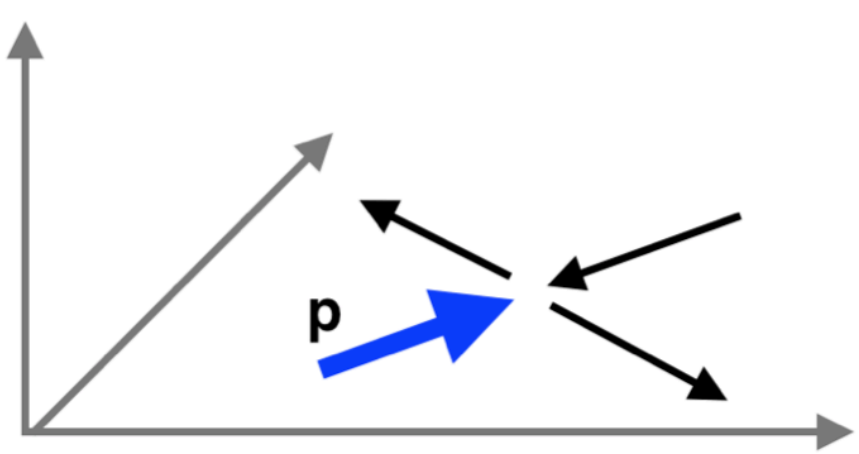
Helicity vs Transversity

$$u^S_{ab} = D^S_{ab}\left(\frac{\pi}{2}, \frac{\pi}{2}, -\frac{\pi}{2}\right)$$

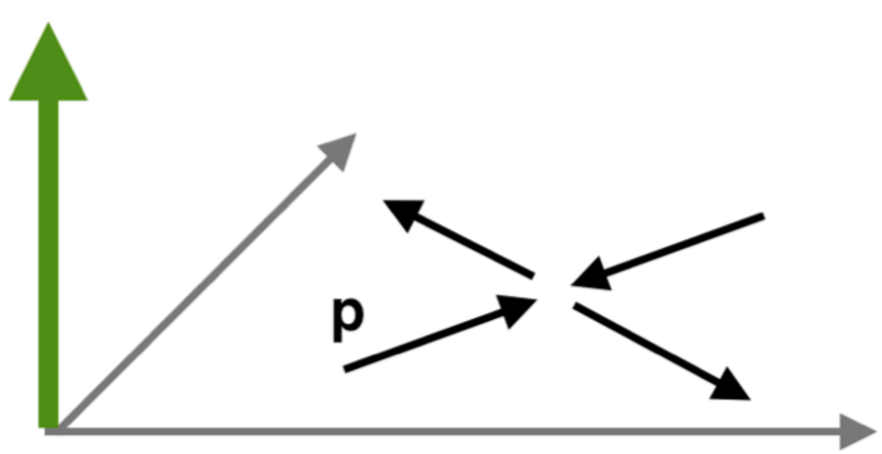
From Helicity to Transversity:

$$\mathcal{T}_{\tau_1\tau_2\tau_3\tau_4} = \sum_{\lambda_1\lambda_2\lambda_3\lambda_4} u^{S_1}_{\lambda_1\tau_1} u^{S_2}_{\lambda_2\tau_2} u^{S_1*}_{\tau_3\lambda_3} u^{S_2*}_{\tau_4\lambda_4} \mathcal{H}_{\lambda_1\lambda_2\lambda_3\lambda_4}$$

Helicity



Transversity



Regularized transversity amplitude

For singularities: $u = 0 \quad s = 4m^2$

$$(\sqrt{-su})^{\xi}(s(s-4m^2))^{S_1+S_2}\mathcal{T}_{\tau_1\tau_2\tau_3\tau_4}$$

$\xi = 1$ for BF scattering, $\xi = 0$ otherwise

For singularities: $\sqrt{stu} = 0$

$\sqrt{stu} \leftrightarrow -\sqrt{stu}$ ⟷ $\theta \leftrightarrow -\theta$

$$\mathcal{T}_{\tau_1\tau_2\tau_3\tau_4}(\theta) + \mathcal{T}_{\tau_1\tau_2\tau_3\tau_4}(-\theta)$$

