Phase structure and critical behavior of black holes and branes in canonical ensemble

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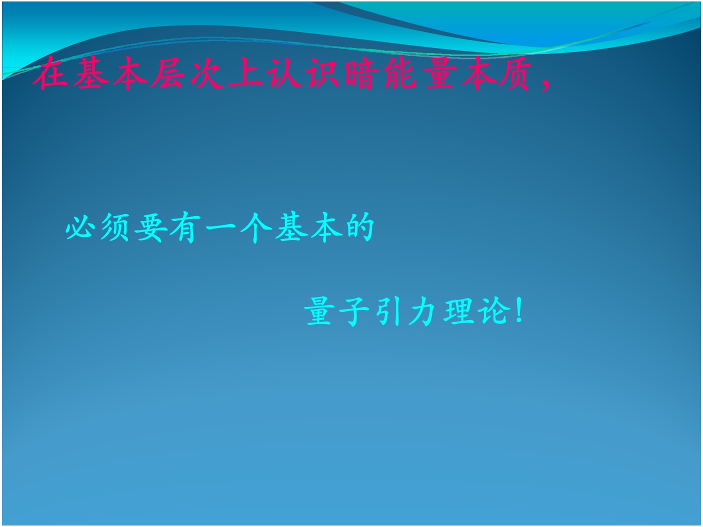
Introduction

Introduction



Introduction

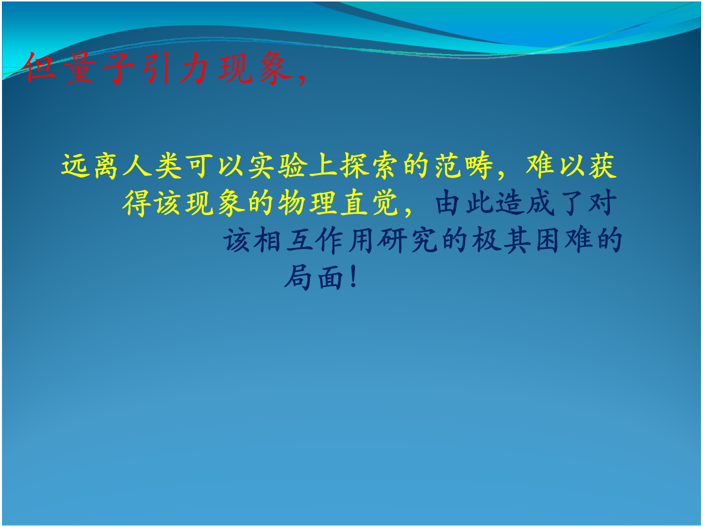
另外，

无论从已知四种相互作用的统一，

还是对如理解宇宙极早期、黑洞奇点等

我们都需要研究引力的量子行为！

Introduction



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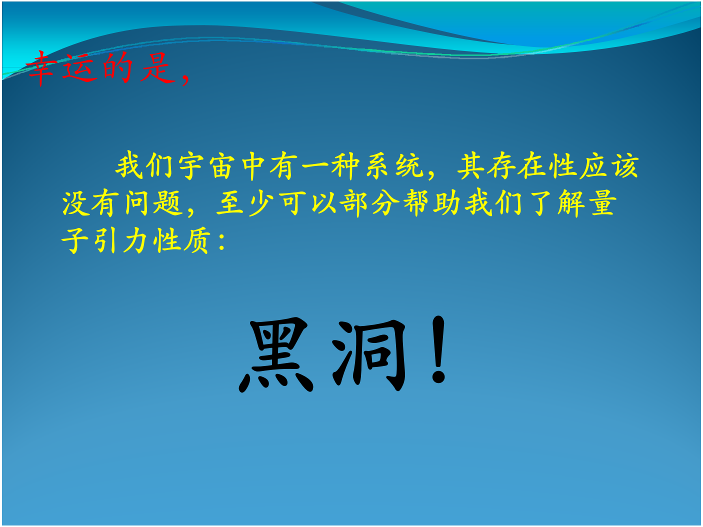

