



# Stochastic Gravitational Wave Background

Shi Pi

Kavli IPMU, University of Tokyo

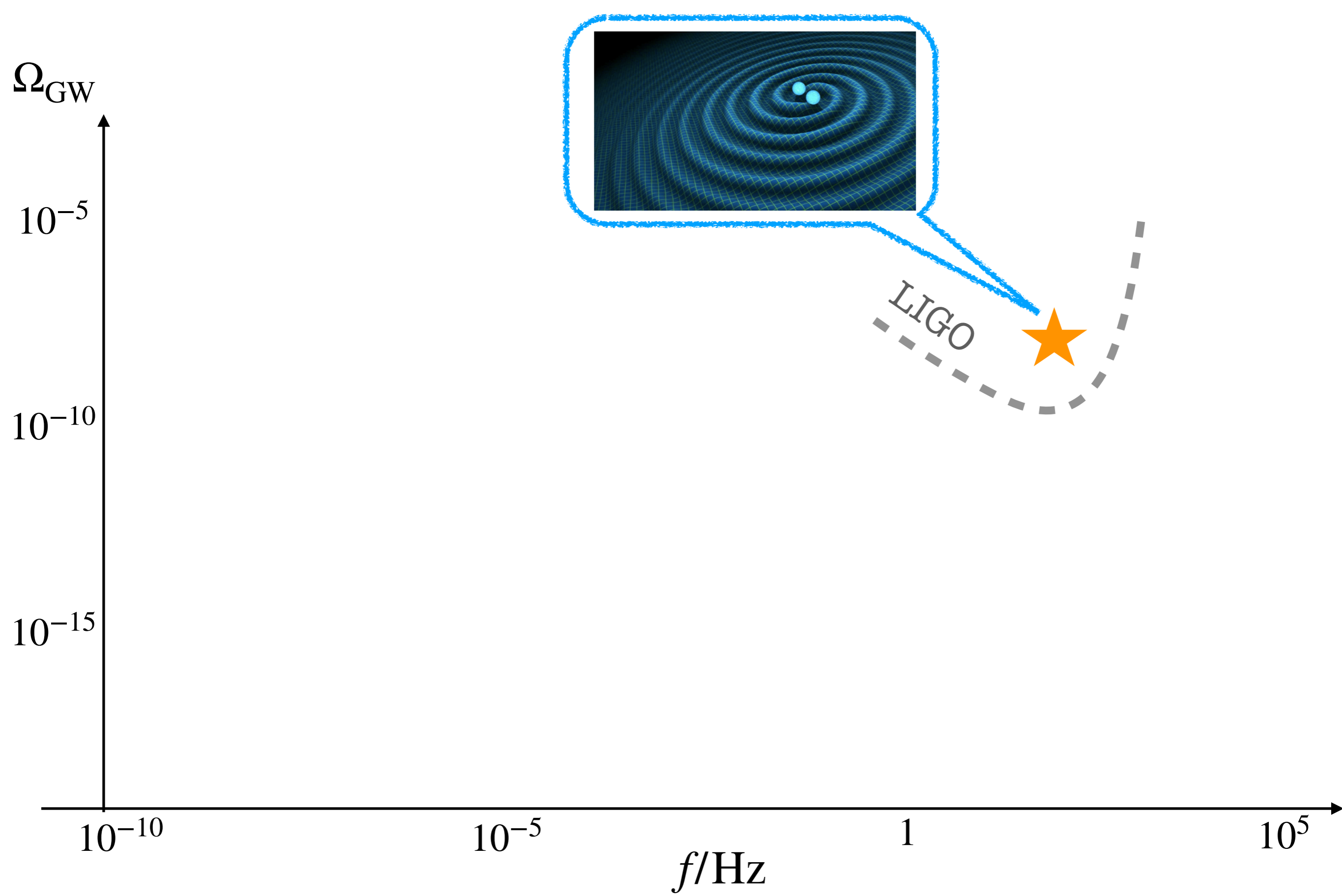
Based on arXiv:1810.11000,  
with Rong-gen Cai and Misao Sasaki

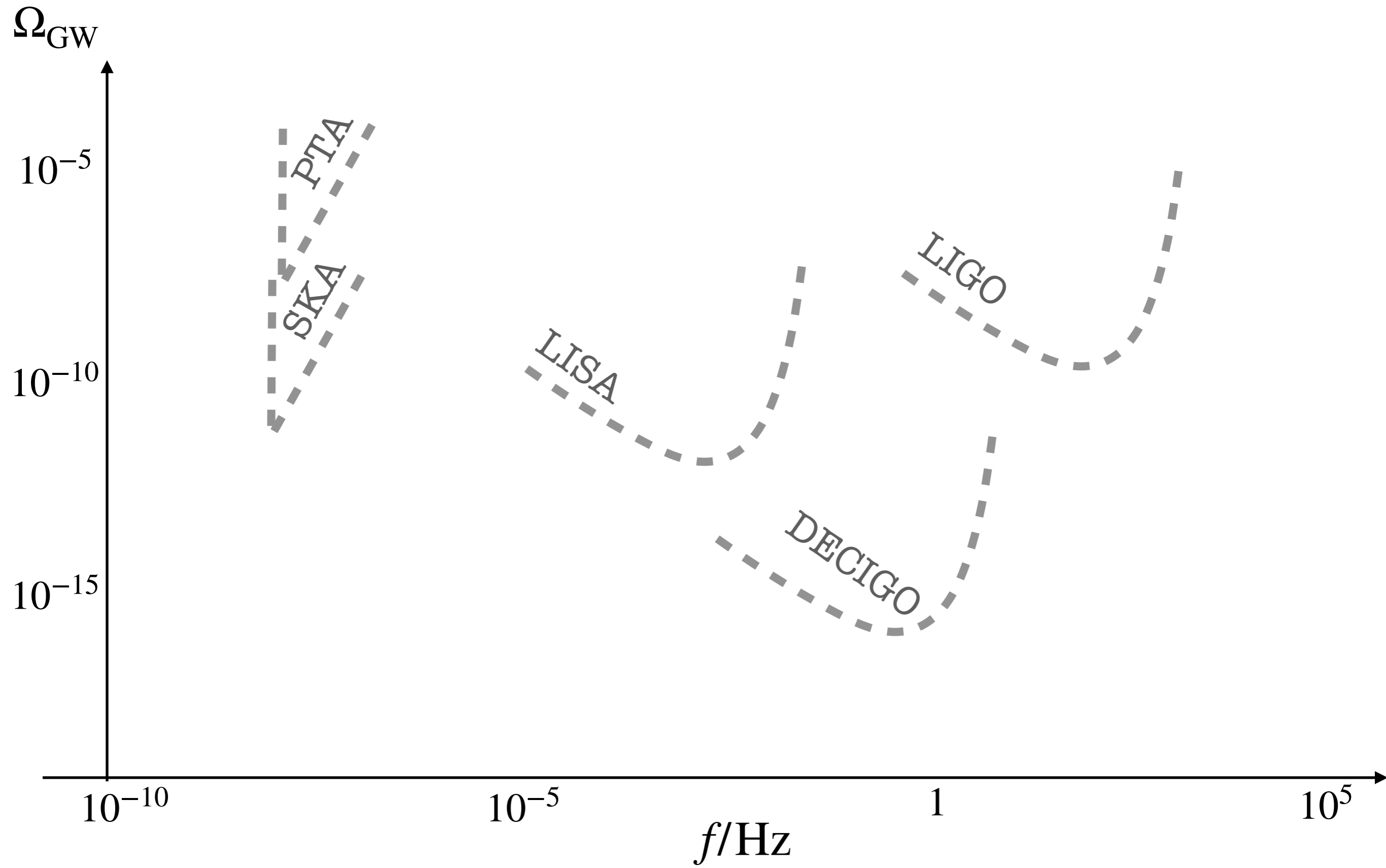
# Content

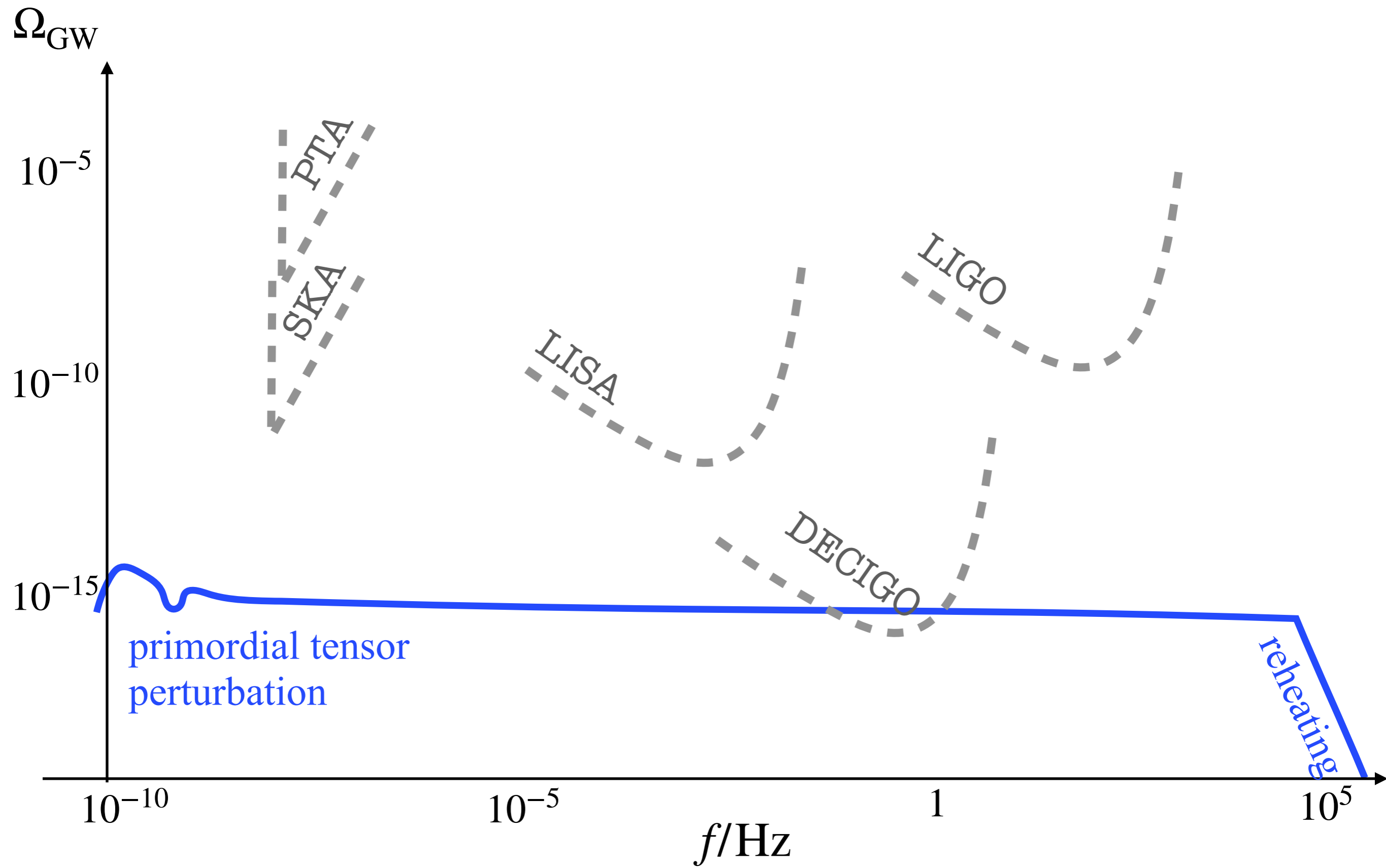
- Mechanism of SGWB
- PBH abundances and GWs
- Induced GWs: A probe for non-Gaussianity
- Conclusion

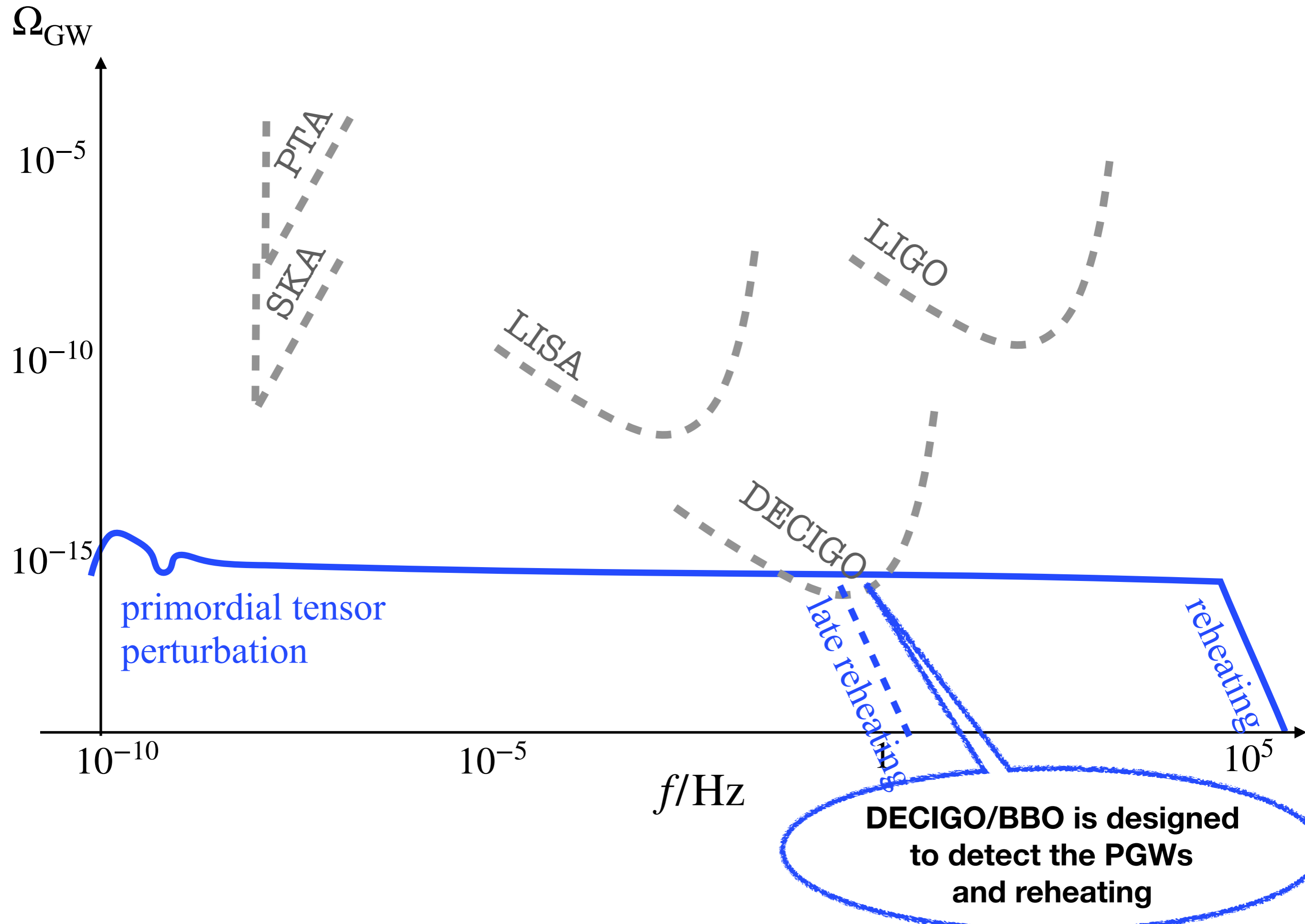
# Content

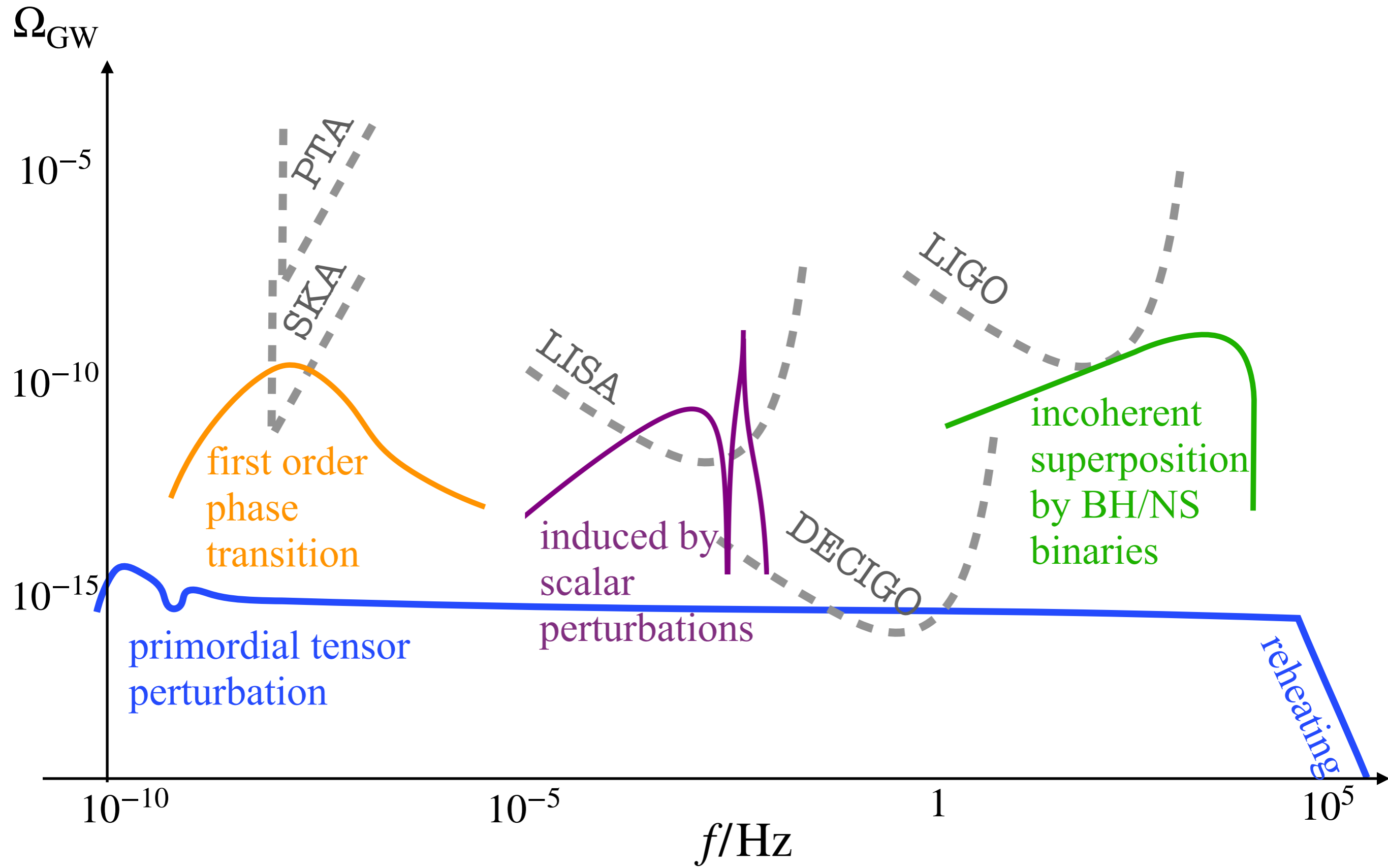
- Mechanism of SGWB
- PBH abundances and GWs
- Induced GWs: A probe for non-Gaussianity
- Conclusion



