BCJ duality as worldsheet Yangian

Chih-Hao Fu

Shaanxi Normal University

Based on work in collaboration with Pierre Vanhove and Yihong Wang
arXiv:1806.09584 JHEP09(2018)141
& work in progress

BCJ duality as worldsheet Yangian

special thanks to Kirill Krasnov

Based on work in collaboration with Pierre Vanhove and Yihong Wang
arXiv:1806.09584 JHEP09(2018)141
& work in progress

Vocabulary translated

	Phys. arXiv:[hep-th]	**Maths.** arXiv:[math.QA]
$\int_{\dots t_1<t_2<t_3\dots} \prod_i dt_i S_1(t_1)S_2(t_2)S_3(t_3)\dots V(x)$	amplitude	(fundamental) reprsn. (highest weight module)
$\int_{C_1,C_2,\dots} \prod_i dt_i S_1(t_1)S_2(t_2)S_3(t_3)\dots V(x)$ $= \dots [\int_{\text{line}} S_2 [\int_{\text{line}} S_1, V(x)]_{\alpha'}]_{\alpha'}$	numerator	adj. reprsn.
$R_{n-1,n}\dots(R_{2n}\dots R_{23}R_{23})(R_{1n}\dots R_{13}R_{12})$	momentum kernel	quantum symmetriser
	kinematic algebra	screening vertex algebra/ Nichols algebra [Andruskiewitsch, Schneider 02]
	double copy ↑ (single copy)	Yetter-Drinfeld bimodule [Semikhatov, Tipunin 11]
	$\Delta(V) = T^a \otimes V^a$	coaction (coproduct on modules)

Outline

- **Hopf algebra, quantum group & Yangian**
  - A very brief Introduction to quantum algebras
  - N=4 SYM, super conformal symmetry, Y(gl_N)
- **Kinematic algebra, Vertex operators**
  - Relation to quantum groups and integrability
- **Quantum algebra approach to amplitudes**
  - Algebraic calculations of YM amplitudes

Yangian (Янгиан) Symmetry

Level 0: $J_a^{(0)} = \sum_{i=1}^{n} J_{i,a}^{(0)}$

total mom/anglr mom
= sum of mom/anglr mom
of each particle

$[J_a^{(0)}, J_b^{(0)}\} = f_{ab}{}^{c} J_c^{(0)}$

superconformal symmetry
SU(2,2|4)

Vladimir Drinfeld

Level 1: $J_a^{(1)} = f_a{}^{bc} \sum_{1 \le i < j \le n} J_{i,b}^{(0)} J_{j,c}^{(0)}$

$[J_a^{(1)}, J_b^{(0)}\} = f_{ab}{}^{c} J_c^{(1)}$

Level 2: $\vdots$

[Drummond, Henn, Plefka 09]

Level 0: $J_a^{(0)} \mathcal{A}(1234) = 0$

mom/anglr mom conservation ⟷ translation/roation invariance

Level 1: $J_a^{(1)} \mathcal{A}(1234) = 0$

$\vdots$ $\vdots$

$\mathcal{N} = 4$ SYM ✓

pure YM ✗

Hopf Algebra
= Bialgebra + Antipode

$(\mathcal{H},\, \cdot,\, I,\, \Delta,\, \epsilon)$ S

$(g_1 \cdot g_2)\,|\phi\rangle = g_1 \cdot (g_2\,|\phi\rangle)$ multiplication (group representation)

$(g_1 + g_2)\,|\phi\rangle = g_1\,|\phi\rangle + g_2\,|\phi\rangle$ superposition

$[J_x, J_y] = i\,J_z$

$\vdots$

algebra

---

comultiplication $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$

bialgebra

$\Delta(g_1 \cdot g_2) = \Delta(g_1) \cdot \Delta(g_2)$

multiparticle representation

$|\phi\rangle \to |\phi_1\rangle \otimes |\phi_2\rangle$

$J \xrightarrow{\Delta} J_1 \otimes I_2 + I_1 \otimes J_2$

$e^{i\theta J} \xrightarrow{\Delta} e^{i\theta J_1} \otimes e^{i\theta J_2}$

Yangian

$$t_{i\,j}^{(r)} = \sum_{s=1}^{r} t_{i\,k}^{(s)} \otimes t_{k\,j}^{(r-s)}$$

---

antipode $S : \mathcal{H} \to \mathcal{H}$

Hopf algebra

$m(S \otimes id)\Delta(g) = \epsilon(g)\, I$

$S(g) \cdot g = I$

$S(g) = g^{-1}$ (generalised inverse)

Quantum Groups

satisfies RTT and Yang-Baxter eqn

$$R\,T_1T_2 = T_2T_1\,R$$

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

$$R(u) = I + u^{-1}P_{12} = I + u^{-1}\sum_{i,j=1}^{N} e_{i\,j}\otimes e_{j\,i}$$

(Yang) universal R-matrix

$$T(u) = \begin{bmatrix} t_{11}(u) & t_{12}(u) & \dots \\ t_{21}(u) & \ddots & \\ \vdots & & t_{NN}(u) \end{bmatrix}$$

$$t_{i\,j}(u) = t^{(0)}_{i\,j} + u^{-1}t^{(1)}_{i\,j} + u^{-2}t^{(2)}_{i\,j} + \dots$$

$$\longrightarrow \quad [t_{i\,j}(u),\, t_{k\,\ell}(u)] \sim 0 + \mathcal{O}(u^{-1})$$

(FRT construction)

[Faddeev, Reshetikhin, Takhtadzhyan 90]

$$e_{i\,j}\cdot v_j = v_i$$

$$e_{i\,j}\cdot v_k = 0 \quad (j\neq k)$$

$$e_{i\,j}\cdot e_{j\,k} = e_{i\,k}$$

$$T_1T_2 = e_{i\,j}\otimes e_{k\,\ell}\otimes t_{i\,j}t_{k\,\ell}$$

$$T_2T_1 = e_{i\,j}\otimes e_{k\,\ell}\otimes t_{k\,\ell}t_{i\,j}$$

Quantum Groups

satisfies RTT and Yang-Baxter eqn

$$R\,T_1T_2 = T_2T_1\,R$$

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

Ex1. quantum matrix group $SL_q(2)$

$$R^{ab}{}_{cd} = \begin{array}{c} \\ 11 \\ 12 \\ 21 \\ 22 \end{array} \begin{array}{c} \begin{array}{cccc} 11 & 12 & 21 & 22 \end{array} \\ \begin{bmatrix} q & & & \\ & 1 & & \\ & q-q^{-1} & 1 & \\ & & & q \end{bmatrix} \end{array}$$

$$T(u) = T = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

$$R^{ab}{}_{cd} \xrightarrow{q\to 1} \delta^a{}_c \delta^b{}_d$$

$\longrightarrow$

$$\alpha\beta = q\,\beta\alpha \qquad \beta\gamma = \gamma\beta$$

$$\alpha\gamma = q\,\gamma\alpha \qquad \alpha\delta - \delta\alpha = (q - q^{-1})\beta\gamma \qquad q \in \mathbb{C}$$

$$\beta\delta = q\,\delta\beta$$

$$\gamma\delta = q\,\delta\gamma$$

(FRT construction)

[Faddeev, Reshetikhin, Takhtadzhyan 90]

Quantum Groups

satisfies RTT and Yang-Baxter eqn

$$R\,T_1T_2 = T_2T_1\,R$$

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

Ex2. quantum enveloping algebra $U_q(sl(2))$

T q-Lie group

↕

L q-Lie algebra

$$R^{ab}{}_{cd} = \begin{array}{c} \\ 11 \\ 12 \\ 21 \\ 22 \end{array} \begin{array}{c} \begin{array}{cccc} 11 & 12 & 21 & 22 \end{array} \\ \left[\begin{array}{cccc} q & & & \\ & 1 & & \\ & q-q^{-1} & 1 & \\ & & & q \end{array}\right] \end{array}$$

$$L^+ = \begin{bmatrix} K^{-1} & q^{\frac{-1}{2}}(q-q^{-1})X^+ \\ & K \end{bmatrix}$$

$$L^- = \begin{bmatrix} K & \\ -q^{\frac{1}{2}}(q-q^{-1})X^- & K^{-1} \end{bmatrix}$$

satisfies RLL and Yang-Baxter eqn (same R)

$$RL_1^{\pm}L_2^{\pm} = L_2^{\pm}L_1^{\pm}R$$

$$RL_1^{+}L_2^{-} = L_2^{-}L_1^{+}R$$

(FRT construction)

[Faddeev, Reshetikhin, Takhtadzhyan 90]

Quantum Groups

satisfies RTT and Yang-Baxter eqn

$$R\,T_1T_2 = T_2T_1\,R$$

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

Ex2. quantum enveloping algebra $U_q(sl(2))$

T q-Lie group

L q-Lie algebra

$$[H,\,X^{\pm}] = \pm 2X^{\pm}$$

$$[X^+,\,X^-] = \frac{K - K^{-1}}{q - q^{-1}}$$

$$K = q^H$$

$q \to 1$

"classical" limit

$$[H,\,X^{\pm}] = \pm 2X^{\pm}$$

$$[X^+,\,X^-] = H$$

$sl(2)$ Lie algebra

$$L^+ = \begin{bmatrix} K^{-1} & q^{\frac{-1}{2}}(q-q^{-1})X^+ \\ & K \end{bmatrix}$$

$$L^- = \begin{bmatrix} K & \\ -q^{\frac{1}{2}}(q-q^{-1})X^- & K^{-1} \end{bmatrix}$$

(FRT construction)

[Faddeev, Reshetikhin, Takhtadzhyan 90]

Quantum Groups

satisfies RTT and Yang-Baxter eqn

$$R\,T_1T_2 = T_2T_1\,R$$

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

Ex2. quantum enveloping algebra $U_q(sl(2))$

T q-Lie group

L q-Lie algebra

$$[H,\, X^{\pm}] = \pm 2X^{\pm}$$

$$[X^{+},\, X^{-}] = \frac{K - K^{-1}}{q - q^{-1}}$$

$$K = q^{H}$$

$$L^{+} = \begin{bmatrix} K^{-1} & q^{\frac{-1}{2}}(q - q^{-1})X^{+} \\ & K \end{bmatrix}$$

$$L^{-} = \begin{bmatrix} K & \\ -q^{\frac{1}{2}}(q - q^{-1})X^{-} & K^{-1} \end{bmatrix}$$

$$\Delta(X^{\pm}) = X^{\pm} \otimes K^{\frac{1}{2}} + K^{-\frac{1}{2}} \otimes X^{\pm}$$

$$\Delta(K) = K \otimes K$$

$$S(X^{\pm}) = -q^{\pm 1}X^{\pm}$$

$$S(K) = K^{-1}$$

Hopf algebra

(FRT construction)

[Faddeev, Reshetikhin, Takhtadzhyan 90]

N=4 SYM & Yangian

$$L_{ab}(u) = \delta_{ab} + \frac{1}{u}\, x_a p_b$$

$$R_{12}(u) = \int_0^\infty \frac{dz}{z^{1-u}} e^{-z\,(p_1 \cdot x_2)}$$

satisfies RLL and Yang-Baxter

$$R\, L_1 L_2 = L_2 L_1 R$$

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

[Chicherin, Derkachov, Kirschner 14]

Yangian for gl(N), $Y(gl_N)$

$$x \to (\lambda_\alpha, \partial_{\bar\lambda_{\dot\alpha}}, \partial_{\eta^A})$$

$$p \to (\partial_{\lambda_\alpha}, -\bar\lambda_{\dot\alpha}, -\eta^A)$$

$$L(u) = \begin{bmatrix} 1 + u^{-1}\lambda \otimes \partial_\lambda & -u^{-1}\lambda \otimes \bar\lambda & -u^{-1}\lambda \otimes \eta \\ u^{-1}\partial_{\bar\lambda} \otimes \partial_\lambda & 1 - u^{-1}\partial_{\bar\lambda} \otimes \bar\lambda & -u^{-1}\partial_{\bar\lambda} \otimes \eta \\ u^{-1}\partial_\eta \otimes \partial_\lambda & -u^{-1}\partial_\eta \otimes \bar\lambda & 1 - u^{-1}\partial_\eta \otimes \eta \end{bmatrix}$$

$$M(u) = L_1(u)\, L_2(u)\, \ldots L_n(u)$$

Monodromy matrix

$$M_{ab}(u) = \sum_{m=0}^{n} u^{-m}\, J_{ab}^{(m)}$$

$$J^{(0)} = I$$

$$J_{ab}^{(1)} = \sum_{1 \le i \le n} x_{a,i} p_{b,i}$$

$$J_{ab}^{(2)} = \sum_{1 \le i < j \le n} x_{a,i} p_{i,c} x_{c,j} p_{b,j}$$

$$M_{ab}(u)\, A_n = 0$$

N=4 SYM & Yangian

$$L_{ab}(u) = \delta_{ab} + \frac{1}{u} x_a p_b$$

satisfies RLL and Yang-Baxter

$$R\, L_1 L_2 = L_2 L_1 R$$
$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

$$M(u) = L_1(u)\, L_2(u)\, \ldots L_n(u)$$

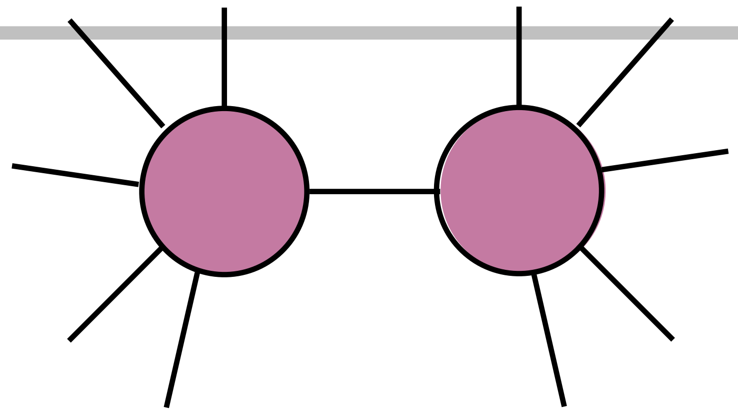
Monodromy matrix

$$M_{ab}(u)\, A_n = 0$$

BCFW recursion

$$A_n(u) = \int d^4\eta\, e^{-z\,\lambda_n \partial_{\lambda_1}} A_L(\eta_1, \lambda_1, \bar\lambda_1; \eta_0, -P_0) \frac{1}{\Pi_i^2(u)} e^{z\,\bar\lambda_1 \partial_{\bar\lambda_n} + z\,\eta_1 \partial_{\eta_n}} A_R(\eta_n, \lambda_n, \bar\lambda_n; \eta_0, P_0)$$
$$= R_{1n}(u) \int d^4\eta_0\, d^4 P_0\, \delta(P_0^2)\, A_L(\eta_1, \lambda_1, \bar\lambda_1; \eta_0, -P_0)\, A_R(\eta_n, \lambda_n, \bar\lambda_n; \eta_0, P_0)$$

[Britto, Cachazo, Feng, Witten 05]



$$L_{n-1} \ldots L_2 L_1 L_n R_{1n}\, A_L A_R$$
$$= R_{1n} L_{n-1} \ldots L_{i+1}\, (L_i \ldots L_2 L_1 A_L)\, L_n A_R$$

[Chicherin, Derkachov, Kirschner 14]

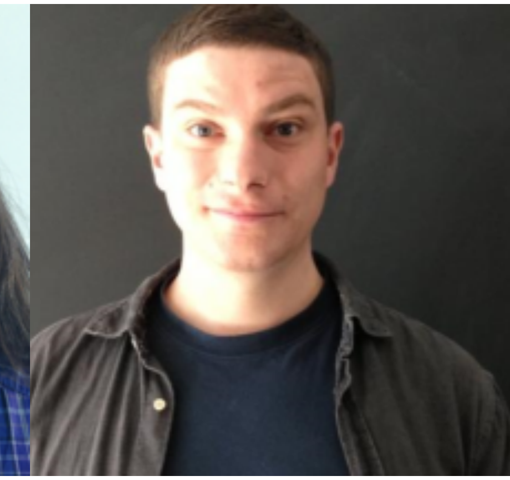
BCJ duality

Bern, Carrasco, Johansson:
"What is the simplest QFT?"

Ans: Klein-Gordon (trivial)

Z. Bern

J. J. M. Carrasco



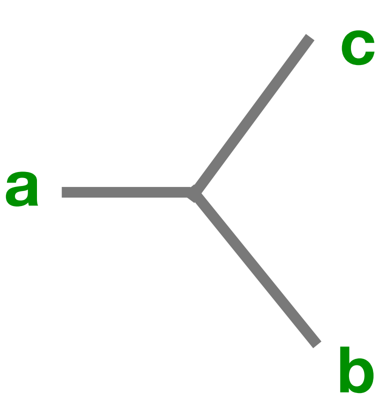
H. Johansson

"What is the next simplest QFT?"

Ans: ϕ^3 theory

$$L \sim \phi^a \partial^2 \phi^a + f^{abc} \phi^a \phi^b \phi^c$$

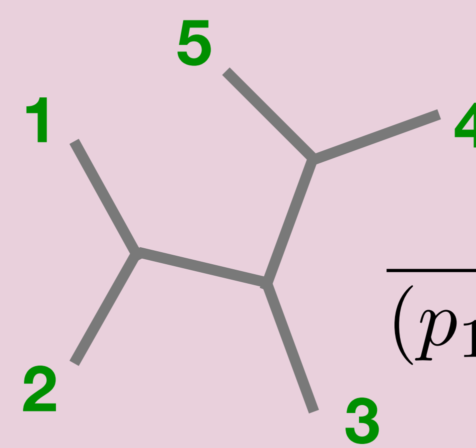
$$\frac{\delta^{ab}}{p^2}$$



$$f^{abc}$$

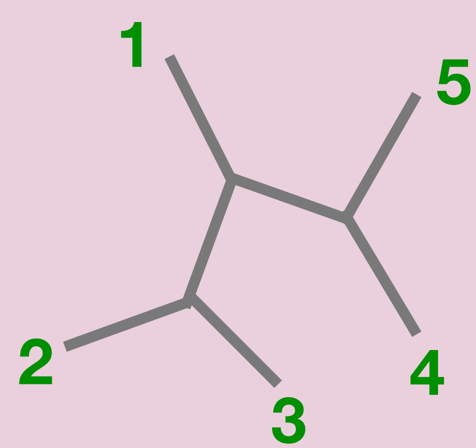
Ex. 5 points

$$\mathcal{A}(12345) =$$



$$\frac{f^{12\sigma} f^{\sigma 3\rho} f^{\rho 45}}{(p_1+p_2)^2 (p_4+p_5)^2}$$

$+$



$+ \ldots$

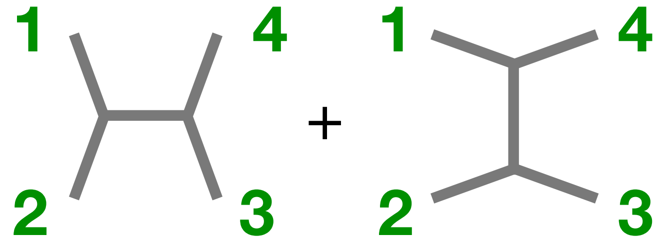
[Bern, Carrasco, Johansson 08]

Key Point: **MANY Theories are Double Copies**

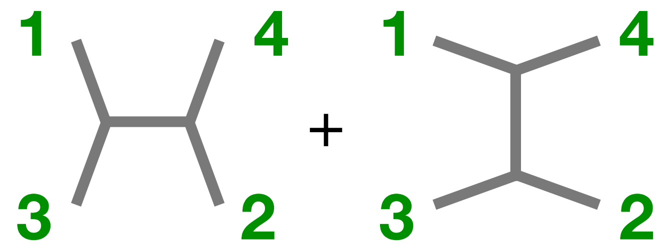
Bi-Adjoint Scalar:	color $\otimes$ color
Bern, de Freitas, Wong ('99); Bern, Dennen, Huang; Du, Feng, Fu; Bjerrum-Bohr, Damgaard, Monteiro, O'Connell	
(S)YM (…(S)QCD…):	color $\otimes$ spin-1
BCJ ('08) Bjerrum-Bohr, Damgaard, Vanhove; Steiberger; Feng et al; Mafra, Schlotterer, ('08-'11); Johansson, Ochirov	
(S)Gr (…(S)Einstein-YM…):	spin-1 $\otimes$ spin-1
KLT('86); BCJ ('08); Chiodaroli, Gunaydin, Johansson, Roiban; Johansson, Ochirov; Johansson, Kälin, Mogull	
NLSM / Chiral Lagrangian:	"color" $\otimes$ even-spin-0
Chen, Du '13 Cachazo, He, Yuan '14 Cheung, Shen '16	
(S)Born-Infeld:	spin-1 $\otimes$ even-spin-0
Cachazo, He, Yuan '14	
Special Galileon:	even-spin-0 $\otimes$ even-spin-0
Cachazo, He, Yuan '14 Cheung, Shen '16	
Open String:	α' $\otimes$ spin-1
Broedel, Schlotterer, Stieberger	
Closed String:	spin-1 $\otimes$ α' corrected spin-1
Broedel, Schlotterer, Stieberger;	
Z-theory:	α' $\otimes$ "color"
Broedel, Schlotterer, Stieberger; JJMC, Mafra, Schlotterer	

