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General Relativity

x^μ, $\mu = 0,1,2,3$. $\eta^{\mu\nu} = \eta_{\mu\nu} = (1,-1,-1,-1)$

$$ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu$$

$x^\mu \to x'^\mu = x'^\mu(x)$: GL(4) group of general coord. transf.

Covariant derivatives :

$$V^\lambda{}_{;\mu} = D_\mu V^\lambda + \Gamma^\lambda{}_{\mu\nu} V^\nu$$
$$V_{\nu;\mu} = D_\mu V_\nu - \Gamma^\lambda{}_{\nu\mu} V_\lambda$$

$$\boxed{g_{\mu\nu}(x)\,, \quad \Gamma^\lambda{}_{\mu\nu}}$$

Riemannian : $\begin{cases} \Gamma^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\nu\mu} \\ g^{\mu\nu}{}_{;\lambda} = g_{\mu\nu;\lambda} = 0 \end{cases}$

$$\Downarrow$$

$\left\{ {\lambda \atop \mu\nu} \right\}$ $\quad \Gamma^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}\left(g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho} \right)$

Christoffel connection

General Relativity as a gauge theory:

$$\begin{cases} \text{Yang-Mills} \quad \vec{A}_\mu(x) \; : \; SU(2) \text{ gauge field} \\ \qquad\qquad\qquad\qquad\quad \text{" connection} \\ \text{GR} \qquad \Gamma^\lambda{}_{\mu\nu} \; : \; GL(4) \text{ connection} \\ \qquad\qquad\qquad\qquad\quad \text{" gauge field} \end{cases}$$

$$\begin{cases} \vec{F}_{\mu\nu} = \vec{A}_{\mu,\nu} - \vec{A}_{\nu,\mu} + \vec{A}_\mu \times \vec{A}_\nu \\ R^{\lambda\rho}{}_{\mu\nu} = g^{\rho\sigma}(\Gamma^\lambda{}_{\sigma\mu,\nu} - \Gamma^\lambda{}_{\sigma\nu,\mu} - \Gamma^\lambda{}_{\alpha\mu}\Gamma^\alpha{}_{\sigma\nu} + \Gamma^\lambda{}_{\alpha\nu}\Gamma^\alpha{}_{\sigma\mu}) \end{cases}$$

Hilbert-Einstein:

$$W = -\frac{c^3}{16\pi G}\int \sqrt{-g}\, R\, d^4x + W_{matter}$$

$$R = R^{\mu\nu}{}_{\mu\nu}, \quad g = \det(g_{\mu\nu})$$

$$\delta W = 0 : \quad R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^3}$$

General Relativity: GL(4) gauge theory

Dirac Equation (1928)

$$(i\hbar \gamma^\mu \partial_\mu - m)\psi = 0. \qquad \hbar = 1.$$

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}, \quad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

Under Lorentz transf. $x^\mu \to x^\mu + \varepsilon^\mu{}_\nu x^\nu$, $\varepsilon^{\mu\nu} = -\varepsilon^{\nu\mu}$.

$$\psi(x) \to \psi'(x) = e^{-i\varepsilon^{\mu\nu}\frac{\sigma_{\mu\nu}}{4}}\psi(x), \left(\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]\right)$$

How to formulate Dirac equation in curved space?

H. Weyl, Z. Phys. 56, 330 (1929)

Proc. Nat. Acad. Sci. 15, 323 (1959)

V. Fock, Z. Phys. 57, 261 (1959).

Dirac spinor $\psi(x)$: 2-valued representation of Lorentz group $SO(1,3)$

$\left\{ \begin{array}{l} \text{In flat space, spinor can be uniformly defined.} \\ \text{In curved space, spinor can only be defined with respect to \underline{local Cartesian frames}} \end{array} \right.$

local Cartesian frame at x:

$e^a{}_\mu(x)$, $a = 0, 1, 2, 3$ Cartesian index

vierbein field

$$g_{\mu\nu} = \eta_{ab}\, e^a{}_\mu\, e^a{}_\nu$$

$\boxed{\begin{array}{l} e_a{}^\mu : \text{inverse of } e^a{}_\mu \\ e_a{}^\mu e^a{}_\nu = \delta^\mu{}_\nu \\ e_a{}^\mu e^b{}_\mu = \delta_a{}^b \end{array}}$

$$\{\gamma_a, \gamma_b\} = 2\eta_{ab} \quad , \quad \eta_{ab} = (1, -1, -1, -1)$$

$$\sigma_{ab} = \frac{i}{2}[\gamma_a, \gamma_b] .$$

The set of $\frac{1}{2}\sigma_{ab}$ satisfy <u>Lorentz algebra</u>:

$$\left[\frac{\sigma_{ab}}{2}, \frac{\sigma_{cd}}{2}\right] = i\left(\eta_{bc}\frac{\sigma_{ad}}{2} - \eta_{ac}\frac{\sigma_{bd}}{2} + \eta_{ad}\frac{\sigma_{bc}}{2} - \eta_{bd}\frac{\sigma_{ac}}{2}\right)$$