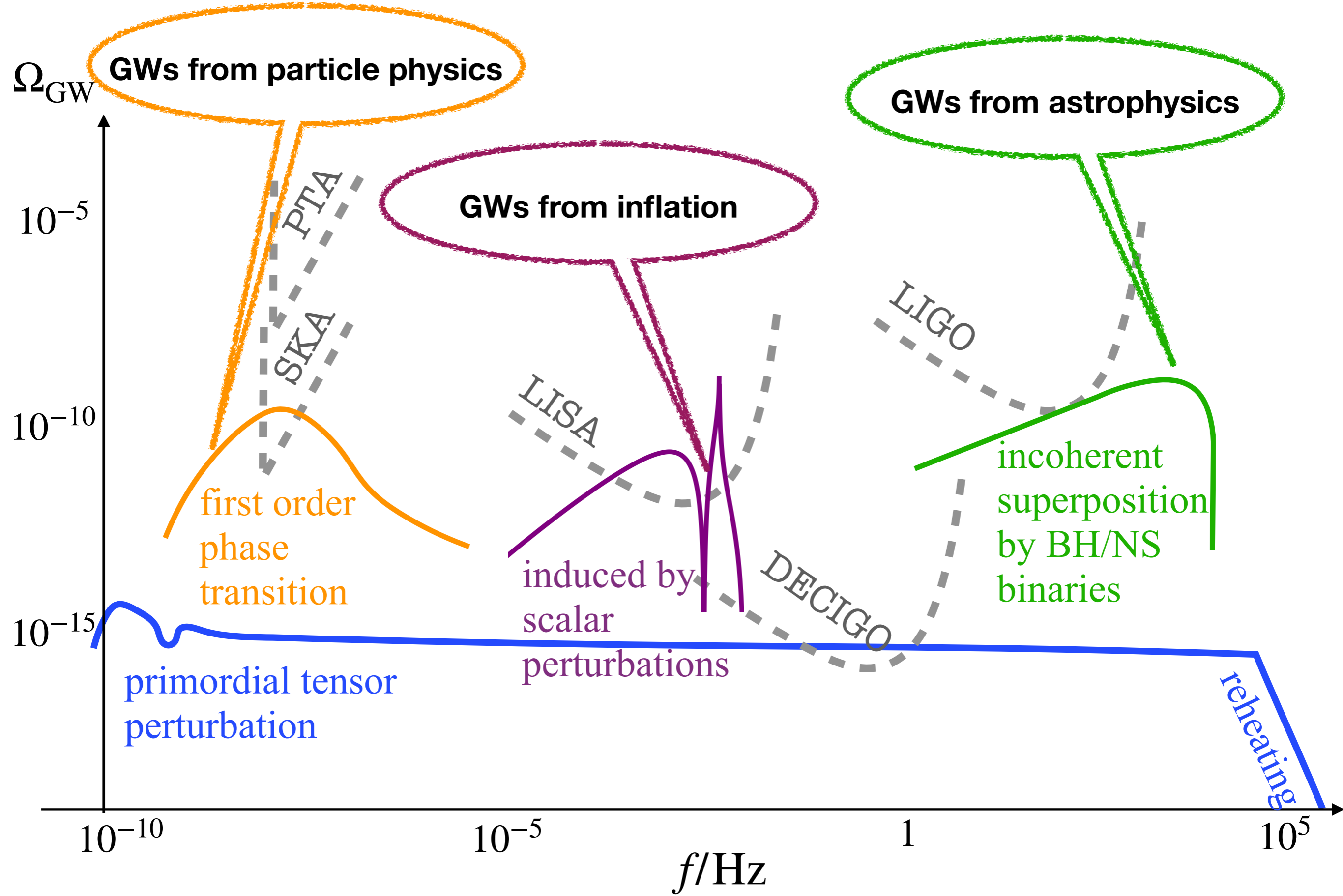


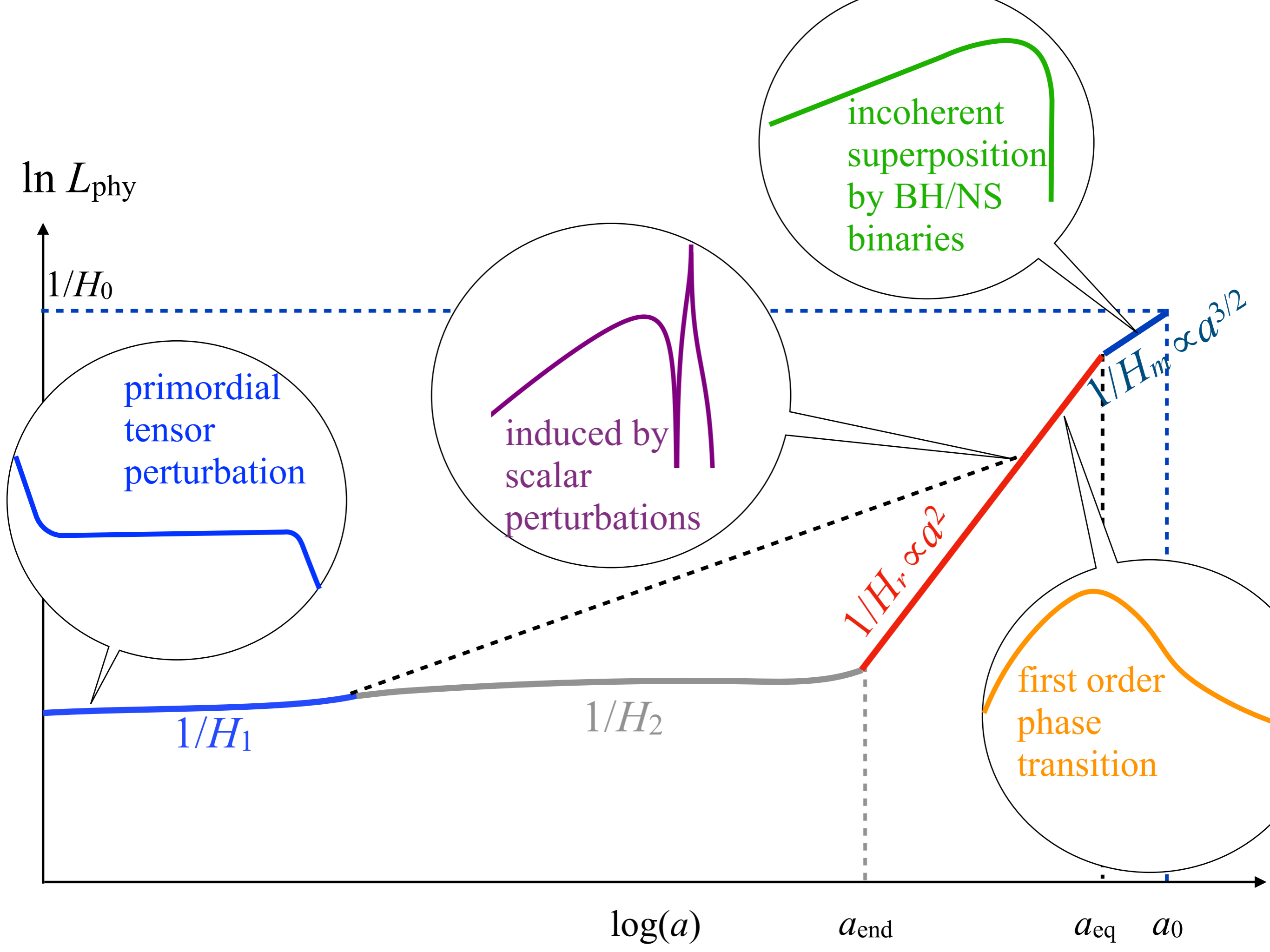






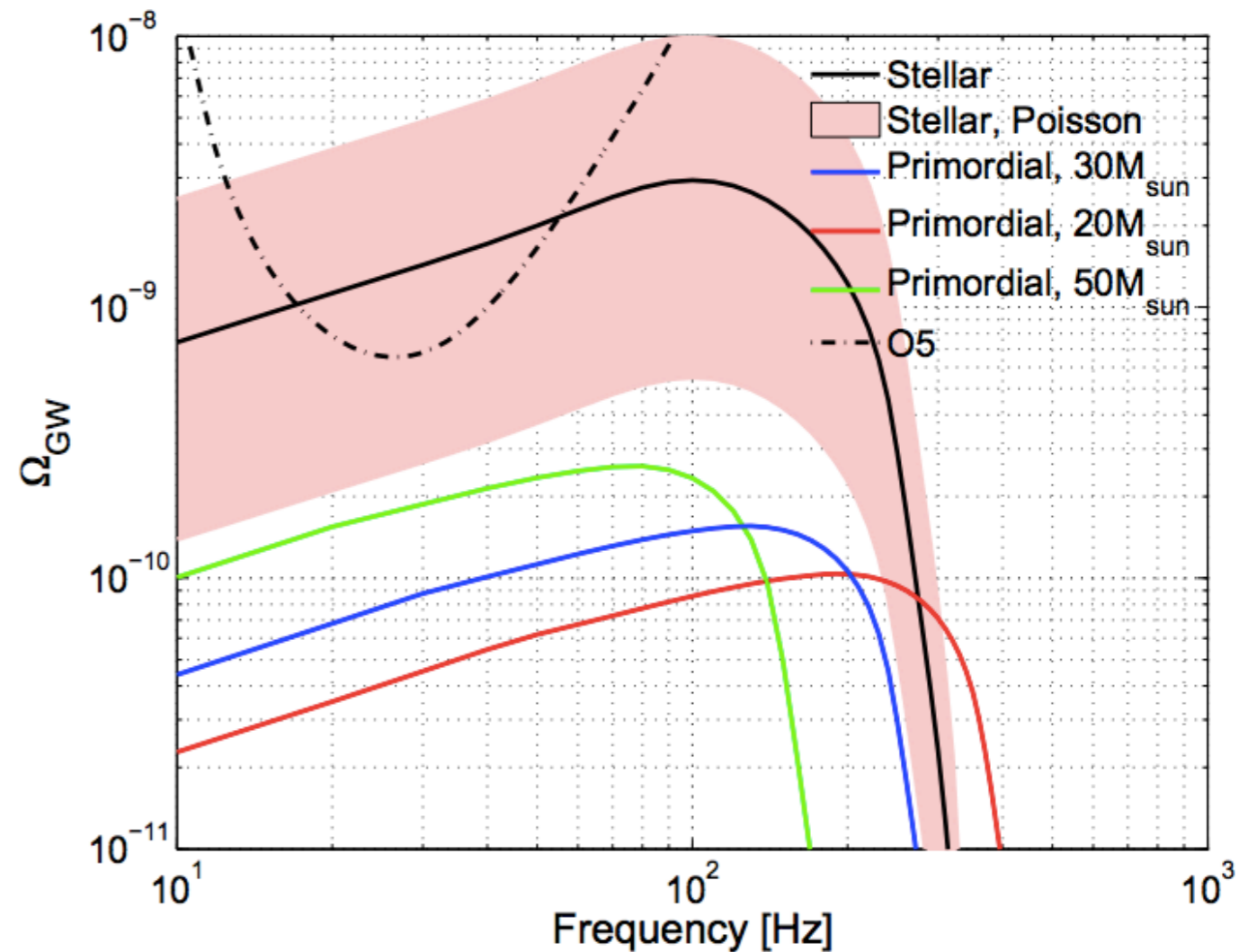






# SGWB from binaries

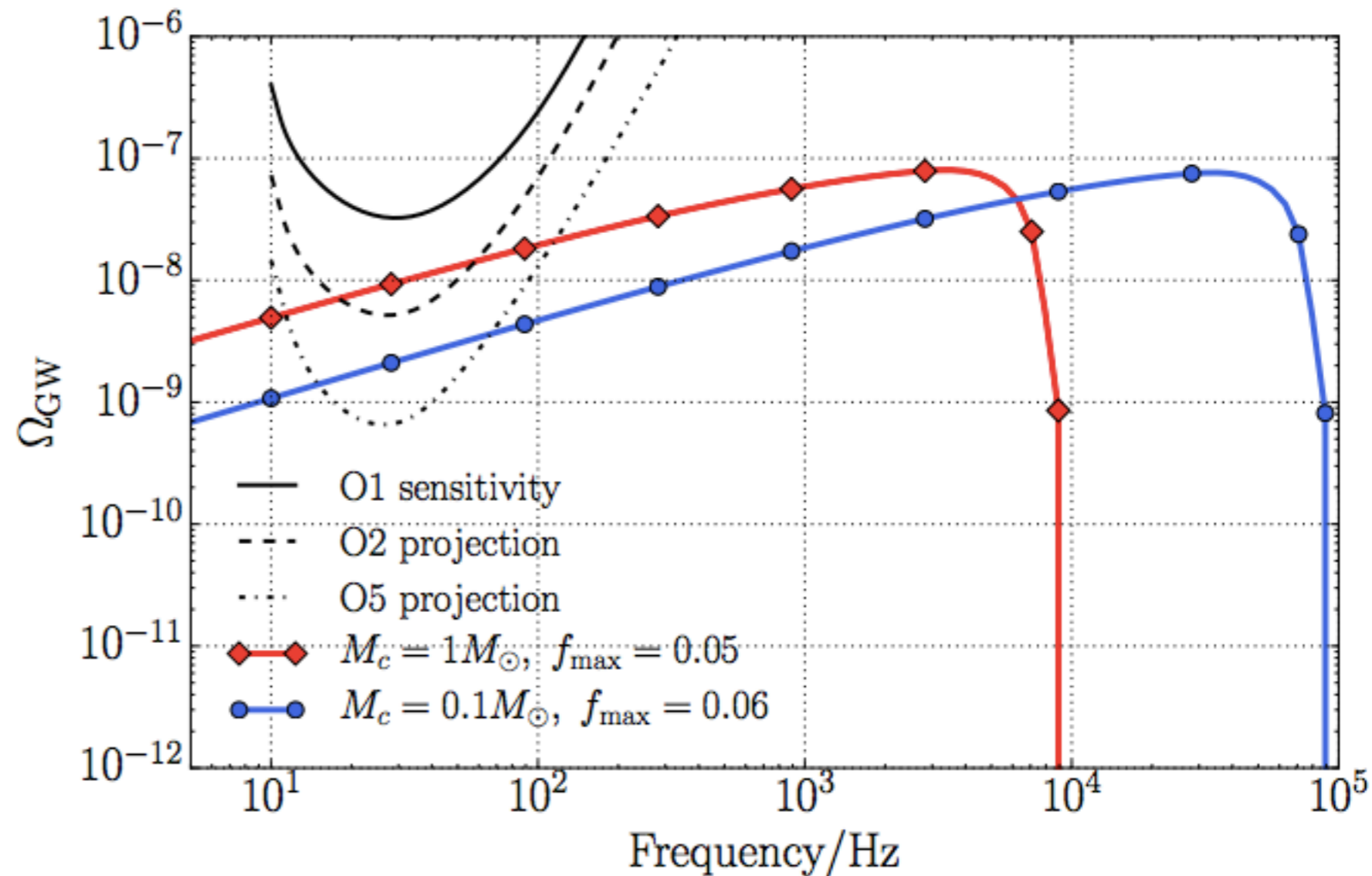
- Origin: incoherent superposition of the GWs emitted by compact star binaries (BH, NS,...)
- Frequencies: 100 Hz (for  $10M_{\odot}$ )
- Amplitude:  $10^{-9}$



1608.06699

# SGWB from PBHB

- PBH binaries
- Frequency: 1000Hz (for  $1M_{\odot}$ )
- Amplitude:  $10^{-9}$
- Can be used to constrain PBH abundances



1610.08725

# SGWB from 1OPT

- 1st order phase transition
- EW (extended) or beyond
- Dynamics: Bubble nucleation and collision, turbulence, sound waves, .....
- Parameters:  $\alpha$  (latent heat) and  $1/\beta$  (duration of the phase transition)

$$f_{\text{peak}} \simeq 10^{-6} \text{Hz} \left( \frac{\beta}{H_*} \right) \left( \frac{T_*}{100 \text{GeV}} \right) \left( \frac{g_*}{100} \right)^{1/6}$$

$$\Omega_{\text{peak}} h^2 \simeq 10^{-4} \left( \frac{\beta}{H_*} \right)^{-2} \left( \frac{g_*}{100} \right)^{-1/3}$$

# SGWB from 1OPT

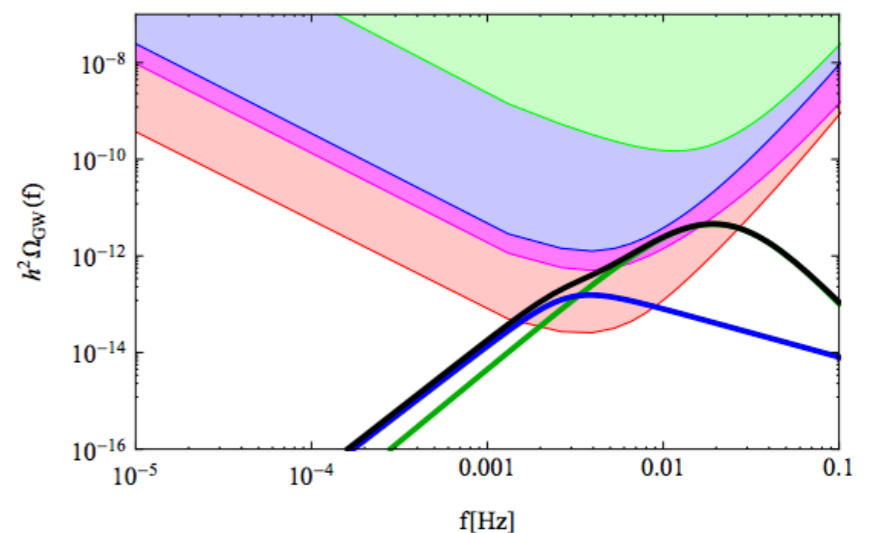
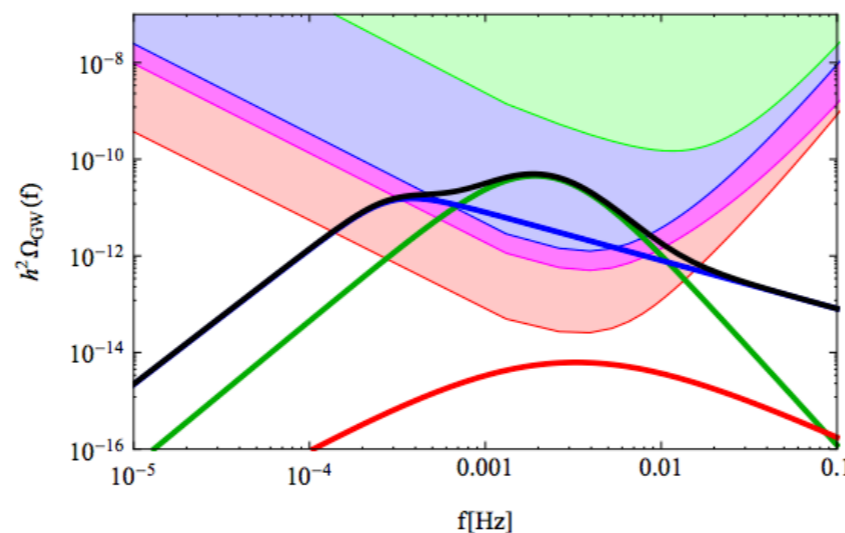
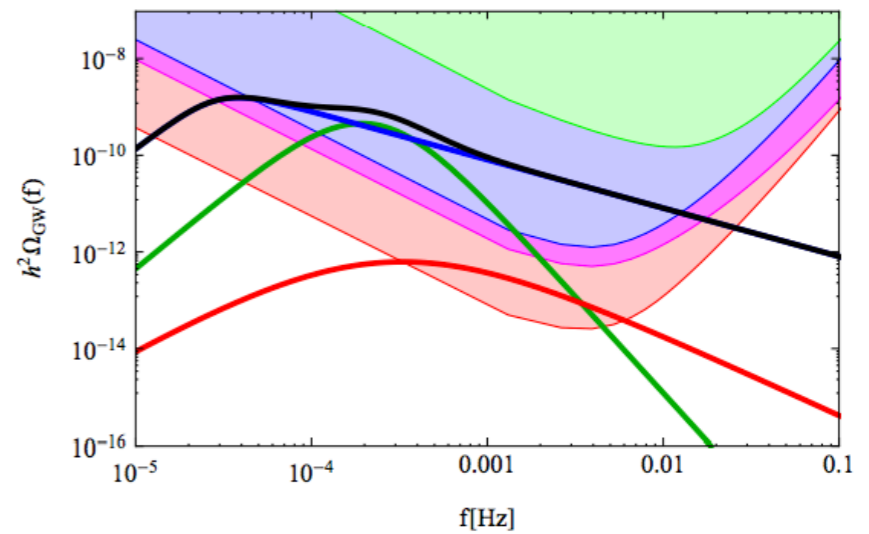
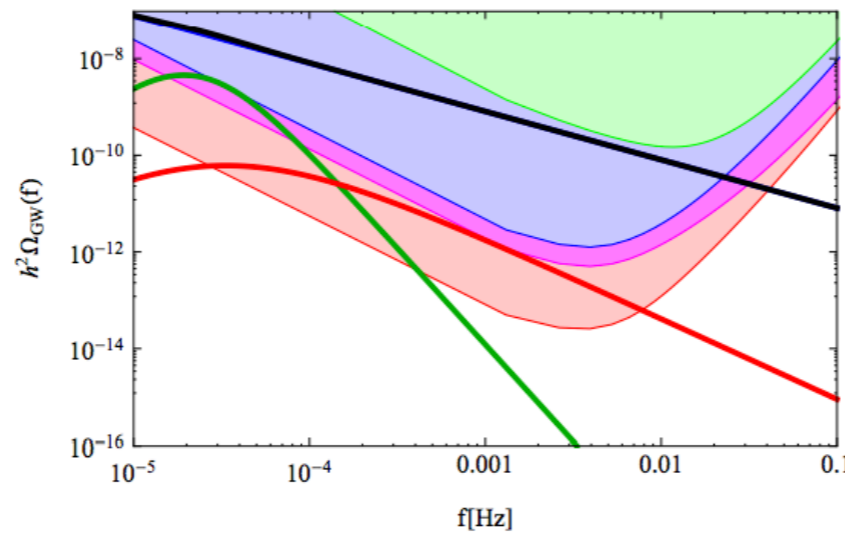
$$f_{\text{peak}} \simeq 10^{-6} \text{Hz} \left( \frac{\beta}{H_*} \right) \left( \frac{T_*}{100 \text{GeV}} \right) \left( \frac{g_*}{100} \right)^{1/6}$$

$$\Omega_{\text{peak}} h^2 \simeq 10^{-6} \left( \frac{\beta}{H_*} \right)^{-2} \left( \frac{g_*}{100} \right)^{-1/3}$$

- Key feature:  $k^2$  increasing,  $k^{-2}$  decreasing.

- For  $\beta/H_* \sim 100$ , frequency is  $10^{-4} \text{Hz}$ , in LISA band, but the peak is only  $10^{-10}$ .

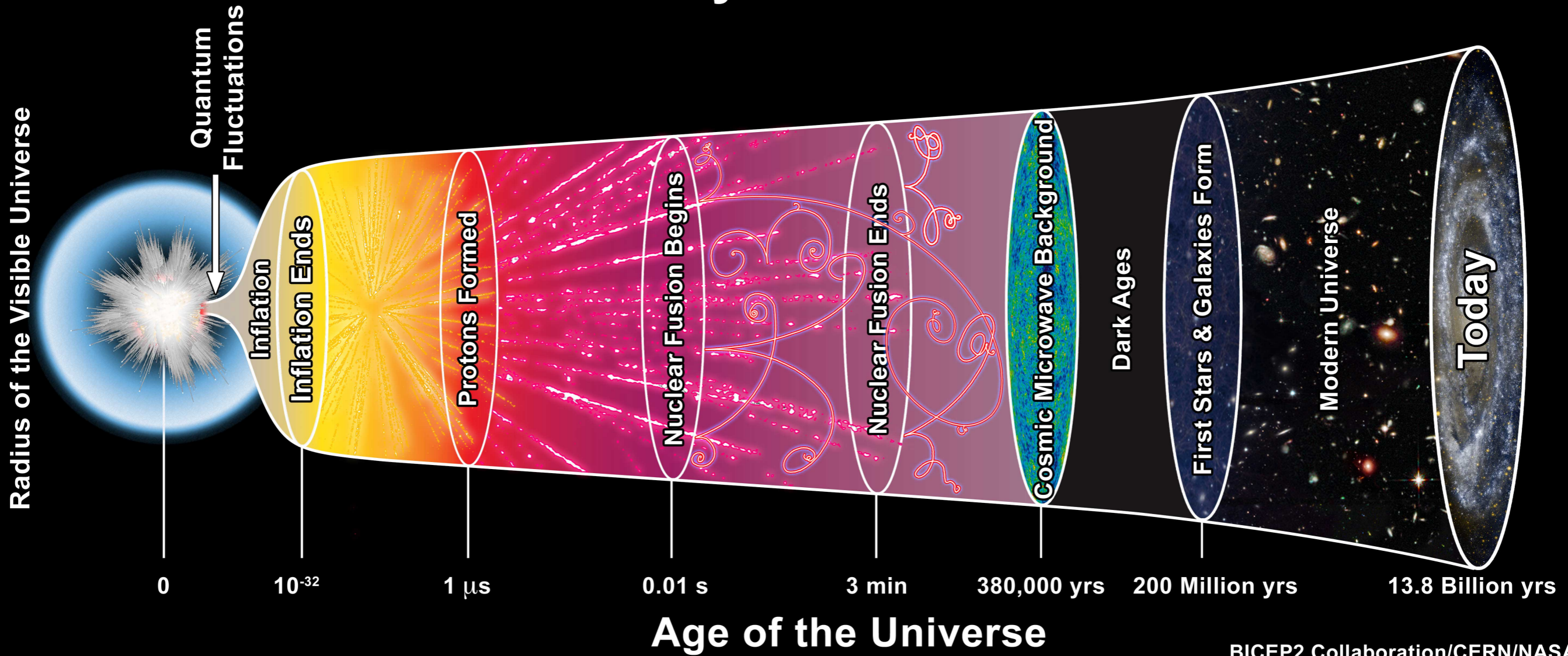
1512.06239



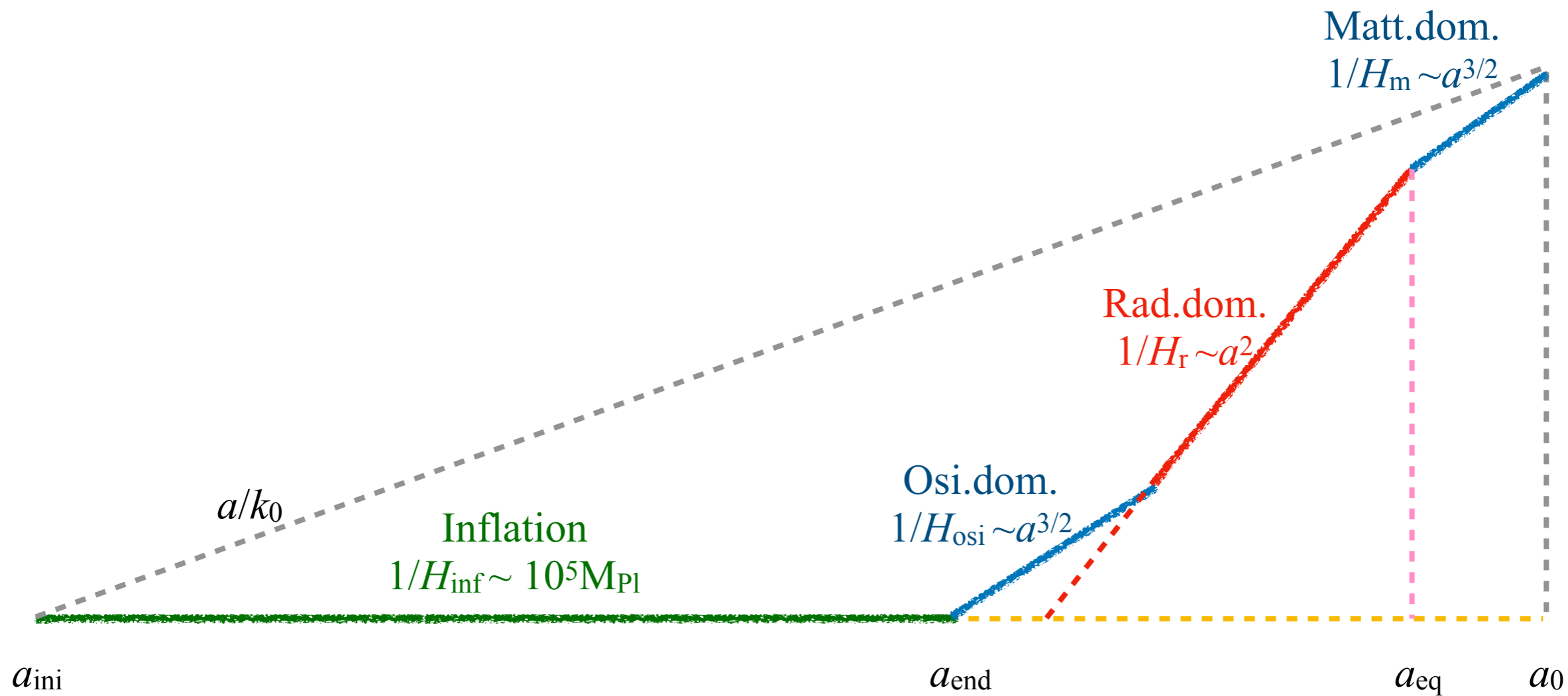
# Content

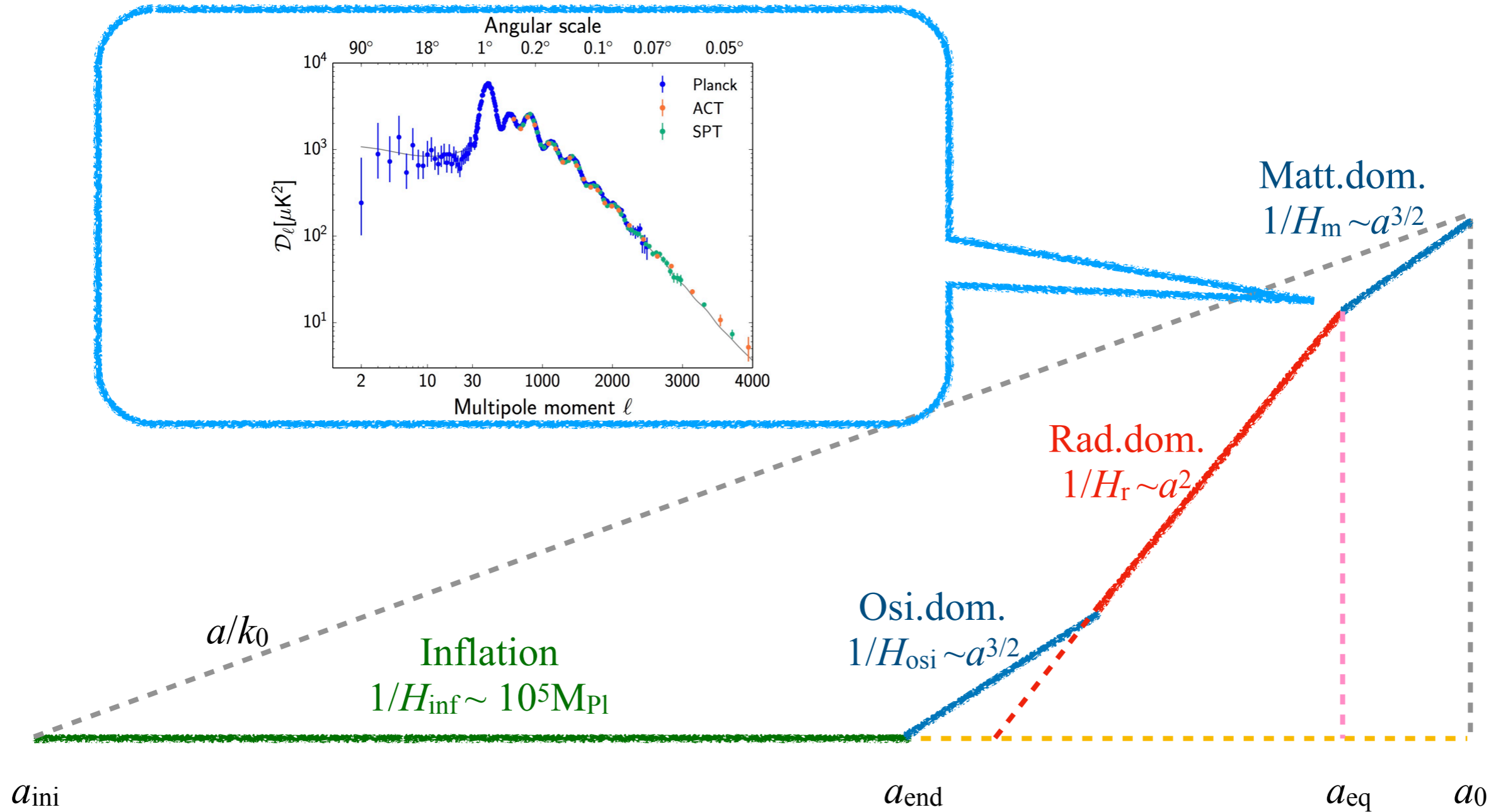
- Mechanism of SGWB
- **PBH abundances and GWs**
- Induced GWs: A probe for non-Gaussianity
- Conclusion