Ex: 4 pts

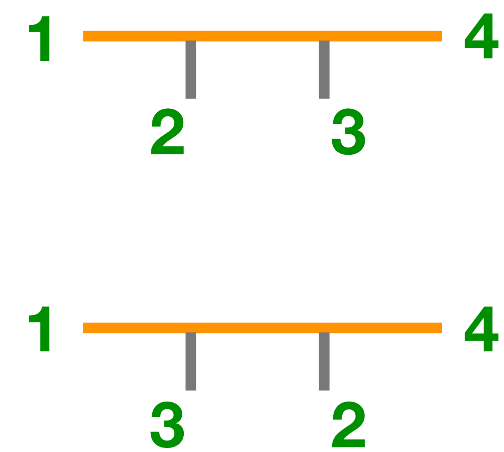
$A(1234)$ =



$A(1324)$ =



$$\begin{bmatrix} A(1234) \\ A(1324) \end{bmatrix} = \begin{bmatrix} \frac{1}{s_{12}} + \frac{1}{s_{23}} & \frac{-1}{s_{23}} \\ \frac{-1}{s_{23}} & \frac{1}{s_{13}} + \frac{1}{s_{23}} \end{bmatrix} \begin{bmatrix} n_{1234} \\ n_{1324} \end{bmatrix}$$



Off-shell

nonsingular

solve by taking inverse

$$n_{1\alpha n} = \frac{1}{k_n^2} S[\alpha^T|\beta]\, A(1\beta n)$$

Momentum kernel

On-shell

$$\begin{bmatrix} A's \\ \; \end{bmatrix} = \begin{bmatrix} 0 & \\ & \lambda \end{bmatrix} \begin{bmatrix} n's \\ \; \end{bmatrix}$$

$$s_{21} A(1234) + (s_{21} + s_{23}) A(1324) = 0$$

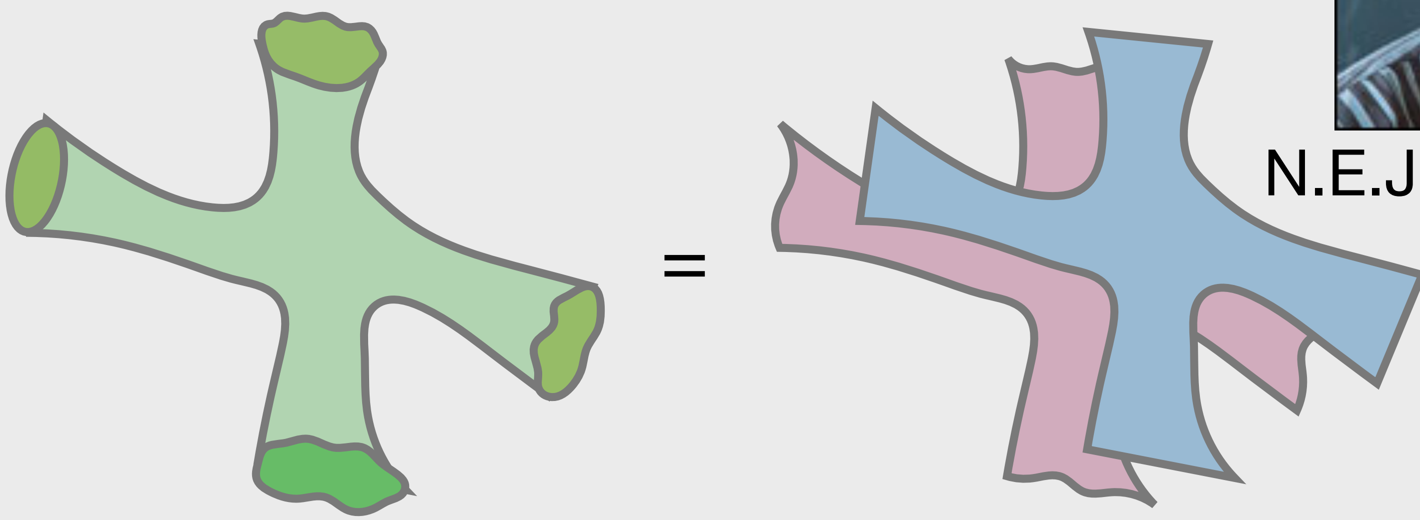
kernel → **BCJ amplitude relations**

String KLT & monodromy


N.E.J.Bjerrum-Bohr


P.H.Damgaard


P.Vanhove



close string

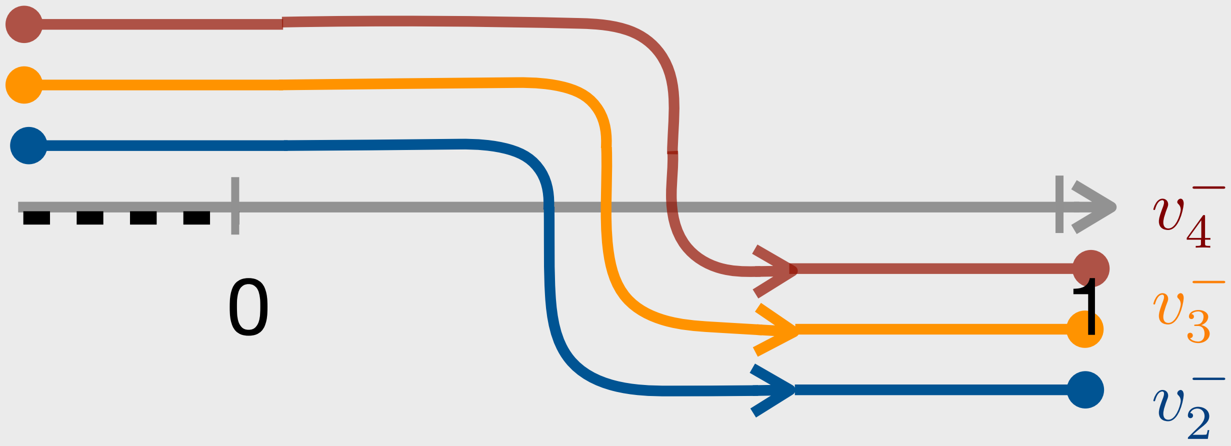
open string x open string

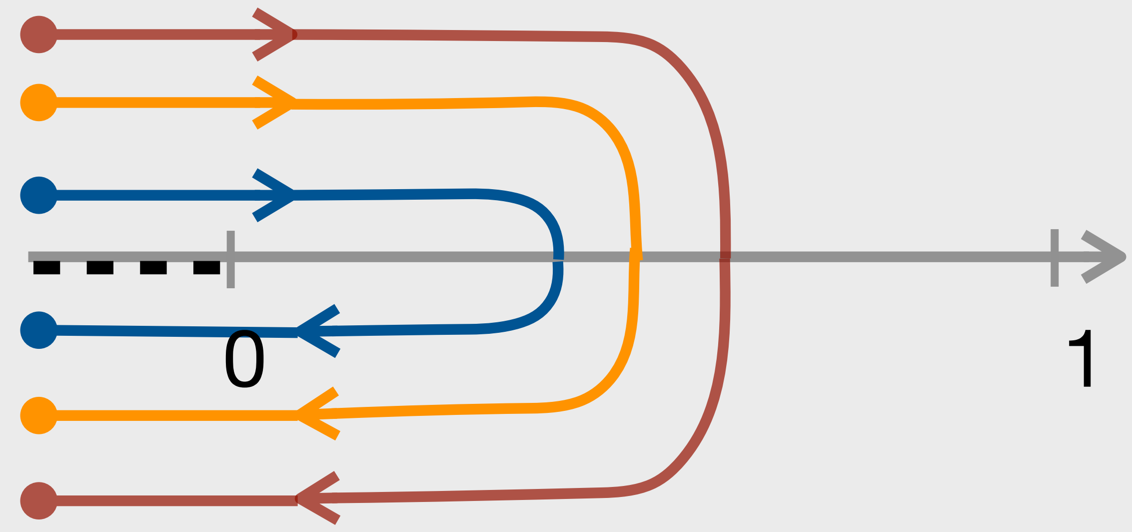
$$\mathcal{M}_n = \sum_{\sigma} \tilde{A}_n(1,\sigma(2,\ldots,n-2),n-1,n) \times \mathcal{I}(1,\sigma(2,\ldots,n-2),n-1,n)$$

$$= \sum_{\sigma,\gamma \in S_{n-3}} \tilde{A}_n(1,\sigma(2,\ldots n-2),n-1,n) \times \mathcal{S}_{\alpha'}[\sigma^T|\gamma] \times \mathcal{A}(n-1,n,\gamma(2,\ldots,n-2),1)$$

$$\mathcal{I} = \int_{-\infty}^{\infty} \prod_{i=2}^{n-2} dv_i^- (v_i^-)^{\alpha' k_i \cdot k_1} (1-v_i^-)^{\alpha' k_{n-1}\cdot k_i} \prod_{j>i} (v_j^- - v_i^-)^{\alpha' k_j \cdot k_i} f(v^-)$$

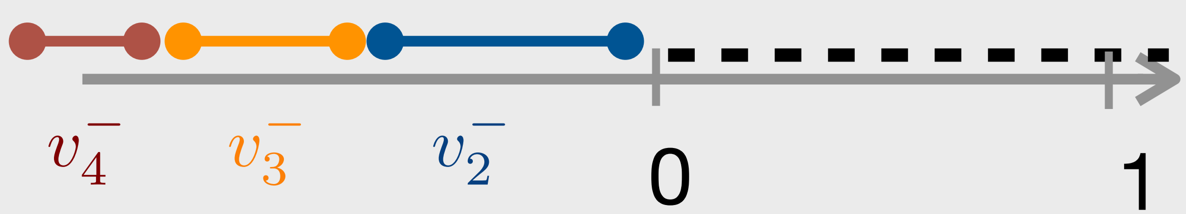
[Bjerrum-Bohr, Damgaard, Vanhove 11]





$$\int_{C_2} dv_2^- (v_2^-)^{\alpha' k_1\cdot k_2} (1-v_2^-)^{\alpha' k_{n-1}\cdot k_2} \prod_{j>2} (v_j^- - v_2^-)^{\alpha' k_j\cdot k_2} f(v^-)$$

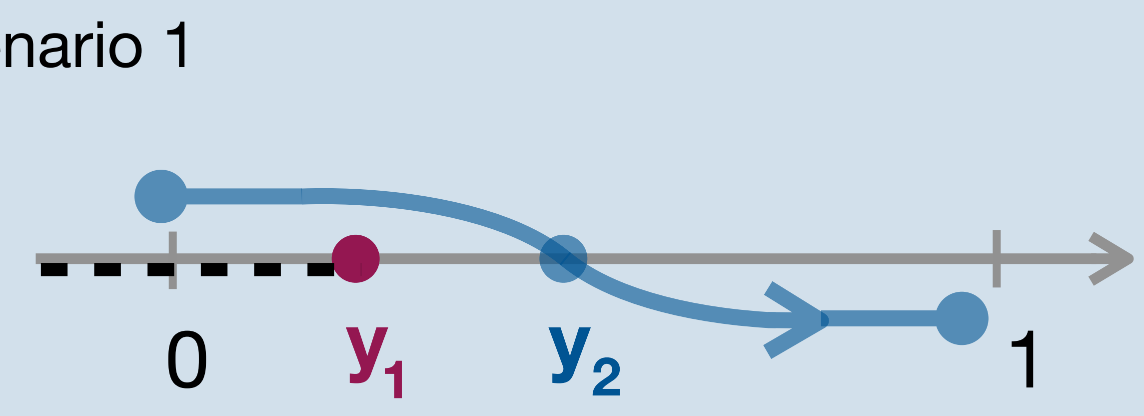
$$= 2i\sin(\pi\alpha' k_1\cdot k_2) \int_{-\infty}^{0} dv_2^- (-v_2^-)^{\alpha' k_1\cdot k_2} (1-v_2^-)^{\alpha' k_{n-1}\cdot k_2} \prod_{j>2} (v_j^- - v_2^-)^{\alpha' k_j\cdot k_2} f(v^-)$$

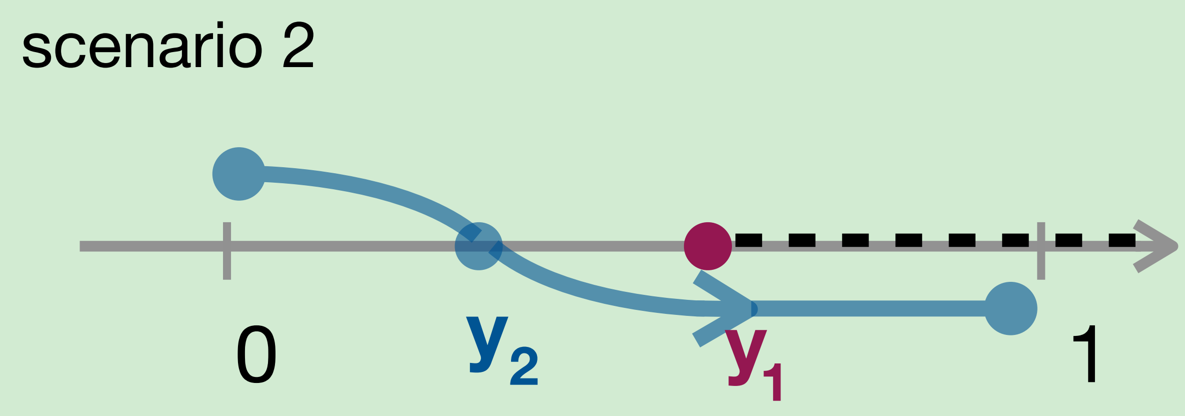


$$\int_{C_3} dv_3^- (v_3^-)^{\alpha' k_1\cdot k_3} (v_3^- - v_2^-)^{\alpha' k_3\cdot k_2} \ldots$$

$$= 2i\sin(\pi\alpha' k_1\cdot k_3) \int_{v_2^- < v_3^- < 0} dv_2^- (-v_3^-)^{\alpha' k_1\cdot k_3} (v_3^- - v_2^-)^{\alpha' k_3\cdot k_2} \ldots$$

$$+ 2i\sin(\pi\alpha' (k_1+k_2)\cdot k_3) \int_{v_3^- < v_2^-} dv_2^- (-v_3^-)^{\alpha' k_1\cdot k_3} (v_2^- - v_3^-)^{\alpha' k_3\cdot k_2} \ldots$$





$$\Big(\int_0^1 dy_2 \frac{V(y_2)}{y_2}\Big)\,\frac{V(y_1)}{y_1}$$

$$\sim e^{\alpha' k_1\cdot k_2\, ln\, y_2} e^{-\alpha' \sum_1^\infty \frac{1}{n}\left(\frac{y_1}{y_2}\right)^n} : V(y_1)V(y_2):$$

$$\sim e^{\alpha' k_1\cdot k_2\, ln(y_2 - y_1)}$$

$$\longrightarrow \quad (y_2 - y_1)^{\alpha' k_1\cdot k_2}$$

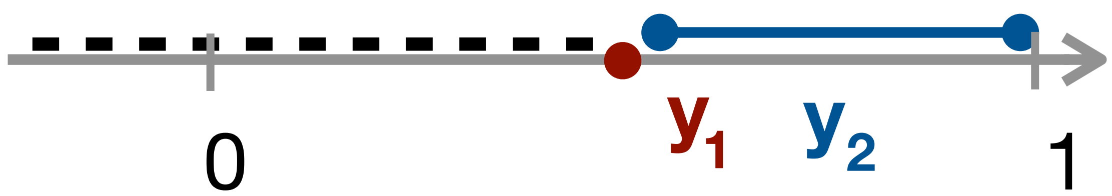
$$\frac{V(y_1)}{y_1}\,\Big(\int_0^1 dy_2 \frac{V(y_2)}{y_2}\Big)$$

$$\sim e^{\alpha' k_1\cdot k_2\, ln\, y_1} e^{-\alpha' \sum_1^\infty \frac{1}{n}\left(\frac{y_2}{y_1}\right)^n} : V(y_1)V(y_2):$$

$$\sim e^{\alpha' k_1\cdot k_2\, ln(y_1 - y_2)}$$

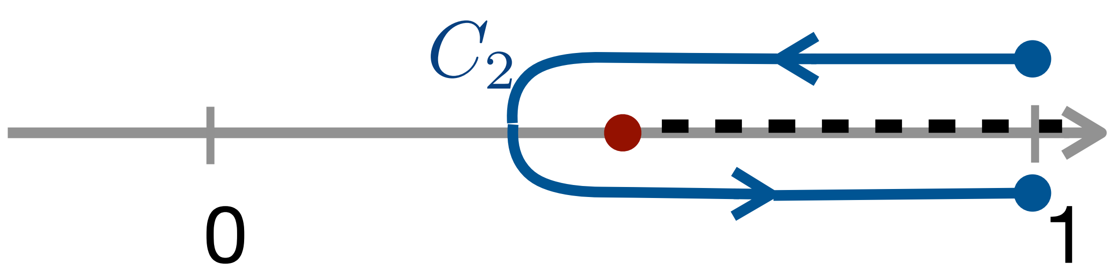
$$\longrightarrow \quad (y_1 - y_2)^{\alpha' k_1\cdot k_2}$$

[CF, Wang, Vanhove 18]