At each point x^μ, and with respect to local Cartesian frame $e^a{}_\mu(x)$, there is the local freedom of defining spinor:

$$\psi(x) \longrightarrow \psi'(x) = e^{-\frac{i}{4}\varepsilon^{ab}(x)\sigma_{ab}}\psi(x)$$

local Lorentz gauge freedom

To define a Lorentz covariant derivative, a connection field $\omega^{ab}{}_\mu(x)$ is introduced:

$$\boxed{D_\mu\psi(x) = \left(\partial_\mu - \frac{i}{4}\omega^{ab}{}_\mu(x)\sigma_{ab}\right)\psi(x)}$$

Under a local Lorentz transformation

$$\psi(x) \longrightarrow e^{-\frac{i}{4}\varepsilon^{ab}(x)\sigma_{ab}}\psi(x),$$

the connection field $\omega^{ab}{}_\mu(x)$ is required to transform in such a way that

$$D_\mu\psi(x) \longrightarrow e^{-\frac{i}{4}\varepsilon^{ab}(x)\sigma_{ab}}D_\mu\psi(x),$$

leading to

$$\omega_\mu(x) \longrightarrow \omega'_\mu(x) = e^{-i\varepsilon(x)}\omega_\mu(x)e^{i\varepsilon(x)} - \left[i\partial_\mu e^{-i\varepsilon(x)}\right]e^{i\varepsilon(x)}.$$

where

$$\omega_\mu(x) \equiv \frac{1}{4}\sigma_{ab}\omega^{ab}{}_\mu(x),$$

$$\varepsilon_\mu(x) \equiv \frac{1}{4}\sigma_{ab}\varepsilon^{ab}(x).$$

Dirac Equation in curved space.

$$D_\mu \psi = \left(\partial_\mu - \frac{i}{4}\omega^{ab}{}_\mu \sigma_{ab}\right)\psi \qquad \text{covariant}$$

γ^a : transf. according to "a".

$\gamma^a e_a{}^\mu$: " " to "μ".

$$\bar{\psi}\, i\gamma^a e_a{}^\mu \left(\partial_\mu - \frac{i}{4}\omega^{ab}{}_\mu \sigma_{ab}\right)\psi$$: invariant under $SO(1,3)$ and $GL(4)$.
but <u>not</u> hermitian

It's hermitian adjoint is

$$-\bar{\psi}\left(\overleftarrow{\partial}_\mu + \frac{i}{4}\omega^{ab}{}_\mu \sigma_{ab}\right) i\gamma^a e_a{}^\mu \psi .$$

$$\boxed{\begin{aligned} \therefore \quad W &= \int d^4x\, h\Big[\tfrac{1}{2}\left(\bar{\psi}\, i\gamma^a e_a{}^\mu D_\mu \psi - \bar{\psi}\,\bar{D}_\mu\, i\gamma^a e_a{}^\mu \psi\right) \\ &\qquad - m\bar{\psi}\psi \Big] \\ D_\mu &= \partial_\mu - \frac{i}{4}\omega^{ab}{}_\mu \sigma_{ab} \\ \bar{D}_\mu &= \overleftarrow{\partial}_\mu + \frac{i}{4}\omega^{ab}{}_\mu \sigma_{ab} \end{aligned}}$$

$$h = \det(e^a{}_\mu) = \sqrt{-g}$$

We have in the theory, in addition to $\psi(x)$,

$$\boxed{e^a{}_\mu(x)\,, \quad \omega^{ab}{}_\mu(x)}$$ independent field variables

In terms of these field variables, what are the geometric entities

$$\boxed{g_{\mu\nu}(x)\,,\ \Gamma^\lambda{}_{\mu\nu}(x)} \qquad ?$$

Already, we have the metric tensor

$$\boxed{g_{\mu\nu}(x) = \eta_{ab}\, e^a{}_\mu(x)\, e^b{}_\nu(x)\,.}$$

What is the affine connection

$$\Gamma^\lambda{}_{\mu\nu}(x) \qquad ?$$

Recall :

$$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}\left(g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho}\right)$$

in General Relativity.

Geometry is Riemannian.