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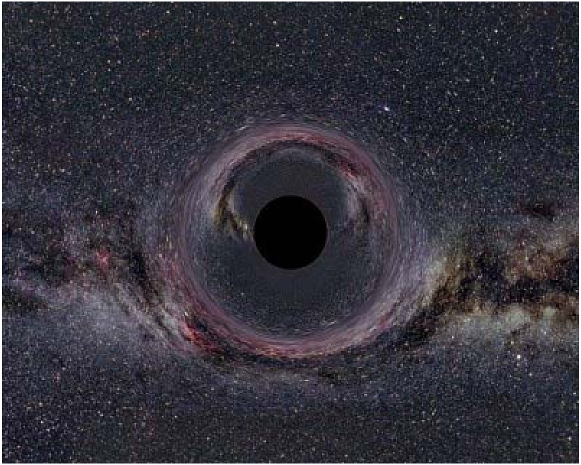

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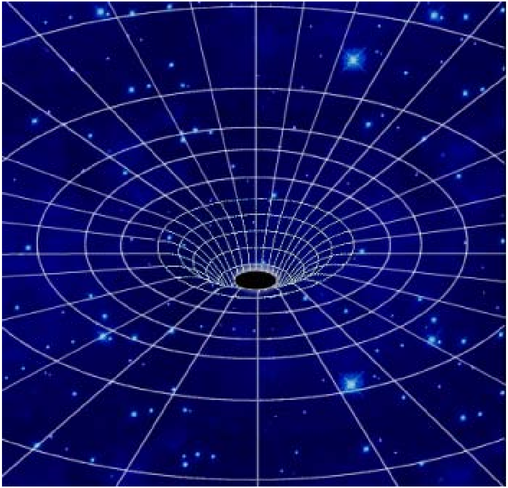


Introduction

Found four laws of black hole mechanics resembling the four laws of thermodynamics for stationary black hole:

Four laws of black hole	Four laws of thermodynamics
0-th: constant κ at horizon,	uniform T
1-th: $dM = \frac{\kappa}{8\pi} dA + \Omega dJ + \Phi dQ$,	$dM = TdS + \Omega dJ + \Phi dQ$
2-nd: $\frac{dA}{dt} \geq 0$,	$\frac{dS}{dt} \geq 0$
3-rd: no black hole for $\kappa = 0$,	T = 0 cannot be reached

In the above, A = area of horizon and κ = the surface gravity of horizon. We also take $k = \hbar = c = G = 1$.

Introduction

Bekenstein (72) proposed:

$$S_{\rm BH} \sim A, \quad T_{\rm BH} \sim \kappa \tag{1.1}$$

while Hawking (74) went a step further to show a black hole radiates and has a temperature

$$T_{\rm BH} = \frac{\kappa}{2\pi} \tag{1.2}$$

. This gives a precise identification:

$$S = \frac{A}{4} \quad \left(= \frac{kc^3 A}{4G\hbar}\right) \tag{1.3}$$

Introduction

While with the above a black hole appears as a well-defined thermodynamical system, there exists a serious issue for asymptotically flat black hole with such an interpretation.

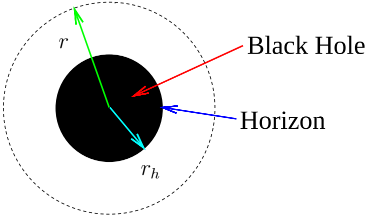
For example, a Schwarzschild black,

$$S_{\rm BH} = 4\pi M^2, \qquad T_{\rm BH} = \frac{1}{8\pi M} \tag{1.4}$$

with M the ADM energy carried by the black hole. This system is actually thermodynamically unstable (the specific heat $C < 0$)!!!

Introduction

$T(r) = T_{\mathrm{BH}}(1 - r_h/r)^{-1/2}$

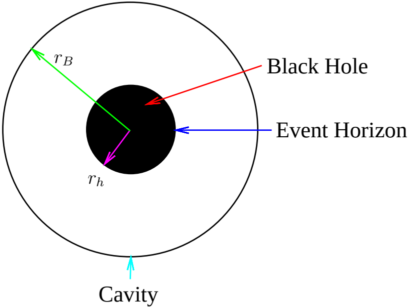


$T(r \to \infty) = T_{\mathrm{BH}} = \frac{1}{8\pi M}$

Introduction

- So in order to give a proper consideration of asymptotically flat black hole thermodynamics, we need first to suitably stabilize the black hole thermally.
- In other words, we need to consider ensembles that include not only the black hole under consideration but also its environment.
- Further, as self-gravitating systems are spatially inhomogeneous, any specification of such ensembles requires not just thermodynamic quantities of interest but also the place at which they take specific values.