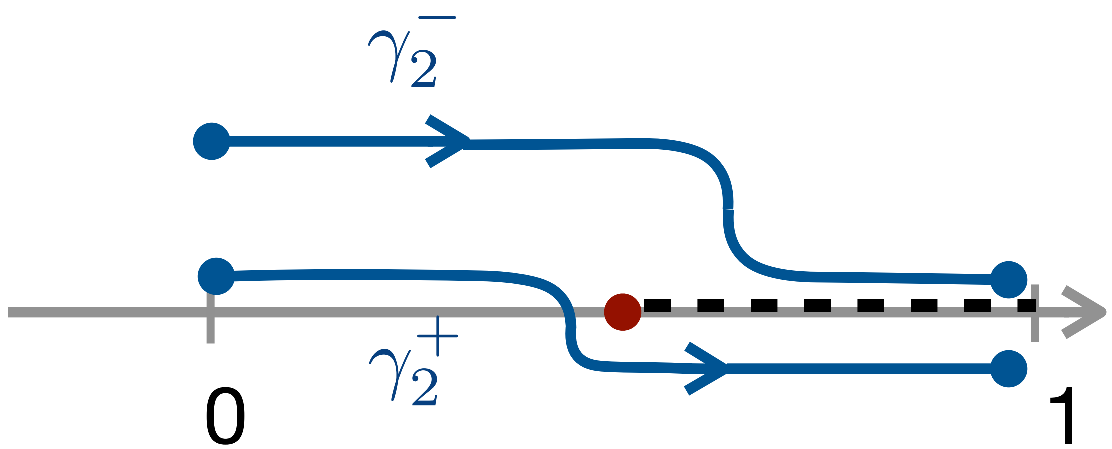


$$2i\sin(\alpha'\pi k_1\cdot k_2)\,A(123)$$

$$\sim\ 2i\sin(\alpha'\pi k_1\cdot k_2)\int_{y_1<y_2}\frac{dy_2}{y_2}(y_2-y_1)^{\alpha' k_1\cdot k_2}$$



$$=\int_{C_2}\frac{dy_2}{y_2}(y_1-y_2)^{\alpha' k_1\cdot k_2}$$



[CF, Wang, Vanhove 18]

$$=\int_{\gamma_2^+}-\int_{\gamma_2^-}$$

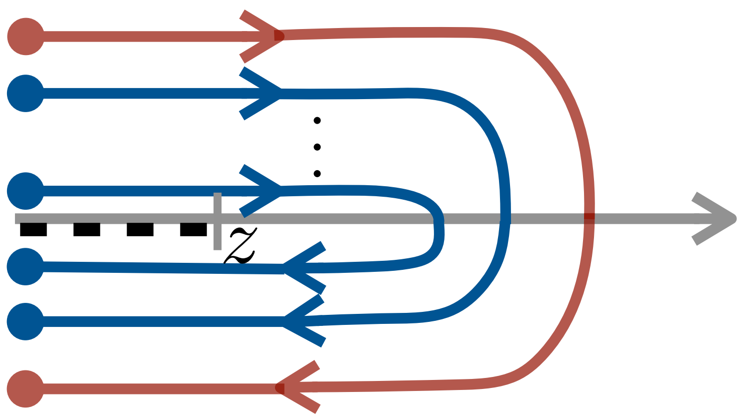
$$=\int_{\gamma_2^+}\frac{dy_2}{y_2}(y_1-y_2)^{\alpha' k_1\cdot k_2}-e^{-i\pi\alpha' k_1\cdot k_2}\int_{\gamma_2^-}\frac{dy_2}{y_2}(y_2-y_1)^{\alpha' k_1\cdot k_2}$$

$$\sim\int_0^1\frac{dy_2}{y_2}\,\frac{1}{y_1}\,V(y_1)V(y_2)-e^{-i\pi\alpha' k_1\cdot k_2}V(y_2)V(y_1)$$

$$[T_1,T_2]_{\alpha'}=T_1T_2-e^{-i\pi\alpha' k_1\cdot k_2}T_2T_1$$

Screening Vertex Operators & Nichols Algebra

Coulomb gas



$F_i \, e_{\{i_1, i_2, \ldots, i_r\}}(z)$

$$S_i(z_1) S_j(z_2) = q^{\Omega_{i\,j}} S_j(z_2) S_i(z_1)$$

$$S_i(z_1) V_a(z_2) = q^{\Omega_{i\,a}} V_a(z_2) S_i(z_1)$$

[Feigin, Fuks 82]

$$e_{\{i_1, i_2, \ldots, i_r\}}(z) = \int_{C_1} dt_1 S_{i_1}(t_1) \ldots \int_{C_r} dt_r S_{i_r}(t_r) \, V_1(z)$$
$$= F_{i_1} \ldots F_{i_r} e_1(z)$$

$$e_1(z) = V_1(z)$$
scree**ned** vertex operator

$$F_i = \int_C dt \, S_i(t)$$
screening

$$H_i = \oint \partial X_i$$
charge (momentum)

$$[H_i, \, F_j] = -\Omega_{i\,j} F_j$$

$$\Omega_{i\,j} = (\alpha_i, \, \alpha_j)$$

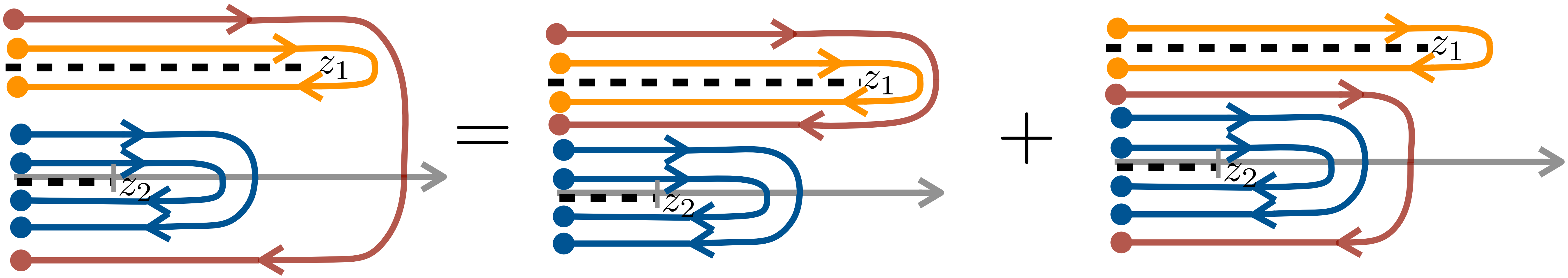
$$K_i = q^{H_i}$$

$$K_i F_j = q^{-\Omega_{i\,j}} F_j K_i$$

$$K_i K_j = K_j K_i$$

Screening Vertex Operators & Nichols Algebra

Coulomb gas



coproduct
2-particle representation

$$\Delta F_i \Big[e_{i_1 \ldots i_r}(z_1)\, e_{j_1 \ldots j_s}(z_2) \Big] = F_i \Big[e_{i_1 \ldots i_r}(z_1) \Big]\, e_{j_1 \ldots j_s}(z_2)$$
$$+ q^{\Omega_{i,\{i_1,\ldots,i_r\}}}\, e_{i_1 \ldots i_r}(z_1)\, F_i \Big[e_{j_1 \ldots j_s}(z_2) \Big]$$

$$\longrightarrow \quad \Delta F_i = F_i \otimes I + K_i^{-1} \otimes F_i$$

$$\Delta K_i = K_i \otimes K_i$$

Screening Vertex Operators & Nichols Algebra

$$[H_i,\, H_j] = 0$$

$$[H_i,\, X_j^{\pm}] = \pm(\alpha_i,\, \alpha_j)\, X_j^{\pm}$$

$$[X_i^{+},\, X_j^{-}] = \delta_{i\,j}\, \frac{q^{H_i} - q^{-H_i}}{q - q^{-1}}$$

$\longleftrightarrow$

$$F_i = X_i^{-} q^{-H_i/2}$$

$$E_i = X_i^{+} q^{H_i/2}$$

$$K_i = q^{H_i}$$

$$\sum_{k=0}^{m} (-1)^k \left(\begin{array}{c} m \\ k \end{array} \right)_q q_i^{-k(m-k)/2} \left(X_i^{\pm}\right)^k X_j^{\pm} \left(X_i^{\pm}\right)^{m-k} = 0$$

q-Serre relation

$m = 1 - A_{i\,j}$

$$\Delta(H_i) = H_i \otimes I + I \otimes H_i$$

$$\Delta(X_i^{\pm}) = X_i^{\pm} \otimes q^{-H_i/2} + q^{H_i/2} \otimes X_i^{\pm}$$

$$S(H_i) = -H_i$$

$$S(X_i^{\pm}) = q^{-\sum_i H_i} X_i^{\pm} q^{\sum_i H_i}$$

If only 1 tachyon screening

$\longrightarrow \quad U_q(sl(2))$

Quantum Groups

satisfies RTT and Yang-Baxter eqn

$$R\,T_1 T_2 = T_2 T_1\,R$$

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

Ex2. quantum enveloping algebra $U_q(sl(2))$

$$[H,\, X^{\pm}] = \pm 2X^{\pm}$$

$$[X^+,\, X^-] = \frac{K - K^{-1}}{q - q^{-1}}$$

$$K = q^H$$

$q \to 1$

"classical" limit

$$[H,\, X^{\pm}] = \pm 2X^{\pm}$$

$$[X^+,\, X^-] = H$$

$sl(2)$ Lie algebra

T q-Lie group

L q-Lie algebra

$$L^+ = \begin{bmatrix} K^{-1} & q^{\frac{-1}{2}}(q - q^{-1})X^+ \\ & K \end{bmatrix}$$

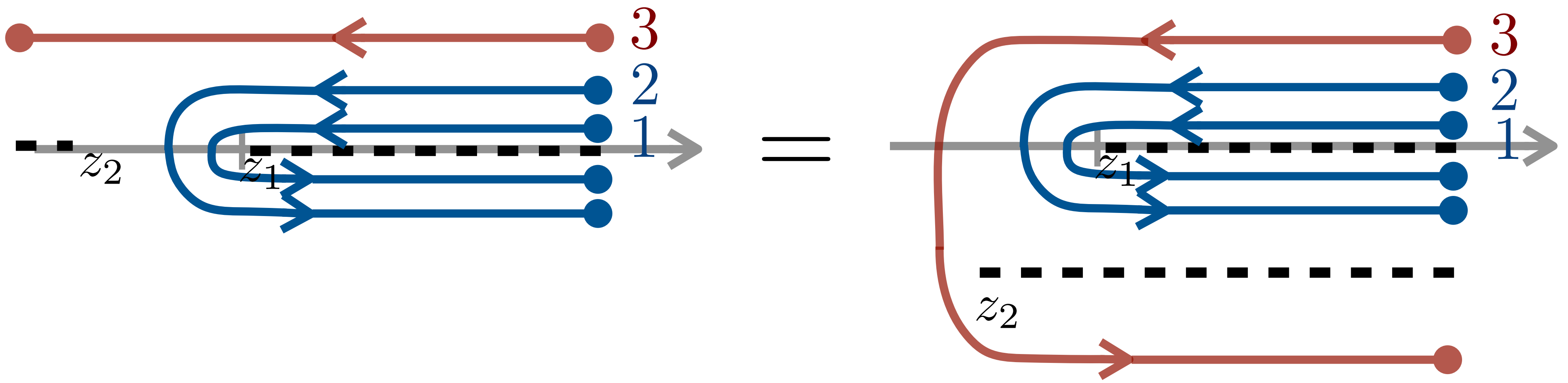
$$L^- = \begin{bmatrix} K & \\ -q^{\frac{1}{2}}(q - q^{-1})X^- & K^{-1} \end{bmatrix}$$

(FRT construction)

[Faddeev, Reshetikhin, Takhtadzhyan 90]

Monodromy relations

Why are they zero?



$$\Big\langle \Big| \quad \Delta F_3 \Big(F_2 F_1 V(z_1) \otimes V(z_2) \Big) \Big| \Big\rangle = 0$$

standard trick in QM:

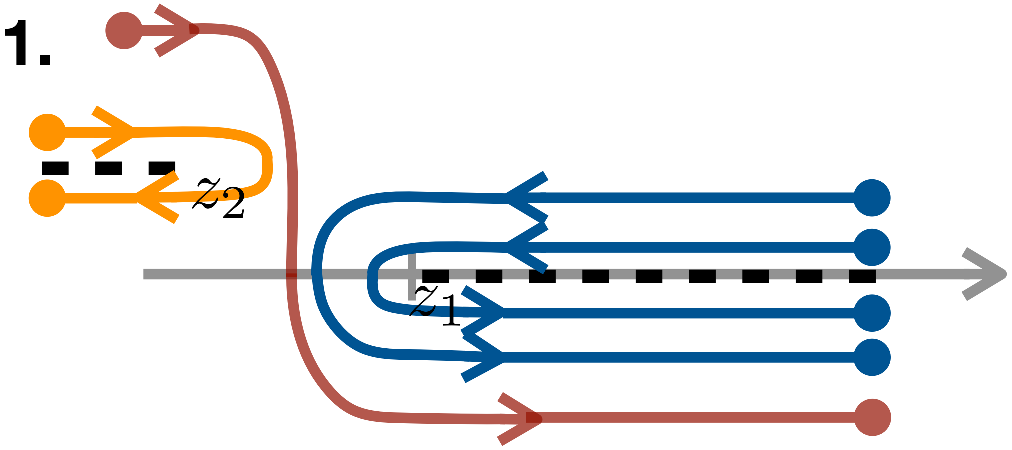
$$\Big\langle \Big| J_+ J_+ \dots \Big| \Big\rangle = 0\,?$$

$$\Big\langle \Big| J_+ J_+ \dots \Big| \Big\rangle = \Big(\Big\langle J_- vac \Big| \Big) \, J_+ \dots \Big| \Big\rangle = 0$$

Monodromy relations

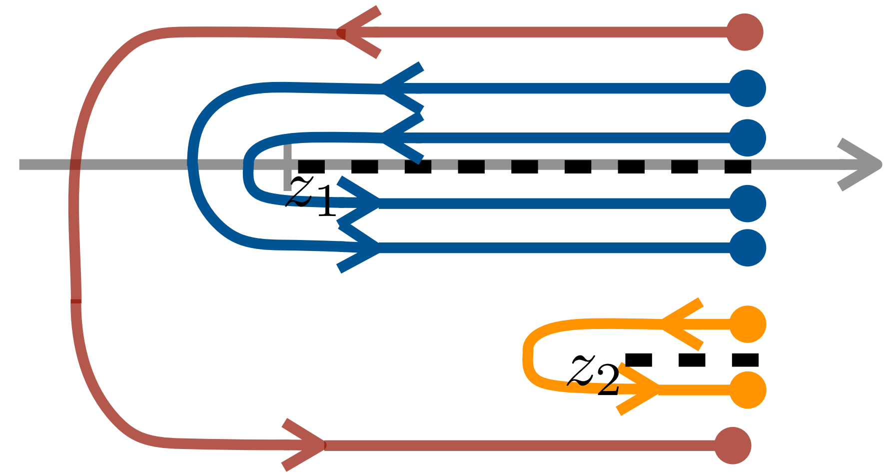
Why are they zero?

1.



$$\left\langle \Big| V(z_2) \ldots E_j \quad F_i \ldots V(z_1) \Big| \right\rangle$$

2.



$$\left\langle \Big| \Delta F_k \Big(F_i \ldots V(z_1) \otimes F_j \ldots V(z_2) \Big) \Big| \right\rangle$$

$$\to S[| \ldots]\, S[| \ldots]\, A(\ldots)$$

[Bjerrum-Bohr, Damgaard, Feng, Sondergaard 10]

Algebraic approach to amplitudes

YM amplitudes → Harmonic oscillator/Hydrogen atom problem

generically:

$$[E_i,\, E_j] = f_{i\,j}{}^{k}\, E_k$$ Hypergeometry functions

[Lentner 17]

Monodromy relations & Yangian

quantum determinant

momentum kernel

$$qdet\, T(u) = \underbrace{R_{n-1,n} \dots (R_{2n} \dots R_{23} R_{23})\,(R_{1n} \dots R_{13} R_{12})}_{\text{momentum kernel}}\, T_1 T_2 \dots T_n$$

$$= \epsilon^q_{\sigma_1 \sigma_2 \dots \sigma_n}\, T_{\sigma_1,1} T_{\sigma_2,2} \dots T_{\sigma_n,n}$$

$$= d^{(0)} + u^{-1} d^{(1)} + u^{-2} d^{(2)} \dots$$

known to generate the center of full Yangian $qdet\, T \to ZY(g)$

Ex. $U_q(sl_2)$ (1 tachyon only)

$$\tilde{L}^+(u) = L^+ - u^{-1} L^-$$

$$\tilde{L}^-(u) = L^- + u\, L^+$$

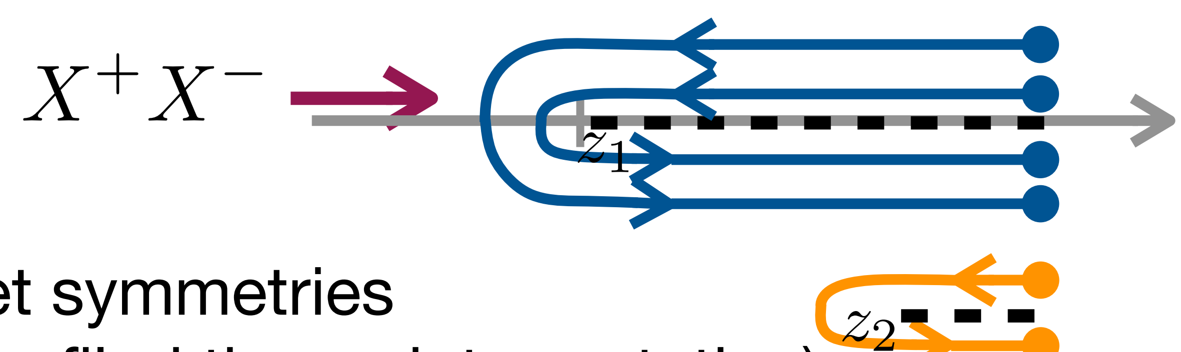
$$L^+ = \begin{bmatrix} K^{-1} & q^{\frac{-1}{2}}(q - q^{-1}) X^+ \\ & K \end{bmatrix}$$

$$L^- = \begin{bmatrix} K & \\ -q^{\frac{1}{2}}(q - q^{-1}) X^- & K^{-1} \end{bmatrix}$$

center:

$$d^{(1)} = \frac{q^{H-1} - q^{-H-1}}{q - q^{-1}} + X^+ X^-$$

$$d^{(0)} = d^{(2)} = I$$



world sheet symmetries
(perhaps no filed theory interpretation)

Vocabulary translated

	Phys. arXiv:[hep-th]	**Maths.** arXiv:[math.QA]
$\int_{\dots t_1<t_2<t_3\dots} \prod_i dt_i S_1(t_1)S_2(t_2)S_3(t_3)\dots V(x)$	amplitude	(fundamental) reprsn. (highest weight module)
$\int_{C_1,C_2,\dots} \prod_i dt_i S_1(t_1)S_2(t_2)S_3(t_3)\dots V(x)$ $= \dots[\int_{\text{line}} S_2[\int_{\text{line}} S_1, V(x)]_{\alpha'}]_{\alpha'}$	numerator	adj. reprsn.
$R_{n-1,n}\dots(R_{2n}\dots R_{23}R_{23})(R_{1n}\dots R_{13}R_{12})$	momentum kernel	quantum symmetriser
	kinematic algebra	screening vertex algebra/ Nichols algebra **[Andruskiewitsch, Schneider 02]**
	double copy	Yetter-Drinfeld bimodule **[Semikhatov, Tipunin 11]**
	↑ $\Delta(V) = T^a \otimes V^a$	coaction (coproduct on modules)
	(single copy)	

Summary & open problems

- Yangian symmetry

  (Hopf algebra satisfying RTT, generators of multiple levels)

- screening vertex algebra

- alpha prime (q undeformed) limit
  root system, Cartan matrix

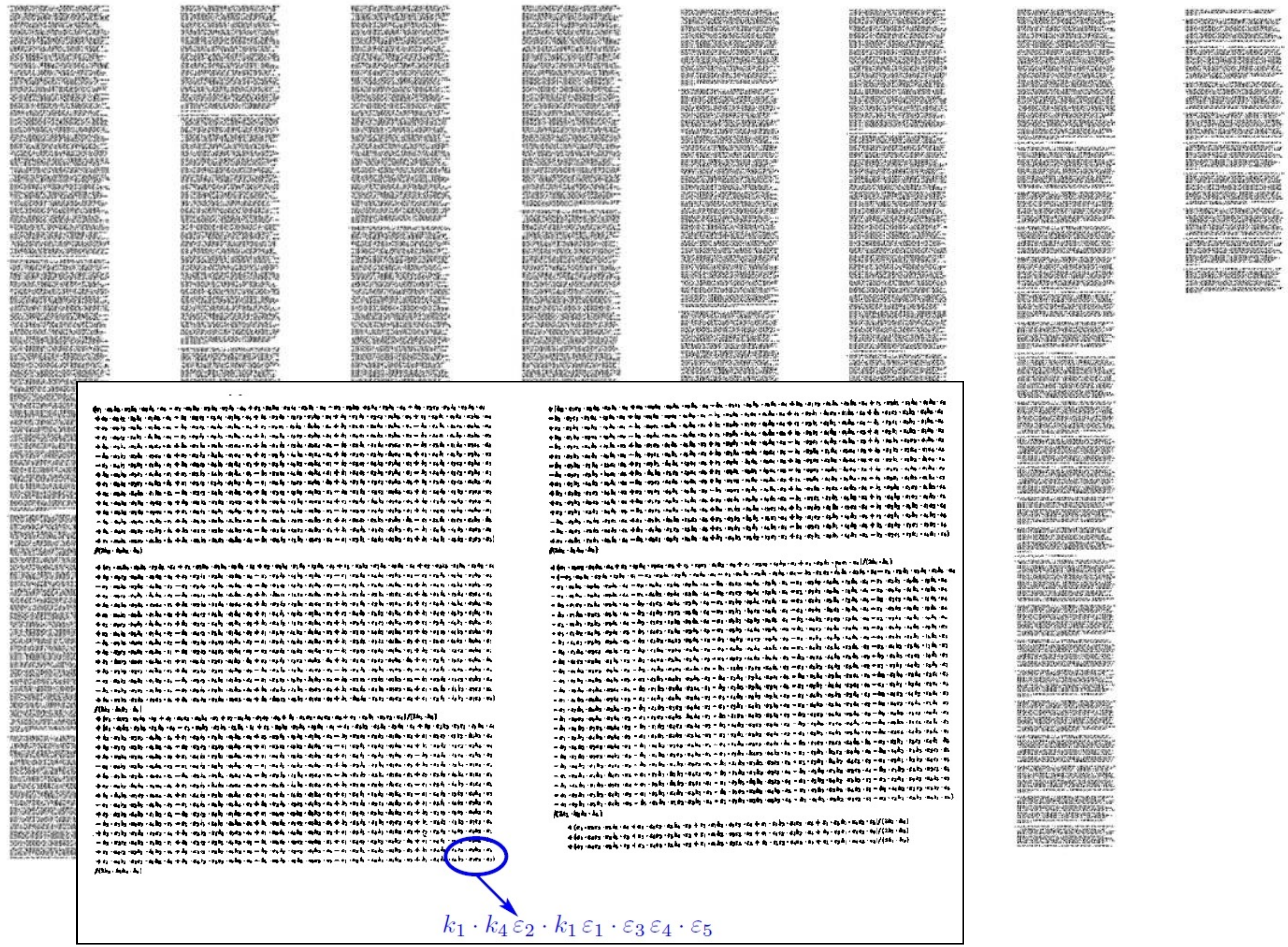
- algebraic construction of YM amplitudes