Regularized transversely amplitude:

$$\mathcal{T}^{+}_{\tau_1\tau_2\tau_3\tau_4}(s,\theta) = (\sqrt{-su})^{\xi}\mathcal{S}^{S_1+S_2}\left(\mathcal{T}_{\tau_1\tau_2\tau_3\tau_4}(s,\theta) + \mathcal{T}_{\tau_1\tau_2\tau_3\tau_4}(s,-\theta)\right),$$

Positivity of the absorptive part

Right hand cut:

$$\frac{\partial^n}{\partial t^n} \mathrm{Abs}_s \mathcal{T}^+_{\tau_1\tau_2\tau_1\tau_2}(s,t,u) > 0 \quad \text{for} \quad 0 \leqslant t < m^2 , \quad s \geqslant 4m^2 ,$$

$$\mathrm{Abs}_s = \frac{1}{2i}\mathrm{Disc}$$

Left hand cut:

$$\mathcal{T}^{s+}_{\tau_1\tau_2\tau_1\tau_2}(s,t,u) = (\sqrt{-su})^{\xi} \mathcal{S}^{S_1+S_2} \left(e^{-i\chi_u \sum_i \tau_i} \mathcal{T}^u_{-\tau_1-\tau_2-\tau_1-\tau_2}(u,t,s) + e^{+i\chi_u \sum_i \tau_i} \mathcal{T}^u_{\tau_1\tau_2\tau_1\tau_2}(u,t,s) \right)$$

after a few technical steps …

$$\frac{\partial^n}{\partial t^n}\Big|_u \mathrm{Abs}_u \mathcal{T}^{s+}_{\tau_1\tau_2\tau_1\tau_2}(s,t,u) > 0 , \quad u > 4m^2 \quad 0 \leqslant t < m^2 ,$$

de Rham, Melville, Tolley & **SYZ**, arXiv:1706.02712

Positivity bounds for spin > 0

Then it reduces to **formally the same** as the spin 0 case.

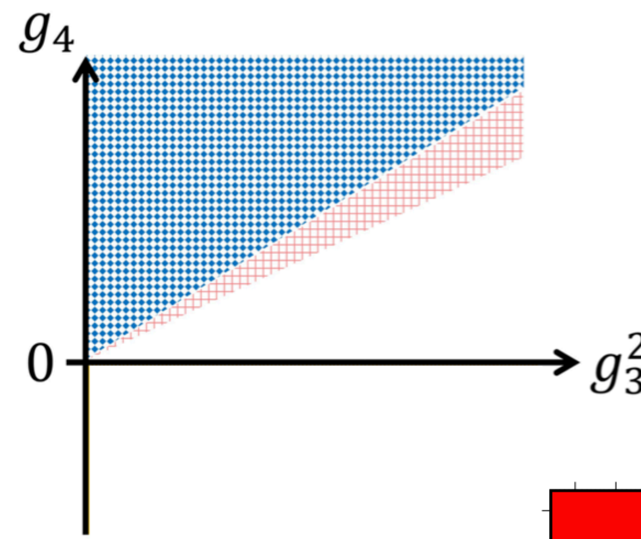
$$Y_{\tau_1\tau_2}^{(2N,M)}(t) = \sum_{r=0}^{M/2} c_r B_{\tau_1\tau_2}^{(2N+2r,M-2r)}(t) + \frac{1}{\mathcal{M}^2} \sum_{\text{even } k=0}^{(M-1)/2} (2N+2k+1)\beta_k Y_{\tau_1\tau_2}^{(2N+2k,M-2k-1)}(t) > 0$$

de Rham, Melville, Tolley & **SYZ**, arXiv:1706.02712 [hep-th]

