Some applications of integral geometry in AdS/CFT

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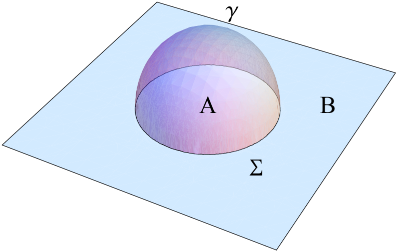
Outline

- Review of integral geometry
- OPE block and reconstruction of bulk operators
- Entanglement renormalization



Entanglement entropy S_A: missing information by restricting to measurement in system A

$$\rho_A = \mathrm{Tr}_B \rho_{AB} = \mathrm{Tr}_B |\psi_{AB}\rangle \langle \psi_{AB}|$$

$$S_{\mathrm{EE}} = -\mathrm{Tr} \rho_A \log \rho_A$$

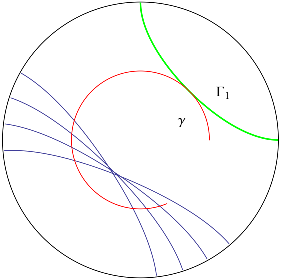
Ryu-Takayanagi formula

$$\boxed{S_{\mathrm{EE}} = \frac{A(\gamma)}{4G}}$$

γ: minimal surface ending on $\Sigma = \partial A$

Crofton's formula

Czech etal. 1505.05515



$$\frac{\sigma(\gamma)}{4G} = \frac{1}{4}\int_{\gamma\cap\Gamma\neq\varnothing} N(\gamma\cap\Gamma)\,\epsilon_{\mathcal{K}}$$

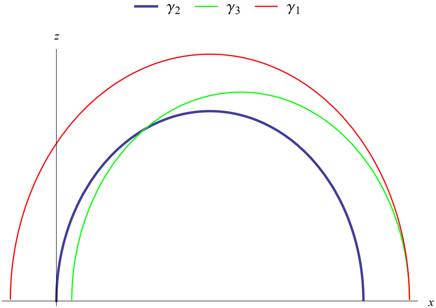
The length $\sigma(\gamma)$ of a curve γ can be expressed in terms of an integral over the geodesics Γ that have nonvanishing intersection number $N(\gamma\cap\Gamma)$ with γ

The measure $\epsilon_{\mathcal{K}}$ is given by the second derivative of the entanglement entropy

$$\epsilon_{\mathcal{K}}(u,v) = \frac{\partial^2 S(u,v)}{\partial u \partial v}\mathrm{d}u\wedge\mathrm{d}v$$

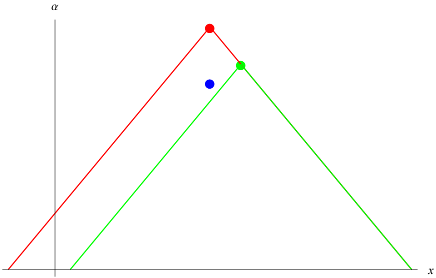
Hence we can obtain the geometry from the entanglement structure of the field theory on the boundary

Kinematic space



Hyperbolic space:

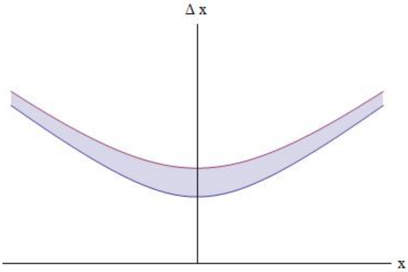
$$\mathrm{d}s^2 = \frac{1}{z^2}\left(\mathrm{d}z^2 + \mathrm{d}x^2\right)$$



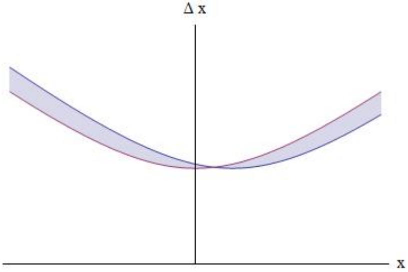
de Sitter:

$$\mathrm{d}s^2 = \frac{1}{\alpha^2}\left(-\mathrm{d}\alpha^2 + \mathrm{d}x^2\right)$$

Point curve and distance



A (blue): (x,z), B (red): (x,z+δ)



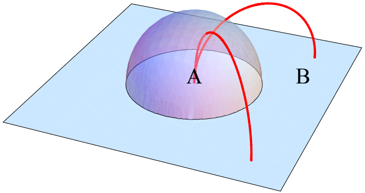
A (blue): (x,z), B (red): (x-δ,z)