  (QM-like techniques: SHO/Hydrogen atom, angular momentum algebra…)


Thank you!
спасибо

5 gluon scattering amplitude

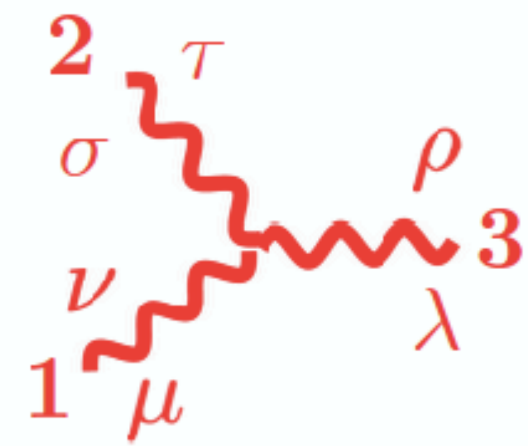
$A_5 =$



Gravity 3-vertex

$$
\begin{aligned}
&\frac{\delta S^3}{\delta\varphi_{\mu\nu}\delta\varphi_{\sigma\tau}\delta\varphi_{\rho\lambda}} \to 2\eta^{\mu\tau}\eta^{\nu\sigma}k_1^{\lambda}k_1^{\rho} + 2\eta^{\mu\sigma}\eta^{\nu\tau}k_1^{\lambda}k_1^{\rho} - 2\eta^{\mu\nu}\eta^{\sigma\tau}k_1^{\lambda}k_1^{\rho} + \\
&2\eta^{\lambda\tau}\eta^{\mu\nu}k_1^{\sigma}k_1^{\rho} + 2\eta^{\lambda\sigma}\eta^{\mu\nu}k_1^{\tau}k_1^{\rho} + \eta^{\mu\tau}\eta^{\nu\sigma}k_2^{\lambda}k_1^{\rho} + \eta^{\mu\sigma}\eta^{\nu\tau}k_2^{\lambda}k_1^{\rho} + \eta^{\lambda\tau}\eta^{\nu\sigma}k_2^{\mu}k_1^{\rho} + \\
&\eta^{\lambda\sigma}\eta^{\nu\tau}k_2^{\mu}k_1^{\rho} + \eta^{\lambda\tau}\eta^{\mu\sigma}k_2^{\nu}k_1^{\rho} + \eta^{\lambda\sigma}\eta^{\mu\tau}k_2^{\nu}k_1^{\rho} + \eta^{\lambda\tau}\eta^{\nu\sigma}k_3^{\mu}k_1^{\rho} + \eta^{\lambda\sigma}\eta^{\nu\tau}k_3^{\mu}k_1^{\rho} - \\
&\eta^{\lambda\nu}\eta^{\sigma\tau}k_3^{\mu}k_1^{\rho} + \eta^{\lambda\tau}\eta^{\mu\sigma}k_3^{\nu}k_1^{\rho} + \eta^{\lambda\sigma}\eta^{\mu\tau}k_3^{\nu}k_1^{\rho} - \eta^{\lambda\mu}\eta^{\sigma\tau}k_3^{\nu}k_1^{\rho} + \eta^{\lambda\nu}\eta^{\mu\tau}k_3^{\sigma}k_1^{\rho} + \\
&\eta^{\lambda\mu}\eta^{\nu\tau}k_3^{\sigma}k_1^{\rho} + \eta^{\lambda\nu}\eta^{\mu\sigma}k_3^{\tau}k_1^{\rho} + \eta^{\lambda\mu}\eta^{\nu\sigma}k_3^{\tau}k_1^{\rho} + 2\eta^{\mu\nu}\eta^{\rho\tau}k_1^{\lambda}k_1^{\sigma} + 2\eta^{\mu\nu}\eta^{\rho\sigma}k_1^{\lambda}k_1^{\tau} - \\
&2\eta^{\lambda\rho}\eta^{\mu\nu}k_1^{\sigma}k_1^{\tau} + 2\eta^{\lambda\nu}\eta^{\mu\rho}k_1^{\sigma}k_1^{\tau} + 2\eta^{\lambda\mu}\eta^{\nu\rho}k_1^{\sigma}k_1^{\tau} + \eta^{\mu\tau}\eta^{\nu\rho}k_1^{\sigma}k_2^{\lambda} + \eta^{\mu\rho}\eta^{\nu\tau}k_1^{\sigma}k_2^{\lambda} + \\
&\eta^{\mu\sigma}\eta^{\nu\rho}k_1^{\tau}k_2^{\lambda} + \eta^{\mu\rho}\eta^{\nu\sigma}k_1^{\tau}k_2^{\lambda} + \eta^{\nu\tau}\eta^{\rho\sigma}k_1^{\lambda}k_2^{\mu} + \eta^{\nu\sigma}\eta^{\rho\tau}k_1^{\lambda}k_2^{\mu} + \eta^{\lambda\tau}\eta^{\nu\rho}k_1^{\sigma}k_2^{\mu} - \\
&\eta^{\lambda\rho}\eta^{\nu\tau}k_1^{\sigma}k_2^{\mu} + \eta^{\lambda\nu}\eta^{\rho\tau}k_1^{\sigma}k_2^{\mu} + \eta^{\lambda\sigma}\eta^{\nu\rho}k_1^{\tau}k_2^{\mu} - \eta^{\lambda\rho}\eta^{\nu\sigma}k_1^{\tau}k_2^{\mu} + \eta^{\lambda\nu}\eta^{\rho\sigma}k_1^{\tau}k_2^{\mu} + \\
&2\eta^{\nu\rho}\eta^{\sigma\tau}k_2^{\lambda}k_2^{\mu} + \eta^{\mu\tau}\eta^{\rho\sigma}k_1^{\lambda}k_2^{\nu} + \eta^{\mu\sigma}\eta^{\rho\tau}k_1^{\lambda}k_2^{\nu} + \eta^{\lambda\tau}\eta^{\mu\rho}k_1^{\sigma}k_2^{\nu} - \eta^{\lambda\rho}\eta^{\mu\tau}k_1^{\sigma}k_2^{\nu} + \\
&\eta^{\lambda\mu}\eta^{\rho\tau}k_1^{\sigma}k_2^{\nu} + \eta^{\lambda\sigma}\eta^{\mu\rho}k_1^{\tau}k_2^{\nu} - \eta^{\lambda\rho}\eta^{\mu\sigma}k_1^{\tau}k_2^{\nu} + \eta^{\lambda\mu}\eta^{\rho\sigma}k_1^{\tau}k_2^{\nu} + 2\eta^{\mu\rho}\eta^{\sigma\tau}k_2^{\lambda}k_2^{\nu} + \\
&2\eta^{\lambda\tau}\eta^{\rho\sigma}k_2^{\mu}k_2^{\nu} + 2\eta^{\lambda\sigma}\eta^{\rho\tau}k_2^{\mu}k_2^{\nu} - 2\eta^{\lambda\rho}\eta^{\sigma\tau}k_2^{\mu}k_2^{\nu} + \eta^{\mu\tau}\eta^{\nu\sigma}k_1^{\lambda}k_2^{\rho} + \eta^{\mu\sigma}\eta^{\nu\tau}k_1^{\lambda}k_2^{\rho} + \\
&\eta^{\lambda\nu}\eta^{\mu\tau}k_1^{\sigma}k_2^{\rho} + \eta^{\lambda\mu}\eta^{\nu\tau}k_1^{\sigma}k_2^{\rho} + \eta^{\lambda\nu}\eta^{\mu\sigma}k_1^{\tau}k_2^{\rho} + \eta^{\lambda\mu}\eta^{\nu\sigma}k_1^{\tau}k_2^{\rho} + 2\eta^{\mu\tau}\eta^{\nu\sigma}k_2^{\lambda}k_2^{\rho} + \\
&2\eta^{\mu\sigma}\eta^{\nu\tau}k_2^{\lambda}k_2^{\rho} - 2\eta^{\mu\nu}\eta^{\sigma\tau}k_2^{\lambda}k_2^{\rho} + 2\eta^{\lambda\nu}\eta^{\sigma\tau}k_2^{\mu}k_2^{\rho} + 2\eta^{\lambda\mu}\eta^{\sigma\tau}k_2^{\nu}k_2^{\rho} + \eta^{\nu\tau}\eta^{\rho\sigma}k_1^{\lambda}k_3^{\mu} + \\
&\eta^{\nu\sigma}\eta^{\rho\tau}k_1^{\lambda}k_3^{\mu} - \eta^{\nu\rho}\eta^{\sigma\tau}k_1^{\lambda}k_3^{\mu} + \eta^{\lambda\tau}\eta^{\nu\rho}k_1^{\sigma}k_3^{\mu} + \eta^{\lambda\nu}\eta^{\rho\tau}k_1^{\sigma}k_3^{\mu} + \eta^{\lambda\sigma}\eta^{\nu\rho}k_1^{\tau}k_3^{\mu} + \\
&\eta^{\lambda\nu}\eta^{\rho\sigma}k_1^{\tau}k_3^{\mu} + \eta^{\nu\tau}\eta^{\rho\sigma}k_2^{\lambda}k_3^{\mu} + \eta^{\nu\sigma}\eta^{\rho\tau}k_2^{\lambda}k_3^{\mu} + \eta^{\lambda\tau}\eta^{\rho\sigma}k_2^{\nu}k_3^{\mu} + \eta^{\lambda\sigma}\eta^{\rho\tau}k_2^{\nu}k_3^{\mu} + \\
&\eta^{\lambda\tau}\eta^{\nu\sigma}k_2^{\rho}k_3^{\mu} + \eta^{\lambda\sigma}\eta^{\nu\tau}k_2^{\rho}k_3^{\mu} + \eta^{\mu\tau}\eta^{\rho\sigma}k_1^{\lambda}k_3^{\nu} + \eta^{\mu\sigma}\eta^{\rho\tau}k_1^{\lambda}k_3^{\nu} - \eta^{\mu\rho}\eta^{\sigma\tau}k_1^{\lambda}k_3^{\nu} + \\
&\eta^{\lambda\tau}\eta^{\mu\rho}k_1^{\sigma}k_3^{\nu} + \eta^{\lambda\mu}\eta^{\rho\tau}k_1^{\sigma}k_3^{\nu} + \eta^{\lambda\sigma}\eta^{\mu\rho}k_1^{\tau}k_3^{\nu} + \eta^{\lambda\mu}\eta^{\rho\sigma}k_1^{\tau}k_3^{\nu} + \eta^{\mu\tau}\eta^{\rho\sigma}k_2^{\lambda}k_3^{\nu} + \\
&\eta^{\mu\sigma}\eta^{\rho\tau}k_2^{\lambda}k_3^{\nu} + \eta^{\lambda\tau}\eta^{\rho\sigma}k_2^{\mu}k_3^{\nu} + \eta^{\lambda\sigma}\eta^{\rho\tau}k_2^{\mu}k_3^{\nu} + \eta^{\lambda\tau}\eta^{\mu\sigma}k_2^{\rho}k_3^{\nu} + \eta^{\lambda\sigma}\eta^{\mu\tau}k_2^{\rho}k_3^{\nu} + \\
&2\eta^{\lambda\tau}\eta^{\rho\sigma}k_3^{\mu}k_3^{\nu} + 2\eta^{\lambda\sigma}\eta^{\rho\tau}k_3^{\mu}k_3^{\nu} - 2\eta^{\lambda\rho}\eta^{\sigma\tau}k_3^{\mu}k_3^{\nu} + \eta^{\mu\tau}\eta^{\nu\rho}k_1^{\lambda}k_3^{\sigma} + \eta^{\mu\rho}\eta^{\nu\tau}k_1^{\lambda}k_3^{\sigma} + \\
&\eta^{\lambda\nu}\eta^{\mu\rho}k_1^{\tau}k_3^{\sigma} + \eta^{\lambda\mu}\eta^{\nu\rho}k_1^{\tau}k_3^{\sigma} + \eta^{\mu\tau}\eta^{\nu\rho}k_2^{\lambda}k_3^{\sigma} + \eta^{\mu\rho}\eta^{\nu\tau}k_2^{\lambda}k_3^{\sigma} - \eta^{\mu\nu}\eta^{\rho\tau}k_2^{\lambda}k_3^{\sigma} + \\
&\eta^{\lambda\tau}\eta^{\nu\rho}k_2^{\mu}k_3^{\sigma} + \eta^{\lambda\nu}\eta^{\rho\tau}k_2^{\mu}k_3^{\sigma} + \eta^{\lambda\tau}\eta^{\mu\rho}k_2^{\nu}k_3^{\sigma} + \eta^{\lambda\mu}\eta^{\rho\tau}k_2^{\nu}k_3^{\sigma} - \eta^{\lambda\tau}\eta^{\mu\nu}k_2^{\rho}k_3^{\sigma} + \\
&\eta^{\lambda\nu}\eta^{\mu\tau}k_2^{\rho}k_3^{\sigma} + \eta^{\lambda\mu}\eta^{\nu\tau}k_2^{\rho}k_3^{\sigma} + 2\eta^{\lambda\rho}\eta^{\nu\tau}k_3^{\mu}k_3^{\sigma} + 2\eta^{\lambda\rho}\eta^{\mu\tau}k_3^{\nu}k_3^{\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}k_1^{\lambda}k_3^{\tau} + \\
&\eta^{\mu\rho}\eta^{\nu\sigma}k_1^{\lambda}k_3^{\tau} + \eta^{\lambda\nu}\eta^{\mu\rho}k_1^{\sigma}k_3^{\tau} + \eta^{\lambda\mu}\eta^{\nu\rho}k_1^{\sigma}k_3^{\tau} + \eta^{\mu\sigma}\eta^{\nu\rho}k_2^{\lambda}k_3^{\tau} + \eta^{\mu\rho}\eta^{\nu\sigma}k_2^{\lambda}k_3^{\tau} - \\
&\eta^{\mu\nu}\eta^{\rho\sigma}k_2^{\lambda}k_3^{\tau} + \eta^{\lambda\sigma}\eta^{\nu\rho}k_2^{\mu}k_3^{\tau} + \eta^{\lambda\nu}\eta^{\rho\sigma}k_2^{\mu}k_3^{\tau} + \eta^{\lambda\sigma}\eta^{\mu\rho}k_2^{\nu}k_3^{\tau} + \eta^{\lambda\mu}\eta^{\rho\sigma}k_2^{\nu}k_3^{\tau} - \\
&\eta^{\lambda\sigma}\eta^{\mu\nu}k_2^{\rho}k_3^{\tau} + \eta^{\lambda\nu}\eta^{\mu\sigma}k_2^{\rho}k_3^{\tau} + \eta^{\lambda\mu}\eta^{\nu\sigma}k_2^{\rho}k_3^{\tau} + 2\eta^{\lambda\rho}\eta^{\nu\sigma}k_3^{\mu}k_3^{\tau} + 2\eta^{\lambda\rho}\eta^{\mu\sigma}k_3^{\nu}k_3^{\tau} - \\
&2\eta^{\lambda\rho}\eta^{\mu\nu}k_3^{\sigma}k_3^{\tau} + 2\eta^{\lambda\nu}\eta^{\mu\rho}k_3^{\sigma}k_3^{\tau} + 2\eta^{\lambda\mu}\eta^{\nu\rho}k_3^{\sigma}k_3^{\tau} - \eta^{\lambda\tau}\eta^{\mu\sigma}\eta^{\nu\rho}k_1\cdot k_2 - \eta^{\lambda\sigma}\eta^{\mu\tau}\eta^{\nu\rho}k_1\cdot \\
&k_2 - \eta^{\lambda\tau}\eta^{\mu\rho}\eta^{\nu\sigma}k_1\cdot k_2 + \eta^{\lambda\rho}\eta^{\mu\tau}\eta^{\nu\sigma}k_1\cdot k_2 - \eta^{\lambda\sigma}\eta^{\mu\rho}\eta^{\nu\tau}k_1\cdot k_2 + \eta^{\lambda\rho}\eta^{\mu\sigma}\eta^{\nu\tau}k_1\cdot k_2 + \\
&2\eta^{\lambda\tau}\eta^{\mu\nu}\eta^{\rho\sigma}k_1\cdot k_2 - \eta^{\lambda\nu}\eta^{\mu\tau}\eta^{\rho\sigma}k_1\cdot k_2 - \eta^{\lambda\mu}\eta^{\nu\tau}\eta^{\rho\sigma}k_1\cdot k_2 + 2\eta^{\lambda\sigma}\eta^{\mu\nu}\eta^{\rho\tau}k_1\cdot k_2 - \\
&\eta^{\lambda\nu}\eta^{\mu\sigma}\eta^{\rho\tau}k_1\cdot k_2 - \eta^{\lambda\mu}\eta^{\nu\sigma}\eta^{\rho\tau}k_1\cdot k_2 - 2\eta^{\lambda\rho}\eta^{\mu\nu}\eta^{\sigma\tau}k_1\cdot k_2 + 2\eta^{\lambda\nu}\eta^{\mu\rho}\eta^{\sigma\tau}k_1\cdot k_2 + \\
&2\eta^{\lambda\mu}\eta^{\nu\rho}\eta^{\sigma\tau}k_1\cdot k_2 - \eta^{\lambda\tau}\eta^{\mu\sigma}\eta^{\nu\rho}k_1\cdot k_3 - \eta^{\lambda\sigma}\eta^{\mu\tau}\eta^{\nu\rho}k_1\cdot k_3 - \eta^{\lambda\tau}\eta^{\mu\rho}\eta^{\nu\sigma}k_1\cdot k_3 + \\
&2\eta^{\lambda\rho}\eta^{\mu\tau}\eta^{\nu\sigma}k_1\cdot k_3 - \eta^{\lambda\sigma}\eta^{\mu\rho}\eta^{\nu\tau}k_1\cdot k_3 + 2\eta^{\lambda\rho}\eta^{\mu\sigma}\eta^{\nu\tau}k_1\cdot k_3 + 2\eta^{\lambda\tau}\eta^{\mu\nu}\eta^{\rho\sigma}k_1\cdot k_3 - \\
&\eta^{\lambda\nu}\eta^{\mu\tau}\eta^{\rho\sigma}k_1\cdot k_3 - \eta^{\lambda\mu}\eta^{\nu\tau}\eta^{\rho\sigma}k_1\cdot k_3 + 2\eta^{\lambda\sigma}\eta^{\mu\nu}\eta^{\rho\tau}k_1\cdot k_3 - \eta^{\lambda\nu}\eta^{\mu\sigma}\eta^{\rho\tau}k_1\cdot k_3 - \\
&\eta^{\lambda\mu}\eta^{\nu\sigma}\eta^{\rho\tau}k_1\cdot k_3 - 2\eta^{\lambda\rho}\eta^{\mu\nu}\eta^{\sigma\tau}k_1\cdot k_3 + \eta^{\lambda\nu}\eta^{\mu\rho}\eta^{\sigma\tau}k_1\cdot k_3 + \eta^{\lambda\mu}\eta^{\nu\rho}\eta^{\sigma\tau}k_1\cdot k_3 - \\
&\eta^{\lambda\tau}\eta^{\mu\sigma}\eta^{\nu\rho}k_2\cdot k_3 - \eta^{\lambda\sigma}\eta^{\mu\tau}\eta^{\nu\rho}k_2\cdot k_3 - \eta^{\lambda\tau}\eta^{\mu\rho}\eta^{\nu\sigma}k_2\cdot k_3 + 2\eta^{\lambda\rho}\eta^{\mu\tau}\eta^{\nu\sigma}k_2\cdot k_3 - \\
&\eta^{\lambda\sigma}\eta^{\mu\rho}\eta^{\nu\tau}k_2\cdot k_3 + 2\eta^{\lambda\rho}\eta^{\mu\sigma}\eta^{\nu\tau}k_2\cdot k_3 + \eta^{\lambda\tau}\eta^{\mu\nu}\eta^{\rho\sigma}k_2\cdot k_3 - \eta^{\lambda\nu}\eta^{\mu\tau}\eta^{\rho\sigma}k_2\cdot k_3 - \\
&\eta^{\lambda\mu}\eta^{\nu\tau}\eta^{\rho\sigma}k_2\cdot k_3 + \eta^{\lambda\sigma}\eta^{\mu\nu}\eta^{\rho\tau}k_2\cdot k_3 - \eta^{\lambda\nu}\eta^{\mu\sigma}\eta^{\rho\tau}k_2\cdot k_3 - \eta^{\lambda\mu}\eta^{\nu\sigma}\eta^{\rho\tau}k_2\cdot k_3 - \\
&2\eta^{\lambda\rho}\eta^{\mu\nu}\eta^{\sigma\tau}k_2\cdot k_3 + 2\eta^{\lambda\nu}\eta^{\mu\rho}\eta^{\sigma\tau}k_2\cdot k_3 + 2\eta^{\lambda\mu}\eta^{\nu\rho}\eta^{\sigma\tau}k_2\cdot k_3
\end{aligned}
$$