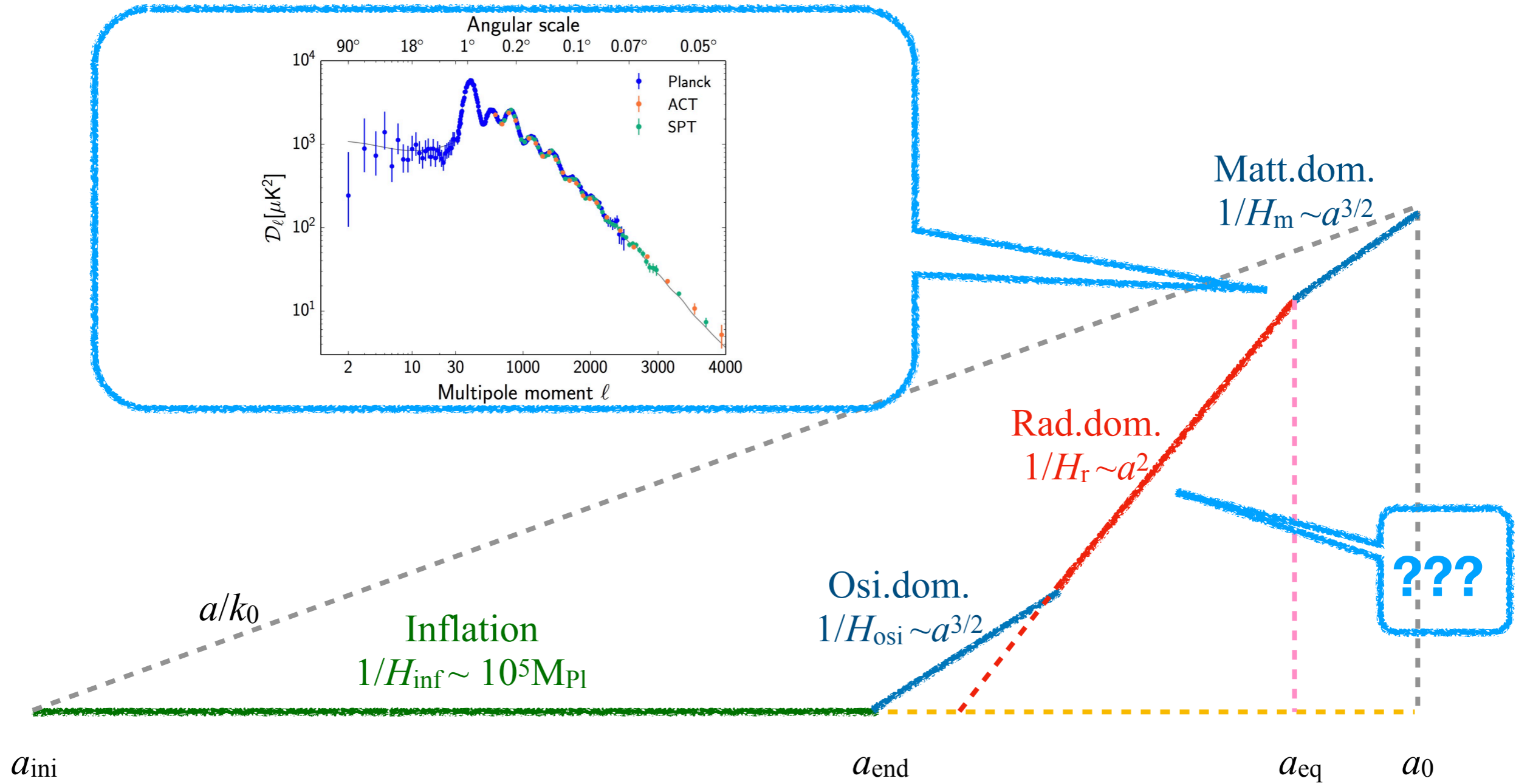
# History of the Universe



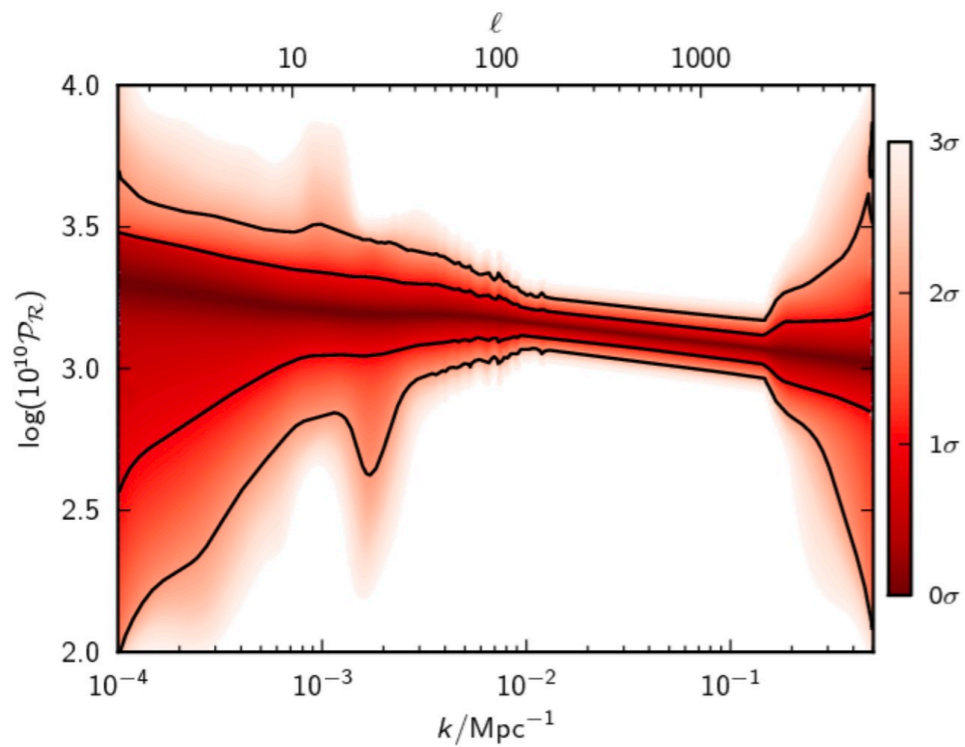




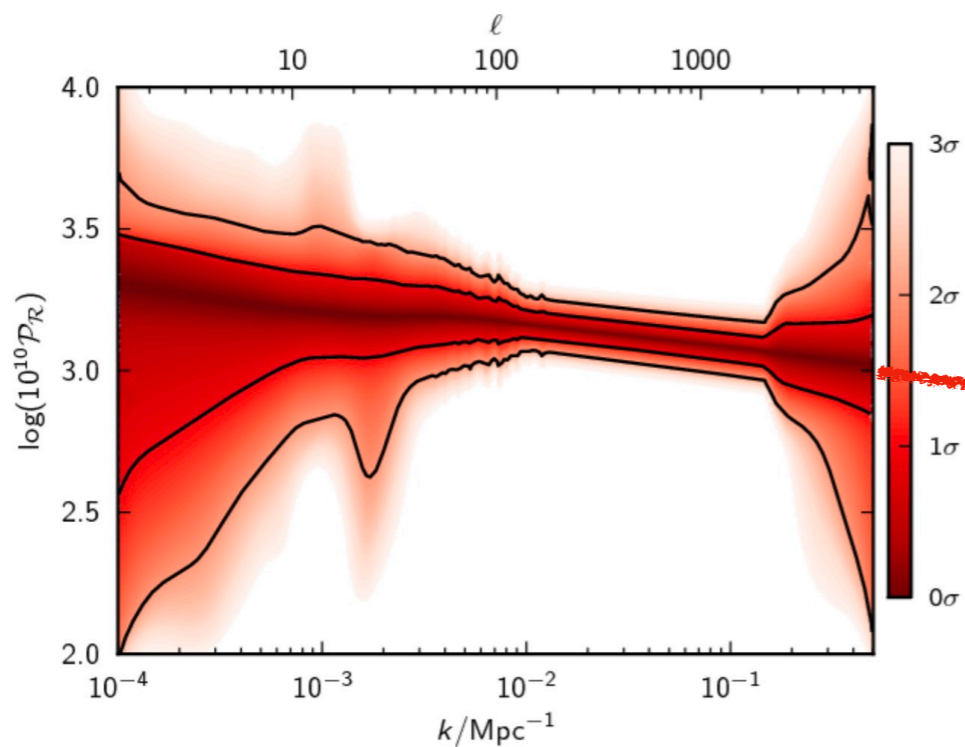




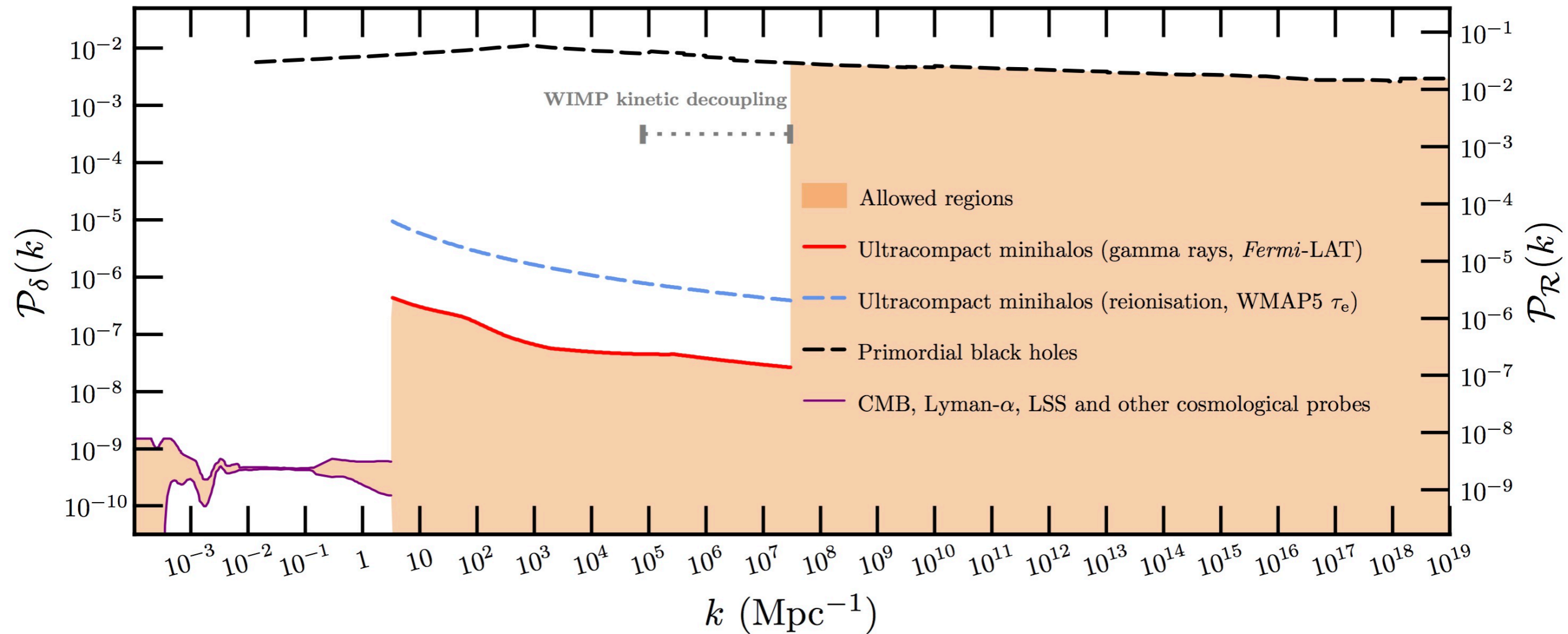
# Bayesian reconstruction of the primordial power spectrum for $l < 2300$ . (Planck 2015)



Bayesian reconstruction  
of the primordial power  
spectrum for  $l < 2300$ .  
(Planck 2015)

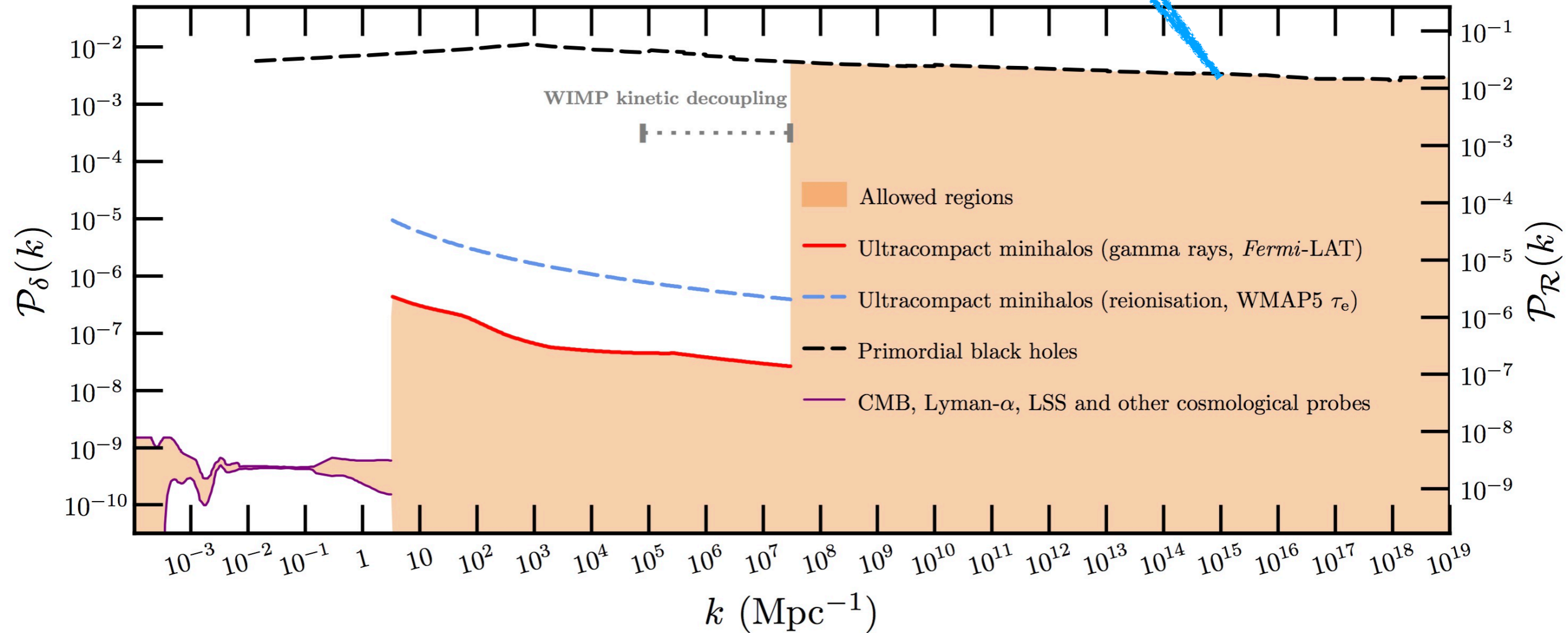


The resolution is  
lacking to say anything  
precise about higher  $l$ .



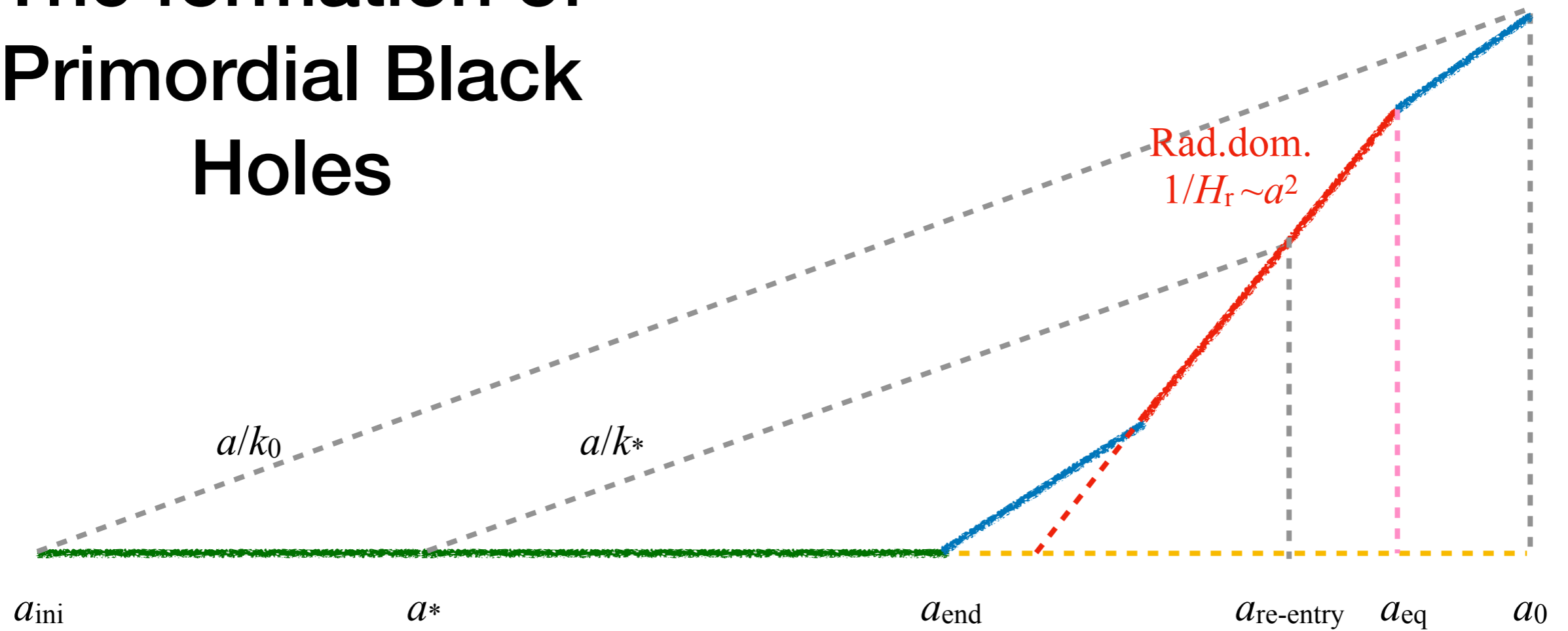
There are some constraints on small scales, but quite loose. (Bringmann et. al. 2011)

The constraints on small scales are mainly from PBHs. (Josan et. al. 2009)



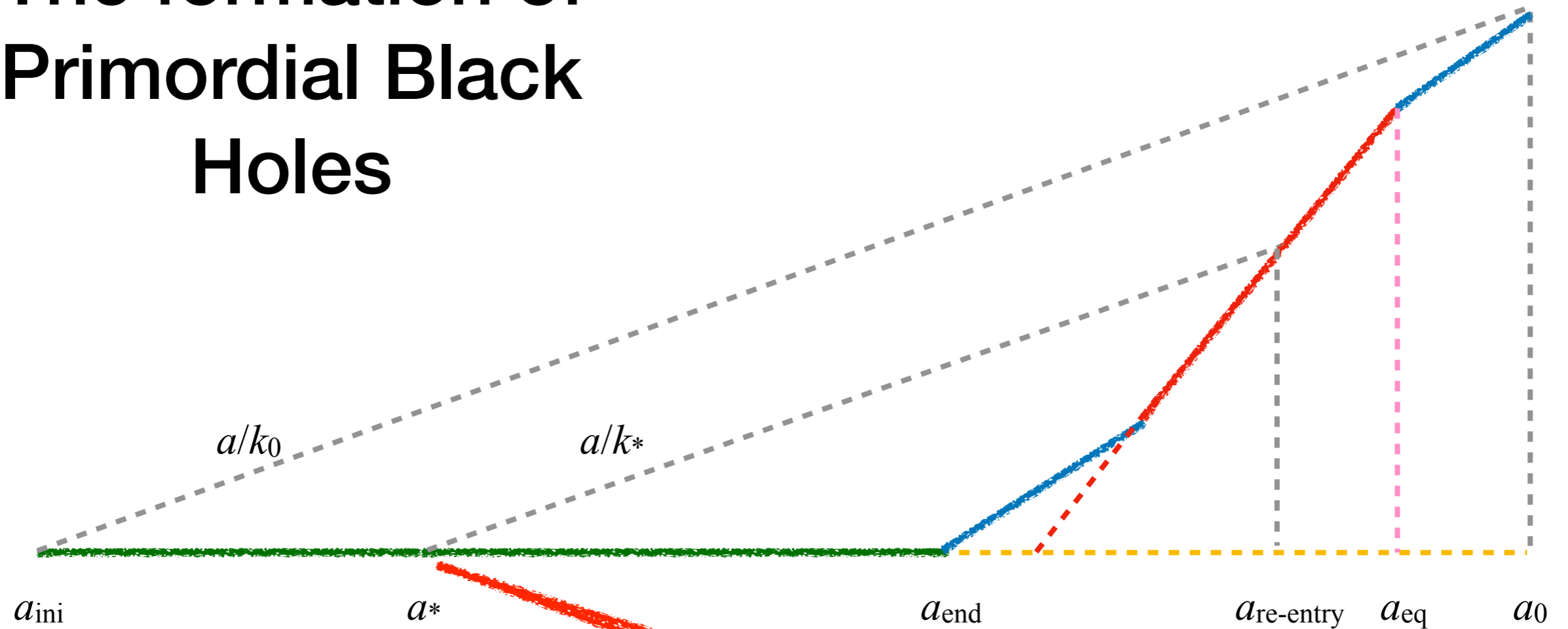
There are some constraints on small scales, but quite loose.