To proceed, we need to define covariant derivatives with respect to local Lorentz transf. and general coordinate transf. for a generic entity $\chi^a{}_\mu$:

$$\boxed{D_\nu \chi^a{}_\mu = \chi^a{}_{\mu;\nu} = \chi^a{}_{\mu,\nu} + \omega^a{}_{b\nu}\chi^b{}_\mu - \Gamma^\lambda{}_{\mu\nu}\chi^a{}_\lambda .}$$

Similarly,

$$\boxed{D_\nu \chi_a{}^\mu = \chi_a{}^\mu{}_{;\nu} = \chi_a{}^\mu{}_{,\nu} - \omega^b{}_{a\nu}\chi_b{}^\mu + \Gamma^\mu{}_{\lambda\nu}\chi_a{}^\lambda .}$$

To construct $\Gamma^\lambda{}_{\mu\nu}$ in terms of $e^a{}_\mu(x)$ and $\omega^{ab}{}_\mu(x)$, let's first impose the condition of "metricity":

$$D_\lambda g^{\mu\nu} = D_\lambda g_{\mu\nu} = 0 ,$$

or, equivalently,

$$\boxed{D_\lambda e^a{}_\mu = D_\lambda e_a{}^\mu = 0 .}$$

$$D_\nu e^a{}_\mu = e^a{}_{\mu,\nu} + \omega^a{}_{b\nu} e^b{}_\mu - \Gamma^\lambda{}_{\mu\nu} e^a{}_\lambda$$

$$\overset{\|}{0} \Rightarrow \boxed{\Gamma^\lambda{}_{\mu\nu} = e_a{}^\lambda \left(e^a{}_{\mu,\nu} + \omega^a{}_{b\nu} e^b{}_\mu\right)}$$

$$\boxed{\omega^{ab}{}_\mu = e^a{}_\lambda e^{b\nu}\Gamma^\lambda{}_{\nu\mu} - e^{b\nu} e^a{}_{\nu,\mu}}$$

$$g_{\mu\nu;\lambda} = g_{\mu\nu,\lambda} - \Gamma^\rho{}_{\mu\lambda} g_{\rho\nu} - \Gamma^\rho{}_{\nu\lambda} g_{\mu\rho} = 0 \qquad \times(-1/2)$$

$$g_{\nu\lambda;\mu} = g_{\nu\lambda,\mu} - \Gamma^\rho{}_{\nu\mu} g_{\rho\lambda} - \Gamma^\rho{}_{\lambda\mu} g_{\nu\rho} = 0 \qquad \times(+1/2)$$

$$g_{\lambda\mu;\nu} = g_{\lambda\mu,\nu} - \Gamma^\rho{}_{\lambda\nu} g_{\rho\mu} - \Gamma^\rho{}_{\mu\nu} g_{\lambda\rho} = 0 \qquad \times(+1/2)$$

Sum :

$$\frac{1}{2}(g_{\lambda\mu,\nu} + g_{\lambda\nu,\mu} - g_{\mu\nu,\lambda})$$
$$- \underbrace{\frac{1}{2}(\Gamma^\rho{}_{\mu\nu} + \Gamma^\rho{}_{\nu\mu})}_{\Gamma^\rho{}_{\mu\nu} - \frac{1}{2}(\Gamma^\rho{}_{\mu\nu} - \Gamma^\rho{}_{\nu\mu})} g_{\lambda\rho} - \frac{1}{2}(\Gamma^\rho{}_{\lambda\nu} - \Gamma^\rho{}_{\nu\lambda}) g_{\mu\rho} - \frac{1}{2}(\Gamma^\rho{}_{\lambda\mu} - \Gamma^\rho{}_{\mu\lambda}) g_{\nu\rho}$$
$$= 0$$

$$\boxed{g^{\alpha\lambda} g_{\lambda\rho} = \delta^\alpha_\rho}$$

$$\therefore \quad \Gamma^\alpha{}_{\mu\nu} = \frac{1}{2} g^{\alpha\lambda}(g_{\lambda\mu,\nu} + g_{\lambda\nu,\mu} - g_{\mu\nu,\lambda})$$
$$+ \frac{1}{2} C^\alpha{}_{\mu\nu} + \frac{1}{2} C^\rho{}_{\nu\lambda} g_{\mu\rho} g^{\alpha\lambda} + \frac{1}{2} C^\rho{}_{\mu\lambda} g_{\nu\rho} g^{\alpha\lambda}$$

with $\boxed{\Gamma^\alpha{}_{\mu\nu} - \Gamma^\alpha{}_{\nu\mu} \equiv C^\alpha{}_{\mu\nu}}$ torsion