171 terms



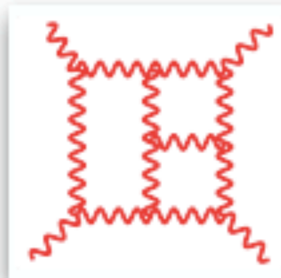
[DeWitt, 1967]

According to standard calculation:

171 terms (at least) / vertex

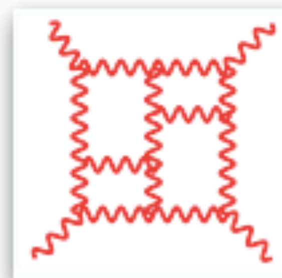
3 terms / internal line

A single 3 loop diagram:



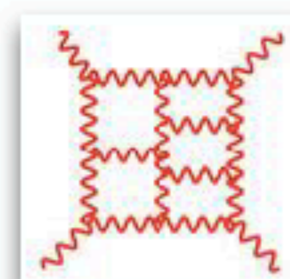
$\sim 10^{20}$ TERMS

4 loop diagram:



$\sim 10^{26}$ TERMS

5 loop diagram:



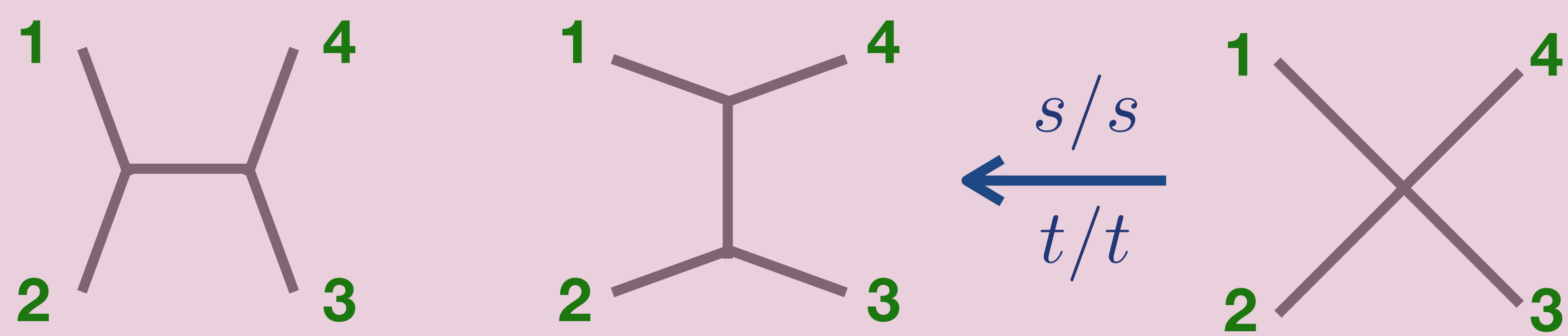
$\sim 10^{31}$ TERMS

The result: very often just one or two terms

BCJ duality - the settings

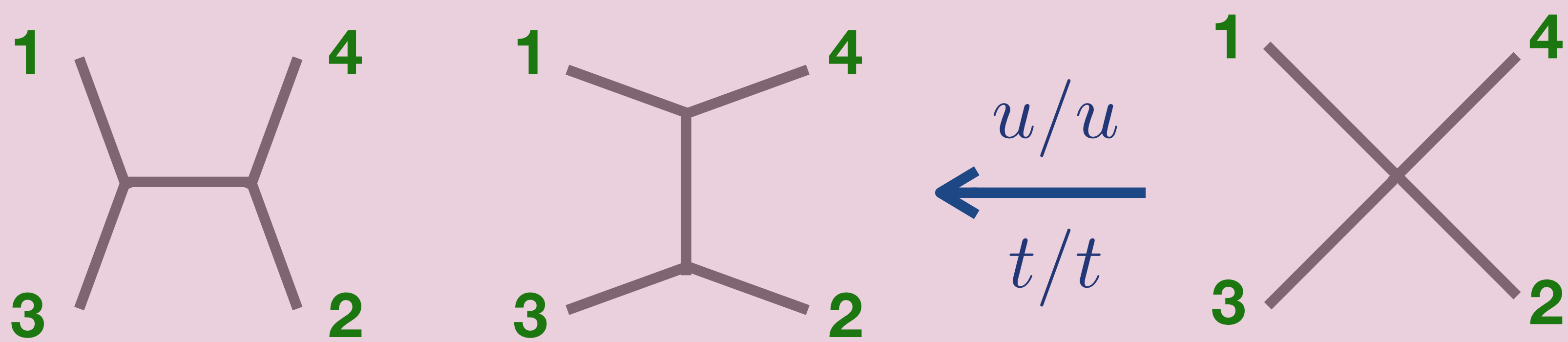
Z. Bern J. J. M. Carrasco H. Johansson

Defining numerators by absorbing contact terms



$$A(1234) = \frac{n_s}{s} - \frac{n_t}{t}$$

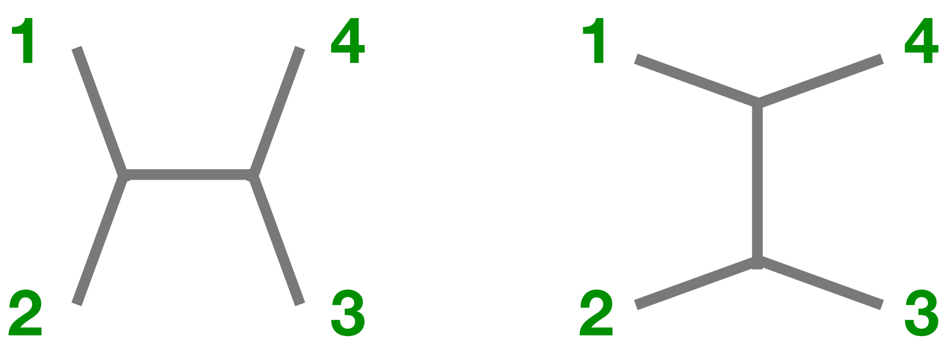
$$A(1324) = -\frac{n_u}{u} + \frac{n_t}{t}$$



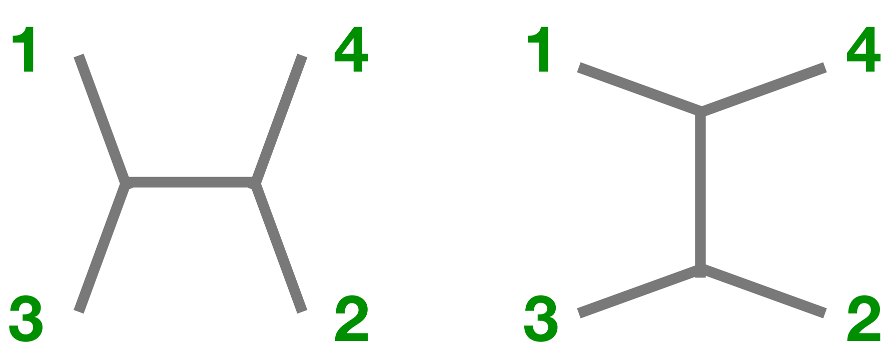
BCJ duality - the settings

Z. Bern J. J. M. Carrasco H. Johansson

Tri-valent description of YM

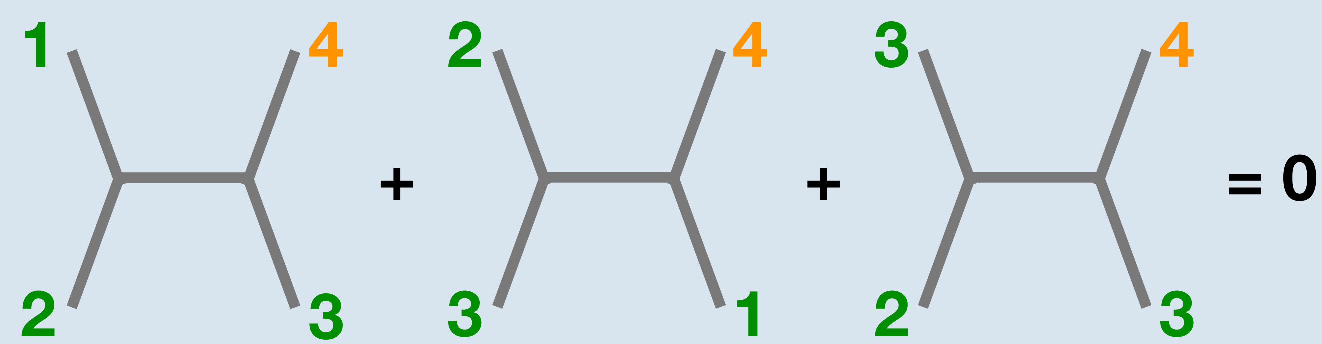


$$A(1234) = \frac{n_s}{s} - \frac{n_t}{t}$$
$$A(1324) = -\frac{n_u}{u} + \frac{n_t}{t}$$



Jacobi identity (& anti-symmetry)

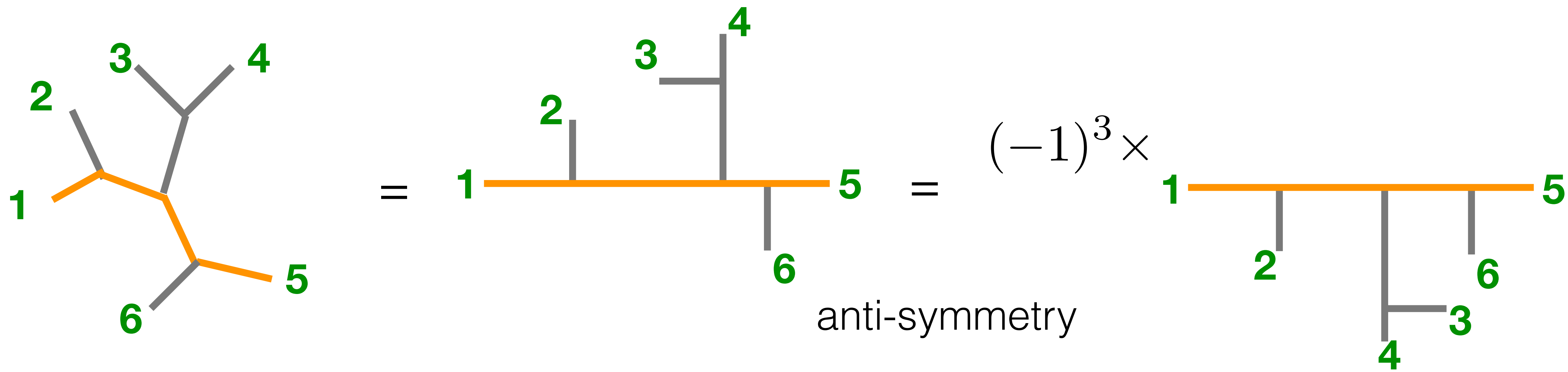
$$n_s + n_t + n_u = 0$$



$$f^{12e} f^{e34} + f^{23e} f^{e14} + f^{31e} f^{e24} = 0$$

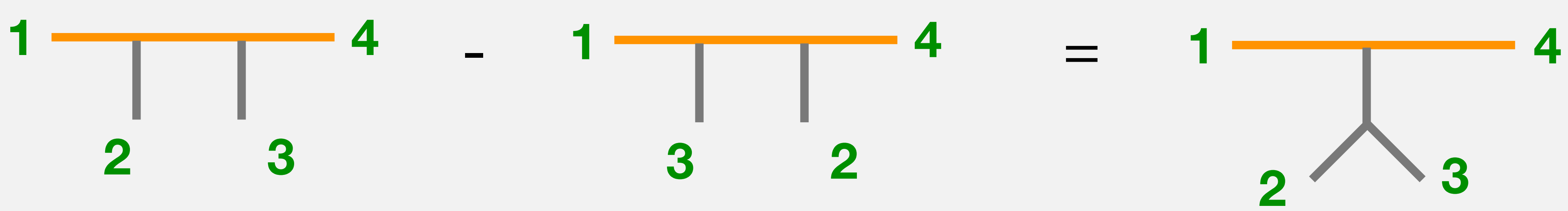
[Bern, Carrasco, Johansson 08]

- **A basis for all cubic graphs**

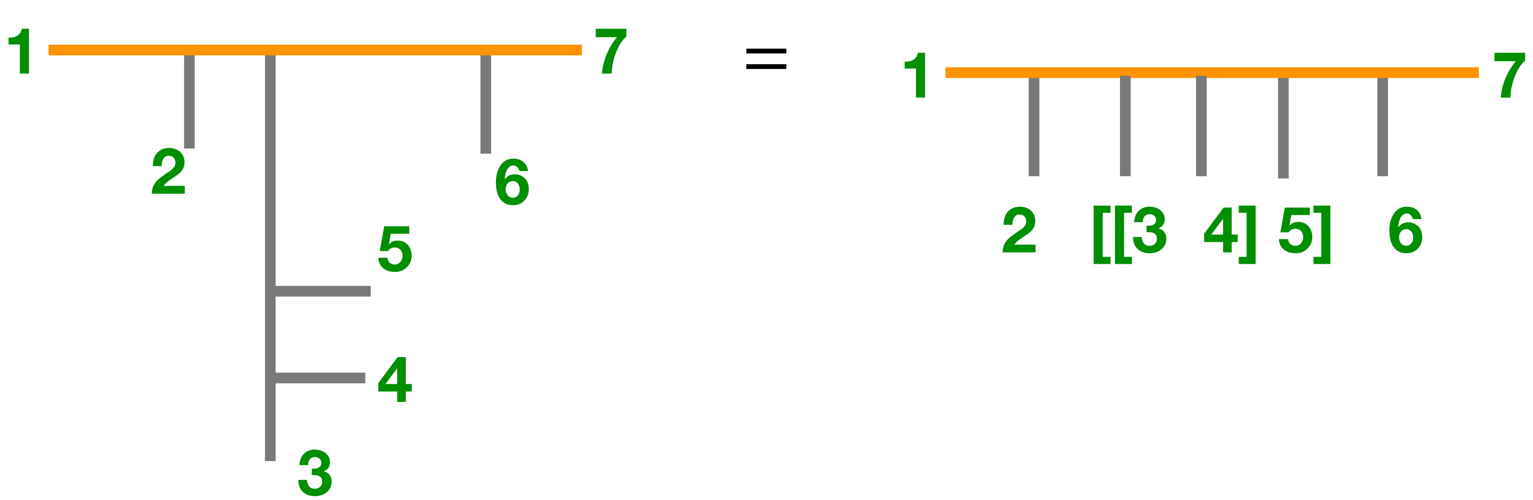


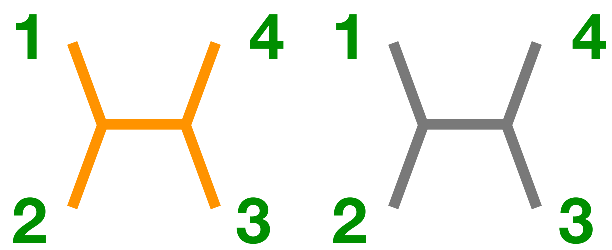
(Legs 1 & 5 at two ends)

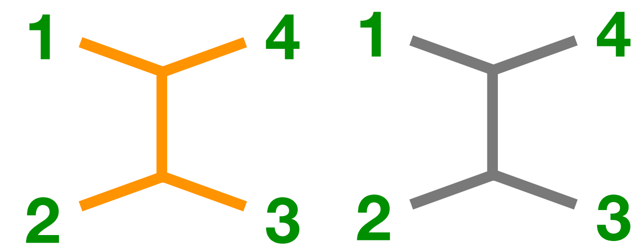
- **Jacobi identity**

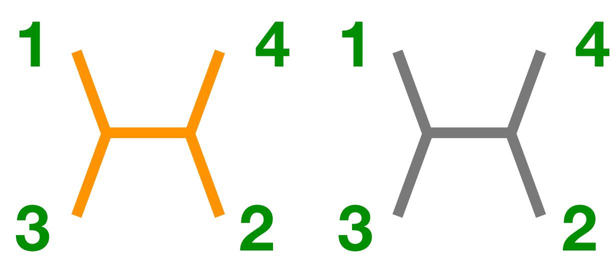


- **Generically any cubic graph can be spanned by (n-2)! half-ladders**

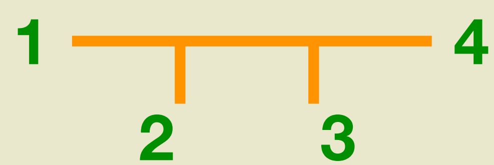


$$\mathcal{M}_{full} = \frac{1}{s}$$


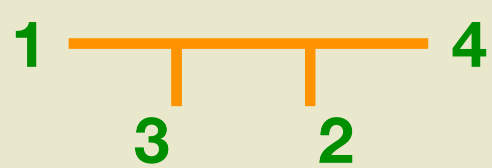
$$+\frac{1}{t}$$


$$+\frac{1}{u}$$


[Bern, Carrasco, Johansson 08]

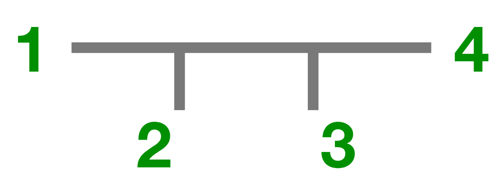
$=$


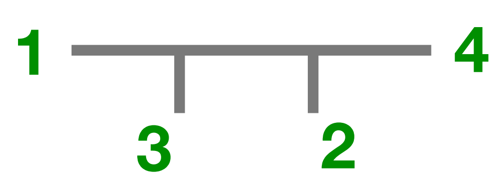
$A(1234)$

$+$


$A(1324)$

[Del Duca, Dixon, Maltoni 00]

$= \tilde{A}(1234)$


$+ \tilde{A}(1324)$


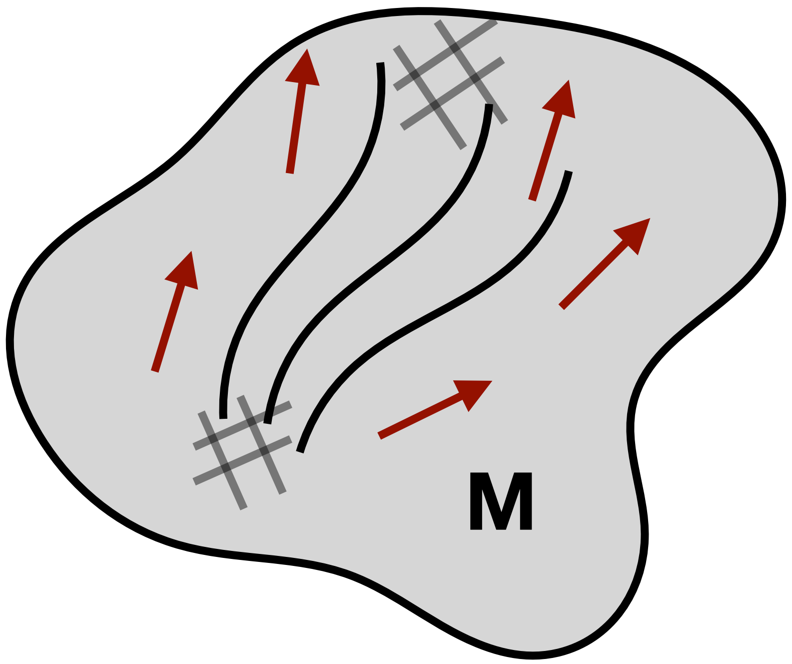
$$= \sum_{\sigma,\gamma\in\text{perm}} \tilde{A}_n(\sigma)\,\mathcal{S}[\sigma|\gamma]\,A_n(\gamma)$$

[Kawai, Lewellen, Tye 86]

understanding the kinematic algebra

SDYM ⟶ diffeomorphism algebra

[Bjerrum-Bohr, Damgaard, Monteiro, O'Connell 13]
[Monteiro, O'Connell 11]
[Du, Feng, CF 12]
[CF, Krasnov 16]
[He, Schlotterer, Zhang 18]