Applications of the Positivity Bounds

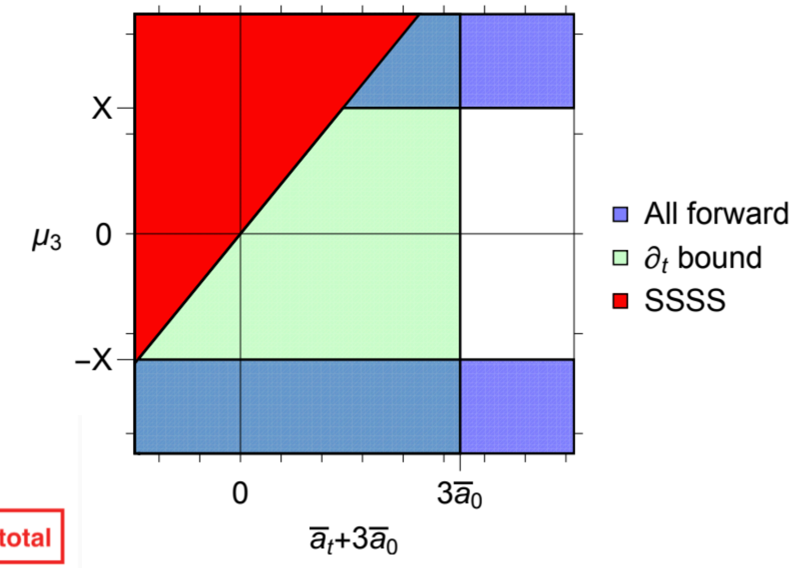
- Massive Galileon theory

  de Rham, Melville, Tolley & **SYZ**, 1702.08577



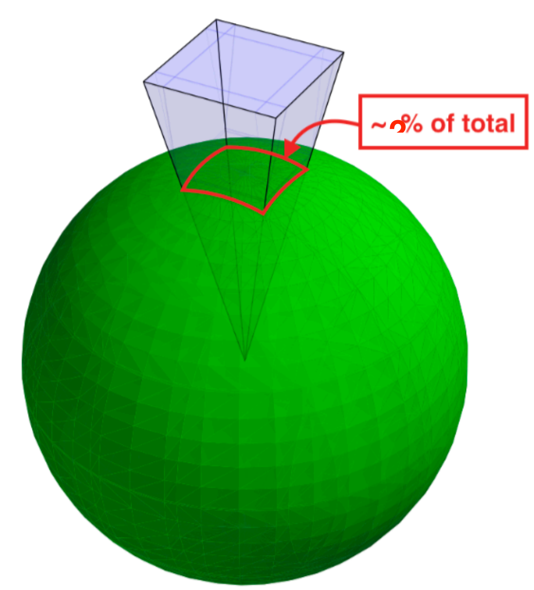
- Proca EFT

  de Rham, Melville, Tolley & **SYZ**, 1804.10624



- Standard Model EFT

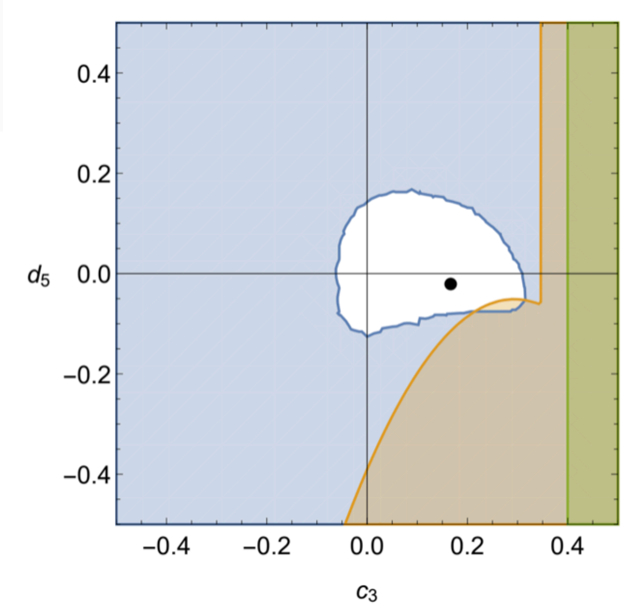
  Zhang & **SYZ**, 1808.00010
  Bi, Zhang & **SYZ**, 1902.08977



cf. Cen Zhang's talk

- Massive gravity

  de Rham, Melville, Tolley & **SYZ**, 1804.10624



Motivations for massive graviton

- Theoretically, does a consistent, interacting, massive spin-2 field theory exist?

Viewed as impossible from early1970s

- Cosmic acceleration: graviton mass ~ Hubble scale?