Introduction



Black hole (r_h) placed in a cavity (r_B) with fixed T and V.

Chargeless case

Consider the simplest spherical symmetric Schwarzschild black hole in Euclidean signature

$$ds_E^2 = \left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega_2^2, \qquad (2.1)$$

with the horizon radius $r_h = 2M$. If we place this black hole in a large spherical hot cavity at a given $r = r_B$ $(r_B > r_h)$ with the temperature at the wall fixed at T, this will define a canonical ensemble *a la* York (PRD33 (1986) 2092) for this hole. Thermal equilibrium says

$$T = T_{BH}(r_B) = T_{BH}\left(1 - \frac{2M}{r_B}\right)^{-1/2} = (8\pi M)^{-1}\left(1 - \frac{2M}{r_B}\right)^{-1/2} \qquad (2.2)$$

Chargeless case

For given T and $r_B > r_h = 2M$, the above equation $\Rightarrow$:

the existence of two black holes, one small and one large.

The large one is at least locally stable thermally while

the small one is not.

Chargeless case

The stability can be analyzed using the Helmholz free energy which can be calculated following Gibbons and Hawking (PRD15(1977)2752) that the partition function Z contains the first-order classical Euclidean Einstein action of the hole as its leading term.

In other words,

$$Z = e^{-\beta F} \approx e^{-I_E} \tag{2.3}$$

$\Rightarrow$

$$I_E(r_B, T; r_h) = \beta F = \beta E(r_B, T; r_h) - S(r_h) \tag{2.4}$$

with $\beta = 1/T$ and E the internal energy of the cavity.

Chargeless case

$$
\begin{aligned}
E(r_B; x) &= r_B \left(1 - (1 - x)^{1/2}\right), \\
S(r_B; x) &= \pi r_B^2 x^2 = 4\pi M^2
\end{aligned}
\tag{2.5}
$$

with

$$
x \equiv \frac{r_h}{r_B} = \frac{2M}{r_B}, \qquad r_B > r_h \tag{2.6}
$$

Note

$$
0 < x < 1. \tag{2.7}
$$

$$
I_E = \beta r_B \left(1 - (1 - x)^{1/2}\right) - \pi r_B^2 x^2, \tag{2.8}
$$

Chargeless case

For simplicity, define

$$\bar{I}_E \equiv \frac{I_E}{4\pi r_B^2}, \quad \bar{b} \equiv \frac{\beta}{4\pi r_B}, \tag{2.9}$$

$$\bar{I}_E = \bar{b}\left(1 - (1-x)^{1/2}\right) - \frac{1}{4}x^2. \tag{2.10}$$

$$\left(\bar{I}_E = 0\,(x=0) \qquad \Leftrightarrow \qquad \text{hot flat space}\right). \tag{2.11}$$

$$\frac{\partial \bar{I}_E}{\partial x} = \frac{1}{2(1-x)^{1/2}}\left(\bar{b} - b(x)\right), \tag{2.12}$$

$$b(x) = x(1-x)^{1/2} > 0. \tag{2.13}$$

Note

$$b(x \to 0) \to 0, \qquad b(x \to 1) \to 0. \tag{2.14}$$

Chargeless case

$$\frac{\partial \bar{I}_E}{\partial x} = 0 \Rightarrow \bar{b} = b(\bar{x}) = \bar{x}(1-\bar{x})^{1/2} \qquad (2.15)$$

$$\Rightarrow T = T(r_B) = (8\pi M)^{-1}\left(1-\frac{2M}{r_B}\right)^{-1/2}. \qquad (2.16)$$

Chargeless case

$$\left.\frac{\partial^2 \bar{I}_E}{\partial x^2}\right|_{x=\bar{x}} \sim -\frac{\partial b(\bar{x})}{\partial \bar{x}}, \qquad (2.17)$$

$$\frac{\partial b(x)}{\partial x} > 0, \qquad \frac{\partial^2 \bar{I}_E}{\partial x^2} < 0 \qquad \text{(unstable)}$$

$$\frac{\partial b(x)}{\partial x} < 0, \qquad \frac{\partial^2 \bar{I}_E}{\partial x^2} > 0 \qquad \text{(stable)} \qquad (2.18)$$

Chargeless case

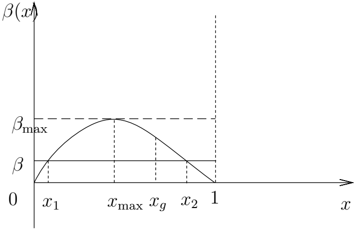


Figure 1: The typical behavior of $\beta(x)$ vs x ($x \equiv r_h/r$).