Lie group manifold

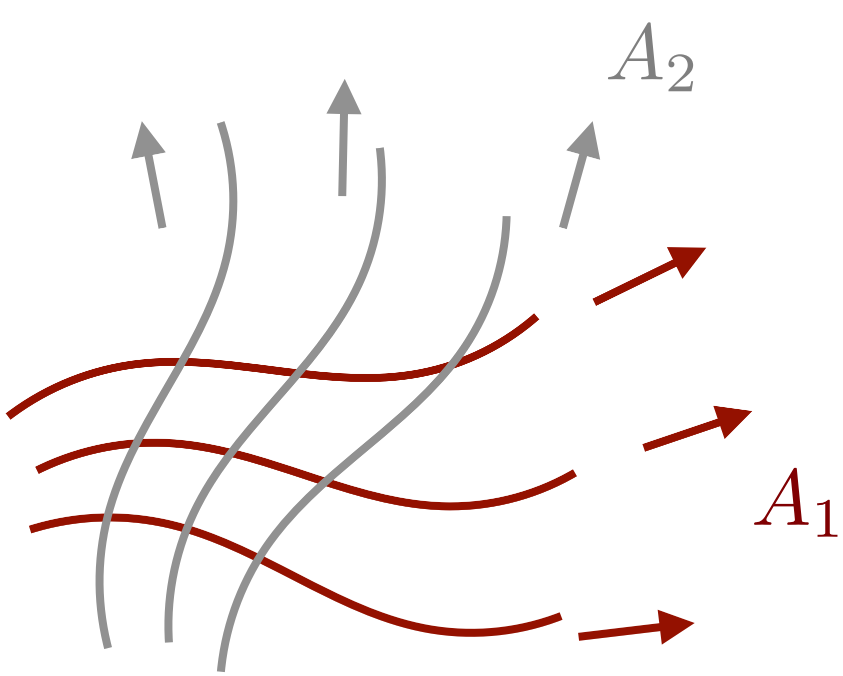
group multiplication ⇄ diffeo

vector field

$$v_3^{\mathrm{YM}}(A_1, A_2, A_3) = [A_1, A_2]A_3 + \text{cyclic}$$

$$[A_1, A_2] = (A_1^\mu \partial_\mu A_2^\nu - A_2^\mu \partial_\mu A_1^\nu)\partial_\nu$$

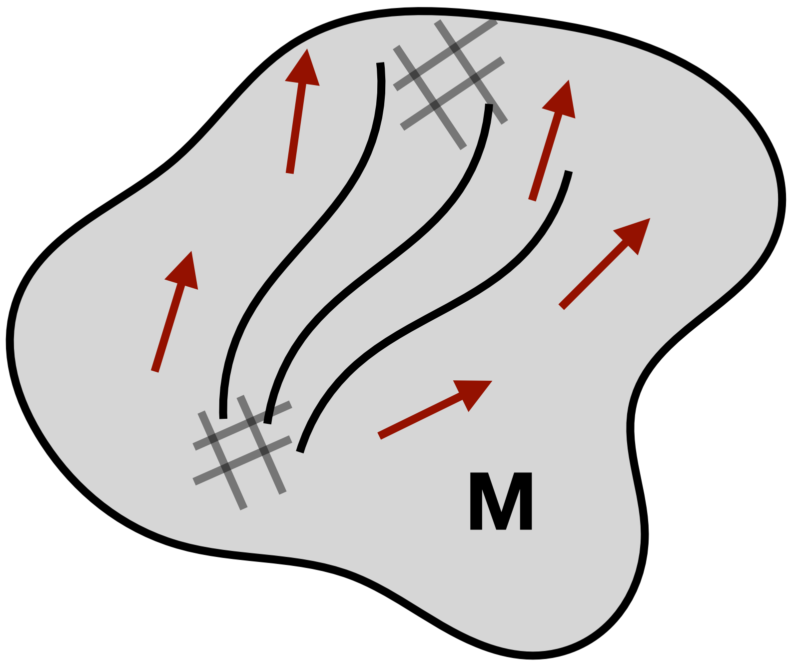
⟶ f^{123}



understanding the kinematic algebra

SDYM ⟶ diffeomorphism algebra

[Bjerrum-Bohr, Damgaard, Monteiro, O'Connell 13]
[Monteiro, O'Connell 11]
[Du, Feng, CF 12]
[CF, Krasnov 16]
[He, Schlotterer, Zhang 18]



Lie group manifold

group multiplication ⇄ diffeo

vector field

$$H^i = \frac{1}{2\pi i}\oint_0 dz\, h^i(z)$$

$$E^i_\pm = \frac{1}{2\pi i}\oint_0 dz\ :Q^i_\pm(z) e^i_\pm(z):$$

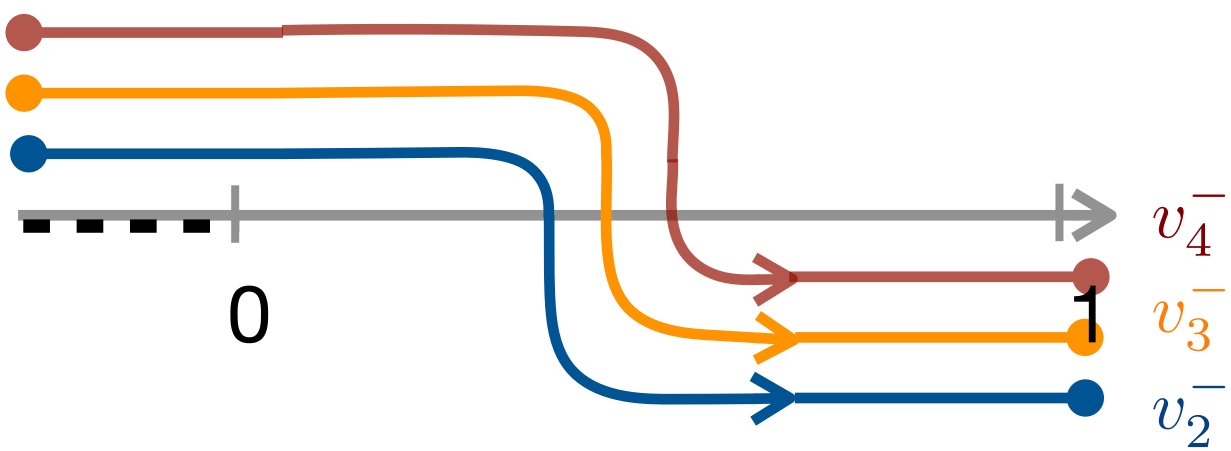
$$Q^i_\pm = q^{\pm\gamma_\pm \int_z^{z_0} dw\, h^i(w)}$$

CFT ⟶ Hopf algebra $(\times, \eta, \Delta, \epsilon, \gamma)$

String

$$\mathcal{M}_n = \sum_{\sigma \in \mathrm{perm}} \tilde{A}_n(\sigma) \times \mathcal{I}(\sigma)$$

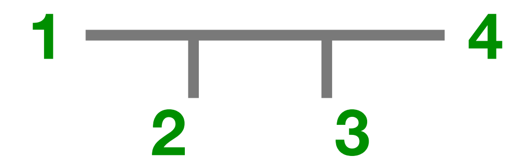
$$= \sum_{\sigma,\gamma \in \mathrm{perm}} \tilde{A}_n(\sigma)\, \mathcal{S}'_{\alpha}[\sigma|\gamma]\, \mathcal{A}_n(\gamma)$$



$$\mathcal{I} = \int_{-\infty}^{\infty} \prod_{i=2}^{n-2} dv_i^- (v_i^-)^{\alpha' k_i \cdot k_1} (1 - v_i^-)^{\alpha' k_{n-1} \cdot k_i} \prod_{j>i} (v_j^- - v_i^-)^{\alpha' k_j \cdot k_i} f(v^-)$$

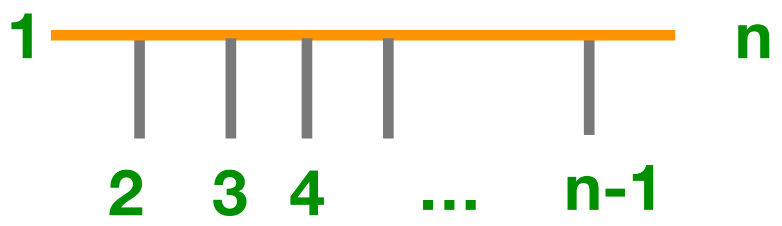
Field theory

$$\mathcal{M}_n = \tilde{A}(1234)$$



$$+ \; \tilde{A}(1324)$$

$$= \sum_{\sigma,\gamma \in \mathrm{perm}} \tilde{A}_n(\sigma)\, \mathcal{S}[\sigma|\gamma]\, A_n(\gamma)$$



$$= \left\langle n \middle| [[[T_1, T_2]_{\alpha'}, T_3]_{\alpha'} \ldots, T_{n-1}]_{\alpha'} \middle| vac \right\rangle$$

$$T_i = \int_0^1 \frac{dy_i}{y_i} V(y_i)$$

$$[T_1, T_2]_{\alpha'} = T_1 T_2 - e^{-i\pi\alpha' k_1 \cdot k_2} T_2 T_1$$

$$\sum_{\sigma \in \mathrm{perm}} Tr(\lambda_1 \lambda_{\sigma(2)} \ldots \lambda_{\sigma(n)})\, \mathcal{A}(1\sigma(2, \ldots, n))$$

$$= \sum_{\rho \in \mathrm{perm}} Tr([[\lambda_1, \lambda_{\rho(2)}]_{\alpha'} \ldots, \lambda_{\rho(n-1)}]_{\alpha'} \lambda_n)\, \mathcal{A}(1\rho(2, \ldots, n-1), n)$$

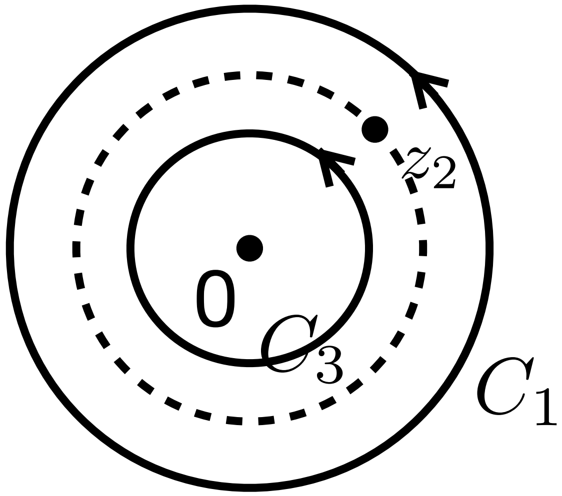
[Ma, Du, Chen 11]

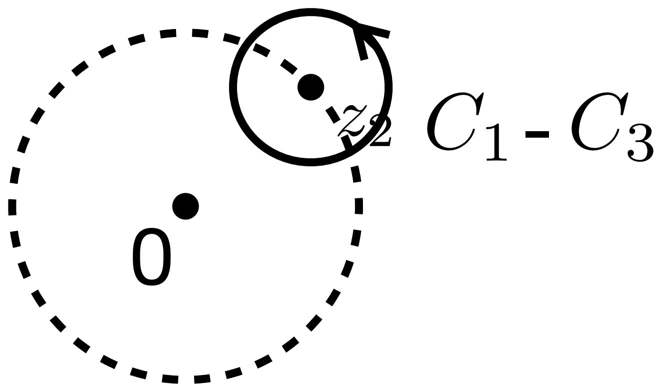
$$[\lambda_1, \lambda_2]_{\alpha'} = \lambda_1 \lambda_2 - e^{-i\pi\alpha' k_1 \cdot k_2} \lambda_2 \lambda_1$$

[Carrasco, Mafra, Schlotterer 16]

vertex operator algebra

CFT → Hopf algebra



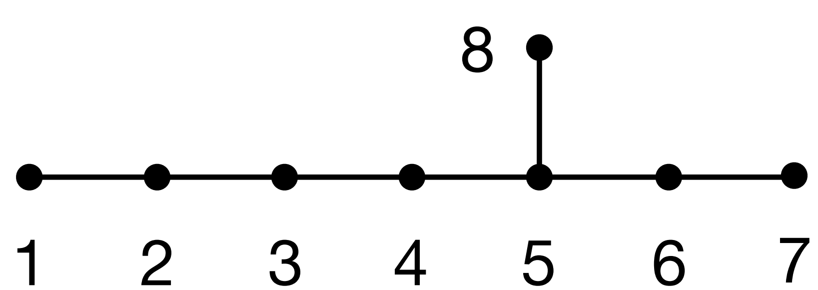


lattice → $k_i \cdot k_j = -1 \text{ or } -2$

$$A_k = \oint \frac{dz}{2\pi i\, z} V(k,z)$$

(cf. Green, Schwarz, Witten Sec. 6)

universal enveloping algebra U(g)
(whole string spectrum)



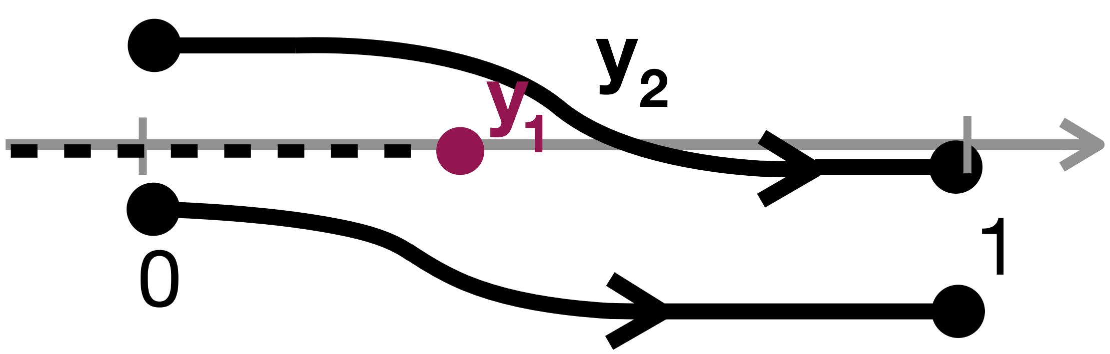
identifying the diffeos

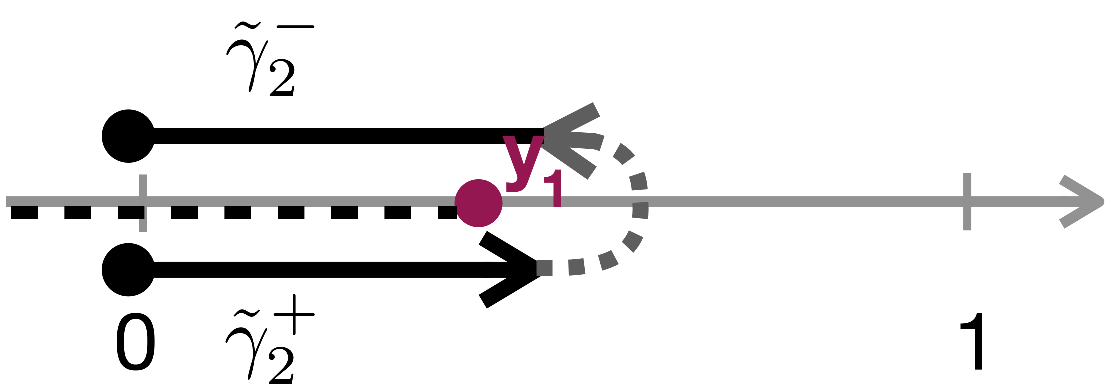
standard OPE. vector $\times$ vector $\rightarrow$ scalar $+$ vector $+$ tensor

$$: e^{ik_1\cdot X+\epsilon_1\cdot\dot{X}} :: e^{ik_2\cdot X+\epsilon_2\cdot\dot{X}} :$$

$$= \left(1+\alpha'\epsilon_2\cdot k_1\frac{y_2}{y_1-y_2}+\alpha'\epsilon_1\cdot k_2\frac{y_1}{y_1-y_2}-\alpha'(\epsilon_1\cdot\epsilon_2)\frac{y_1y_2}{(y_1-y_2)^2}+\dots\right)$$

$$: e^{ik_1\cdot X+\epsilon_1\cdot\dot{X}} e^{ik_2\cdot X+\epsilon_2\cdot\dot{X}} :$$





$$\int_{\tilde{\gamma}_2^\pm} dy_2\,(y_2-y_1)^n e^{\alpha' k_1\cdot k_2\, ln(y_2-y_1)}$$

$$= \frac{1}{n+\alpha' k_1\cdot k_2+1} e^{\alpha' k_1\cdot k_2}$$

$$\left[0 - e^{\mp i\pi\alpha' k_1\cdot k_2} e^{\alpha' k_1\cdot k_2\, ln\, y_1}\right]$$

$$\sim 2i\, sin(\pi\alpha' k_1\cdot k_2)$$

identifying the diffeos

standard OPE. vector $\times$ vector $\rightarrow$ scalar $+$ vector $+$ tensor

$$[[V_1, V_2]_{\alpha'}, V_3]_{\alpha'}$$

$$\longrightarrow \int_{\tilde{\gamma}_3^+ - \tilde{\gamma}_3^-} dy_3 \int_{\tilde{\gamma}_2^+ - \tilde{\gamma}_2^-} dy_2 \frac{1}{(y_2 - y_3)^2} e^{\alpha' k_2 \cdot k_3 ln(y_2 - y_3)} e^{\alpha' k_1 \cdot k_3 ln(y_1 - y_3)} e^{\alpha' k_1 \cdot k_2 ln(y_1 - y_2)}$$

$$= sin(\pi\alpha' k_2 \cdot k_1) sin(\pi\alpha' k_3 \cdot k_1) I(1324) + sin(\pi\alpha' k_2 \cdot k_1) sin(\pi\alpha' k_3 \cdot (k_1 + k_2)) I(1234)$$

$$I(1234) = \int_0^1 dy_3 \int_0^{y_3} dy_2 \frac{1}{(y_2 - y_3)^2} \; e^{\alpha' k_2 \cdot k_3 ln(y_2 - y_3)} e^{\alpha' k_1 \cdot k_3 ln(y_1 - y_3)} e^{\alpha' k_1 \cdot k_2 ln(y_1 - y_2)}$$

generically: hypergeometric functions

Summary & open problems

- conflict between monodromy
  & algebraic setting

- vertex operator algebra

- identified the diffeos

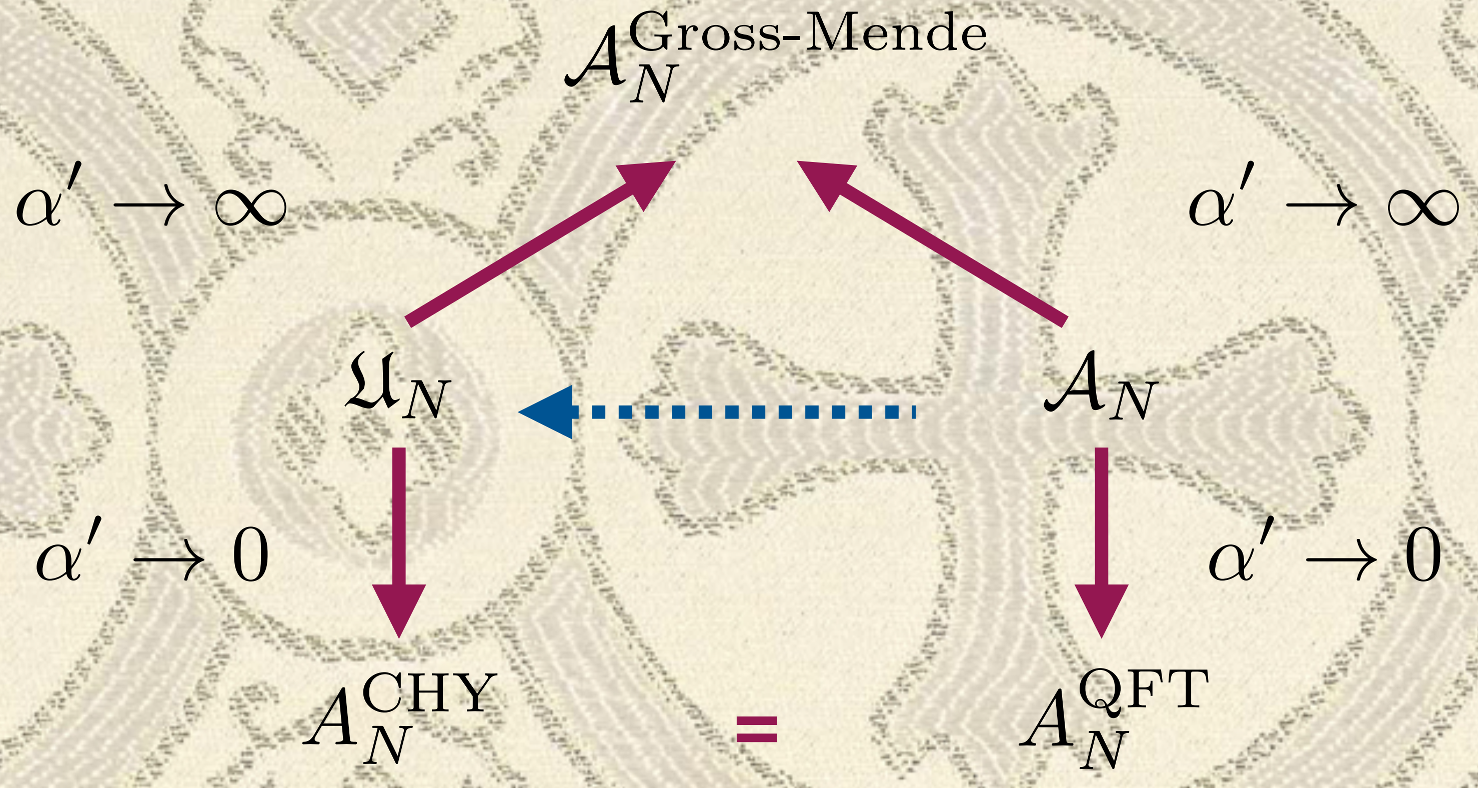
- a family of KLT equivalent relation?
  (string analogue)