# The formation of Primordial Black Holes





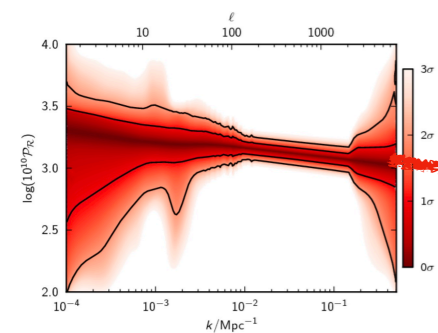
# The formation of Primordial Black Holes



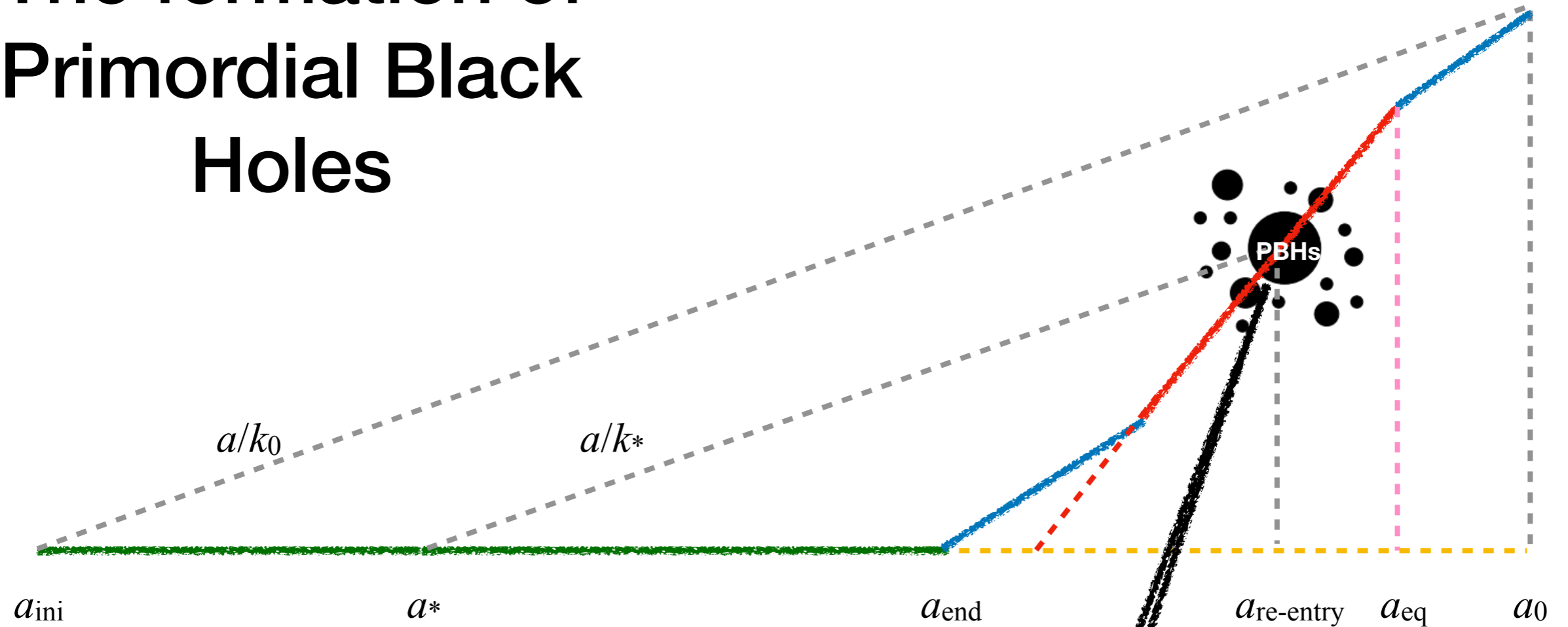
There is a peak on the primordial density perturbation, which leaves horizon and gets frozen at  $a^*$ .

$$k^* = Ha^*$$

SP, Zhang, Huang & Sasaki 1712.09896

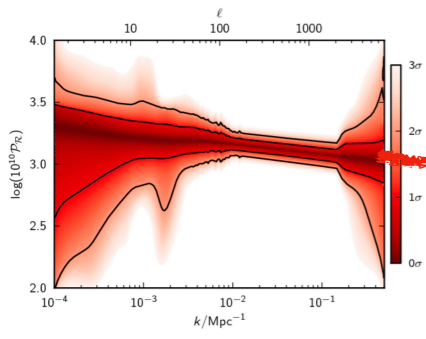


# The formation of Primordial Black Holes

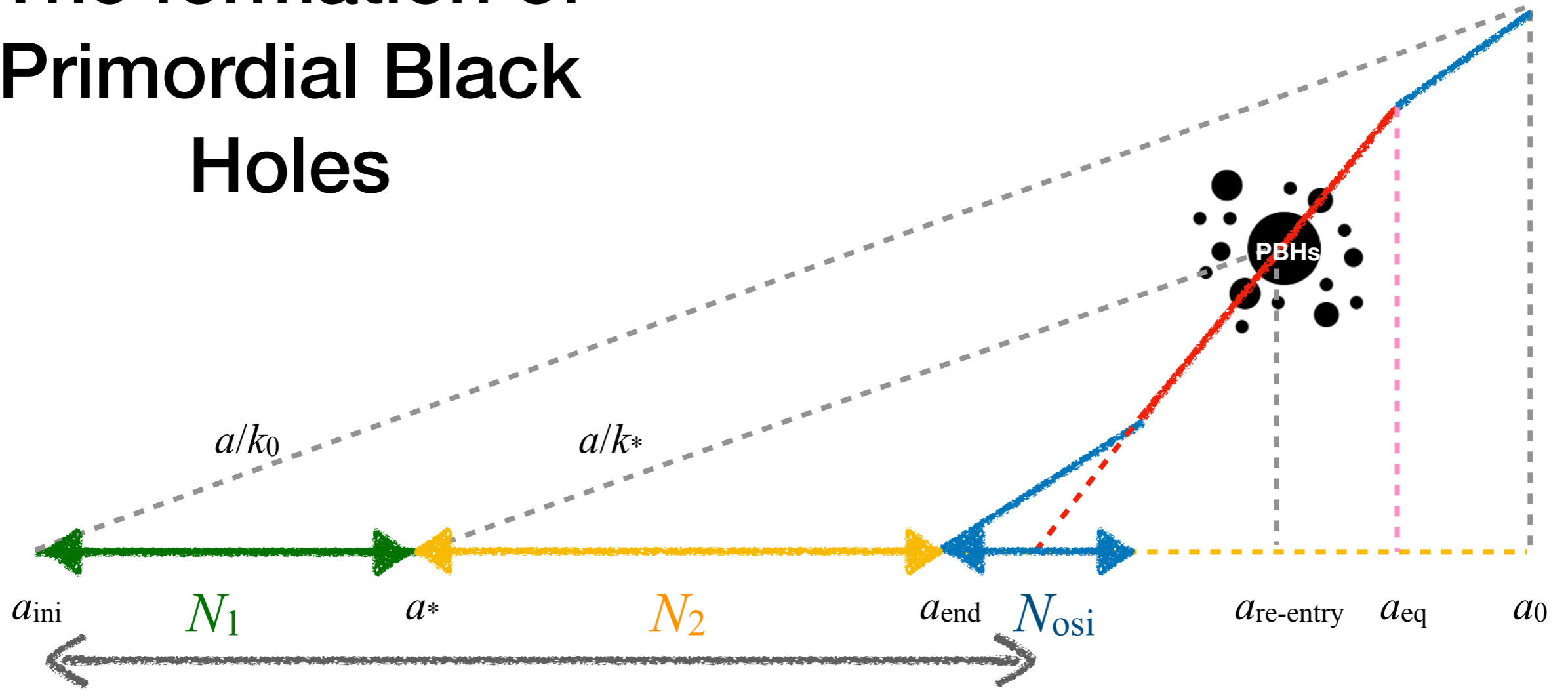


The peak scale re-enters the horizon at radiation dominated era. If it exceeded some critical value  $O(0.1)$ , PBH will form. Its mass is  $O(M_H)$ .

$$k^* = Ha^*$$



# The formation of Primordial Black Holes



$$N_1 + N_2 + N_{osi}/4 \approx 60$$

PBH mass:  $M_{PBH} \sim M_H \sim \frac{M_{Pl}^2}{H_*} e^{2(N_2 + N_{osi}/4)} = \frac{M_{Pl}^2}{H_*} e^{2(60 - N_1)}$

Inverse relation:  $N_1 = 44.4 - \frac{1}{2} \ln \left( \frac{M_{PBH}}{10^{16} \text{ g}} \right)$ .

# The Press-Schechter Mass Function

- Press and Schechter assumed the density distribution is Gaussian:

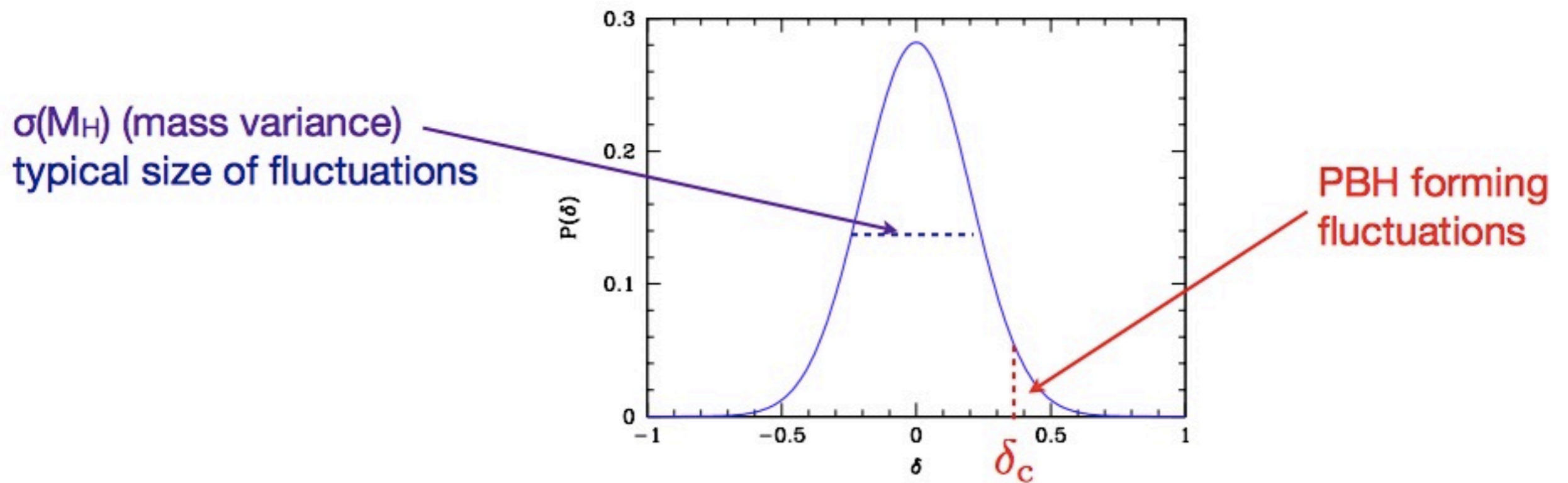
$$P(\delta_M)d\delta_M = \frac{1}{\sqrt{2\pi}\sigma_M} \exp\left(-\frac{\delta_M^2}{2\sigma_M^2}\right) d\delta_M$$

- $\sigma_M$  is the variance of the density perturbation at mass  $M$ .
- The PBH only collapse when  $\delta_M > \delta_c$ , so the total probability is

$$P_{>\delta_c}(M) = 2 \int_{\delta_c}^{\infty} P(\delta_M)d\delta_M = \text{erfc}\left(\frac{\delta_c}{\sqrt{2}\sigma_M}\right)$$

- Ad hoc factor of 2. Highly suppressed when  $\sigma_M \ll \delta_c$ . This is the mass fraction at formation, denoted by  $\beta$ .

# The Press-Schechter Mass Function



- When  $\sigma_M \ll \delta_c$ ,  $\beta$  can be approximated by exponential:

$$\beta \approx \sqrt{\frac{2}{\pi}} \frac{\sigma_M}{\delta_c} \exp\left(-\frac{\delta_c^2}{2\sigma_M^2}\right)$$

# The Press-Schechter Mass Function

- The current PBH mass measured in critical mass is

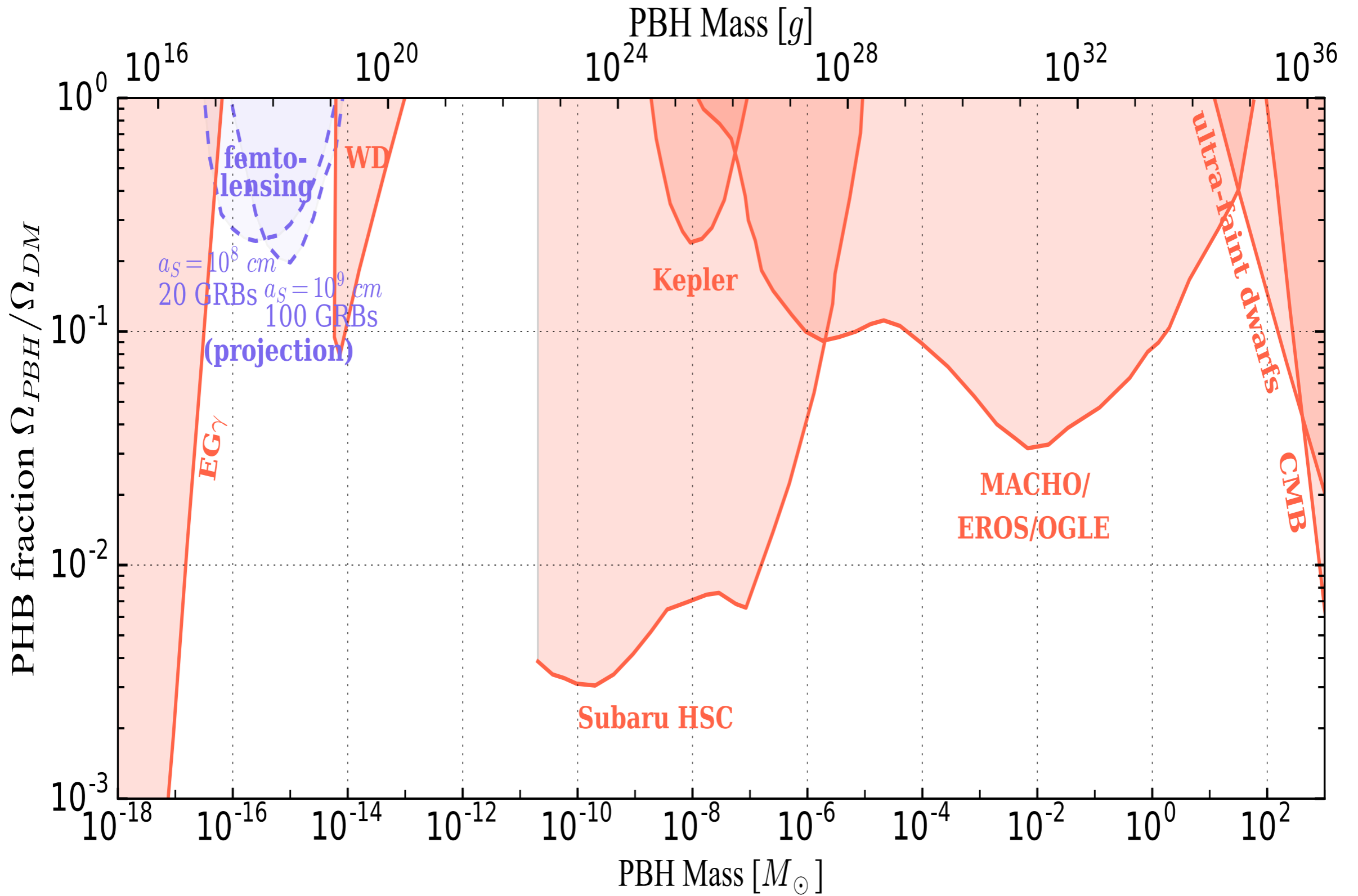
$$\Omega_{\text{PBH}} = \beta \frac{a_{\text{eq}}}{a_{\text{re}}} = \beta \frac{a_{\text{eq}}}{a_0} \frac{a_0}{a_{\text{re}}} \simeq \beta \Omega_r (1 + z_{\text{re}}(M))$$

- where “eq” means equality and “re” means re-entry for the peak of the variance of the density perturbation at mass  $M$ .
- It is easy to estimate  $z(M)$  relation at horizon reentry

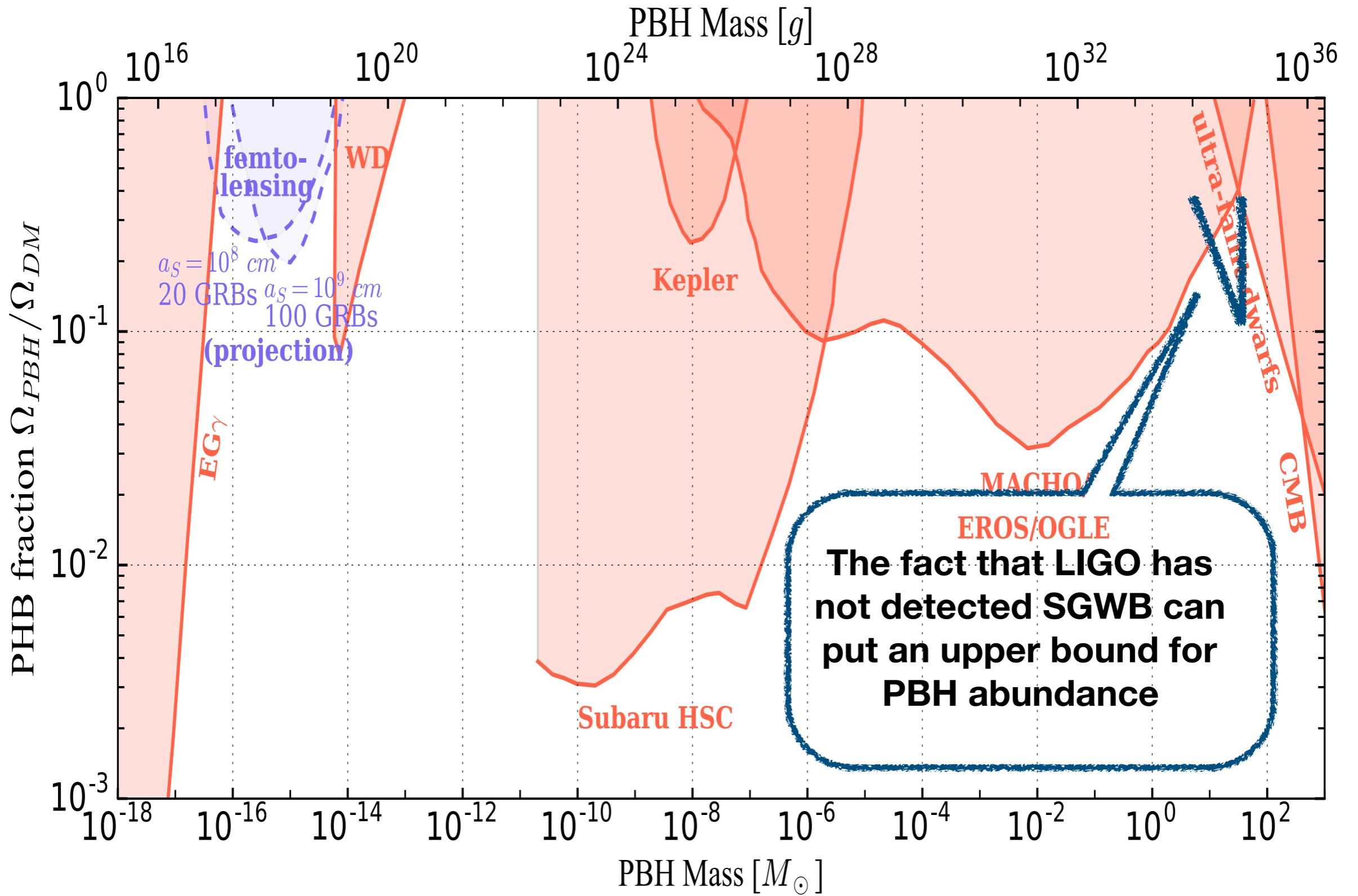
$$M = \frac{c^3}{GH_{\text{re}}} = \frac{c^3}{G\Omega_r^{1/2} (1+z)^2 H_0}$$

- Therefore we have

$$f \equiv \frac{\Omega_{\text{PBH}}}{\Omega_{\text{CDM}}} \approx 4.11 \times 10^8 \beta(M) \left( \frac{M}{M_\odot} \right)^{-1/2}$$