Charged case

The charged (Reissner-Nordström) black hole is

$$ds_E^2 = V(r)dt^2 + \frac{dr^2}{V(r)} + r^2 d\Omega_2^2, \tag{3.1}$$

with

$$V(r) = 1 - \frac{2M}{r} + \frac{e^2}{r^2}, \qquad \Phi = \frac{e}{r}. \tag{3.2}$$

It has two horizons given at (V (r) = 0)

$$r_\pm = M \pm \sqrt{M^2 - e^2}, \tag{3.3}$$

which implies

$$M \geq e, \quad \text{(BPS Bound)} \tag{3.4}$$

Charged case

By the same token,

$$\begin{aligned} I_E(\beta, r_B, e; r_+) &= \beta E(r_B, e; r_+) - S(r_+) \\ &= \beta\left(1 - \sqrt{\left(1 - \frac{r_+}{r_B}\right)\left(1 - \frac{e^2}{r_+ r_B}\right)}\right) - \pi r_+^2 \end{aligned} \quad (3.5)$$

Define,

$$\bar{I}_E \equiv \frac{I_E}{4\pi r_B^2}, \qquad x \equiv \frac{r_+}{r_B}, \quad q \equiv \frac{e}{r_B}, \quad \bar{b} \equiv \frac{\beta}{4\pi r_B},$$
$$q < x < 1, \quad (r_+ > e, r_B > r_+). \quad (3.6)$$

Charged case

$$\bar{I}_E(\bar{b}, q; x) = \bar{b}\left(1 - \sqrt{(1-x)\left(1 - \frac{q^2}{x}\right)}\right) - \frac{1}{4}x^2. \tag{3.7}$$

$$\frac{\partial \bar{I}_E}{\partial x} = \frac{1 - \frac{q^2}{x^2}}{2(1-x)^{1/2}\left(1 - \frac{q^2}{x}\right)^{1/2}}\left(\bar{b} - b_q(x)\right), \tag{3.8}$$

where

$$b_q(x) = \frac{x(1-x)^{1/2}\left(1 - \frac{q^2}{x}\right)^{1/2}}{1 - \frac{q^2}{x^2}}. \tag{3.9}$$

Note

$$b_q(x \to q) \to \infty, \quad b_q(x \to 1) \to 0. \tag{3.10}$$

Charged case

$$\frac{\partial \bar{I}_E}{\partial x} = 0 \qquad \Rightarrow \qquad \bar{b} = b_q(\bar{x}). \tag{3.11}$$

$$\Downarrow$$

$$T = T(r_B) = (4\pi r_+)^{-1}\left(1 - \frac{e^2}{r_+^2}\right)\left(1 - \frac{r_+}{r_B}\right)^{-1/2}\left(1 - \frac{e^2}{r_+ r_B}\right)^{-1/2}. \tag{3.12}$$

Charged case

Once again,

$$\left.\frac{\partial^2 \bar{I}_E}{\partial x^2}\right|_{x=\bar{x}} \sim -\frac{\partial b(\bar{x})}{\partial \bar{x}}, \qquad (3.13)$$

The locally stable black hole requires

$$\frac{\partial b(\bar{x})}{\partial \bar{x}} < 0. \qquad (3.14)$$

Charged case

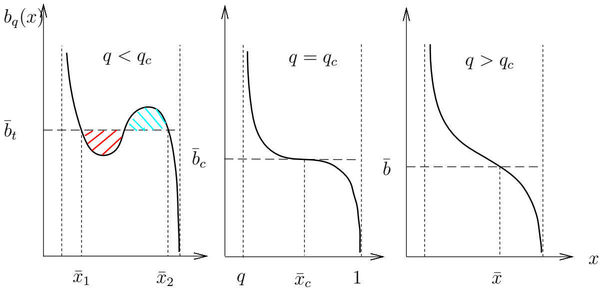
Note that

$$b_q(x \to q) \to \infty, \quad b_q(x \to 1) \to 0, \tag{3.15}$$

there exists a critical charge $q_c = \sqrt{5} - 2(\bar{x}_c = 5 - 2\sqrt{5}, \bar{b}_c = 0.429)$ and we actually have three cases to consider:

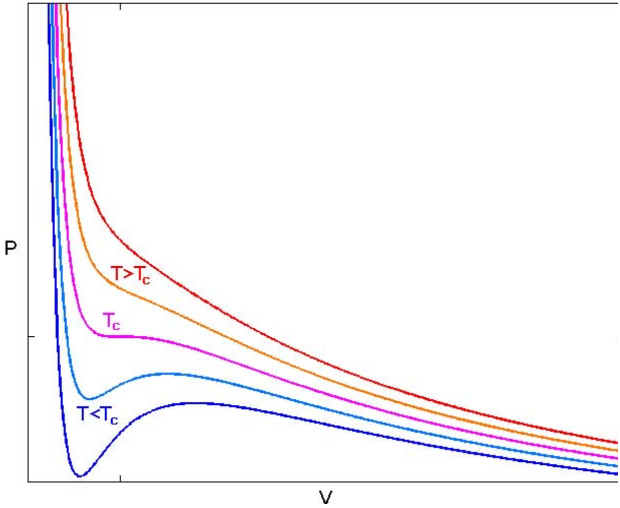
- $q < q_c$, there exists a unique temperature $T_t(q)$ for each given q at which there exists a first order phase transition between a small and large black holes. We have a line of this first-order phase transition, depending on $q < q_c$ and ending at a second-order phase transition point at $q = q_c$;
- $q = q_c$, this is a second-order critical point at which there exists no distinction between small and large black holes. The critical exponent can be read from $c_v \sim (T - T_c)^{-2/3}$ as $-2/3$;
- $q > q_c$ for each given temperature T there exists a unique global stable black hole with size $r_+ = r_B \bar{x}$.

Charged case



The typical behaviors of $b(x)$ vs x for $q < q_c, q = q_c, q > q_c$.

Van der Waals isotherm



So much for the usual black holes!

In string/M-theory, the basic objects are the so-called p-branes and the black correspondences are the asymptotically flat black p-branes with each having a horizon.

Then what happen to the thermodynamical behavior of these branes and the phase structure? (Lu et al JHEP 1101:133(2011))

p-brane

A p-brane is a p-dimensional hyperspace $(p = 0, 1, \cdots, 9)$ residing at the bulk spacetime with dimension D ($D \geq p + 1)$ and can carry either electric-like $d + 1$-form charge with $d = p + 1$ as

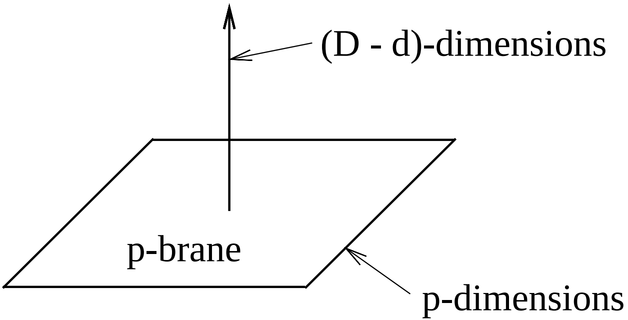
$$e_d \sim \int {}^*F_{d+1} \tag{4.1}$$

or magnetic-like $\tilde{d} + 1$-form charge with $\tilde{d} = D - 2 - d$ as

$$g_{\tilde{d}} \sim \int F_{\tilde{d}+1}. \tag{4.2}$$

p-brane

The spatial dimensions transverse to the p-brane is $D - d = \tilde{d} + 2$ and note $1 \leq \tilde{d} \leq 7$.