- Experiments only give upper bounds on graviton mass

de Rham, Deskins, Tolley & **SYZ**,
Graviton mass bounds, Rev. Mod. Phys., 2016

Linear spin-2 theory

Fierz-Pauli theory

Fierz & Pauli, 1930s

$$\mathcal{L} = -\frac{M_P^2}{4} h^{\mu\nu} \mathcal{E}_{\mu\nu}^{\rho\sigma} h_{\rho\sigma} - \frac{m^2 M_P^2}{8} (h_\nu^\mu h_\mu^\nu - h^2) + \frac{1}{2} h_{\mu\nu} T^{\mu\nu}$$

linearized GR **mass term** **matter coupling**

The unique, linear, Lorentz invariant, ghost-free spin-2 theory

vDVZ discontinuity

van Dam, Veltman & Zakharov, 1970s

scattering amplitude between $T^{\mu\nu}_{(1)}$ and $T^{\mu\nu}_{(2)}$

$$\mathcal{A} \propto \frac{M_P^{-2}}{(k^2 - m^2 + i\epsilon)} \left(2T^{\mu\nu}_{(1)} T_{(2)\mu\nu} - \frac{2}{3} T_{(1)} T_{(2)} \right)$$

$m \to 0$ **GR value = 1**

Nonlinear spin-2 theory

Generic massive gravity $S = \frac{m_{\rm Pl}^2}{2}\int d^4x\sqrt{-g}\left[R - \frac{m^2}{4}V(g,h)\right]$

$$V(g,h) = V_2(g,h) + V_3(g,h) + V_4(g,h) + \cdots \qquad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$V_2(g,h) = +b_1\langle h^2\rangle + b_2\langle h\rangle^2$$
$$V_3(g,h) = +c_1\langle h^3\rangle + c_2\langle h\rangle^2\langle h\rangle + c_3\langle h\rangle^3$$
$$V_4(g,h) = +d_1\langle h^4\rangle + d_2\langle h^3\rangle\langle h\rangle + d_3\langle h^2\rangle^2 + d_4\langle h^2\rangle\langle h\rangle^2 + d_5\langle h\rangle^4$$

Fierz-Pauli tuning
$b_1 = -b_2 = 1$

Boulware-Deser ghost

There is a nonlinear ghost mode!

Boulware & Deser, 1970s

Stueckelberg formulation

Arkani-Hamed, Georgi & Schwartz, 2002

To restore diffeomorphism invariance

$$\eta_{\rho\nu} \to \partial_\rho \phi^a \partial_\nu \phi^a \eta_{ab} \qquad \phi^a : \text{Stueckelberg fields}$$

Helicity decomposition

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad \phi^a = x^a + A^a + \partial^a \pi$$

helicity-2 **helicity-1** **helicity-0**

vDVZ discontinuity

$$\mathcal{L} \supset -\frac{M_P^2}{4} h^{\mu\nu} \mathcal{E}^{\rho\sigma}_{\mu\nu} h_{\rho\sigma} + m^2 M_P^2 h^{\mu\nu} (\Box \pi \eta_{\mu\nu} - \partial_\mu \partial_\nu \pi) + \frac{1}{2} h_{\mu\nu} T^{\mu\nu}$$

$$\xrightarrow{h_{\mu\nu} = \tilde{h}_{\mu\nu} + 2m^2 \pi \eta_{\mu\nu}} \mathcal{L} \supset -\frac{M_P^2}{4} \tilde{h}^{\mu\nu} \mathcal{E}^{\rho\sigma}_{\mu\nu} \tilde{h}_{\rho\sigma} - 3m^4 M_P^2 \partial_\mu \pi \partial^\mu \pi + \frac{1}{2} \tilde{h}_{\mu\nu} T^{\mu\nu} + m^2 \pi T$$

Cutoff of massive gravity

BD ghost

$$\mathscr{L} \supset \frac{\partial^q \left(\partial^2 \hat{\pi}\right)^p}{\Lambda_5^{3p+q-4}} + \ldots$$

Generic massive gravity cutoff: $\Lambda_5 = \left(M_P m^4\right)^{1/5} \sim (10^8 \mathrm{km})^{-1}$

Raising the cutoff

de Rham & Gabadzdze, 2010

make use of the Galileon interactions

$$\mathcal{L} \supset -(\partial\hat{\pi})^2 + \sum_{n=1}^{3} \frac{1}{\Lambda_3^{3n}} \partial^\mu \hat{\pi} \partial_\mu \hat{\pi} \partial^{[\mu_1} \partial_{\mu_1} \hat{\pi} \cdots \partial^{\mu_n} \partial_{\mu_n]} \hat{\pi} + \frac{1}{M_P} \left(\hat{\pi} T + \frac{1}{\Lambda_3^3} \partial \hat{\pi} \partial \hat{\pi} T^{\mu\nu} \right)$$