The distance is given by integration over the region between the point curves

$$\frac{\mathrm{d}(A,B)}{4G} = \frac{1}{4}\int_{(\tilde{p}_A\cup\tilde{p}_B)-(\tilde{p}_A\cap\tilde{p}_B)} \epsilon_{\mathcal{K}}$$

Kinematic space of geodesics in general dimensions

XH and Lin 1507.04633



$$\epsilon_{\mathcal{K}} = \frac{1}{4G} \det\left[\frac{\partial^2 S(\vec{x}_1, \vec{x}_2)}{\partial \vec{x}_1 \partial \vec{x}_2}\right] \prod_{i=1}^{d-1} \mathrm{d}x_1^i \wedge \mathrm{d}x_2^i$$

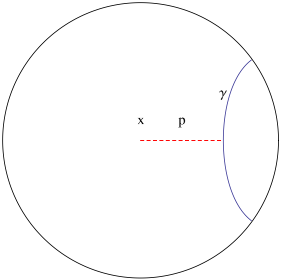
Crofton's formula in higher dimensions says that the area is equal to flux of geodesics

$$\int_{M^q \cap L_r \neq \varnothing} \sigma_{q+r-d}(M^q \cap L_r)\, \epsilon_{\mathcal{K}} = \frac{O_d \ldots O_{d-r} O_{q+r-d}}{O_r \ldots O_1 O_0 O_q} \sigma_q(M^q)$$

- The density is given by the second derivatives of the length of the geodesic
- RT-formula can be reproduced

Radon Transform and its inverse

Duality: $f(x) \leftrightarrow \hat{f}(\gamma)$



Radon transform $f(x) \to \hat{f}(\gamma)$

$$\hat{f}(\gamma) = \int_{x\in\gamma} f(x) d\sigma(\gamma)$$

Dual transform $\varphi(\gamma) \to \check{\varphi}(x)$

$$\check{\varphi}_p(x) = \int_{d(x,\gamma)=p} \varphi(\gamma)$$

The inverse transform:

$$f(x) = -\frac{1}{\pi}\int_0^\infty \frac{1}{\sinh p}\frac{d}{dp}(\hat{f})^\vee_p(x)dp\,.$$

Conformal block

Conformal partial waves (Δ_i all equal) $W_{\Delta,\ell}(x_i)$ from 4-point function

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = \sum_{\mathcal{O}} C_{12\mathcal{O}}\, C^{\mathcal{O}}{}_{34} W_{\Delta,\ell}(x_i)$$

$W_{\Delta,\ell}(x_i)$ is related to the (global for 2d) conformal block $G_{\Delta,\ell}(u,v)$

$$W_{\Delta,\ell}(x_i) \equiv \frac{G_{\Delta,\ell}(u,v)}{(x_{12}^2)^{\frac{1}{2}(\Delta_1+\Delta_2)}(x_{34}^2)^{\frac{1}{2}(\Delta_3+\Delta_4)}}$$

where u, v are the cross ratios:

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\ , \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

Conformal block from Radon transform

Conformal block can be computed holographically from geodesic Witten diagram Hijano etal. 1508.00501

$$W_{\Delta,0}(x_i) \sim \int_{x_0\in\gamma_{12}}\int_{x_0'\in\gamma_{34}} G_{b\partial}(x_0,x_1)G_{b\partial}(x_0,x_2)\times G_{bb}(x_0,x_0';\Delta)\times G_{b\partial}(x_0',x_3)G_{b\partial}(x_0',x_4)$$

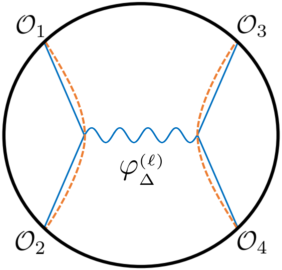


figure courtesy of Hijano etal. arXiv:1508.00501

Taking $\Delta_i = 0$, the conformal block becomes the Radon transform of the bulk two-point function

$$\hat{G}_{bb}(\gamma_{12},\gamma_{34}) = G_{\Delta,0}(u,v)\,.$$

Bulk field from geodesic operator

Each pair of operators gives the Radon transform of a bulk operator $\hat{\phi}$ Czech etal. 1604.03110

$$\hat{\phi}(x_1, x_2) = (x_{12})^{2\Delta_i} C_{ii\mathcal{O}}(x_{12}, \partial)\mathcal{O}(x_2)$$

$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) = \sum_{\mathcal{O}} C_{ij\mathcal{O}}(x_{12}, \partial)\mathcal{O}(x_2)$$