Black p-brane configuration

The black -brane configuration in Euclidean signature is

$$
\begin{aligned}
ds_E^2 &= \Delta_+ \Delta_-^{-\frac{d}{D-2}} dt^2 + \Delta_-^{\frac{\tilde{d}}{D-2}} \sum_{i=1}^{d-1} (dx^i)^2 + \Delta_+^{-1} \Delta_-^{\frac{a^2}{2\tilde{d}}-1} d\rho^2 \\
&\quad + \rho^2 \Delta_-^{\frac{a^2}{2\tilde{d}}} d\Omega_{\tilde{d}+1}^2, \\
A_{[p+1]} &= -i e^{a\phi_0/2} \left[\left(\frac{r_-}{r_+} \right)^{\tilde{d}/2} - \left(\frac{r_- r_+}{\rho^2} \right)^{\tilde{d}/2} \right] dt \wedge dx^1 \wedge \ldots \wedge dx^p, \\
F_{[p+2]} &\equiv dA_{[p+1]} = -i e^{a\phi_0/2} \tilde{d} \frac{(r_- r_+)^{\tilde{d}/2}}{\rho^{\tilde{d}+1}} d\rho \wedge dt \wedge dx^1 \wedge \ldots \wedge dx^p, \\
e^{2(\phi-\phi_0)} &= \Delta_-^a, \qquad (4.3)
\end{aligned}
$$

where

$$
\Delta_\pm = 1 - \frac{r_\pm^{\tilde{d}}}{\rho^{\tilde{d}}}. \qquad (4.4)
$$

Black p-brane configuration

In the above, the horizon occurs at $\rho = r_+$ with a curvature singularity at $\rho = r_-$ $(r_+ > r_-)$. ϕ is the dilaton with ϕ_0 its asymptotic value, related to the string coupling as $g_s = e^{\phi_0}$. a is the dilaton coupling given by

$$a^2 = 4 - \frac{2d\tilde{d}}{D-2}. \tag{4.5}$$

The Euclidean time is periodic with periodicity β^* given as,

$$\beta^* = \frac{4\pi r_+}{\tilde{d}} \left(1 - \frac{r_-^{\tilde{d}}}{r_+^{\tilde{d}}}\right)^{\frac{1}{\tilde{d}} - \frac{1}{2}} \tag{4.6}$$

This is the inverse of temperature T_∞ at $\rho = \infty$.

Black p-brane configuration

The local $\beta(\rho)$ is given as,

$$\beta(\rho) = \Delta_+^{1/2} \Delta_-^{-\frac{d}{2(d+\tilde{d})}} \beta^* \tag{4.7}$$

which is the inverse of local temperature $T(\rho)$ at ρ. The black p-brane will be placed in a cavity with its wall at $\rho = \rho_B$. It is clear from the metric that the physical radius of the cavity is

$$\bar{\rho}_B = \Delta_-^{\frac{a^2}{4\tilde{d}}} \rho_B, \tag{4.8}$$

while ρ_B is merely the coordinate radius. It is this $\bar{\rho}_B$ which we should fix in the following discussion and not ρ_B

Black p-brane configuration

In addition,

- fix the dilaton $\phi = \bar{\phi}$ at $\bar{\rho}_B$,
- $\bar{r}_\pm = \Delta_-^{\frac{a^2}{4\tilde{d}}} r_\pm$ are now the proper parameters. Using the 'barred' variables, $\Delta_\pm$ remain the same as before,

$$\Delta_\pm = 1 - \frac{r_\pm^{\tilde{d}}}{\rho_B^{\tilde{d}}} = 1 - \frac{\bar{r}_\pm^{\tilde{d}}}{\bar{\rho}_B^{\tilde{d}}}. \qquad (4.9)$$

- For the canonical ensemble, fix at $\bar{\rho} = \bar{\rho}_B$ also the cavity temperature T, the brane volume $V_p = \Delta_-^{\frac{\tilde{d}(d-1)}{2(D-2)}} V_p^*$ and the charge defined as,

$$\begin{aligned} Q_d &= \frac{i}{2\sqrt{\kappa}} \int e^{-a(d)\phi} * F_{[p+2]} = \frac{\Omega_{\tilde{d}+1}}{2\sqrt{\kappa}} e^{-a\phi_0/2} \tilde{d} (r_+ r_-)^{\tilde{d}/2} \\ &= \frac{\Omega_{\tilde{d}+1} \tilde{d}}{\sqrt{2}\kappa} e^{-a\bar{\phi}/2} (\bar{r}_+ \bar{r}_-)^{\tilde{d}/2}. \qquad (4.10) \end{aligned}$$