Summary & open problems

- CHY (dual model obtained by changing integration measure)

$$\mathfrak{U}_4(1234) = g_{YM}^2 \int \mathrm{Pf}'(\Psi)\, z_2^{2\alpha' k_1\cdot k_2 - 1}(1 - z_2)^{2\alpha' k_2\cdot k_3 - 1}\delta(S_2)\, dz_2$$

[Bjerrum-Bohr, Damgaard, Toukine, Vanhove 14]



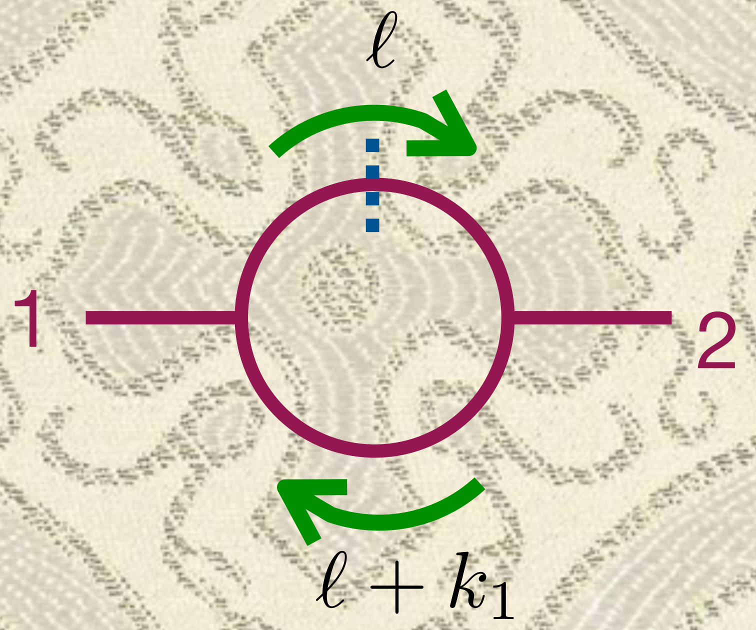
Alexander Ochirov Piotr Tourkine

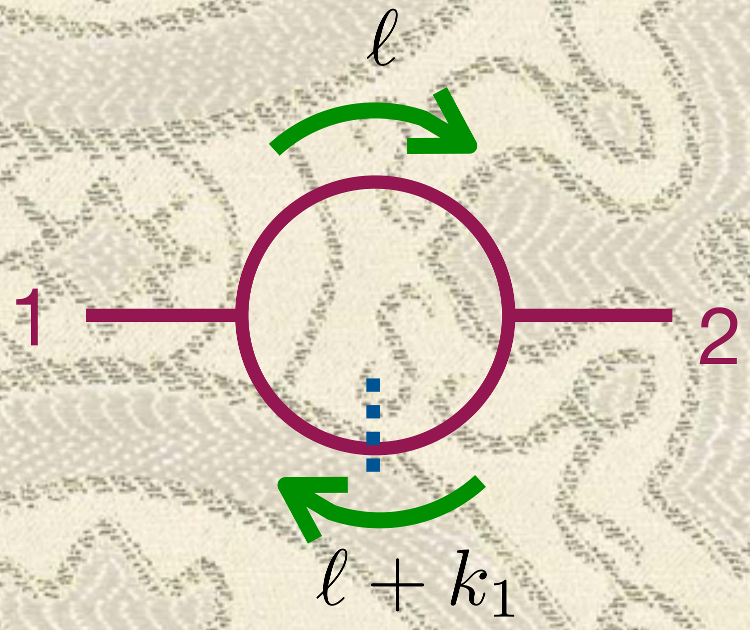
Summary & open problems

Song He

Oliver Schlotterer

- BCJ at 1-loop





$$\int d^D\ell \frac{1}{\ell^2(\ell+k_1)^2} = \int d^D\ell \frac{1}{\ell^2}\,\frac{1}{2\ell\cdot k_1} + \frac{1}{(\ell+k_1)^2}\,\frac{1}{2\ell\cdot k_1}$$

$$= \int d^D\ell \frac{1}{\ell^2}\,\frac{1}{2\ell\cdot k_1} + \frac{1}{\ell^2}\,\frac{1}{2\ell\cdot k_2}$$

1-loop KLT relations **[He, Schlotterer 17]**

off-shell continuation

string analogue of current

$$\mathcal{J}(123\ldots n) \quad := \quad \alpha'^{n-3} g^{n-2} \frac{1}{\hat{k}_n^2} \Big\langle \hat{f} \Big| V_{n-1}(1) \frac{1}{L_0 - I} V_{n-2}(1) \ldots \frac{1}{L_0 - I} V_2(1) \Big| \hat{i} \Big\rangle$$

$$= \quad \alpha'^{n-3} g^{n-2} \Big\langle \hat{f} \Big| \frac{1}{L_0 - I} V_{n-1}(1) \frac{1}{L_0 - I} V_{n-2}(1) \ldots \frac{1}{L_0 - I} V_2(1) \Big| \hat{i} \Big\rangle$$

$$\sim \Big\langle \tilde{f} \Big| \left(z_{n-1}^{L_0 - I} V_{n-1}(1) z_{n-1}^{-(L_0 - I)} \right) \left((z_{n-1} z_{n-2})^{L_0 - I} V_{n-2}(1) (z_{n-1} z_{n-2})^{-(L_0 - I)} \right) \ldots$$

$$\ldots \left((z_2 z_3 \ldots z_{n-1})^{L_0 - I} V_2(1) (z_2 z_3 \ldots z_{n-1})^{-(L_0 - I)} \right) \Big| i \Big\rangle$$

$$\sim \int_{0 < y_2 < y_3 < \cdots < y_{n-1} < 1} \prod_{i=2}^{n-1} dy_i \Big\langle \hat{f} \Big| \frac{V_{n-1}(y_{n-1})}{y_{n-1}} \ldots \frac{V_3(y_3)}{y_3} \frac{V_2(y_2)}{y_2} \Big| \hat{i} \Big\rangle$$

explicit generators

$$n(123) = \lim_{k_3^2, \alpha' \to 0} sin(\pi \alpha' k_2 \cdot k_1) \Big\langle \tilde{f} \Big| : \int_0^1 \frac{dy_2}{y_2} \epsilon_2 \cdot \dot{X}(y_2) e^{ik_2 \cdot X} : \Big| i \Big\rangle$$

$$\sim \; V_3^{YM} \int_0^1 dy_2 \frac{1}{y_2} e^{\alpha' k_1 \cdot k_2 \, ln \, y_2}$$

$$\int_0^1 dy_2 (y_2)^{\alpha' k_1 \cdot k_2 - 1} = \frac{1}{\alpha' k_1 \cdot k_2}$$

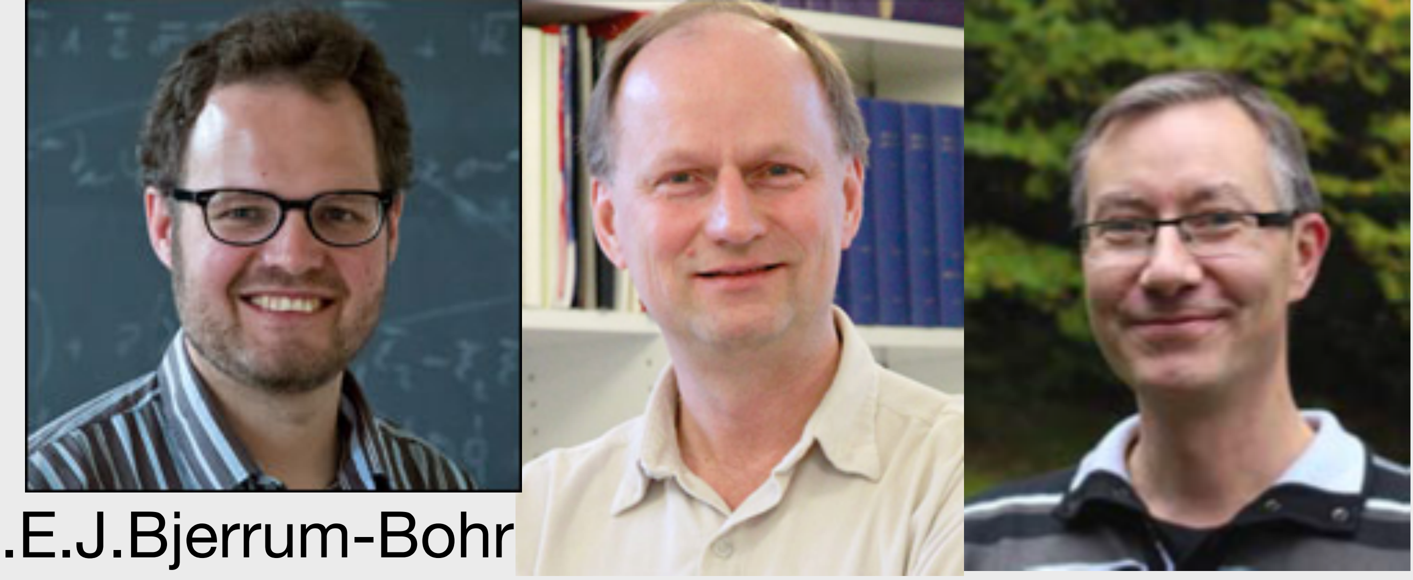
explicit generators

$$V(y) = \sum_{n=-\infty}^{\infty} e^{ik\cdot x} a_n e^{\ell k\cdot p\, ln\, y}$$

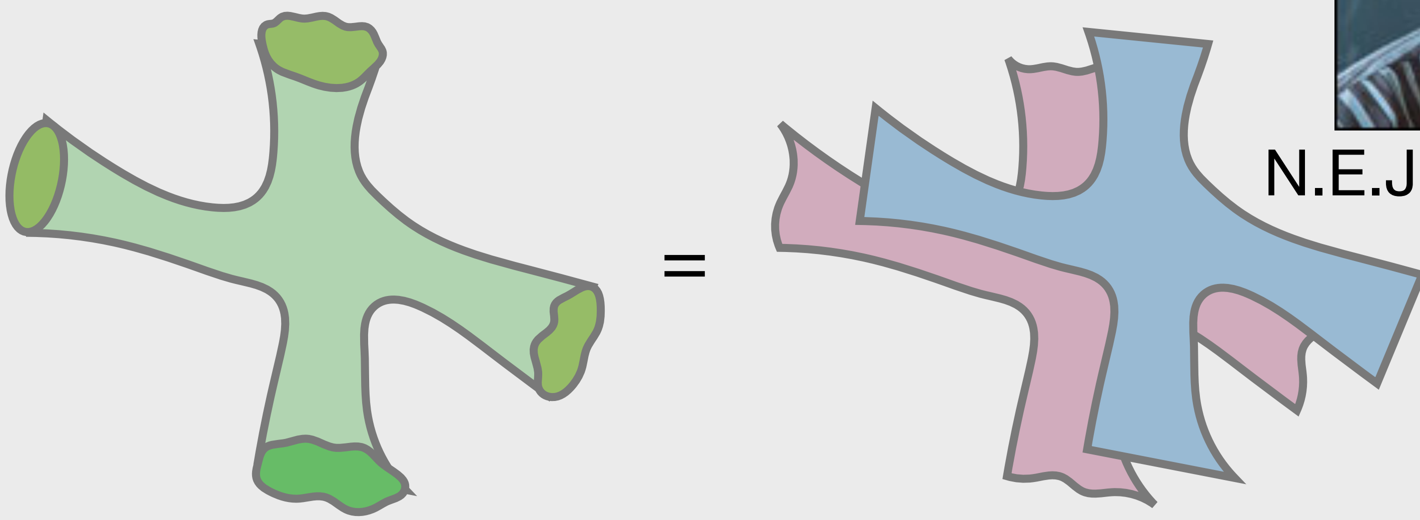
$$\int_0^1 \frac{dy}{y} V(y) = \sum_{n=-\infty}^{\infty} e^{ik\cdot x} a_n \frac{1}{k\cdot\alpha_o + n}$$

$$\begin{aligned} a_0 \quad = \quad & \cdots + \epsilon\cdot\alpha_1\Big(\ell k\cdot\alpha_{-1} - \ell^2\sum_{n=1}^{\infty}\left(\frac{k\cdot\alpha_{-(n+1)}}{n+1}\right)\left(\frac{k\cdot\alpha_n}{n}\right) + \mathcal{O}(\ell^3)\Big) \\ & +\epsilon\cdot\alpha_0\Big(1 - \ell^2\sum_{n=1}^{\infty}\left(\frac{k\cdot\alpha_{-n}}{n}\right)\left(\frac{k\cdot\alpha_n}{n}\right) + \mathcal{O}(\ell^3)\Big) \\ & +\epsilon\cdot\alpha_{-1}\Big(-\ell k\cdot\alpha_1 - \ell^2\sum_{n=1}^{\infty}\left(\frac{k\cdot\alpha_{-n}}{n}\right)\left(\frac{k\cdot\alpha_{n+1}}{n+1}\right) + \mathcal{O}(\ell^3)\Big) + \ldots \end{aligned}$$

String KLT & monodromy



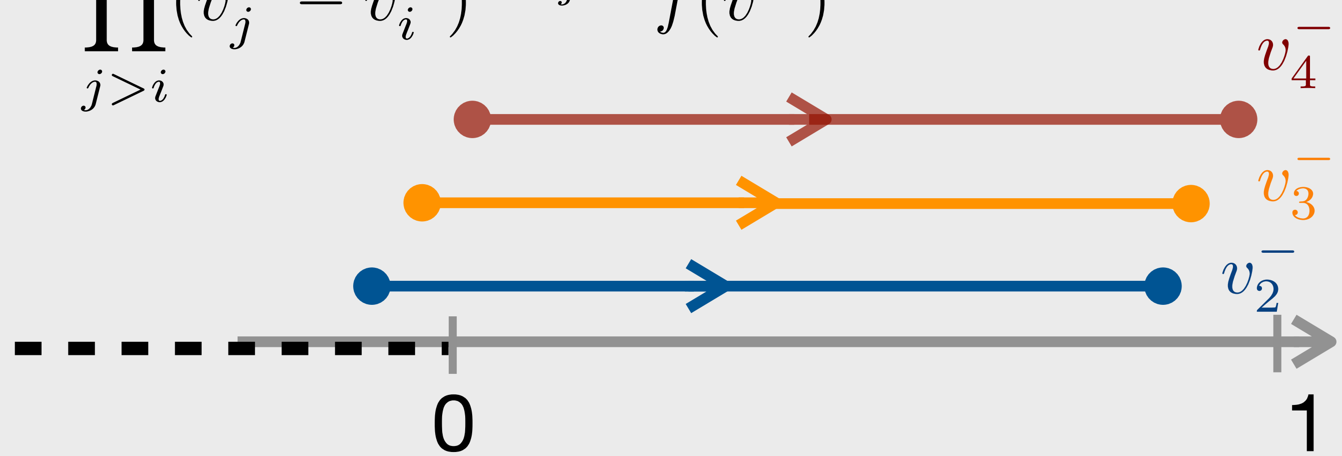
N.E.J.Bjerrum-Bohr P.H.Damgaard P.Vanhove



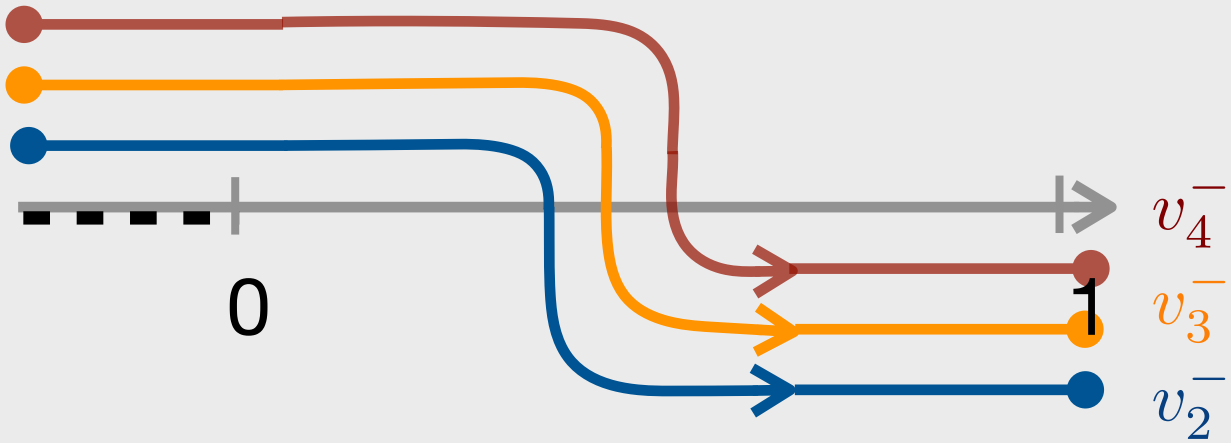
close string open string x open string

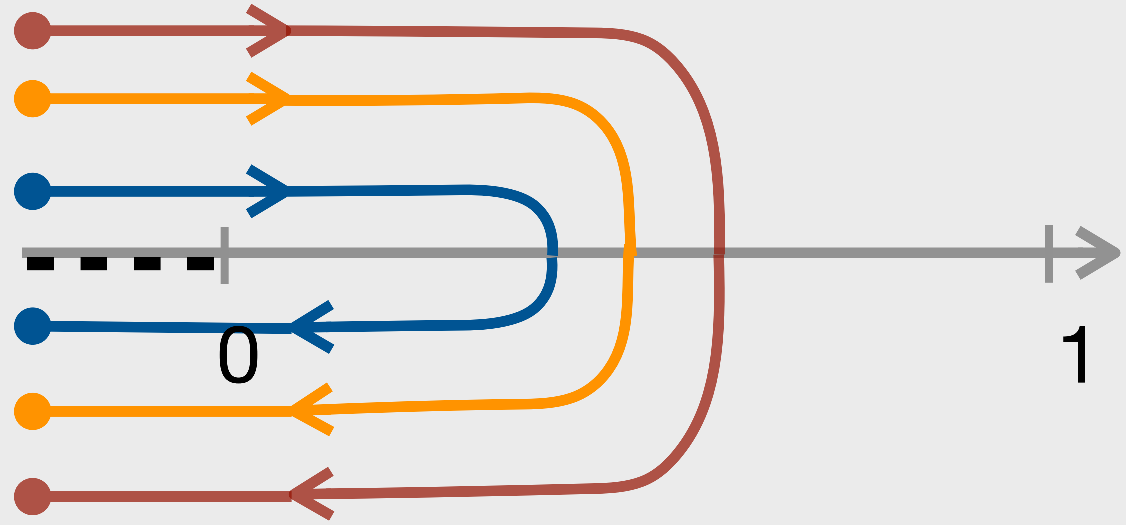
$$\mathcal{M}_n = \sum_{\sigma} \tilde{A}_n(1, \sigma(2, \ldots, n-2), n-1, n) \times \mathcal{I}(1, \sigma(2, \ldots, n-2), n-1, n)$$

$$\mathcal{I} = \int_{-\infty}^{\infty} \prod_{i=2}^{n-2} dv_i^- (v_i^-)^{\alpha' k_i \cdot k_1} (1 - v_i^-)^{\alpha' k_{n-1} \cdot k_i} \prod_{j>i} (v_j^- - v_i^-)^{\alpha' k_j \cdot k_i} f(v^-)$$