Special massive gravity cutoff: $\Lambda_3 = \left(M_P m^2\right)^{1/2}$

dRGT Massive Gravity

de Rham, Gabadzdze & Tolley, 2010

$$\mathcal{L} = M_P^2\sqrt{-g}\left(\frac{R}{2} + m^2\left(\mathcal{K}^\mu_{[\mu}\mathcal{K}^\nu_{\nu]} + \alpha_3\mathcal{K}^\mu_{[\mu}\mathcal{K}^\nu_\nu\mathcal{K}^\rho_{\rho]} + \alpha_4\mathcal{K}^\mu_{[\mu}\mathcal{K}^\nu_\nu\mathcal{K}^\rho_\rho\mathcal{K}^\sigma_{\sigma]}\right) + \mathcal{L}_m\right)$$

$$\mathcal{K}^\mu_\nu = \delta^\mu_\nu - \mathcal{X}^\mu_\nu \qquad \mathcal{X} = \sqrt{g^{-1}\eta},\ g^{-1} = (g^{\mu\nu}),\ \eta = (\eta_{\mu\nu})$$

The **unique** graviton potential to eliminate the BD ghost!

de Rham & Gabadzdze, 2010
de Rham, Gabadzdze & Tolley, 2010
Hassan & Rosen, 2011

massive spin-2 EFT with cutoff $\Lambda_3 = \left(M_P m^2\right)^{1/2}$

dRGT in Vierbein form

Hinterbichler & Rosen, 2012

$$S_{\mathrm{dRGT}} = \frac{M_P^2}{4}\int \epsilon_{abcd} R^{ab} \wedge e^a \wedge e^b + m^2 V(e, I)$$

$$\begin{aligned} V(e,I) = \epsilon_{abcd}\big(& c_0 e^a \wedge e^b \wedge e^c \wedge e^d + c_1 e^a \wedge e^b \wedge e^c \wedge I^d \\ &+ c_2 e^a \wedge e^b \wedge I^c \wedge I^d + c_3 e^a \wedge I^b \wedge I^c \wedge I^d \\ &+ c_4 I^a \wedge I^b \wedge I^c \wedge I^d\big) \end{aligned}$$