The action

Again by the same token, $I_E = \beta E - S$ can be given as

$$I_E = -\frac{\beta V_p \Omega_{\tilde{d}+1}}{2\kappa^2}\bar{\rho}_B^{\tilde{d}}\left[(\tilde{d}+2)\left(\frac{\Delta_+}{\Delta_-}\right)^{1/2} + \tilde{d}(\Delta_+\Delta_-)^{1/2} - 2(\tilde{d}+1)\right] - \frac{4\pi V_p \Omega_{\tilde{d}+1}}{2\kappa^2}\bar{r}_+^{\tilde{d}+1}\Delta_-^{-\frac{1}{2}-\frac{1}{\tilde{d}}}\left(1 - \frac{\bar{r}_-^{\tilde{d}}}{\bar{r}_+^{\tilde{d}}}\right)^{\frac{1}{2}+\frac{1}{\tilde{d}}} \quad (4.11)$$

One can check,

$$\frac{\partial I_E}{\partial \bar{r}_+} = 0 \Rightarrow \beta = \beta(\bar{\rho}_B) = \frac{4\pi \bar{r}_+}{\tilde{d}}\Delta_+^{\frac{1}{2}}\Delta_-^{-\frac{1}{\tilde{d}}}\left(1 - \frac{\bar{r}_-^{\tilde{d}}}{\bar{r}_+^{\tilde{d}}}\right)^{\frac{1}{\tilde{d}}-\frac{1}{2}}. \quad (4.12)$$

The equation of state

Note

$$\bar{r}_- = \left(\frac{\sqrt{2}\kappa Q_d}{\Omega_{\tilde{d}+1}\tilde{d}} e^{a\bar{\phi}/2}\right)^{\frac{2}{\tilde{d}}} \frac{1}{\bar{r}_+} = \frac{(Q_d^*)^2}{\bar{r}_+}, \tag{4.13}$$

where $Q_d^* \equiv [(\sqrt{2}\kappa Q_d e^{a\bar{\phi}/2})/(\Omega_{\tilde{d}+1}\tilde{d})]^{1/\tilde{d}}$, also a fixed quantity.

With this,

$$\beta(\bar{\rho}_B) = \frac{4\pi\bar{r}_+}{\tilde{d}} \left(1 - \frac{Q_d^{*\,2\tilde{d}}}{\bar{r}_+^{2\tilde{d}}}\right)^{\frac{1}{\tilde{d}} - \frac{1}{2}} \left(1 - \frac{\bar{r}_+^{\tilde{d}}}{\bar{\rho}_B^{\tilde{d}}}\right)^{1/2} \left(1 - \frac{Q_d^{*\,2\tilde{d}}}{\bar{r}_+^{\tilde{d}}\bar{\rho}_B^{\tilde{d}}}\right)^{\frac{1}{\tilde{d}}}, \tag{4.14}$$

The equation of state

Define,

$$x = \left(\frac{\bar{r}_+}{\bar{\rho}_B}\right)^{\tilde{d}} < 1, \qquad \bar{b} = \frac{\beta}{4\pi\bar{\rho}_B}, \qquad q = \left(\frac{Q_d^*}{\bar{\rho}_B}\right)^{\tilde{d}} < x \quad (4.15)$$

Using these, we have from the above

$$\bar{b} = b_q(x) \equiv \frac{x^{1/\tilde{d}}(1-x)^{1/2}}{\tilde{d}\left(1-\frac{q^2}{x^2}\right)^{\frac{1}{2}-\frac{1}{\tilde{d}}}\left(1-\frac{q^2}{x}\right)^{\frac{1}{\tilde{d}}}}, \quad (4.16)$$

Note

$$b_q(x \to q) \to \infty(\tilde{d} > 2), \quad b_q(x \to 1) \to 0 \quad (4.17)$$

The reduced action

and the reduced action

$$\bar{I}_E(\bar{b}, q; x) \equiv \frac{2\kappa^2 I_E}{4\pi \bar{\rho}_B^{\tilde{d}+1} V_p \Omega_{\tilde{d}+1}}$$
$$= -\bar{b}\left[(\tilde{d}+2)\left(\frac{1-x}{1-\frac{q^2}{x}}\right)^{1/2} + \tilde{d}(1-x)^{1/2}\left(1-\frac{q^2}{x}\right)^{1/2} -2(\tilde{d}+1)\right] - x^{1+\frac{1}{\tilde{d}}}\left(\frac{1-\frac{q^2}{x^2}}{1-\frac{q^2}{x}}\right)^{\frac{1}{2}+\frac{1}{\tilde{d}}}. \quad (4.18)$$