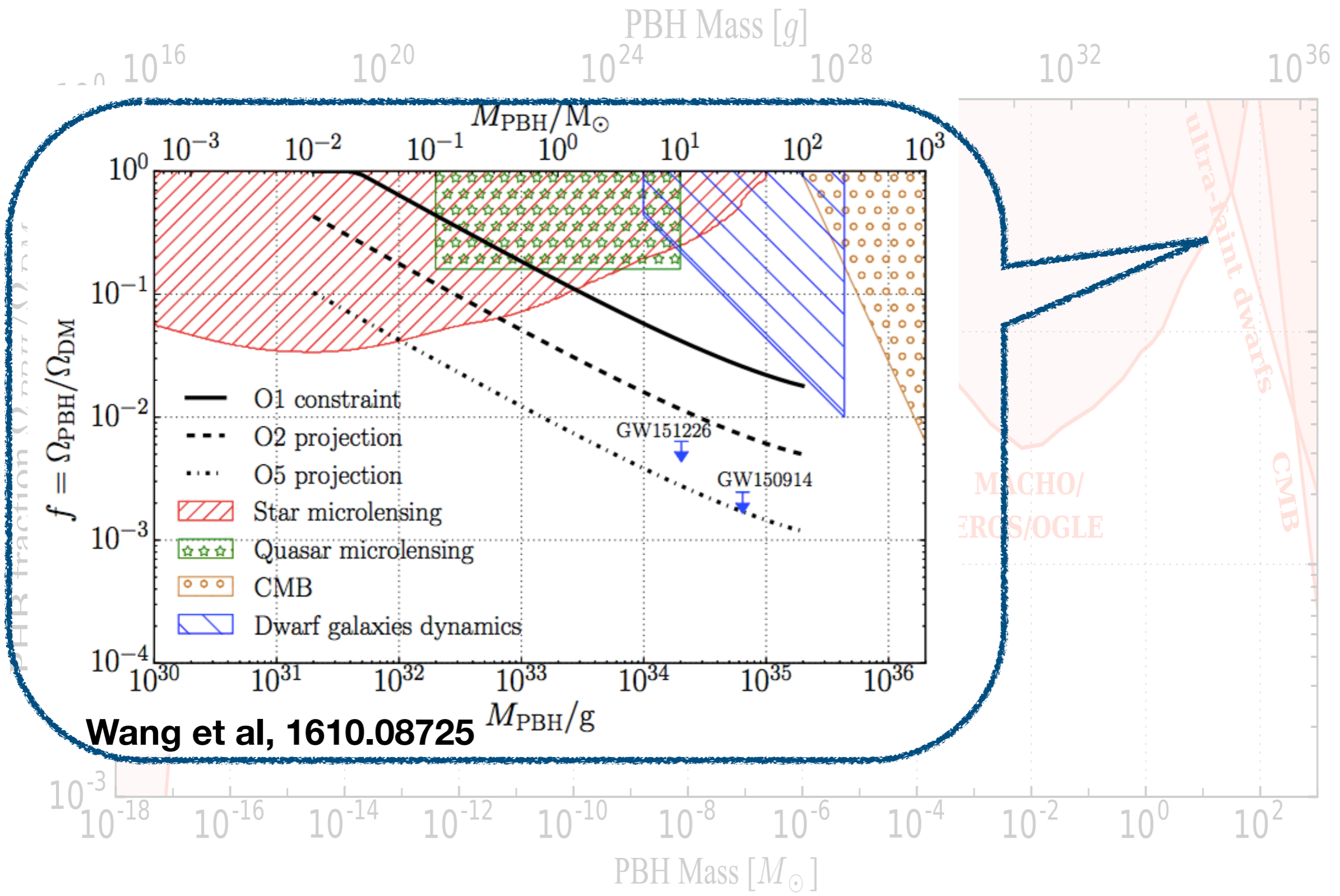


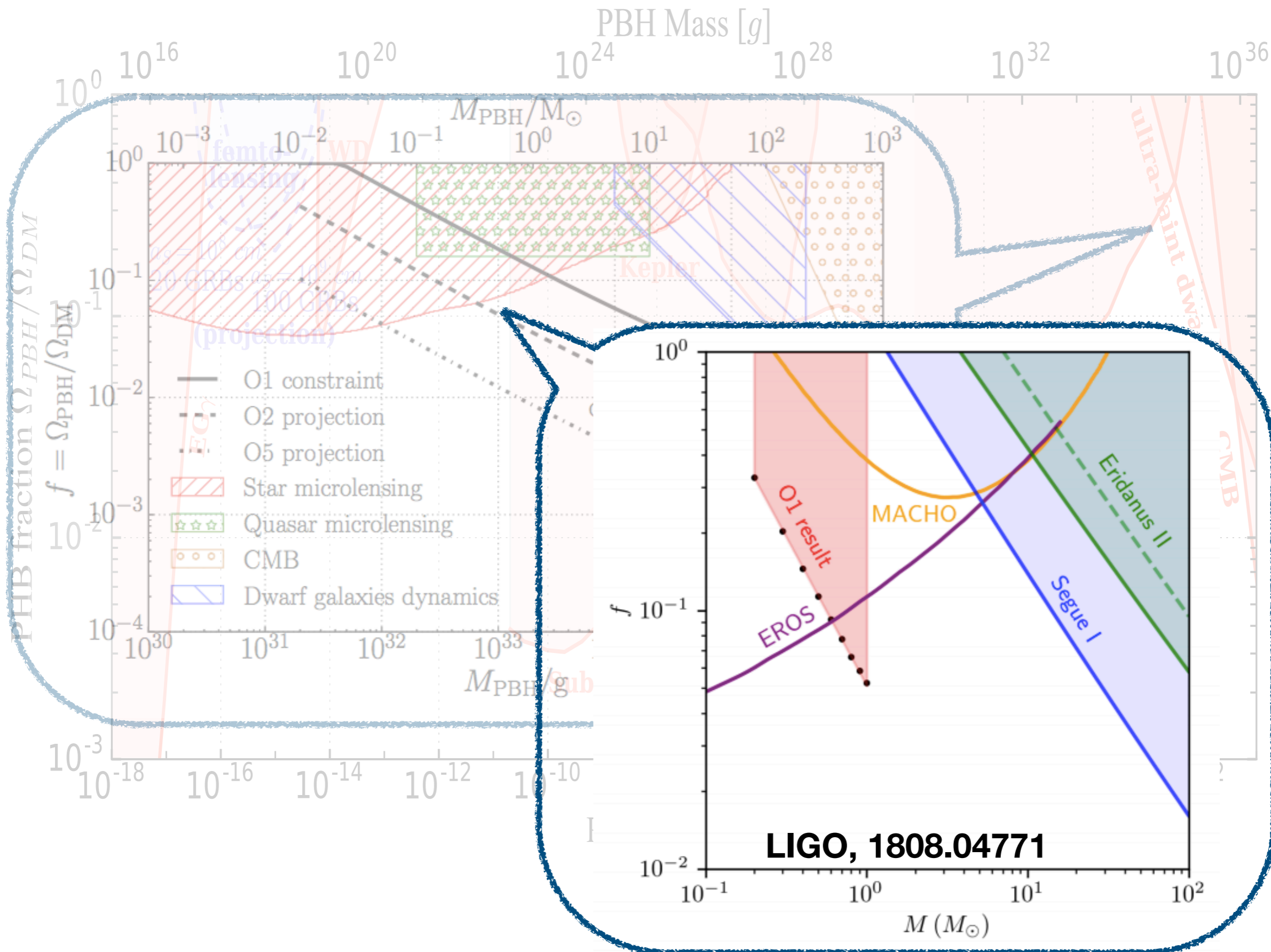
**1807.11495**

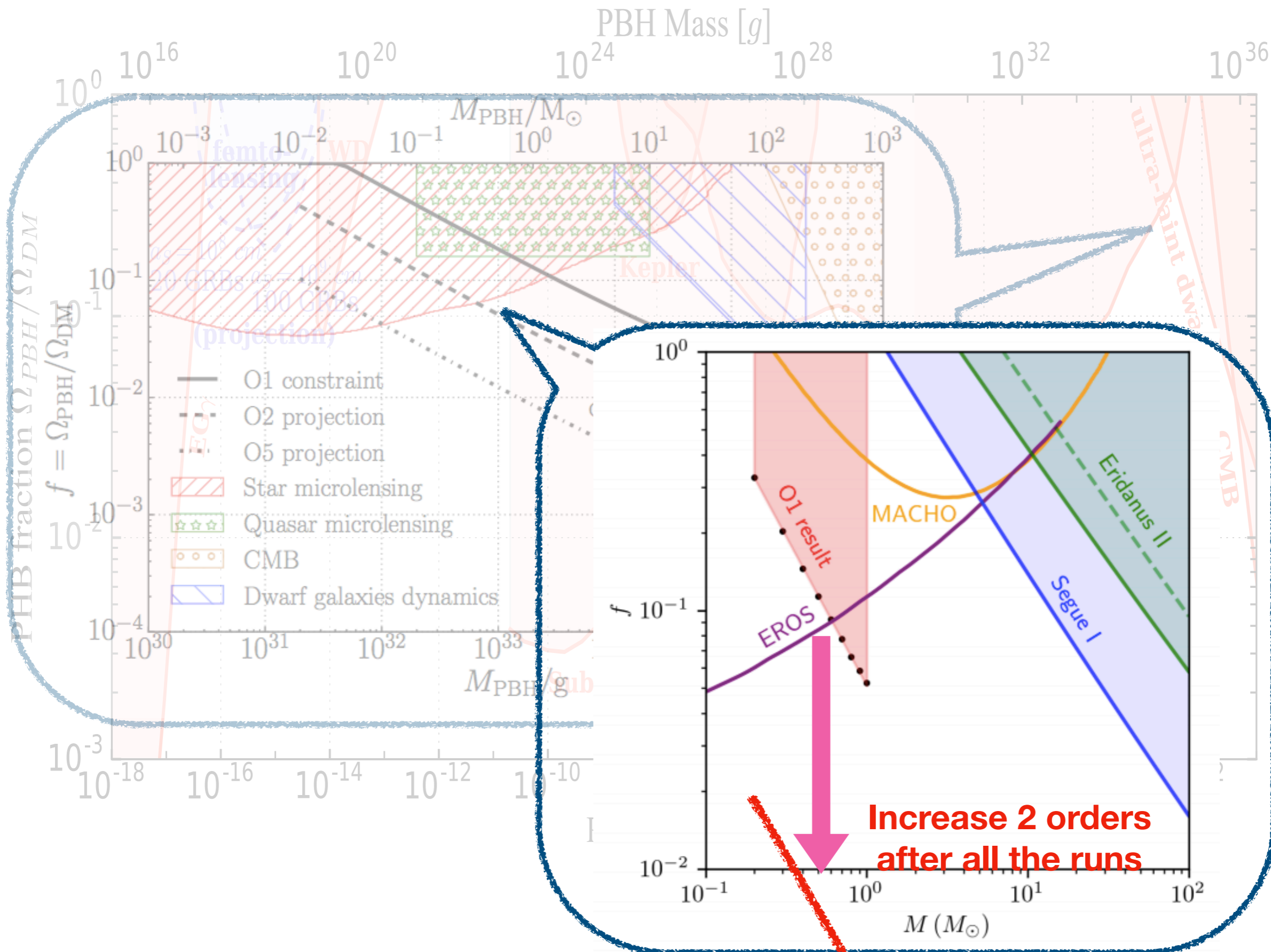


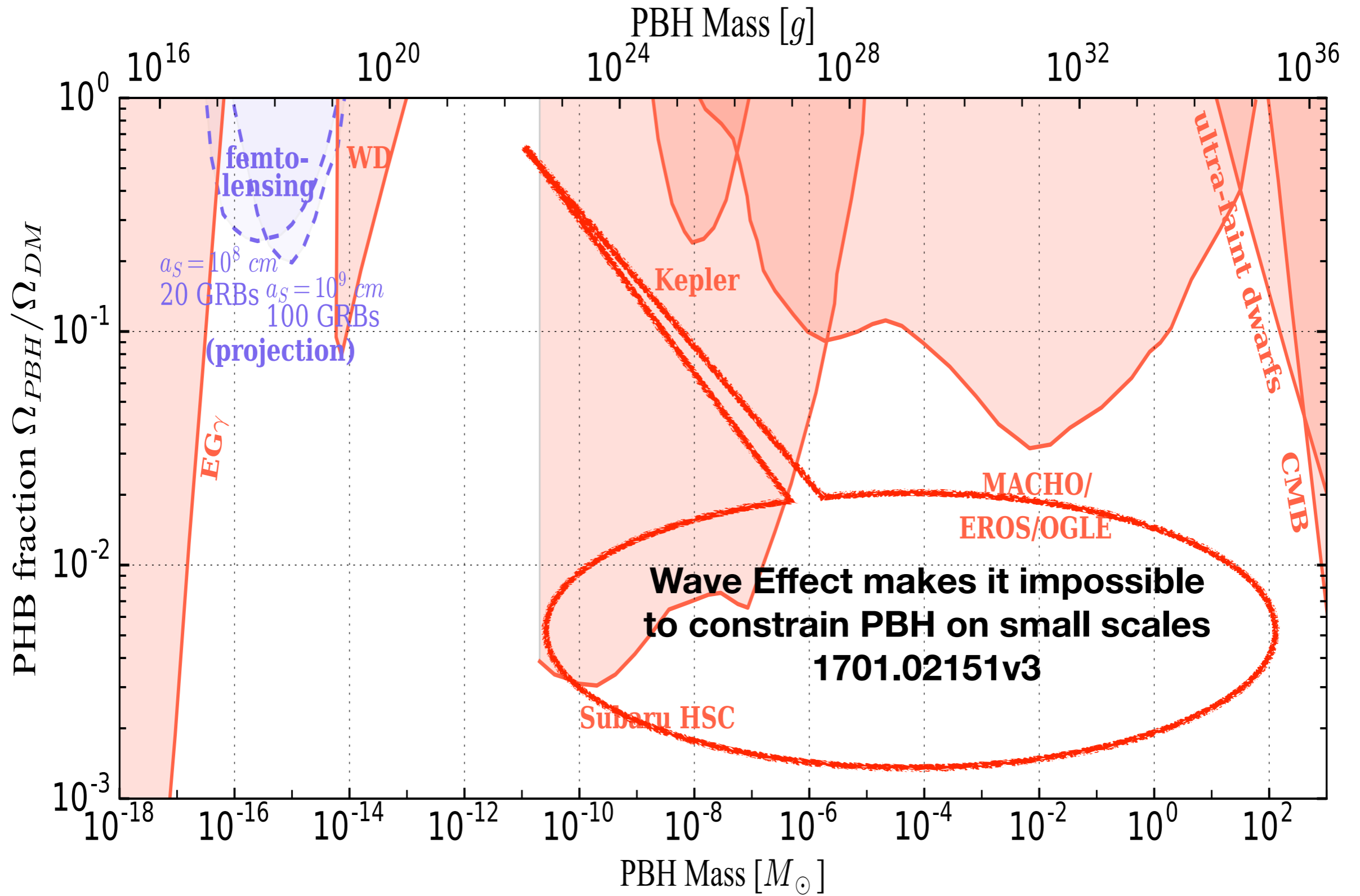
1807.11495



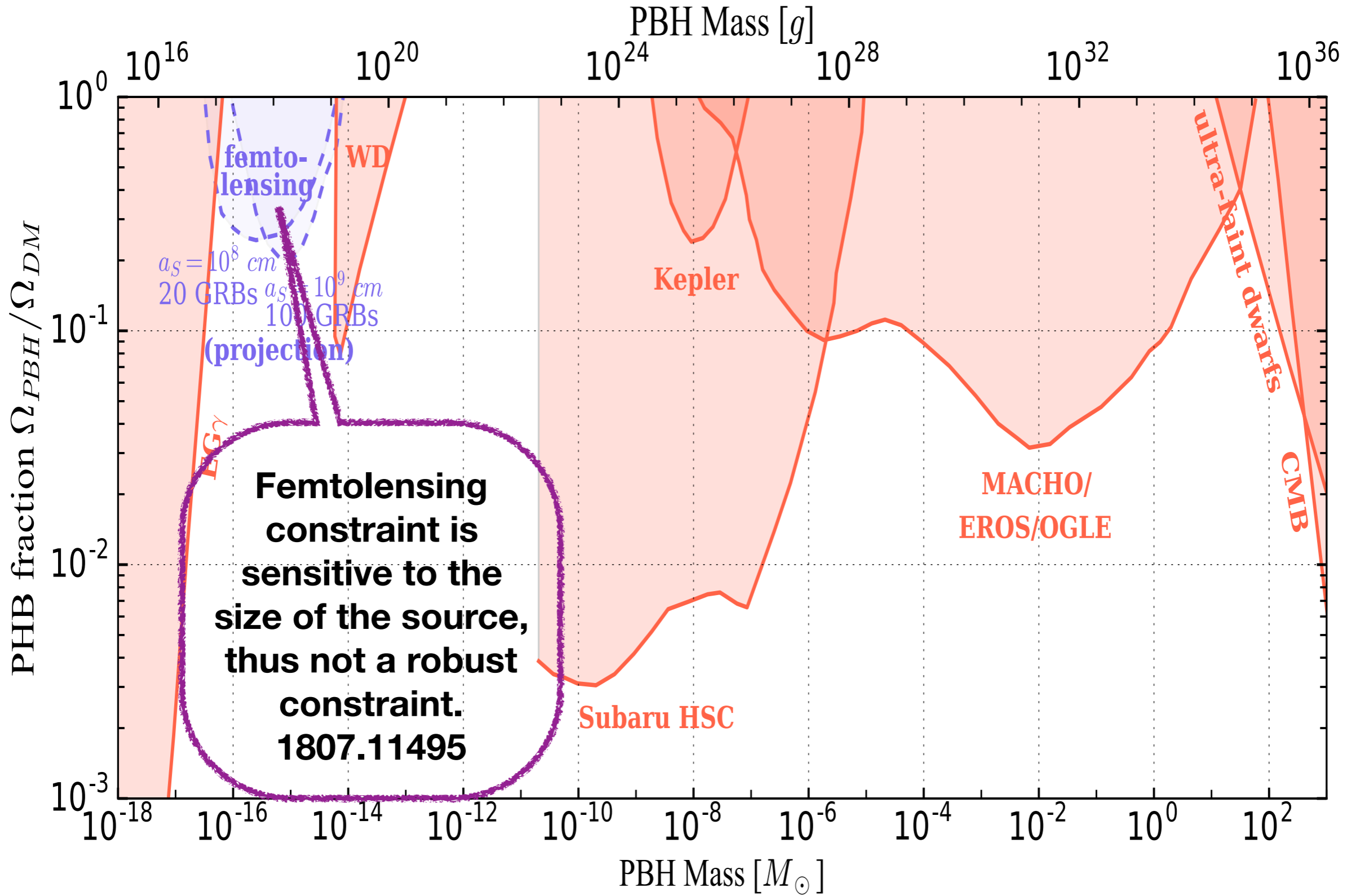








1807.11495



**1807.11495**

# Content

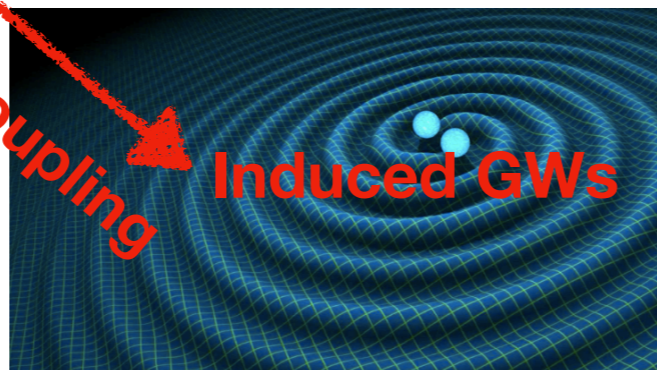
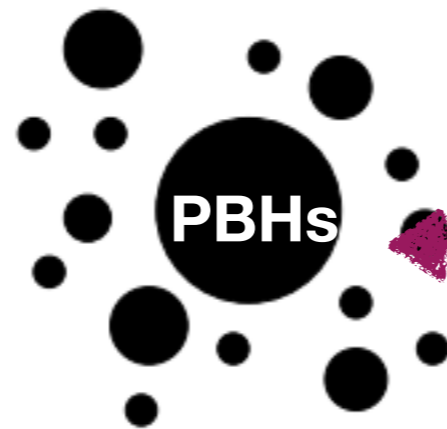
- Mechanism of SGWB
- PBH abundances and GWs
- **Induced GWs: A probe for non-Gaussianity**
- Conclusion

# Induced GWs

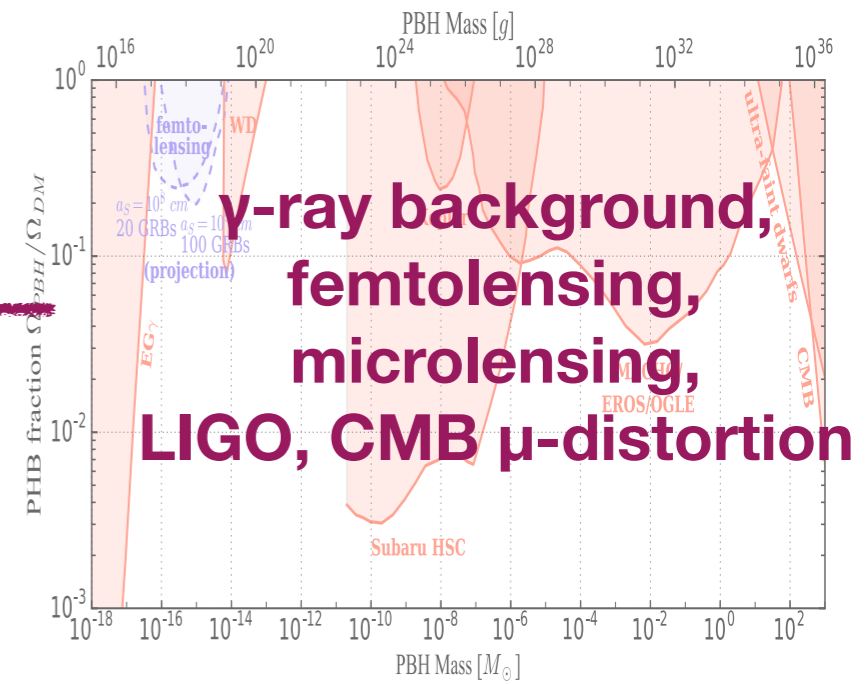
Peak of scalar perturbation on small scales

Peak Theory

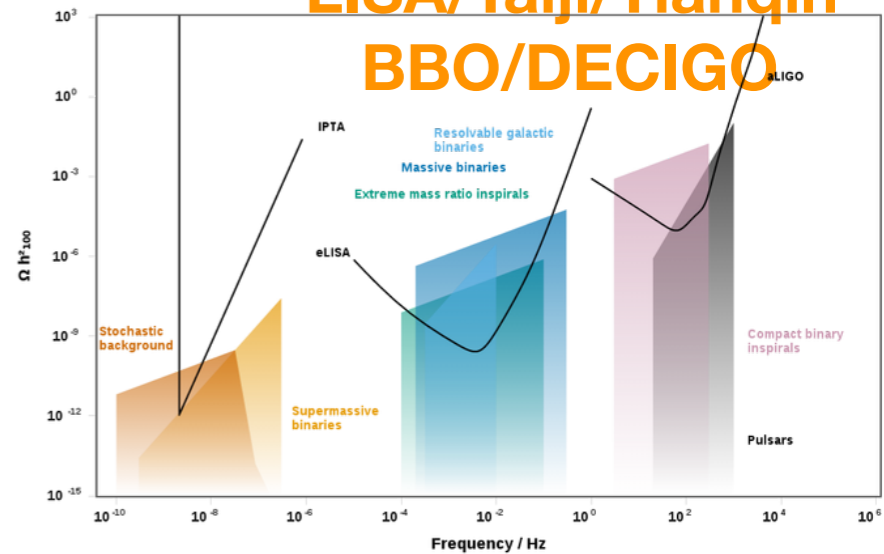
Secondary coupling



Induced GWs

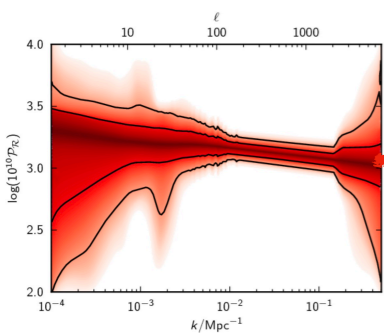
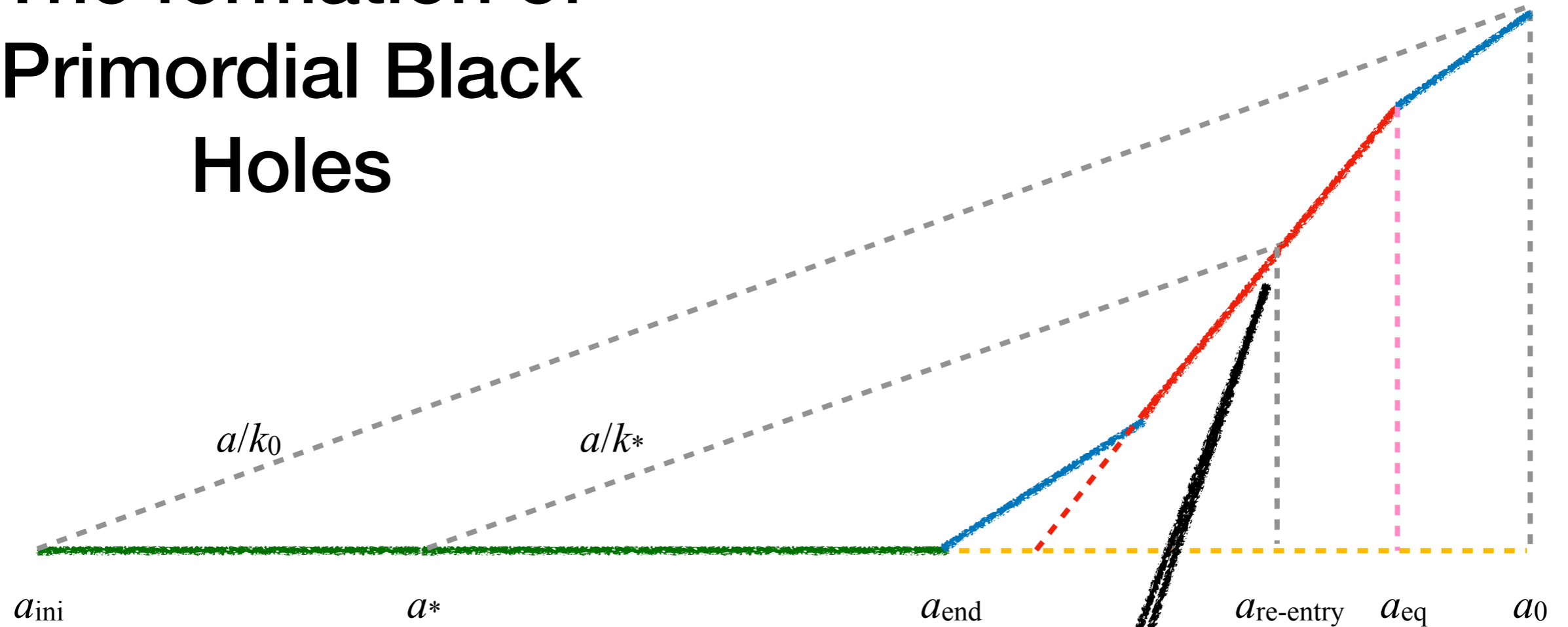


LIGO/VIRGO/KAGRA  
LISA/Taiji/Tianqin  
BBO/DECIGO



Saito & Yokoyama,  
0812.4339

# The formation of Primordial Black Holes



$$k^* = Ha^*$$

The peak scale re-enters the horizon at radiation dominated era. Induced GWs will form at the horizon re-entry.



# Induced GWs

- The metric is

$$ds^2 = a(\eta)^2 \left[ -(1 - 2\Phi) d\eta^2 + \left( 1 + 2\Phi + \frac{1}{2}h_{ij} \right) dx^i dx^j \right].$$

- From the nonlinear equation of motion for the tensor perturbation

$$h_{\mathbf{k}}'' + 2\mathcal{H}h_{\mathbf{k}}' + k^2 h_{\mathbf{k}} = \mathcal{S}(\mathbf{k}, \eta)$$

- where the source term is

$$\begin{aligned} \mathcal{S}(\mathbf{k}, \eta) = & 36 \int \frac{d^3 l}{(2\pi)^{3/2}} \frac{l^2}{\sqrt{2}} \sin^2 \theta \begin{pmatrix} \cos 2\varphi \\ \sin 2\varphi \end{pmatrix} \Phi_l \Phi_{\mathbf{k}-l} \\ & \times \left[ j_0(ux)j_0(vx) - 2\frac{j_1(ux)j_0(vx)}{ux} - 2\frac{j_0(ux)j_1(vx)}{vx} + 3\frac{j_1(ux)j_1(vx)}{uvx^2} \right]. \end{aligned}$$

# Induced GWs

- The quantity we want to calculate is

$$\Omega_{\text{GW}}(k) \equiv \frac{1}{12} \left( \frac{k}{Ha} \right)^2 \frac{k^3}{\pi^2} \overline{\langle h_{\mathbf{k}}(\eta) h_{\mathbf{k}}(\eta) \rangle}.$$

- Then we know that  $\Omega_{\text{GW}} \sim \langle hh \rangle \sim \langle SS \rangle \sim \langle \Phi\Phi\Phi\Phi \rangle \sim P_{\Phi}^2$ .
- It is naive to believe that  $\Phi$  stays Gaussian when it becomes very large on small scales.
- Therefore we want to consider the local-type non-Gaussian scalar induced GWs.

- $$\mathcal{R}(\mathbf{x}) = \mathcal{R}_g(\mathbf{x}) + F_{\text{NL}} \left[ \mathcal{R}_g^2(\mathbf{x}) - \langle \mathcal{R}_g^2(\mathbf{x}) \rangle \right].$$

# Induced GWs

- Then the 2pt of  $\Phi$  is

$$\langle \Phi_{\mathbf{k}} \Phi_{\mathbf{p}} \rangle = (2\pi)^3 \delta^{(3)}(\mathbf{k} + \mathbf{p}) \frac{4}{9} \left( P_{\mathcal{R}}(k) + 2F_{\text{NL}}^2 \int d^3l P_{\mathcal{R}}(|\mathbf{k} - \mathbf{l}|) P_{\mathcal{R}}(l) \right).$$

- And we have to specify the power spectrum of the primordial curvature perturbation. As we mentioned, we suppose there is a narrow peak at around  $k^*$ .

$$P_{\mathcal{R}}(k) = \frac{\mathcal{A}_{\mathcal{R}}}{(2\pi)^{3/2} 2\sigma k_*^2} \exp\left(-\frac{(k - k_*)^2}{2\sigma^2}\right).$$

- Narrow means  $\sigma \ll k^*$ . This is for simplicity.