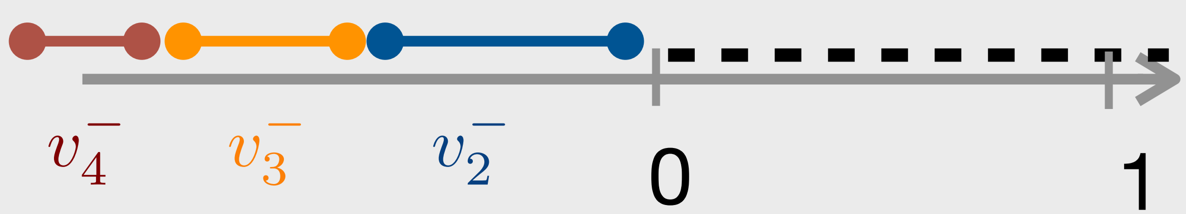
[Bjerrum-Bohr, Damgaard, Vanhove 11]





$$\int_{C_2} dv_2^- (v_2^-)^{\alpha' k_1 \cdot k_2} (1 - v_2^-)^{\alpha' k_{n-1} \cdot k_2} \prod_{j>2} (v_j^- - v_2^-)^{\alpha' k_j \cdot k_2} f(v^-)$$

$$= 2i \sin(\pi \alpha' k_1 \cdot k_2) \int_{-\infty}^{0} dv_2^- (-v_2^-)^{\alpha' k_1 \cdot k_2} (1 - v_2^-)^{\alpha' k_{n-1} \cdot k_2} \prod_{j>2} (v_j^- - v_2^-)^{\alpha' k_j \cdot k_2} f(v^-)$$



$$\int_{C_3} dv_3^- (v_3^-)^{\alpha' k_1 \cdot k_3} (v_3^- - v_2^-)^{\alpha' k_3 \cdot k_2} \ldots$$

$$= 2i \sin(\pi \alpha' k_1 \cdot k_3) \int_{v_2^- < v_3^- < 0} dv_2^- (-v_3^-)^{\alpha' k_1 \cdot k_3} (v_3^- - v_2^-)^{\alpha' k_3 \cdot k_2} \ldots$$

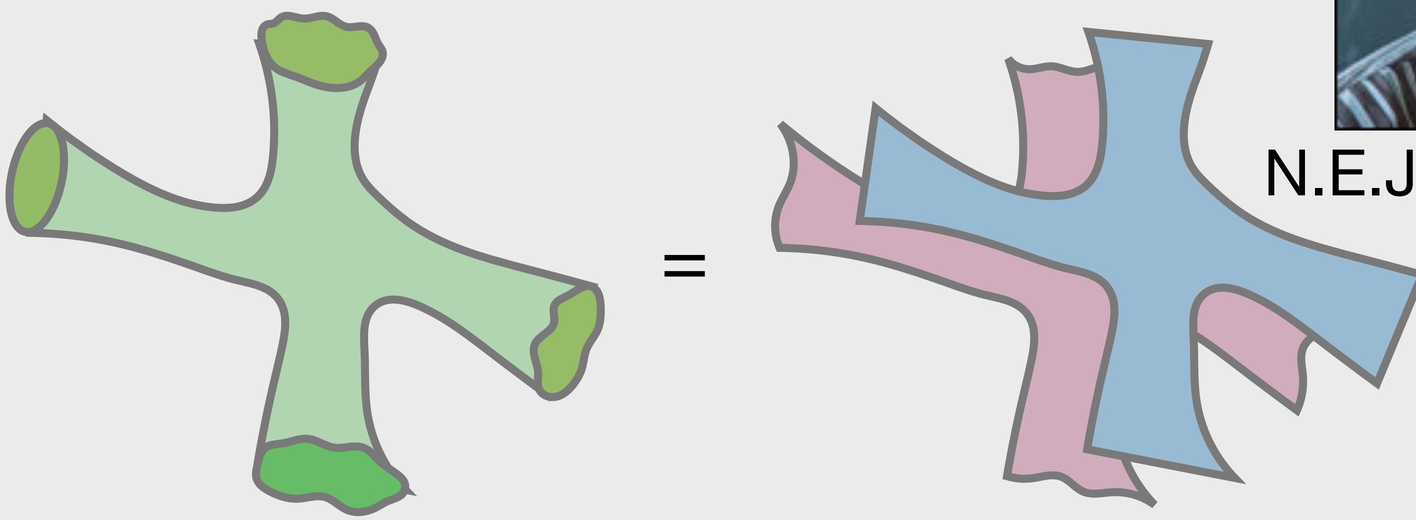
$$+ 2i \sin(\pi \alpha' (k_1 + k_2) \cdot k_3) \int_{v_3^- < v_2^-} dv_2^- (-v_3^-)^{\alpha' k_1 \cdot k_3} (v_2^- - v_3^-)^{\alpha' k_3 \cdot k_2} \ldots$$

String KLT & monodromy

N.E.J.Bjerrum-Bohr

P.H.Damgaard

P.Vanhove

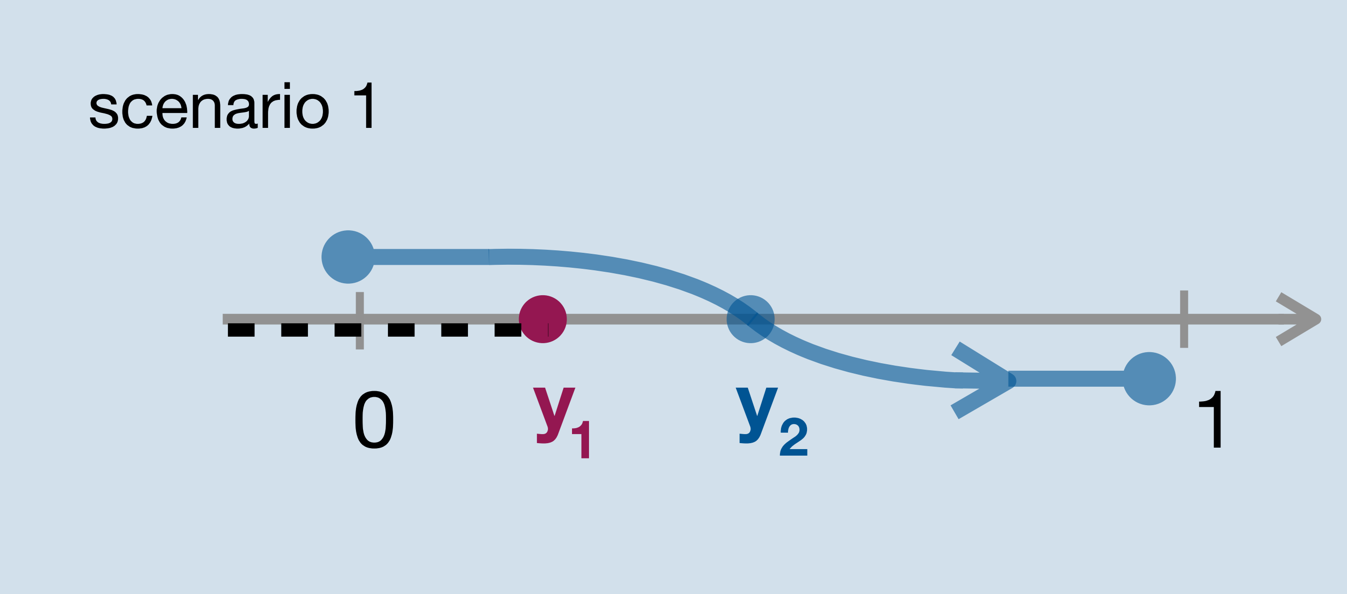


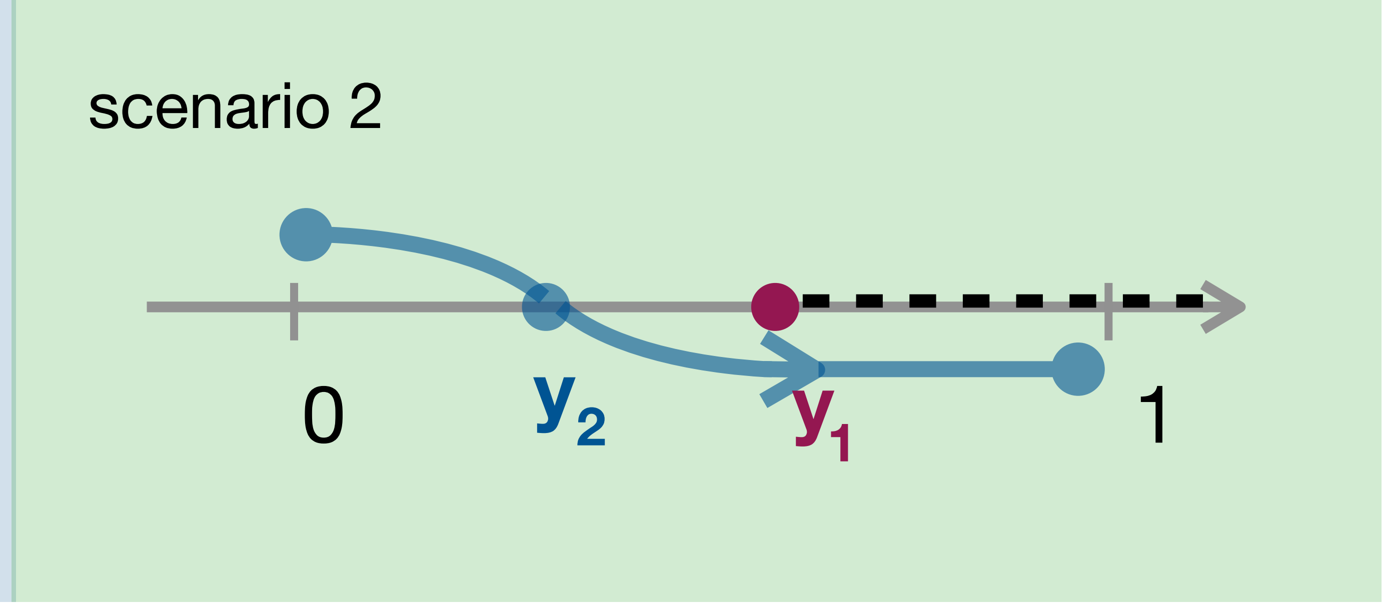
close string   open string x open string

$$\mathcal{M}_n = \sum_\sigma \tilde{A}_n(1,\sigma(2,\ldots,n-2),n-1,n) \times \mathcal{I}(1,\sigma(2,\ldots,n-2),n-1,n)$$

$$= \sum_{\sigma,\gamma\in S_{n-3}} \tilde{A}_n(1,\sigma(2,\ldots n-2),n-1,n) \times \mathcal{S}_{\alpha'}[\sigma^T|\gamma] \times \mathcal{A}(n-1,n,\gamma(2,\ldots,n-2),1)$$

[Bjerrum-Bohr, Damgaard, Vanhove 11]





$$\left(\int_0^1 dy_2 \frac{V(y_2)}{y_2}\right) \frac{V(y_1)}{y_1}$$

$$\sim e^{\alpha' k_1 \cdot k_2 \, ln \, y_2} e^{-\alpha' \sum_1^\infty \frac{1}{n}\left(\frac{y_1}{y_2}\right)^n} : V(y_1)V(y_2) :$$

$$\sim e^{\alpha' k_1 \cdot k_2 \, ln(y_2 - y_1)}$$

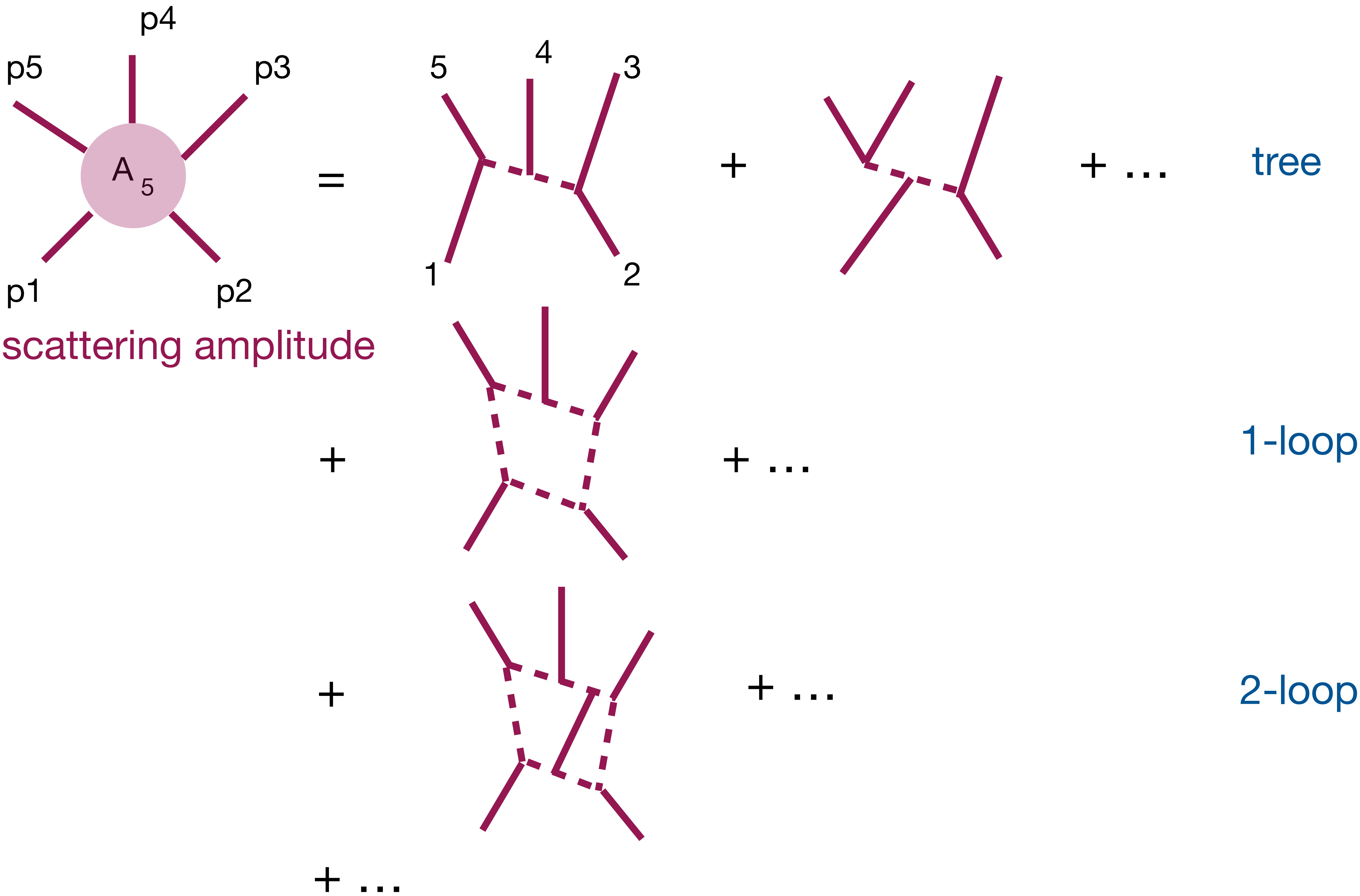
$$\longrightarrow \quad (y_2 - y_1)^{\alpha' k_1 \cdot k_2}$$

$$\frac{V(y_1)}{y_1} \left(\int_0^1 dy_2 \frac{V(y_2)}{y_2}\right)$$

$$\sim e^{\alpha' k_1 \cdot k_2 \, ln \, y_1} e^{-\alpha' \sum_1^\infty \frac{1}{n}\left(\frac{y_2}{y_1}\right)^n} : V(y_1)V(y_2) :$$

$$\sim e^{\alpha' k_1 \cdot k_2 \, ln(y_1 - y_2)}$$

$$\longrightarrow \quad (y_1 - y_2)^{\alpha' k_1 \cdot k_2}$$

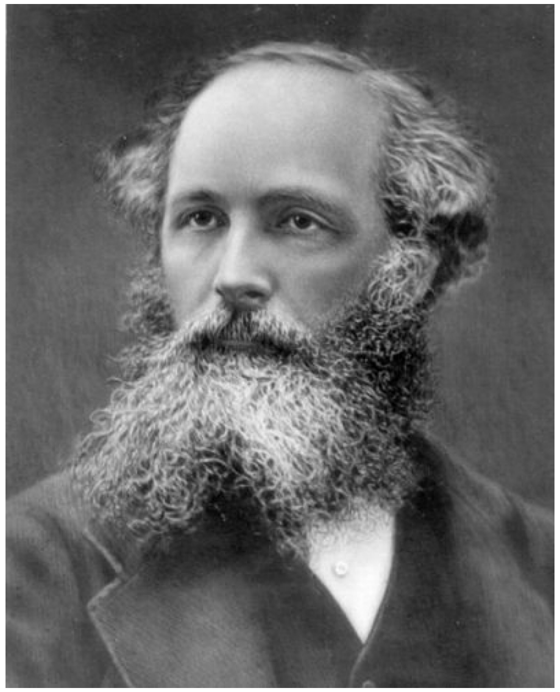


p4
p5
p3
A 5
p1
p2
scattering amplitude
=
5
4
3
1
2
+
+ ...
tree
+
+ ...
1-loop
+
+ ...
2-loop
+ ...

A Dynamical Theory of the Electromagnetic Field

J. Clerk Maxwell

Phil. Trans. R. Soc. Lond. 1865 **155**, 459-512, published 1 January 1865



J. C. Maxwell

vector analysis

$$\nabla \cdot \mathbb{E} = 0 \qquad \nabla \times \mathbb{E} = -\frac{\partial \mathbb{B}}{\partial t}$$

$$\nabla \cdot \mathbb{B} = 0 \qquad \nabla \times \mathbb{B} = \frac{1}{c^2}\frac{\partial \mathbb{E}}{\partial t}$$

differential geometry

$$dF = 0$$

quantities, namely,

For Electromagnetic Momentum	F	G	H
„ Magnetic Intensity	α	β	γ
„ Electromotive Force	P	Q	R
„ Current due to true conduction	p	q	r
„ Electric Displacement	f	g	h
„ Total Current (including variation of displacement)	p'	q'	r'
„ Quantity of free Electricity	e		
„ Electric Potential	Ψ		

Between these twenty quantities we have found twenty equations, viz.

Three equations of Magnetic Force	(B)
„ Electric Currents	(C)
„ Electromotive Force	(D)
„ Electric Elasticity	(E)
„ Electric Resistance	(F)
„ Total Currents	(A)
One equation of Free Electricity	(G)
„ Continuity	(H)

These equations are therefore sufficient to determine all the quantities which occur in them, provided we know the conditions of the problem. In many questions, however, only a few of the equations are required.

Intrinsic Energy of the Electromagnetic Field.

(71) We have seen (33) that the intrinsic energy of any system of currents is found by multiplying half the current in each circuit into its electromagnetic momentum. This is equivalent to finding the integral

$$E=\tfrac{1}{2}\Sigma(Fp'+Gq'+Hr')dV \quad . \quad . \quad . \quad . \quad (37)$$

over all the space occupied by currents, where p, q, r are the components of currents, and F, G, H the components of electromagnetic momentum.

Substituting the values of p', q', r' from the equations of Currents (C), this becomes

$$\frac{1}{8\pi}\Sigma\left\{F\left(\frac{d\gamma}{dy}-\frac{d\beta}{dz}\right)+G\left(\frac{d\alpha}{dz}-\frac{d\gamma}{dx}\right)+H\left(\frac{d\beta}{dx}-\frac{d\alpha}{dy}\right)\right\}dV.$$

Integrating by parts, and remembering that α, β, γ vanish at an infinite distance, the expression becomes

$$\frac{1}{8\pi}\Sigma\left\{\alpha\left(\frac{dH}{dy}-\frac{dG}{dz}\right)+\beta\left(\frac{dF}{dz}-\frac{dH}{dx}\right)+\gamma\left(\frac{dG}{dx}-\frac{dF}{dy}\right)\right\}dV,$$

where the integration is to be extended over all space. Referring to the equations of Magnetic Force (B), p. 482, this becomes

$$E=\frac{1}{8\pi}\Sigma\{\alpha.\mu\alpha+\beta.\mu\beta+\gamma.\mu\gamma\}dV, \quad . \quad . \quad . \quad . \quad (38)$$