The extremal behavior of action

One can show

$$\frac{\partial \bar{I}_E}{\partial x} = f_q(x)\left[\bar{b} - b_q(x)\right] = 0 \Rightarrow \bar{b} = b_q(\bar{x}), \tag{4.19}$$

where $b_q(x)$ is as given above and

$$\begin{aligned} f_q(x) &= (1-x)^{-1/2}\left(1-\frac{q^2}{x}\right)^{-1/2}\left[\tilde{d}+2-\frac{\tilde{d}+2}{2}\left(\frac{1-\frac{q^2}{x^2}}{1-\frac{q^2}{x}}\right)\right. \\ &\left. +\frac{\tilde{d}}{2}\left(1-\frac{q^2}{x^2}\right)\right] > 0. \end{aligned} \tag{4.20}$$

Also

$$\left.\frac{\partial^2 \bar{I}_E}{\partial x^2}\right|_{\bar{b}=b_q(\bar{x})} = -f_q(\bar{x})\frac{\partial b_q(\bar{x})}{\partial \bar{x}}. \tag{4.21}$$

These equations are the basis for the discussion of phase structure and critical behavior of black p-brane in canonical ensemble.

Phase structure and transition

For each given $\tilde{d} > 2$, the phase structure here is the same as the charged black hole though the detail is different. There exists a critical charge q_c, depending on $\tilde{d}$, and we have also the three cases for each given $\tilde{d} > 2$,

- $q < q_c$, there exists a unique temperature $T_t(q)$ for each given q at which there is a line of first-order phase transition between small and large black holes, ending at a second-order phase transition point at $q = q_c$,
- $q = q_c$, this is a second-order critical point at which there exists no distinction between small and large black holes. The critical exponent can be read from $c_v \sim (T - T_c)^{-2/3}$ as $-2/3$,
- $q > q_c$ for each given temperature T there exists a unique global stable black hole with size $r_+ = \rho_B \bar{x}^{1/\tilde{d}}$.

Critical quantities and exponents

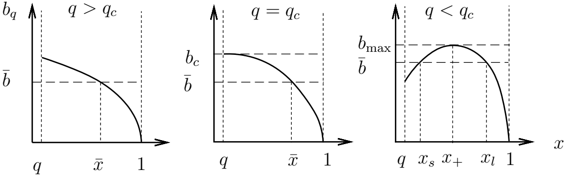
The relevant quantities at the critical point can be calculated explicitly for each allowed value of $\tilde{d}$ as:

$\tilde{d}$	q_c	x_c	b_c
3	0.141626	0.292656	0.199253
4	0.090672	0.238800	0.159921
5	0.064944	0.202012	0.134632
6	0.049599	0.175176	0.116698
7	0.039529	0.154691	0.103210

The critical exponents α of $c_v \sim (T - T_c)^\alpha$ can be calculated straightforward and take a universal value of $-2/3$, independent of $\tilde{d}$.

Phase structure

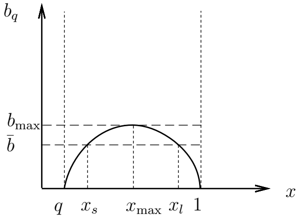
The $\tilde{d} = 2$ case ($q_c = 1/3$):



The typical behaviors of $b_q(x)$ vs x for $\tilde{d} = 2$.

Phase structure

The $\tilde{d} = 1$ case:



The typical behavior of $b_q(x)$ vs x for $\tilde{d} = 1$.

The conclusion

Conclusion

- We have found van der Waals-Maxwell like phase structure and transition for black p-branes in canonical ensemble when $q < q_c$.
- There exists a first order phase transition line when the charge moves from $q < q_c$ towards $q = q_c$, ending up at a second order phase transition point (critical point) when $q = q_c$.
- We calculated explicitly the critical quantities (q_c, x_c, b_c) and found that they all decrease when $\tilde{d}$ increases. The critical exponent is calculated to be $-2/3$, independent of $\tilde{d}$, in this ensemble.
- We also found that branes with the same value of $\tilde{d}$ share the same phase behavior at least to the leading approximation employed in this work.

THANK YOU!