# Induced GWs

- Then the result is the integral:

$$\begin{aligned}
 \Omega_{\text{GW}} &= 6\mathcal{A}_{\mathcal{R}}^2 \frac{k^2}{2\pi\sigma^2} \left(\frac{k}{k_*}\right)^4 \int_0^\infty dv \int_{|1-v|}^{1+v} du uv \mathcal{T}(u, v) \\
 &\times \left( e^{-\frac{(vk-k_*)^2}{2\sigma^2}} + 2\mathcal{A}_{\mathcal{R}} F_{\text{NL}}^2 \frac{\sigma}{vk} \sqrt{\frac{\pi}{2}} \Theta(2k_* - vk) \right) \\
 &\times \left( e^{-\frac{(uk-k_*)^2}{2\sigma^2}} + 2\mathcal{A}_{\mathcal{R}} F_{\text{NL}}^2 \frac{\sigma}{uk} \sqrt{\frac{\pi}{2}} \Theta(2k_* - uk) \right), \\
 \mathcal{T}(u, v) &= \frac{1}{4} \left( \frac{4v^2 - (1 + v^2 - u^2)^2}{4uv} \right)^2 \left( \frac{u^2 + v^2 - 3}{2uv} \right)^2 \\
 &\times \left\{ \left( -2 + \frac{u^2 + v^2 - 3}{2uv} \ln \left| \frac{3 - (u+v)^2}{3 - (u-v)^2} \right| \right)^2 \right. \\
 &\quad \left. + \pi^2 \left( \frac{u^2 + v^2 - 3}{2uv} \right)^2 \Theta(u + v - \sqrt{3}) \right\}.
 \end{aligned}$$

# Induced GWs

- Then the result is the integral:

$$\Omega_{\text{GW}} = 6\mathcal{A}_{\mathcal{R}}^2 \frac{k^2}{2\pi\sigma^2} \left(\frac{k}{k_*}\right)^4 \int_0^\infty dv \int_{|1-v|}^{1+v} du uv \mathcal{T}(u, v)$$

$$\times \left( e^{-\frac{(vk-k_*)^2}{2\sigma^2}} + 2\mathcal{A}_{\mathcal{R}} F_{\text{NL}}^2 \frac{\sigma}{vk} \sqrt{\frac{\pi}{2}} \Theta(2k_* - vk) \right)$$

$$\times \left( e^{-\frac{(uk-k_*)^2}{2\sigma^2}} + 2\mathcal{A}_{\mathcal{R}} F_{\text{NL}}^2 \frac{\sigma}{uk} \sqrt{\frac{\pi}{2}} \Theta(2k_* - uk) \right),$$

$$\mathcal{T}(u, v) = \frac{1}{4} \left( \frac{4v^2 - (1 + v^2 - u^2)^2}{4uv} \right)^2 \left( \frac{u^2 + v^2 - 3}{2uv} \right)^2$$

$$\times \left\{ \left( -2 + \frac{u^2 + v^2 - 3}{2uv} \ln \left| \frac{3 - (u+v)^2}{3 - (u-v)^2} \right| \right)^2 \right.$$

$$\left. + \pi^2 \left( \frac{u^2 + v^2 - 3}{2uv} \right)^2 \Theta(u + v - \sqrt{3}) \right\}.$$

Kohri & Tareda,  
1804.08577

# Induced GWs

- Then the result is the integral:

$$\Omega_{\text{GW}} = 6\mathcal{A}_{\mathcal{R}}^2 \frac{k^2}{2\pi\sigma^2} \left(\frac{k}{k_*}\right)^4 \int_0^\infty dv \int_{|1-v|}^{1+v} du uv \mathcal{T}(u, v)$$

Saito & Yokoyama,  
0812.4339

$$\times \left( e^{-\frac{(vk-k_*)^2}{2\sigma^2}} + 2\mathcal{A}_{\mathcal{R}} F_{\text{NL}}^2 \frac{\sigma}{vk} \sqrt{\frac{\pi}{2}} \Theta(2k_* - vk) \right) \\ \times \left( e^{-\frac{(uk-k_*)^2}{2\sigma^2}} + 2\mathcal{A}_{\mathcal{R}} F_{\text{NL}}^2 \frac{\sigma}{uk} \sqrt{\frac{\pi}{2}} \Theta(2k_* - uk) \right),$$

Kohri & Tareda,  
1804.08577

$$\mathcal{T}(u, v) = \frac{1}{4} \left( \frac{4v^2 - (1 + v^2 - u^2)^2}{4uv} \right)^2 \left( \frac{u^2 + v^2 - 3}{2uv} \right)^2 \\ \times \left\{ \left( -2 + \frac{u^2 + v^2 - 3}{2uv} \ln \left| \frac{3 - (u+v)^2}{3 - (u-v)^2} \right| \right)^2 \right. \\ \left. + \pi^2 \left( \frac{u^2 + v^2 - 3}{2uv} \right)^2 \Theta(u + v - \sqrt{3}) \right\}.$$

# Induced GWs

non-Gaussian  
contributions

- Then the result is the integral:

$$\Omega_{\text{GW}} = 6\mathcal{A}_{\mathcal{R}}^2 \frac{k^2}{2\pi\sigma^2} \left(\frac{k}{k_*}\right)^4 \int_0^\infty dv \int_{|1-v|}^{1+v} du uv \mathcal{T}(u, v)$$

$$\times \left( e^{-\frac{(vk-k_*)^2}{2\sigma^2}} + 2\mathcal{A}_{\mathcal{R}} F_{\text{NL}}^2 \frac{\sigma}{vk} \sqrt{\frac{\pi}{2}} \Theta(2k_* - vk) \right)$$

$$\times \left( e^{-\frac{(uk-k_*)^2}{2\sigma^2}} + 2\mathcal{A}_{\mathcal{R}} F_{\text{NL}}^2 \frac{\sigma}{uk} \sqrt{\frac{\pi}{2}} \Theta(2k_* - uk) \right),$$

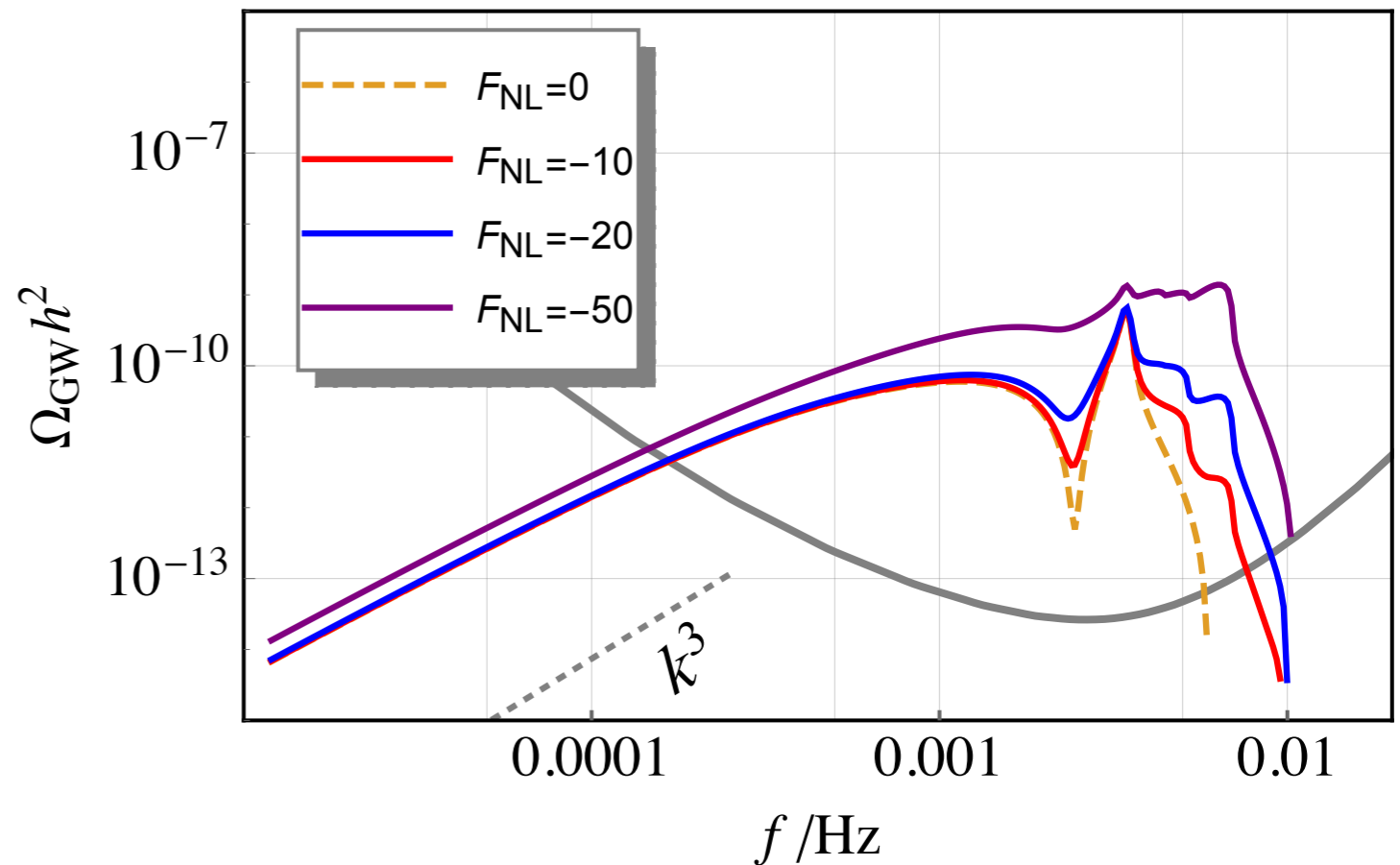
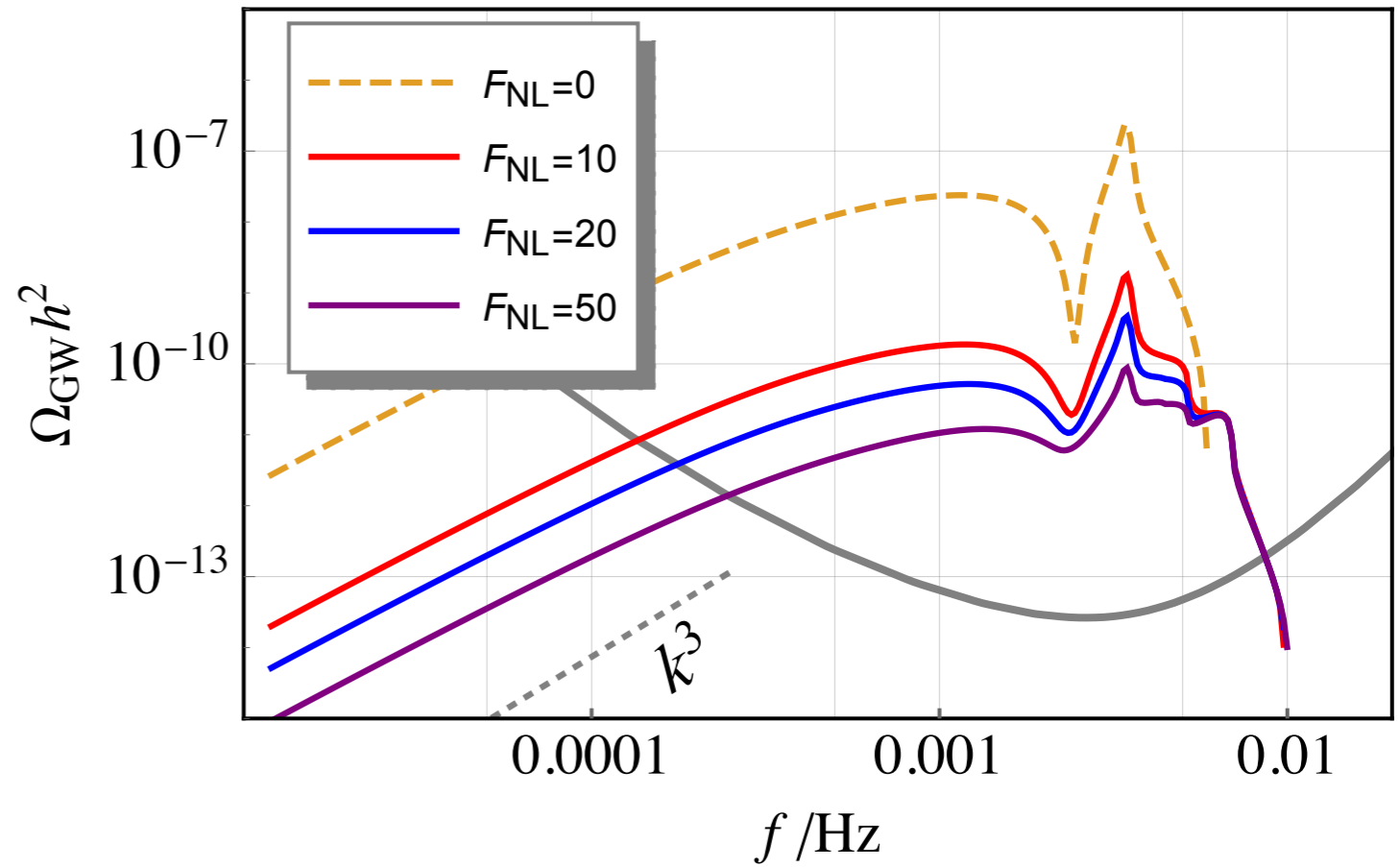
$$\mathcal{T}(u, v) = \frac{1}{4} \left( \frac{4v^2 - (1 + v^2 - u^2)^2}{4uv} \right)^2 \left( \frac{u^2 + v^2 - 3}{2uv} \right)^2$$

$$\times \left\{ \left( -2 + \frac{u^2 + v^2 - 3}{2uv} \ln \left| \frac{3 - (u+v)^2}{3 - (u-v)^2} \right| \right)^2 \right.$$

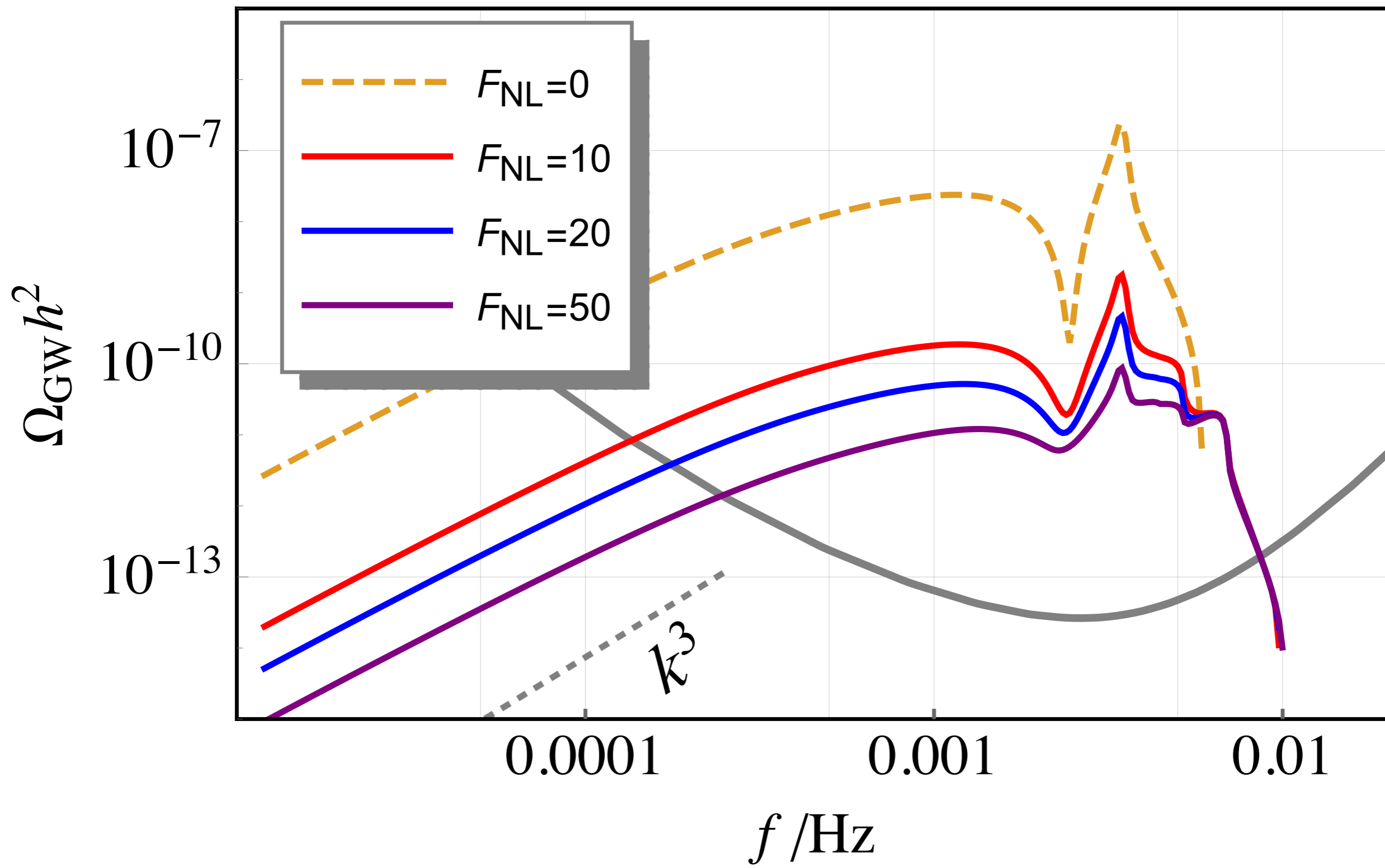
$$\left. + \pi^2 \left( \frac{u^2 + v^2 - 3}{2uv} \right)^2 \Theta(u + v - \sqrt{3}) \right\}.$$

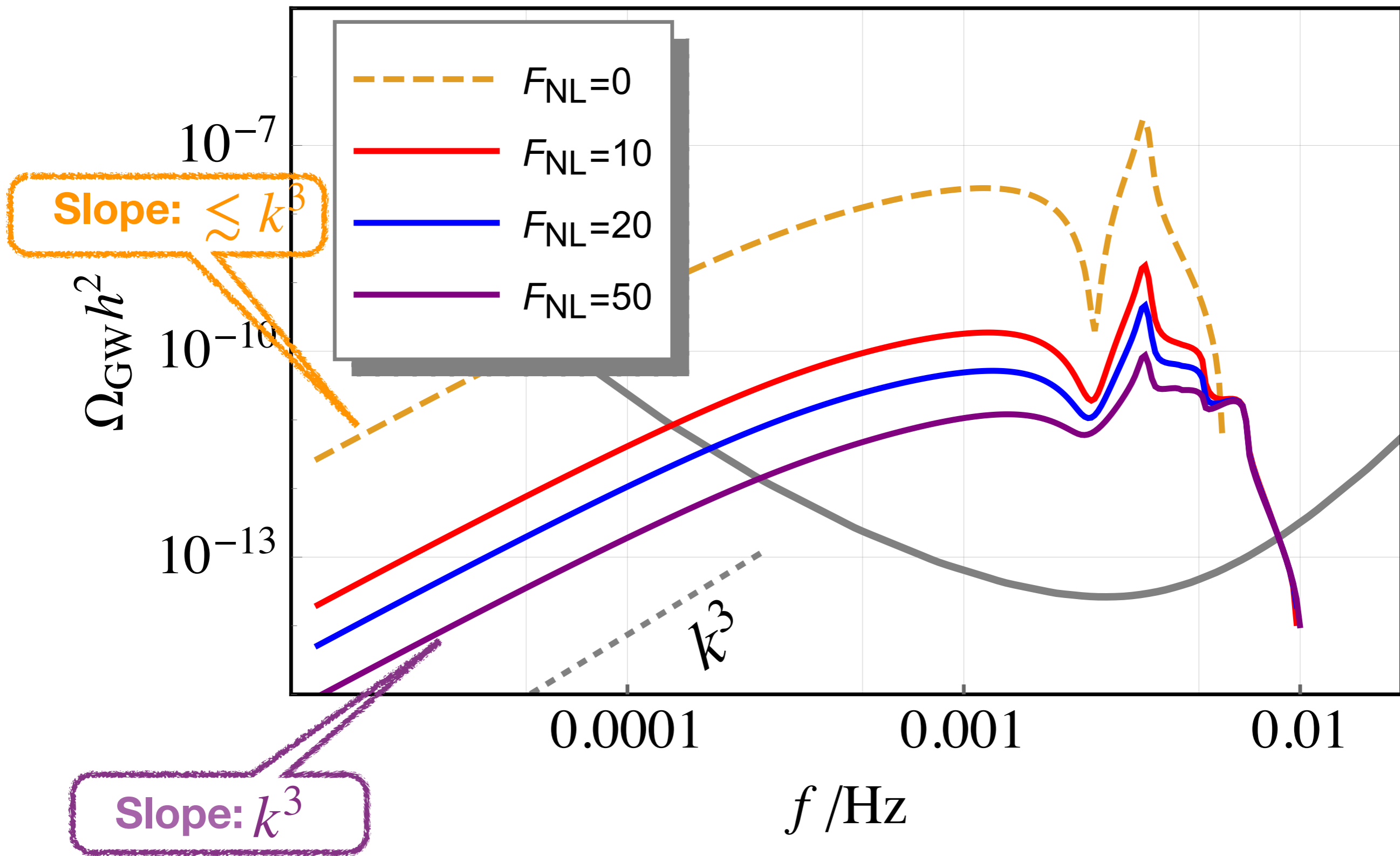
Kohri & Tareda,  
1804.08577

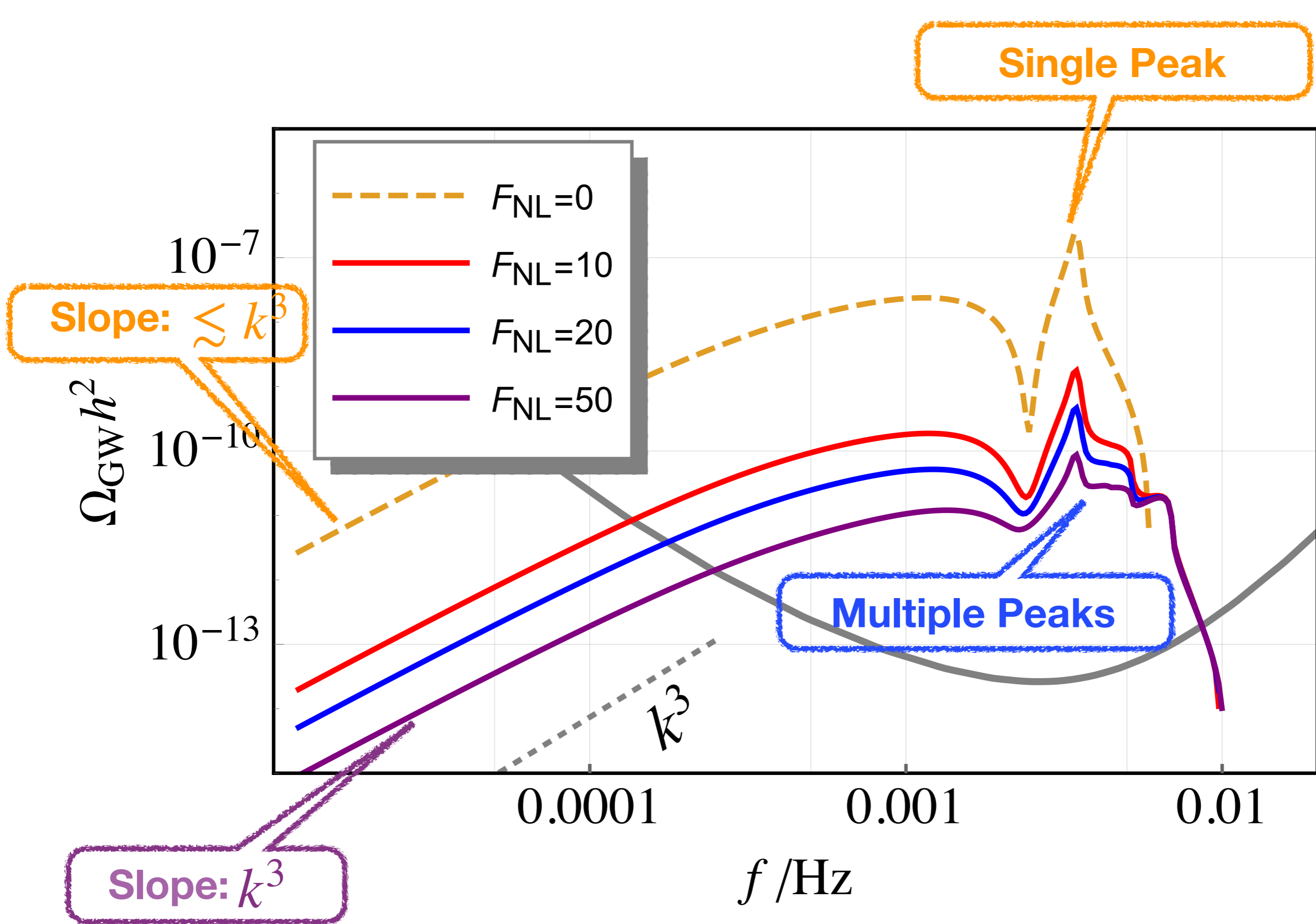
- Up:  $F_{NL} > 0$ , and we fix the PBH abundance to be 1.
- Down:  $F_{NL} < 0$ , and we fix the peak amplitude to be  $\mathcal{A}_{\mathcal{R}} = 10^{-2}$
- Gray curve: LISA
- Frequency: PBH window  $\langle - \rangle$  LISA band
- Coincidence, but fortunate for our universe.