Generalized bounds on generic massive gravity

Generic massive gravity

relevant parameters to leading order $\{c_1, d_1, \Delta c, \Delta d\}$

dRGT massive gravity: $\Delta c = \Delta d = 0$

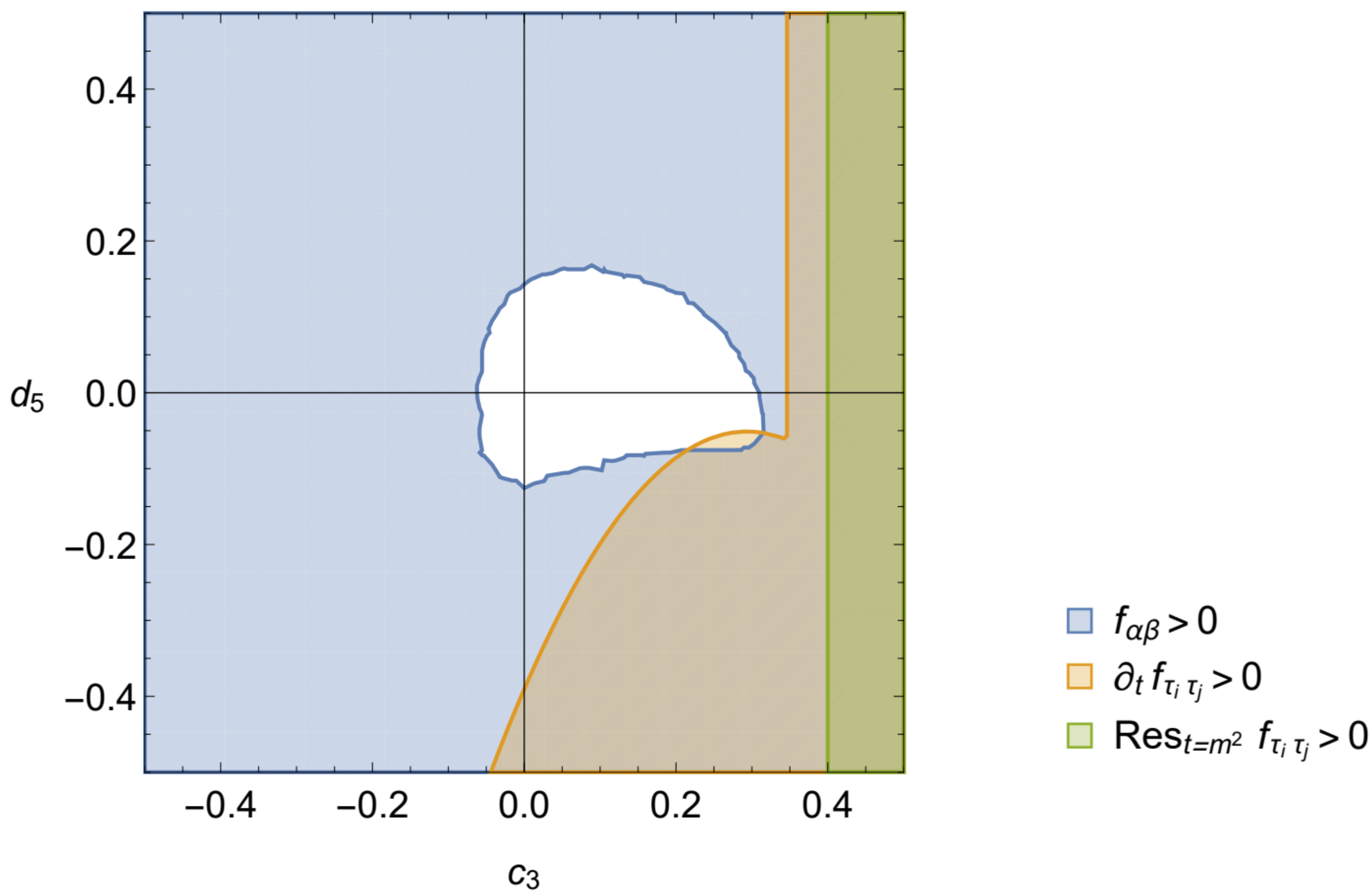
generalized bounds → $\Delta c = \Delta d = 0$

↓

dRGT massive gravity

de Rham, Melville, Tolley & **SYZ**, arXiv:1804.10624

Positivity bounds on dRGT



Cheung & Remmen, 1601.04068
de Rham, Melville, Tolley & **SYZ,** 1804.10624

Summary

Not all low energy EFTs have a UV completion!