Turning on a source of $\phi(x_3)$ (coupled to a primary $\mathcal{O}(x_3)$) gives the 3-point function

$$\frac{(x_{12}^2)^{\frac{1}{2}\Delta_3}}{(x_{13}^2)^{\frac{1}{2}\Delta_3}(x_{23}^2)^{\frac{1}{2}\Delta_3}}$$

which is the Radon transform of the bulk-to-boundary propagator

Casimir

Covariant kinematic space specified by end points
$z_{L,R} = t_{L,R} - x_{L,R}$, $\bar{z}_{L,R} = t_{L,R} + x_{L,R}$

$$\mathrm{d}s^2 = \frac{L}{4G}\left[\frac{1}{(z_L - z_R)^2}\mathrm{d}z_L\mathrm{d}z_R + \frac{1}{(\bar{z}_L - \bar{v}_R)^2}\mathrm{d}\bar{z}_L\mathrm{d}\bar{z}_R\right].$$

The holomorphic and anti-holomorphic coordinates decouple and the metric takes the form of $\mathrm{dS}_2 \times \mathrm{dS}_2$

The descendants of a certain primary $\mathcal{O}_k$ all have eigenvalue C_k and we have $(\Delta_i = 0)$

$$\left[(L^2 + \bar{L}^2), \mathcal{B}_k(x_1, x_2)\right] = C_k\,\mathcal{B}_k(x_1, x_2) = \mathcal{L}_{12}^2\mathcal{B}_k(x_1, x_2)$$

Casimir and equation of motion

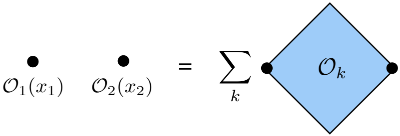
The Casimir in the position space (from $\mathcal{L}_n = z^{n+1}\partial_z$) gives

$$2\left(\Box_{\mathrm{dS}_2} + \Box_{\overline{\mathrm{dS}}_2}\right)\mathcal{B}_k\left(x_1, x_2\right) = C_k \mathcal{B}_k\left(x_1, x_2\right)\ ,$$

which implies

$$\left(\Box_{\mathrm{AdS}} - m^2\right)\phi\left(x\right) = 0 \quad \Longrightarrow \quad \left(2\left(\Box_{\mathrm{dS}_2} + \Box_{\overline{\mathrm{dS}}_2}\right) + m^2\right)\tilde{\phi}\left(\gamma\right) = 0$$

One can solve for the OPE block



Interaction

$$\tilde{\phi}(\gamma_{12}) = \mathcal{B}_{\mathcal{O}_\Delta}(x_1, x_2) + \frac{1}{N} \sum_{\{i,j\},n} a_n^{ij} \mathcal{B}_{[\mathcal{O}_i \mathcal{O}_j]_n}(x_1, x_2) + O(1/N^2)$$

- Multi-trace operators are needed to reproduce interaction
- Bulk interaction expressed in terms of geodesics Witten diagram gives rise to exchange of multi-trace operators
- Interacting geodesic operator (with multi-trace operators involving arbitrary number of T) follows from Virasoro conformal block Guica 1610.08952

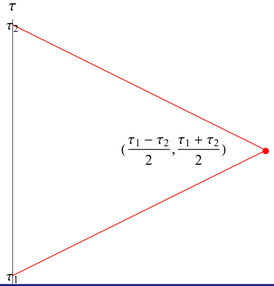
Bulk operator reconstruction in SYK model

4-point function in SYK model reads

$$H_{\mathrm{SYK}} = \sum_{1\le i_1 \le i_2 \cdots \le i_q \le N} j_{i_1 i_2 \dots i_q} \psi_{i_1}\psi_{i_2}\dots\psi_{i_q}, \quad \mathcal{F} = \mathcal{F}_c + \mathcal{F}_{h=2}$$

The CFT_1 contribution $\mathcal{F}_c$ admits the conformal block expansion

$$\mathcal{F}_c(\tau_1,\tau_2,\tau_3,\tau_4) = G(\tau_{12})G(\tau_{34})\sum_{n=1}^{\infty} c_n^2 z^{h_n}{}_2F_1(h_n,h_n;2h_n,z)\,,$$



w/ Chen-Te Ma, in progress

A pair time-like separated determines a bulk point

$$y = \left(\frac{1}{2}(\tau_1+\tau_2), \frac{1}{2}|\tau_1-\tau_2|\right)$$

$$\hat{G}_{bb}(y_{12}, y_{34}) = G_{h_n}(z)\,.$$

Extension to Schwarzian theory

The leading order non-conformal part is given by the Schwarzian

$$\int \{f(\tau), \tau\} \mathrm{d}\tau \,,$$

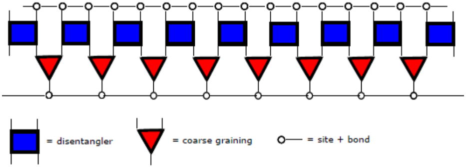
The following operator in the Schwarzian can have the bulk correspondence Mertens etal 1705.08408