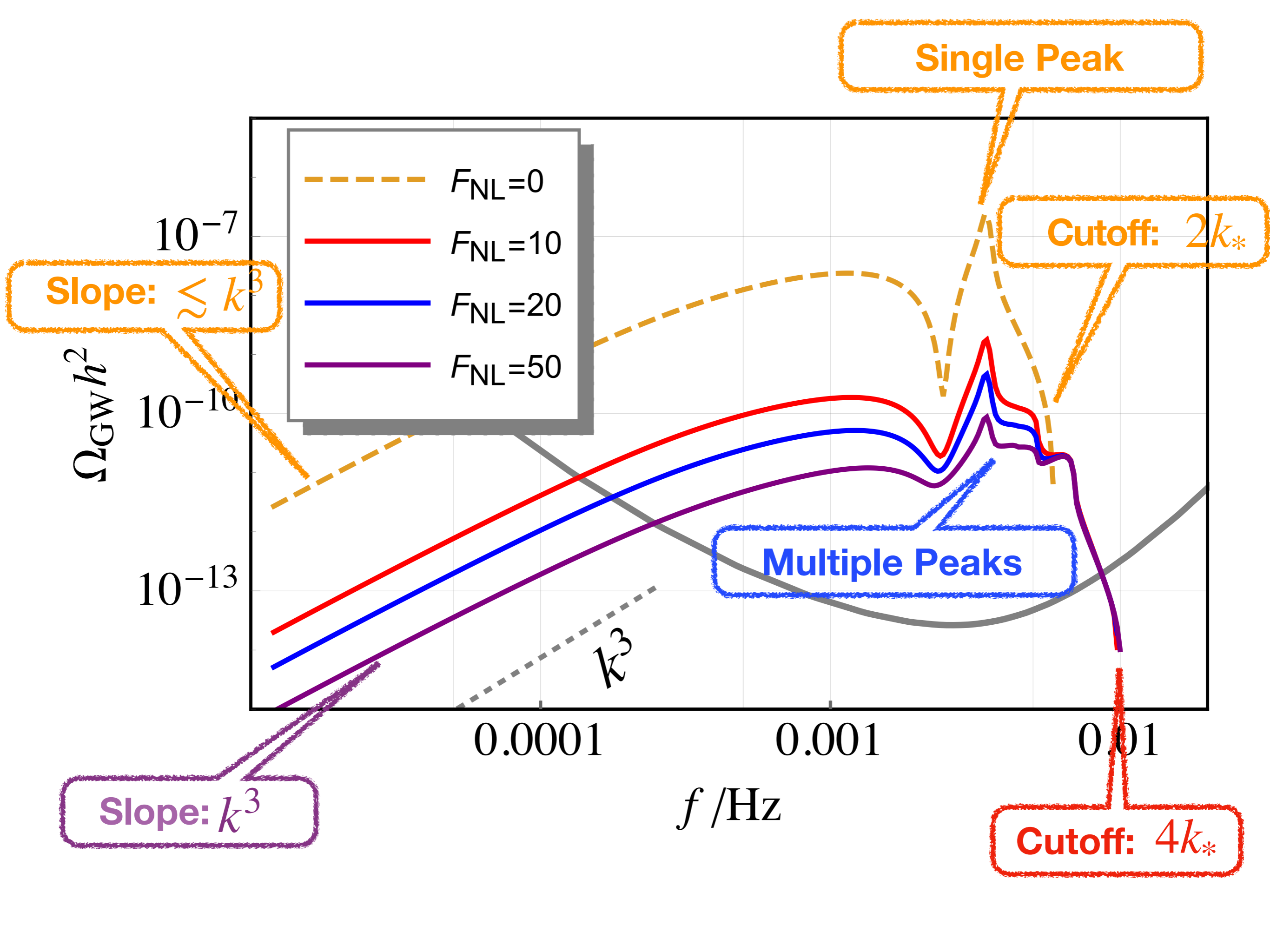


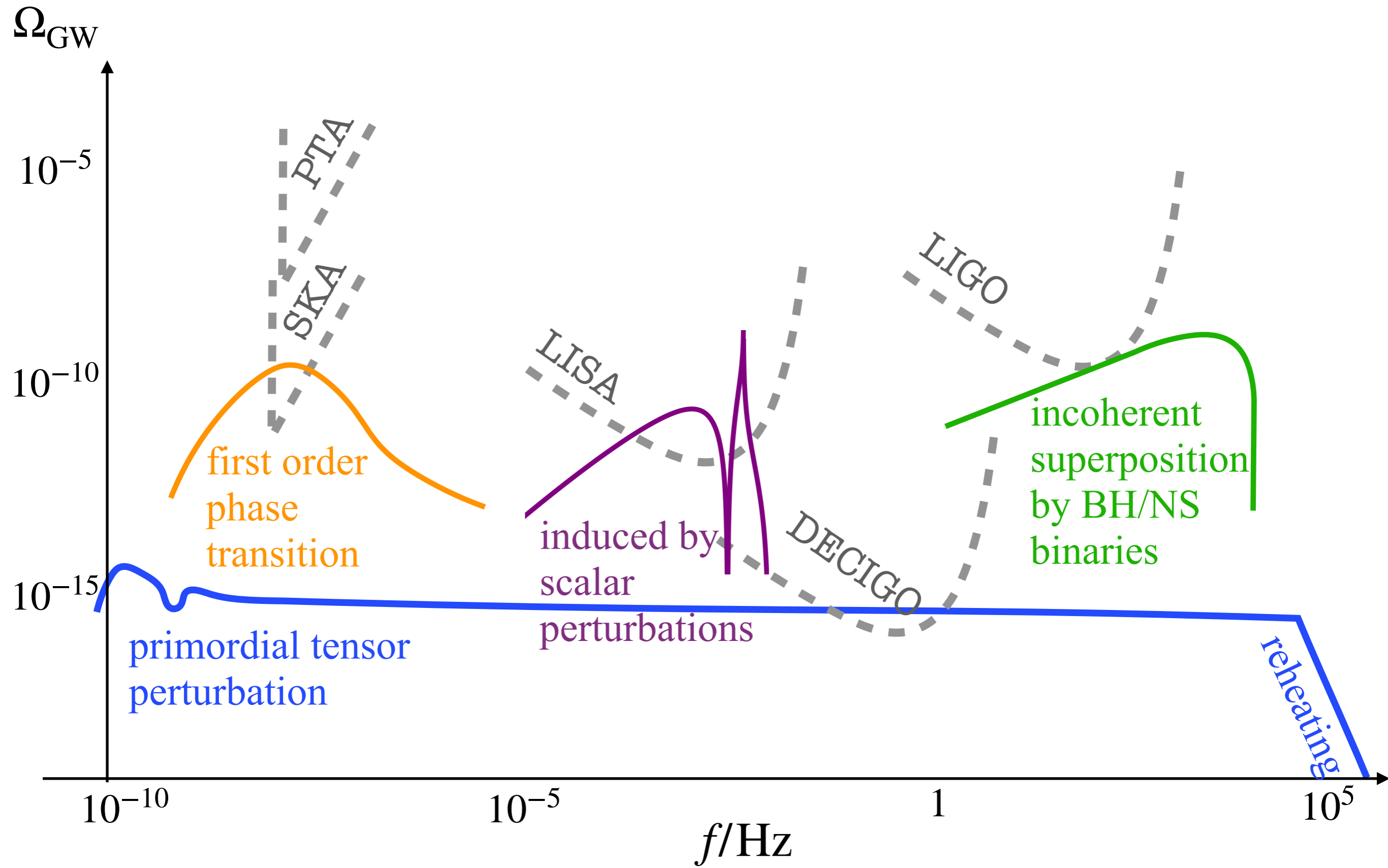




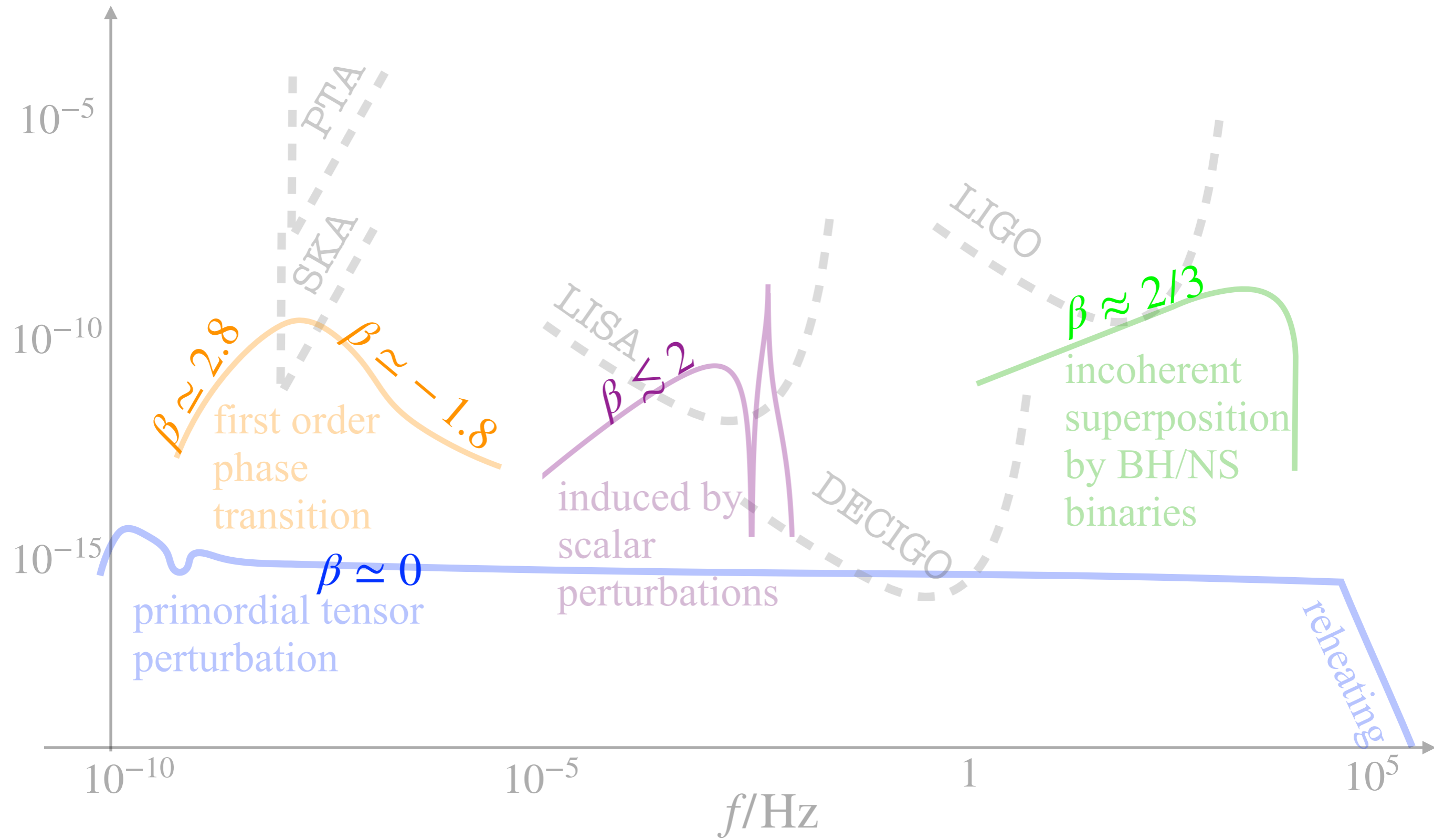




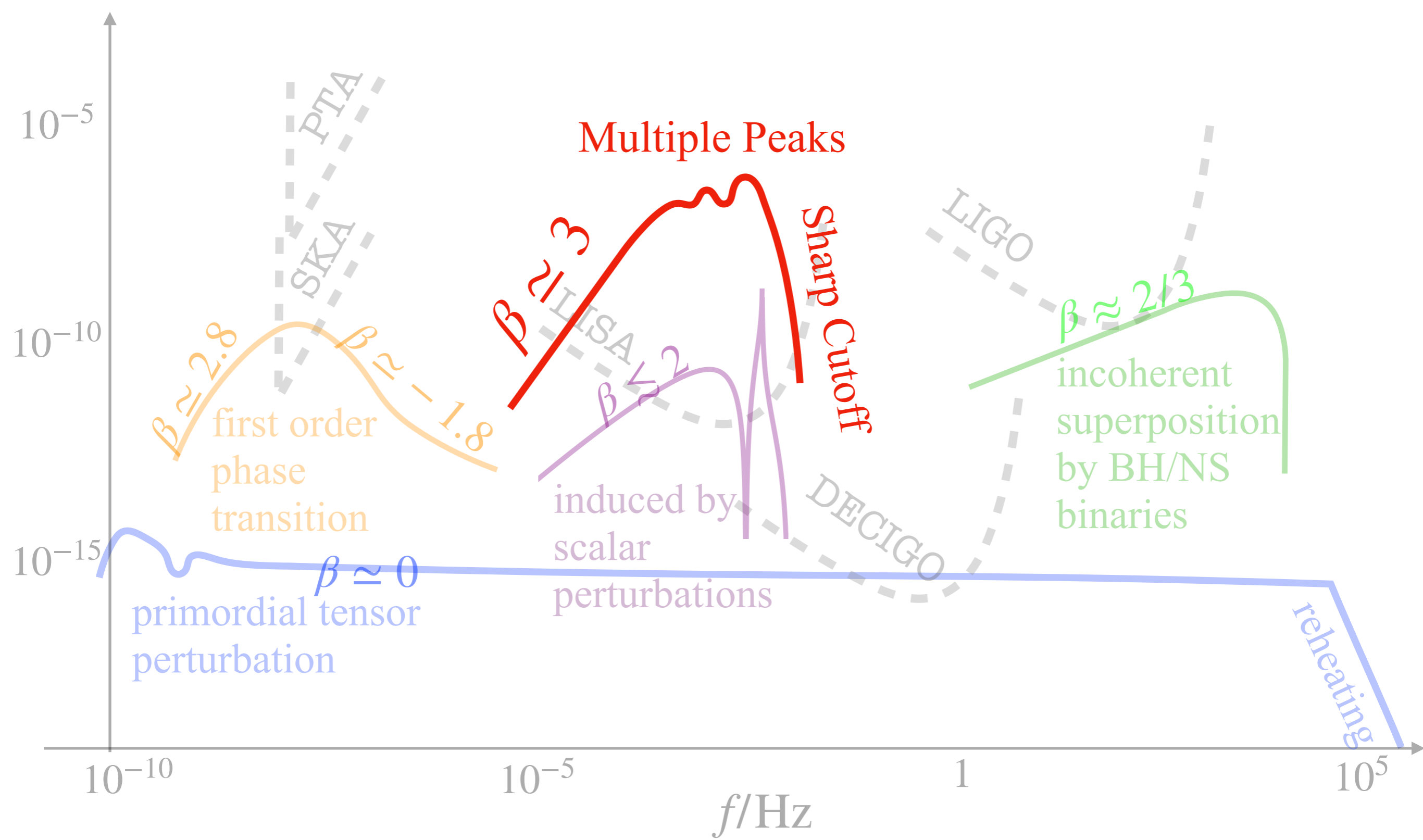


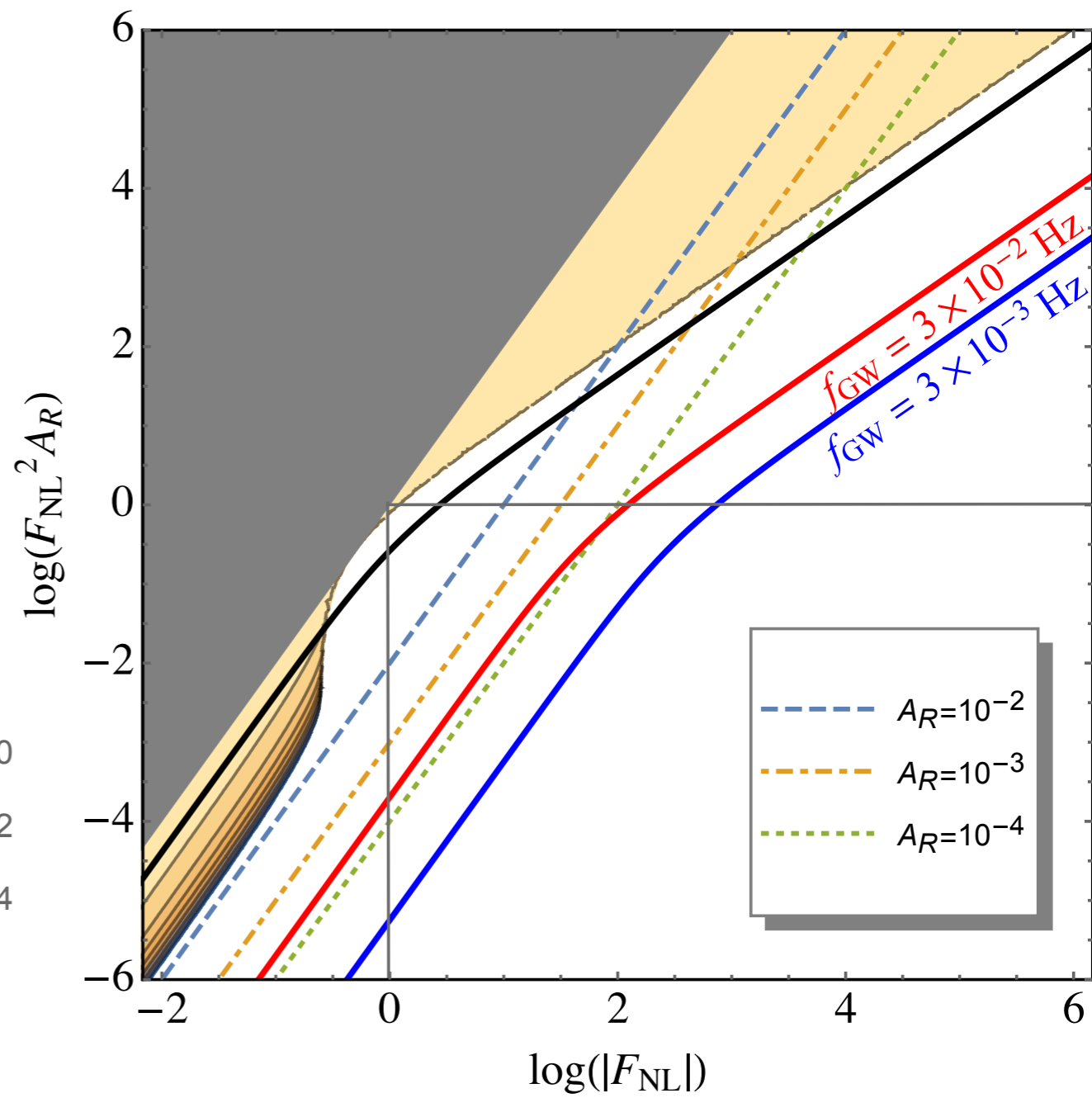
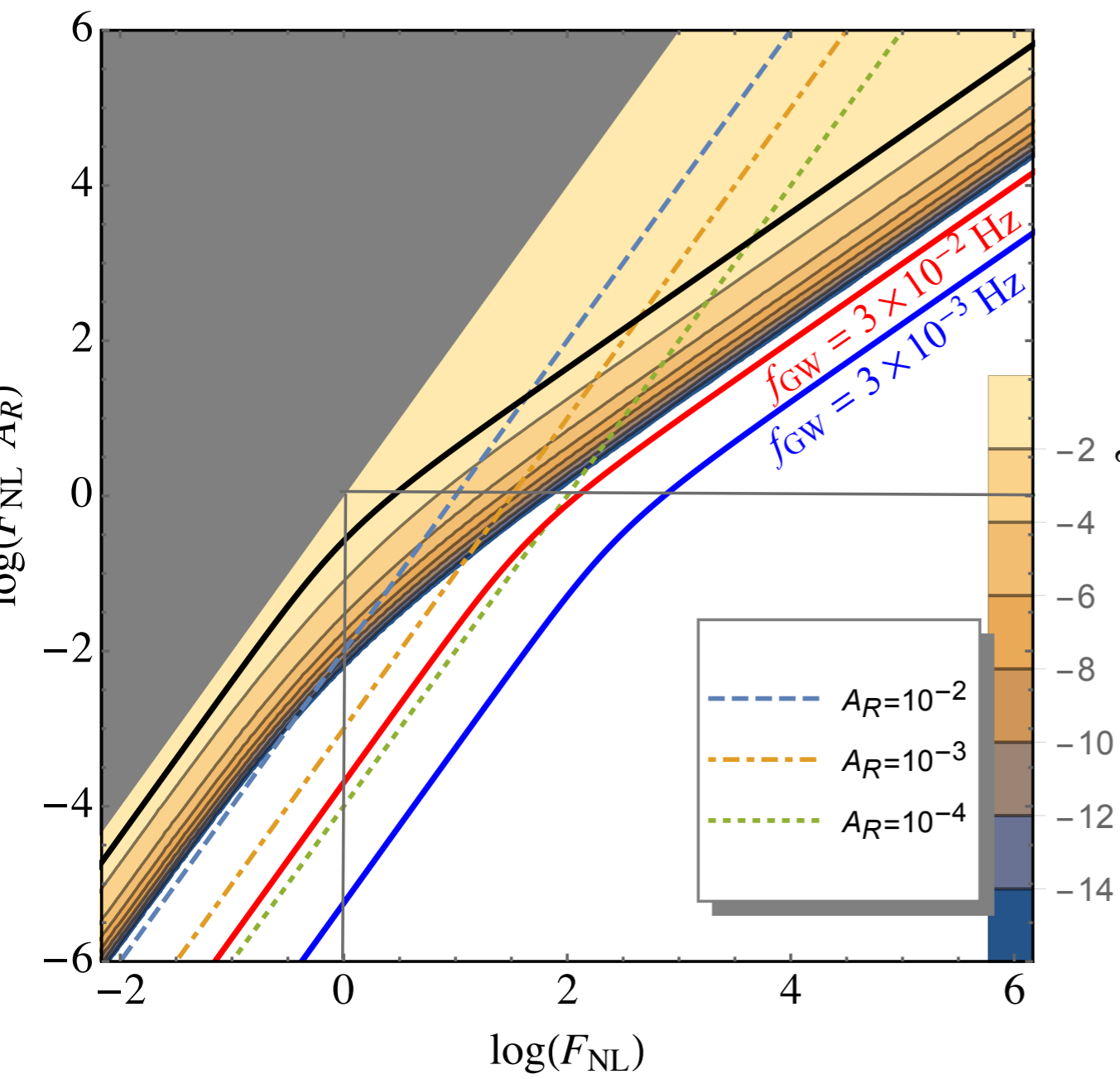