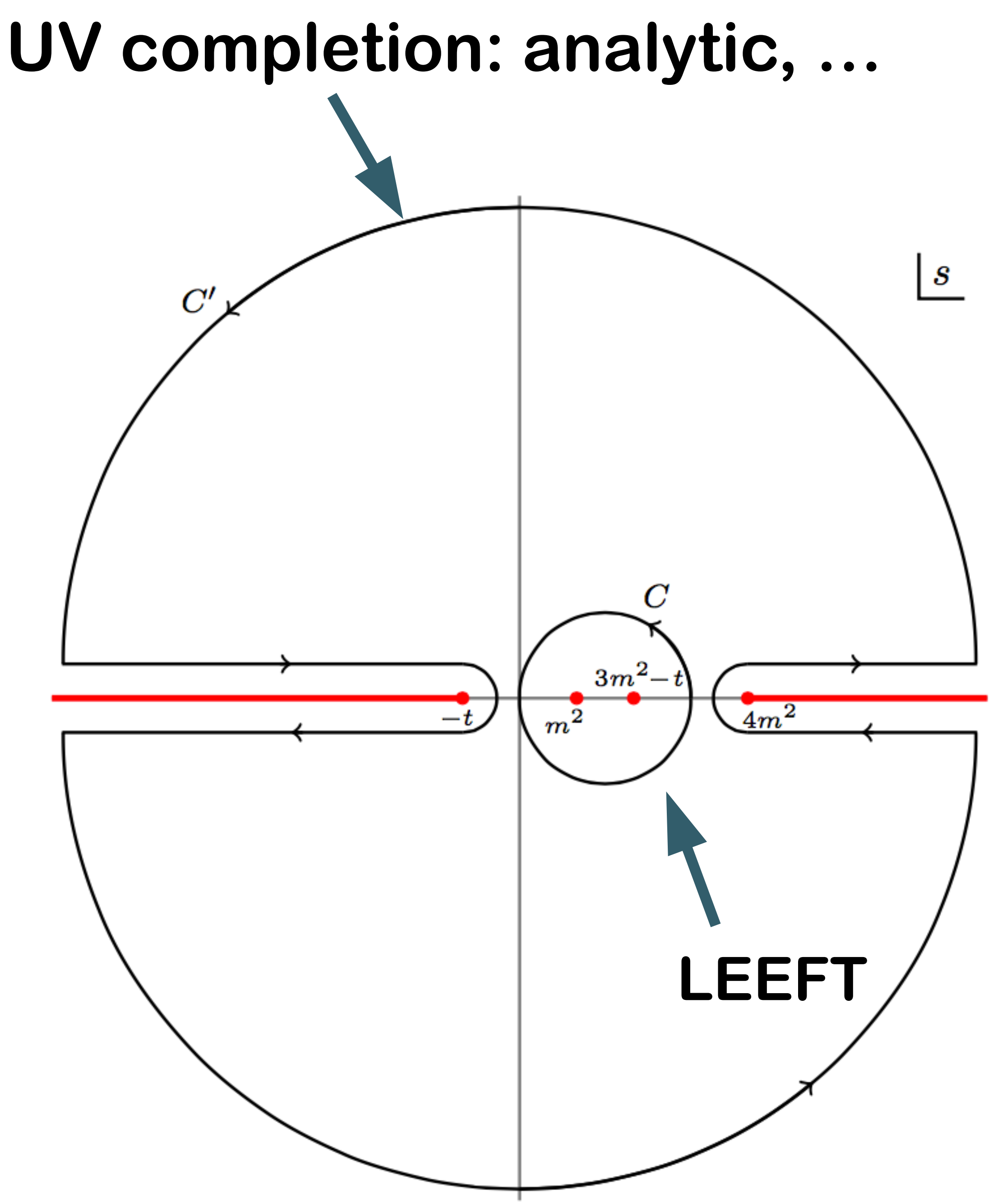
Positivity bounds: constraints on Wilson coefficients

Generalized positivity bounds can often improve the bounds!

Massive gravity may be UV completed in the standard way.

Generalized bounds → dRGT model

Low energy? Think positive! (pun intended)



Input: Amplitudes in LEEFT

↓

Positivity inequalities

↓

Output: Bounds on operators' coeffs

Thank you!

Martin’s assumptions

The essential unitarity ingredient of the whole proof is the following: let $A_s(s, \cos\theta)$ be the absorptive part of the amplitude in the s-channel. Then

$$(3) \quad \begin{cases} \left(\dfrac{d}{d\cos\theta}\right)^n A_s(s,\cos\theta)\Big|_{\cos\theta=1} \geqslant 0 \,, \\ \left|\left(\dfrac{d}{d\cos\theta}\right)^n A_s(s,\cos\theta)\right| \leqslant \left(\dfrac{d}{d\cos\theta}\right)^n A_s(s,\cos\theta)\Big|_{\cos\theta=1} \,, \end{cases} \qquad -1 \leqslant \cos\theta \leqslant 1 \,.$$

The analyticity ingredients are the following:

1) existence of dispersion relations for $-t_M < t \leqslant 0$;

2) for any fixed physical energy s the amplitude and the absorptive part are analytic in the Lehmann ellipse ([4]);

3) from the results of Bros, Epstein and Glaser and Lehmann ([5,6]) we know that in the neighbourhood of any point s_0, t_0, $-t_M < t_0 \leqslant 0$, s_0 outside the cuts, there is analyticity in both s and t in

$$|s - s_0| < \eta(s_0, t_0) \,, \qquad |t - t_0| < \eta(s_0, t_0) \,.$$

A priori the size of this neighbourhood can vary with s_0, t_0.