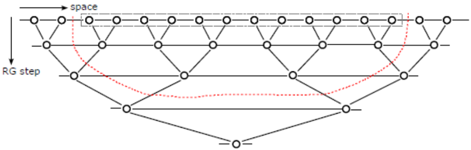
$$\mathcal{O}_\ell(\tau_1, \tau_2) = \left(\frac{f'(\tau_1) f'(\tau_1)}{f(\tau_1) - f(\tau_2)} \right)^{2\ell} \leftrightarrow e^{2\ell\phi(\tau_1,\tau_2)}$$

The OPE block of the stress tensor corresponds to the conformal factor in the bulk metric

$$f \circ B_T^{ii}(\tau_1, \tau_2) = 2[\phi(\tau_1, \tau_2) - \phi_0(\tau_1, \tau_2)]$$

Multi-scale entanglement renormalization ansatz





LUs (disentangler, isometry) modify the entanglement structure

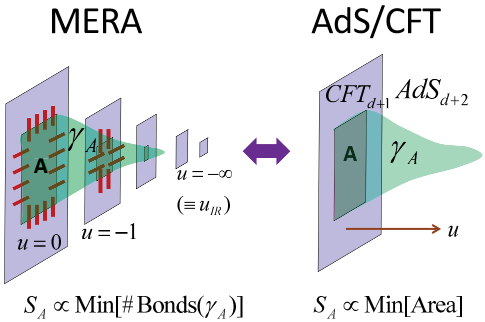
MERA version of RT-formula

$$S_A \leq \#(\text{LUs cut})$$

figures courtesy of Swingle arXiv:1209.3304

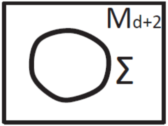
AdS/MERA

MERA = Discretized AdS Swingle 0905.1317

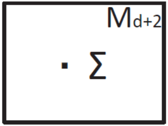


figures courtesy of Nozaki etal. arXiv:1208.3469

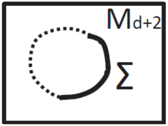
Surface/state correspondence



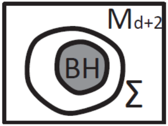
Pure State $|\Phi(\Sigma)\rangle$



Trivial State $|\Omega\rangle$

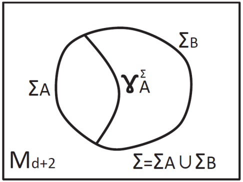


Mixed State $\rho(\Sigma)$



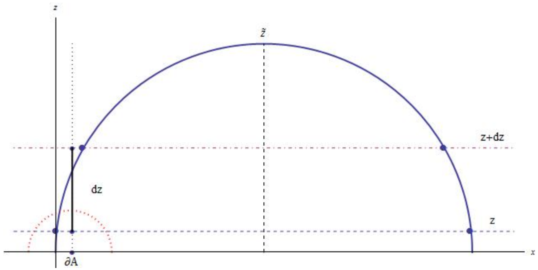
Mixed State $\rho(\Sigma)$

- AdS/CFT applies to $\Phi(\Sigma)$, an excited state
- RT remains valid for $\Phi(\Sigma)$



figures courtesy of Miyaji and Takayanagi arXiv:1503.03542

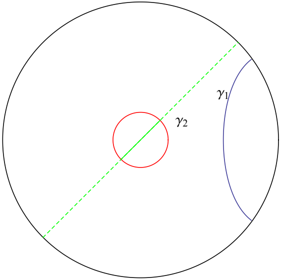
RG flow in the kinematic space



The short-distance entanglement (red) is removed while the long-distance (blue) entanglement is invariant under RG flow but it is reshuffled to shorter scale.

Ishibashi state

- The UV state $\phi(x)|0\rangle$ can be expressed in terms of "global" Ishibashi state *i.e.*, $\sum_n L^n_{-1}\tilde{L}^n_{-1}\mathcal{O}$ Miyaji etal. 1506.01353
- Realization of error correction using the covariant version of inverse Radon transform. Ishibashi state in a local form
  Goto&Takayanagi 1704.00053



- The IR (trivial) state should be an Ishibashi state of the full Virasoro algebra
- The IR operator only consists of geodesic operators going through point x

Summary

- The punch line is that we have a duality between the kinematic space $\leftrightarrow$ real space. We can reconstruct the kinematic space from field theory.
- The construction based on geodesic applies to any space and any boundary.
- Bulk operator can be constructed using Radon transform. The dual operator (geodesic operator) follows from OPE
- The approach in principle can go beyond the conformal case and could have a wide range of applications.