$$\Omega_{\text{GW}} \propto k^\beta$$

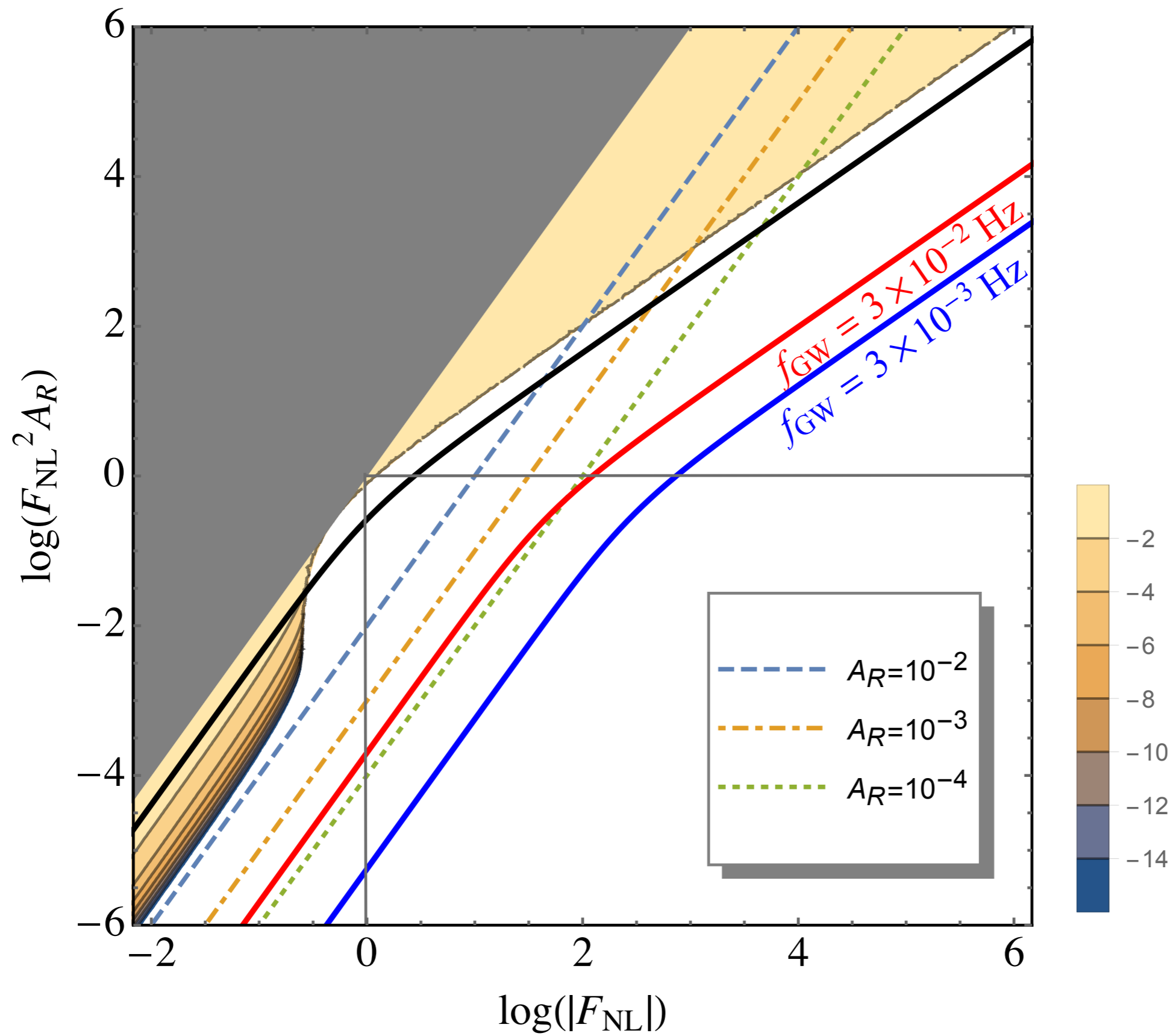


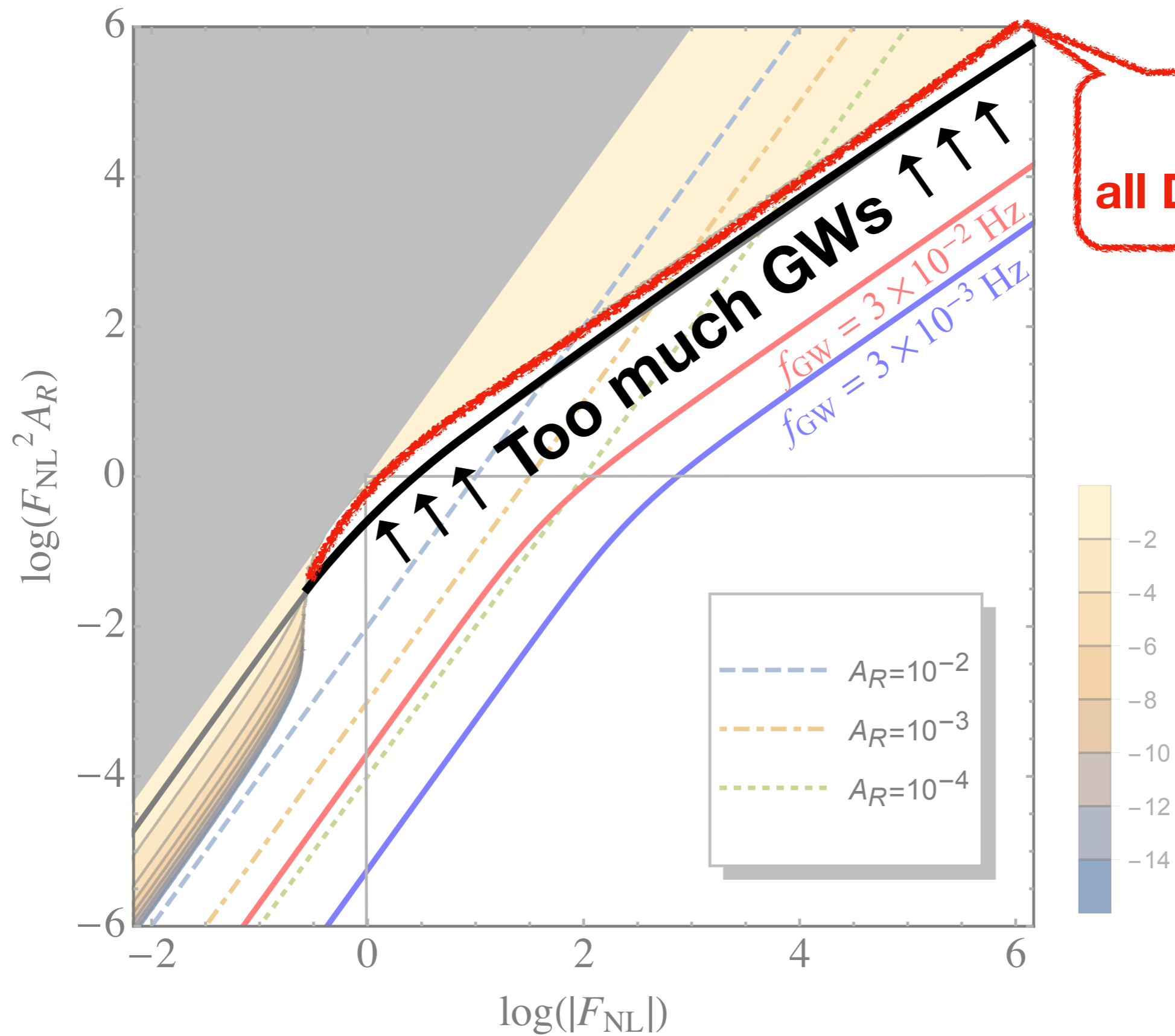
$$\Omega_{\text{GW}} \propto k^\beta$$

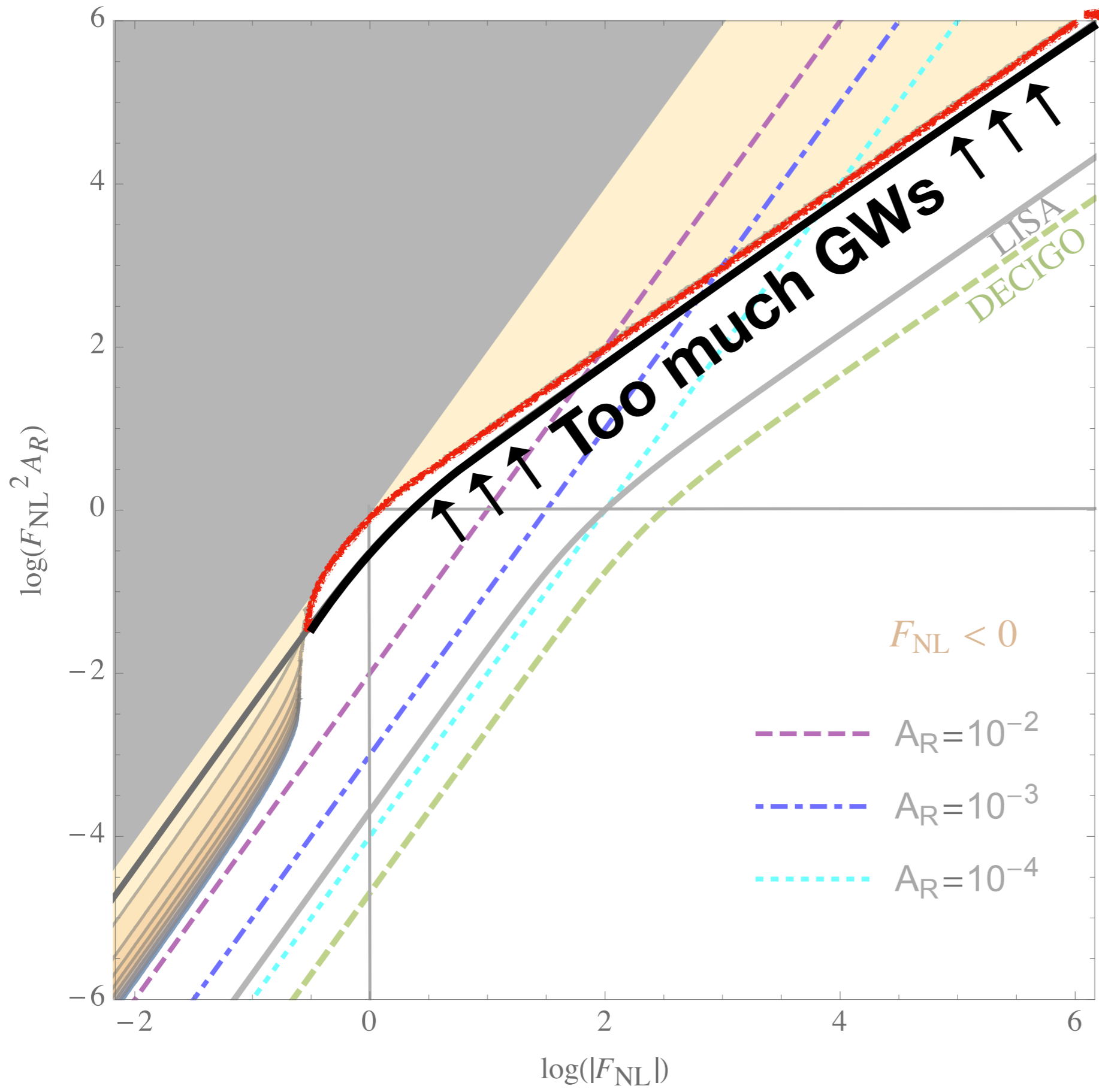


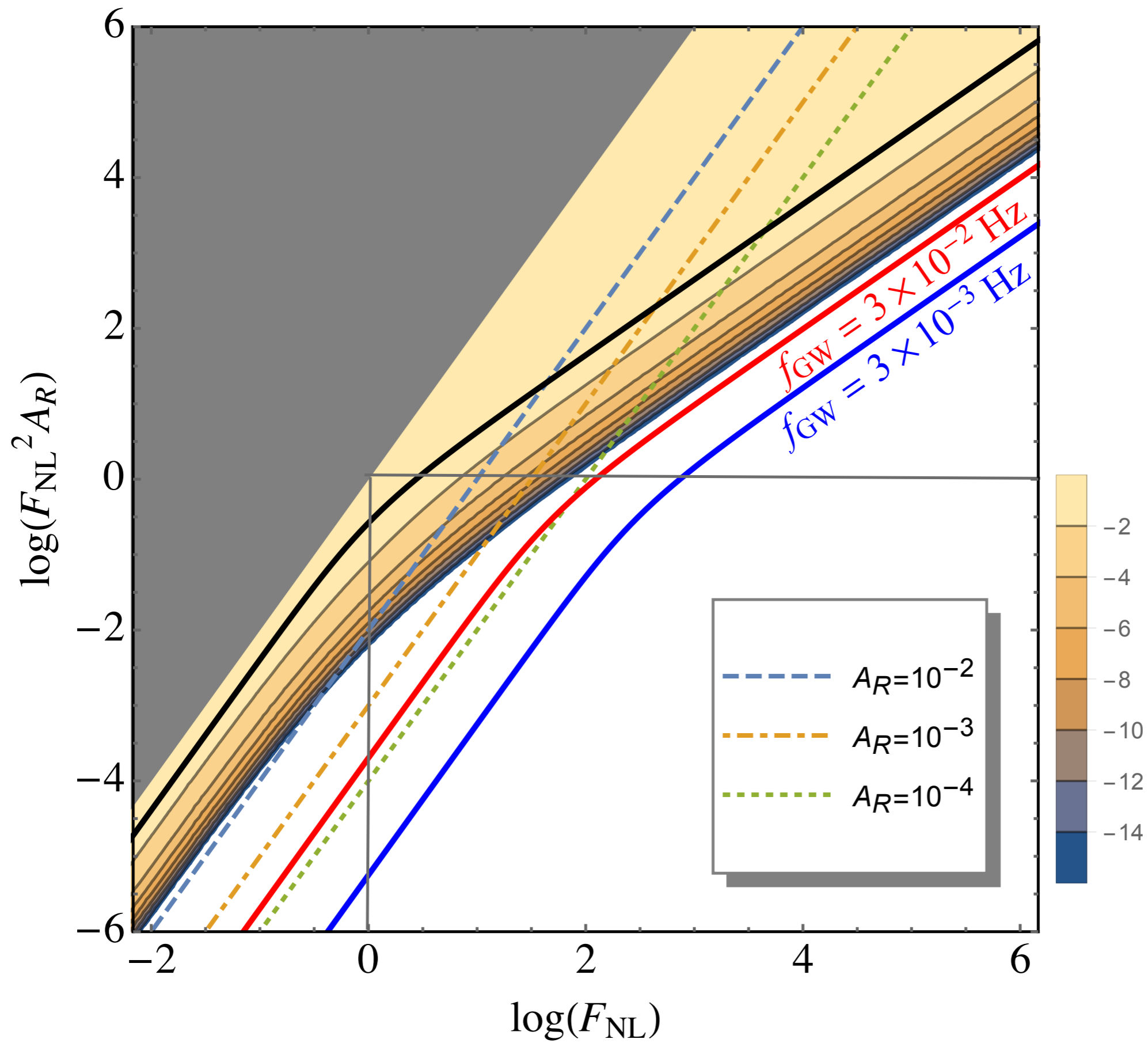


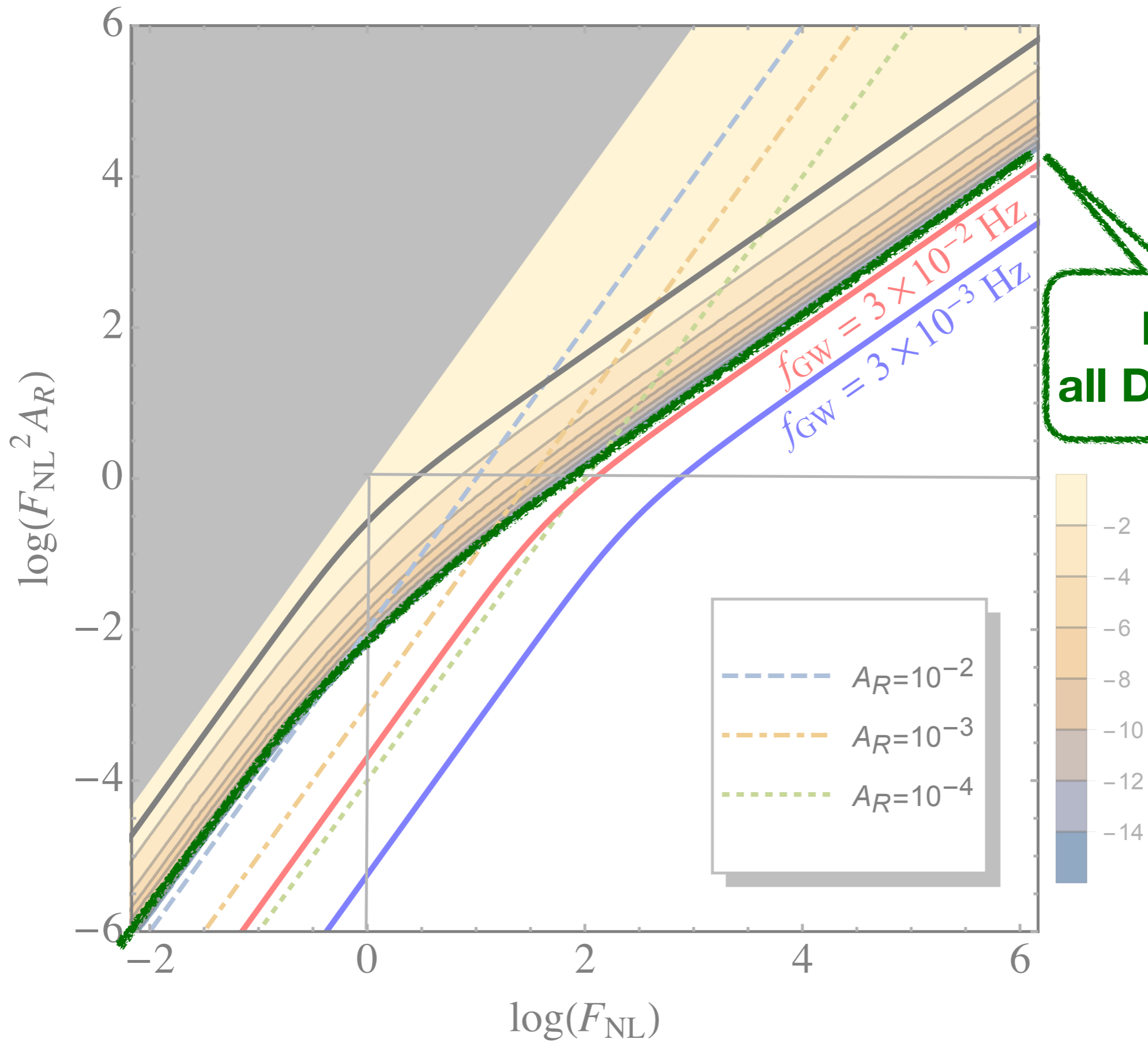




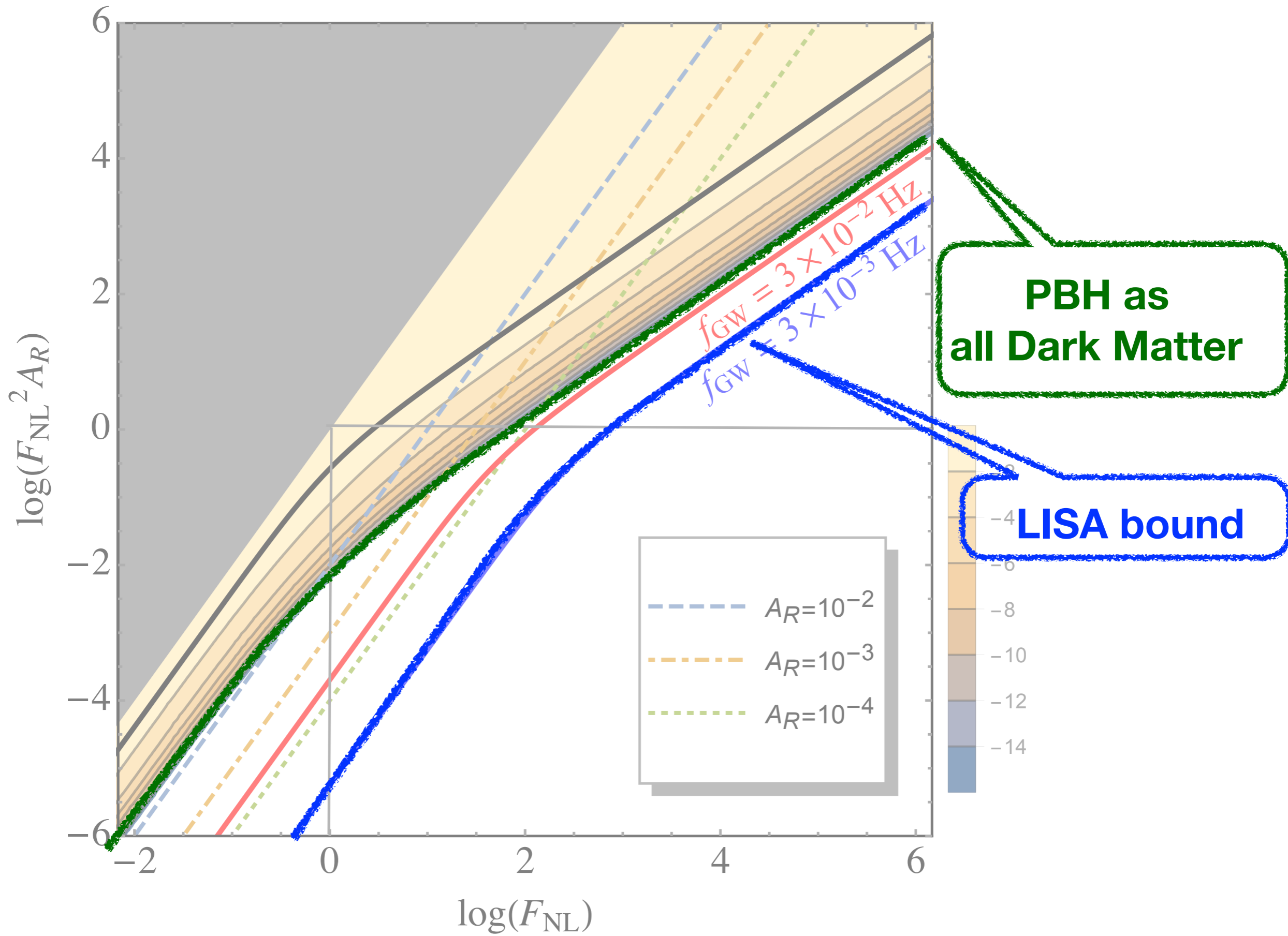


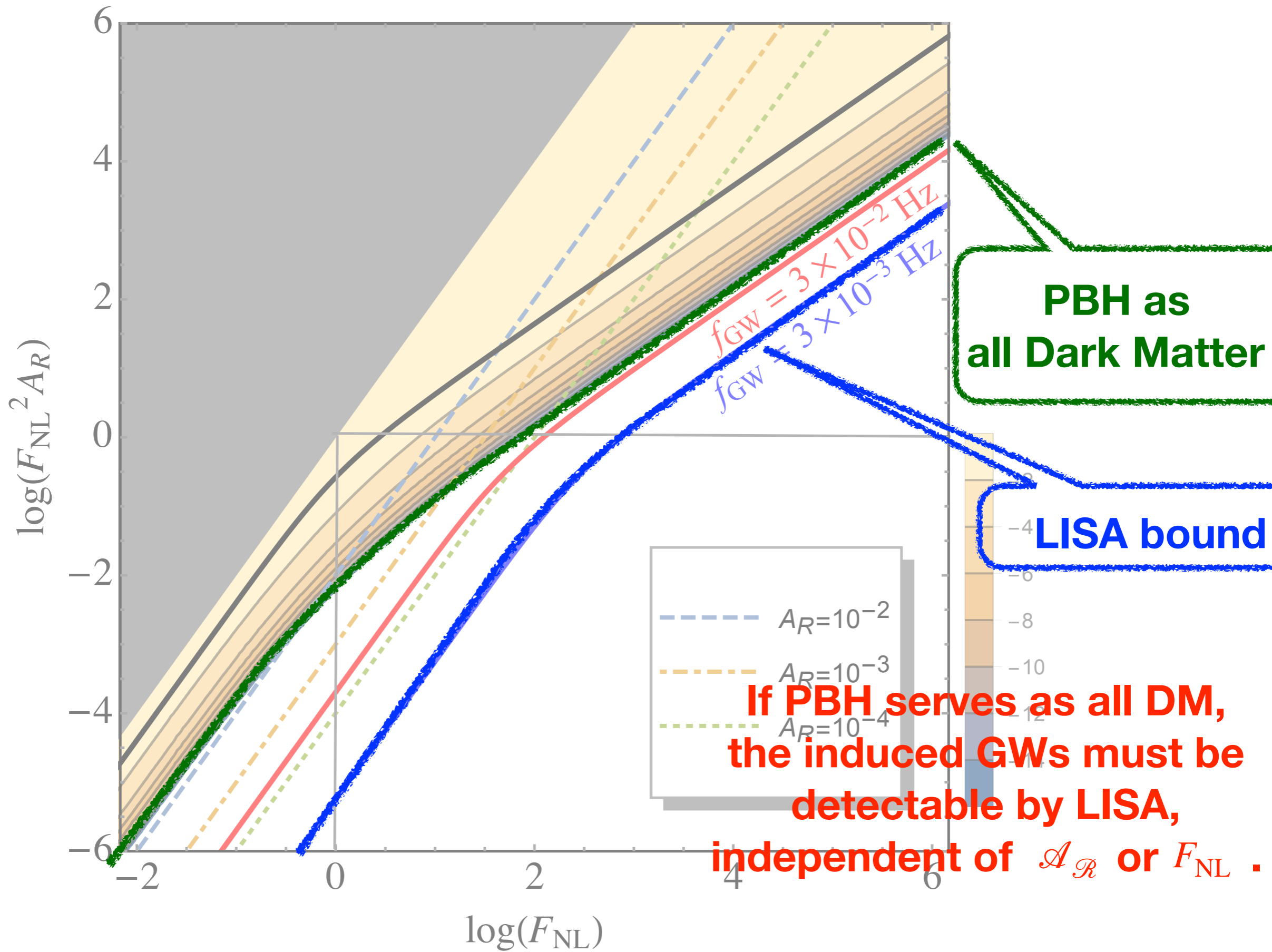


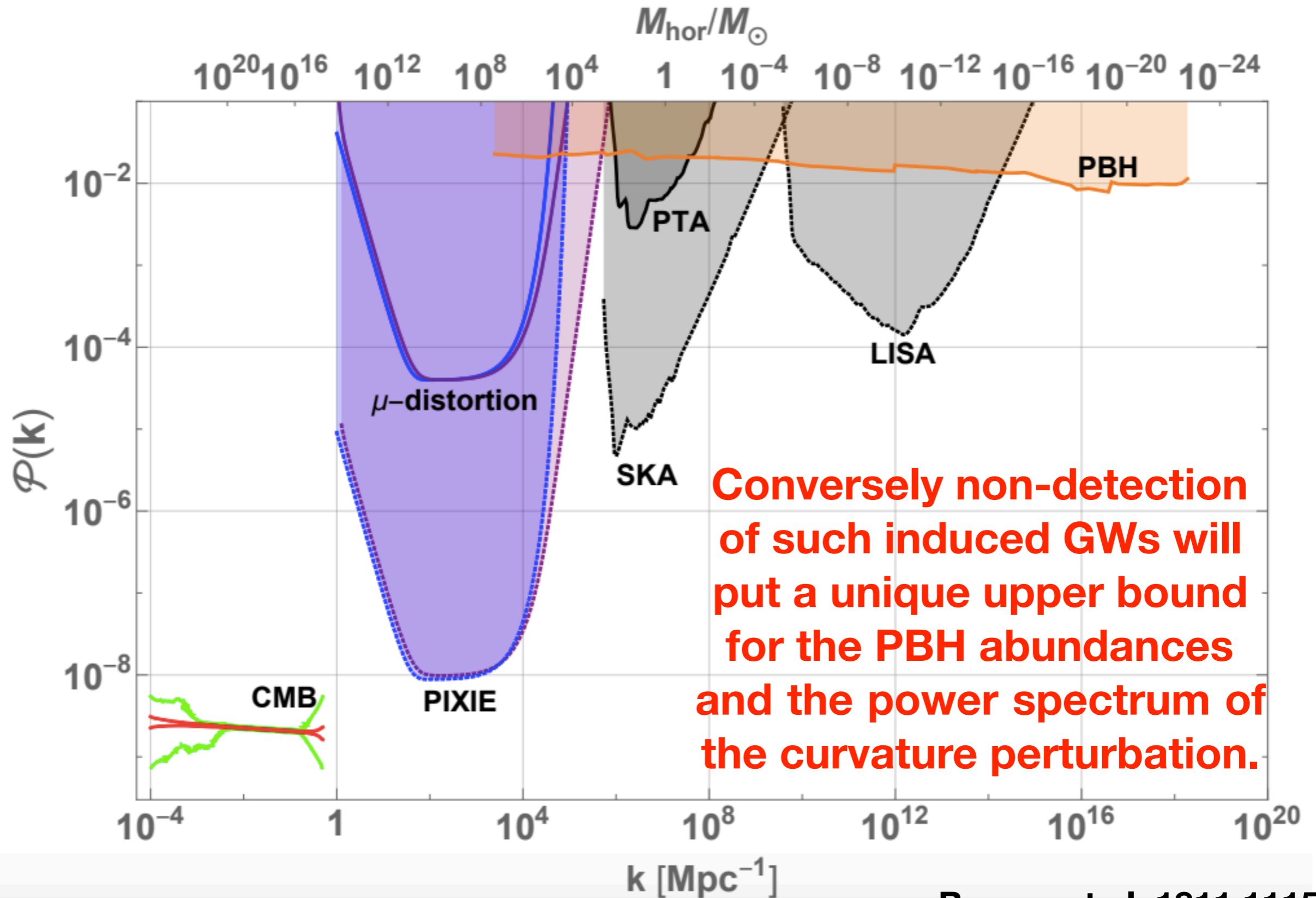




**PBH as  
all Dark Matter**







**Conversely non-detection of such induced GWs will put a unique upper bound for the PBH abundances and the power spectrum of the curvature perturbation.**



# Summary

- Induced GW is a very important source of SGWB.
- LISA detection and crosscheck with PBH abundances.
- Our work shows that induced GW is also a good probe of primordial non-Gaussianity.
- Our work reveals the importance of how to discriminate different sources of SGWB.

*Thank you!*