Martin, 1965; Extension of the Axiomatic Analyticity Domain of Scattering Amplitudes by Unitarity

Froissart bound (1)

Froissart, 1961

Ingredients:

1) $$A(s,z) = 16\pi\sqrt{\frac{s}{s-4m^2}}\sum_{l=0}^{\infty}(2l+1)P_l(z)a_l(s)$$ $z = \cos\theta = 1 + \frac{2t}{s-4m^2}$

expansion converges within $|t| < 4m^2,\ \ z - 1 < \frac{8m^2}{s-4m^2}$

2) Partial wave unitary bounds $0 \leq |a_l(s)|^2 \leq \mathrm{Im}\, a_l(s) \leq 1$

3) Polynomial boundedness $A(s,z) \leq C' s^N$

Locality of real space correlation functions

Froissart bound (2)

Large l limit of P_l

$$P_l(z) \sim \frac{C}{\sqrt{l}} e^{2l\sqrt{(z-1)/2}} \sim \frac{C}{\sqrt{l}} e^{2l\sqrt{4m^2/s}}$$

Bound on this large l mode

$\mathrm{Im}\, a_l(s) > 0$

$$(2l+1)\frac{C}{\sqrt{l}} e^{2l\sqrt{4m^2/s}} \mathrm{Im} a_l(s) < A(s,z) < C' s^N$$

$$\Rightarrow \quad \mathrm{Im} a_l(s) < \frac{1}{\sqrt{l}} C'' e^{N \ln s - 2l\sqrt{4m^2/s}}$$

Largest l for which $\mathrm{Im} a_l$ is significant

$$l_{\max} \sim \frac{1}{2\sqrt{4m^2}} N \sqrt{s} \ln s$$

Froissart bound (3)

$$A(s,z) = 16\pi\sqrt{\frac{s}{s-4m^2}}\sum_{l=0}^{\infty}(2l+1)P_l(z)a_l(s)$$

In the forward limit $t = \theta = 0$ or $z = 1$

$$P_l(z=1) = 1$$

Choose $\mathrm{Im}\, a_l \sim 1$ for $l < l_{\mathrm{max}}$

$$\mathrm{Im}A(s,0) < \sum_{0}^{l_{\mathrm{max}}}(2l+1) \sim l_{\mathrm{max}}^2 \sim \frac{N^2}{16m^2}s(\ln(s))^2$$

Cross section $\sigma_{\mathrm{tot}} = \dfrac{\mathrm{Im}A(s,0)}{s} < \mathrm{constant}\,(\ln(s))^2$

Problems of Nonlinear FP

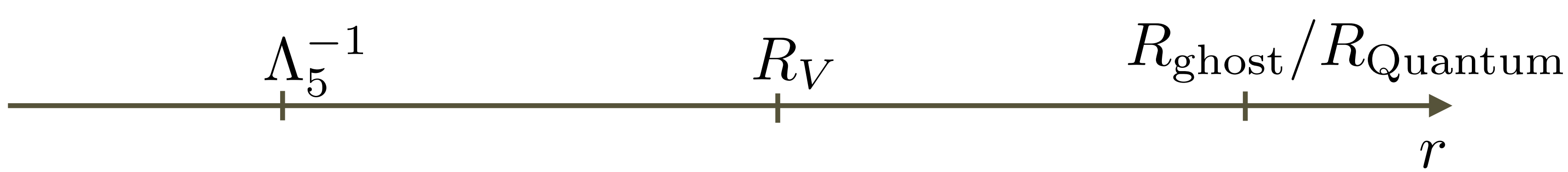
- **Boulware-Deser ghost** — Boulware & Deser, 1970s

6th polarization
Hamiltonian unbounded from below

- **Low strong coupling scale** — Arkani-Hamed, Georgi & Schwarz, 2003

$$\Lambda_5 = (M_P m^4)^{\frac{1}{5}} \sim (10^{16}\mathrm{cm})^{-1} \qquad \Lambda_n \equiv (M_P m^{n-1})^{\frac{1}{n}}$$

$$R_{\mathrm{V}} = \left(\frac{M}{M_P}\right)^{\frac{1}{5}} \Lambda_5^{-1} \qquad R_{\mathrm{ghost}} \sim R_{\mathrm{Quantum}} = \left(\frac{M}{M_P}\right)^{\frac{1}{3}} \Lambda_5^{-1}$$



Nonlinear Fierz-Pauli theory is inconsistent!

Strong coupling in dRGT

- **Strong coupling scale and Vainshtein radius**

$$\Lambda_3 = (M_P m^2)^{\frac{1}{3}} \sim (10\mathrm{km})^{-1} \qquad R_V = \left(\frac{M}{M_P}\right)^{\frac{1}{3}} \Lambda_3^{-1}$$

GR analogy:

- **“Vainshtein on top of Vainshtein”**

$$\Lambda_{true} \gg \Lambda_3$$