Note that $\phi_{;\mu\nu} - \phi_{;\nu\mu} = (\Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu})\phi_{;\lambda}$

$$= C^\lambda{}_{\mu\nu}\, \phi_{;\lambda}$$

$\therefore \quad C^\lambda{}_{\mu\nu}$: tensor

$$\boxed{\Gamma^\lambda{}_{\mu\nu} = \underbrace{\left\{ {\lambda \atop \mu\nu} \right\}}_{\text{Christoffel}} + \frac{1}{2}(C^\lambda{}_{\mu\nu} + C_{\mu\nu}{}^\lambda + C_{\nu\mu}{}^\lambda)}$$

$$\boxed{\begin{array}{l} \text{Riemannian}: \quad \Gamma^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\nu\mu} \\ \qquad\qquad \therefore \; C^{\lambda}{}_{\mu\nu} = 0 \\ \qquad\qquad \Gamma^{\lambda}{}_{\mu\nu} = \left\{ {\lambda \atop \mu\nu} \right\} \quad \text{Christoffel} \\ \omega^{ab}{}_{\mu} = e^{a}{}_{\lambda} e^{b\nu} \left\{ {\lambda \atop \nu\mu} \right\} - e^{b\nu} e^{a}{}_{\nu,\mu} \end{array}}$$

$$\chi^{a}{}_{\lambda;\mu} = \chi^{a}{}_{\lambda,\mu} + \omega^{a}{}_{b\mu} \chi^{b}{}_{\lambda} - \Gamma^{\rho}{}_{\lambda\mu} \chi^{a}{}_{\rho}$$

$$\chi^{a}{}_{\lambda;\mu\nu} - \chi^{a}{}_{\lambda;\nu\mu} = \cdots\cdots\cdots$$

$$= \left(\omega^{a}{}_{b\mu,\nu} - \omega^{a}{}_{b\nu,\mu} + \omega^{a}{}_{c\nu} \omega^{c}{}_{b\mu} - \omega^{a}{}_{c\mu} \omega^{c}{}_{b\nu} \right) \chi^{b}{}_{\lambda}$$

$$- \left(\Gamma^{\rho}{}_{\lambda\mu,\nu} - \Gamma^{\rho}{}_{\lambda\nu,\mu} - \Gamma^{\sigma}{}_{\lambda\nu} \Gamma^{\rho}{}_{\sigma\mu} + \Gamma^{\sigma}{}_{\lambda\mu} \Gamma^{\rho}{}_{\sigma\nu} \right) \chi^{a}{}_{\rho}$$

$$- \left(\Gamma^{\rho}{}_{\mu\nu} - \Gamma^{\rho}{}_{\nu\mu} \right) \chi^{a}{}_{\lambda;\rho}$$

$$\equiv R^{a}{}_{b\mu\nu} \chi^{b}{}_{\lambda} - R^{\rho}{}_{\lambda\mu\nu} \chi^{a}{}_{\rho} - C^{\rho}{}_{\mu\nu} \chi^{a}{}_{\lambda;\rho}$$

$$\boxed{\begin{array}{c} \chi^{a}{}_{\lambda;\mu\nu} - \chi^{a}{}_{\lambda;\nu\mu} = R^{a}{}_{b\mu\nu} \chi^{b}{}_{\lambda} - R^{\rho}{}_{\lambda\mu\nu} \chi^{a}{}_{\rho} - C^{\rho}{}_{\mu\nu} \chi^{a}{}_{\lambda;\rho} \\ R^{a}{}_{b\mu\nu} = \omega^{a}{}_{b\mu,\nu} - \omega^{a}{}_{b\nu,\mu} + \omega^{a}{}_{c\nu} \omega^{c}{}_{b\mu} - \omega^{a}{}_{c\mu} \omega^{c}{}_{b\nu} \\ R^{\rho}{}_{\lambda\mu\nu} = \Gamma^{\rho}{}_{\lambda\mu,\nu} - \Gamma^{\rho}{}_{\lambda\nu,\mu} - \Gamma^{\sigma}{}_{\lambda\nu} \Gamma^{\rho}{}_{\sigma\mu} + \Gamma^{\sigma}{}_{\lambda\mu} \Gamma^{\rho}{}_{\sigma\nu} \end{array}}$$

$$\boxed{e_{a}{}^{\lambda} e_{b}{}^{\rho} R^{ab}{}_{\mu\nu} = \cdots = g^{\rho\sigma} R^{\lambda}{}_{\sigma\mu\nu}}$$

T.W. Kibble, JMP $\underline{2}$, 212 (1961)
D.W. Sciama, "Recent Development in G.R., 1962"

$$R = e_a{}^\mu e_b{}^\nu R^{ab}{}_{\mu\nu}$$

$$h = \det(e^a{}_\mu) = \sqrt{-g}$$

$$W = -\frac{1}{16\pi G}\int d^4x\, h\, R + W_{Dirac} \qquad \begin{array}{l} c = 1 \\ \hbar = 1 \end{array}$$

$$W_{Dirac} = \int d^4x\, h\left[\frac{1}{2}\left(\bar\psi i\gamma^a e_a{}^\mu D_\mu\psi - \bar\psi \bar{D}_\mu i\gamma^a e_a{}^\mu\psi\right) - m\bar\psi\psi\right].$$

(i) Invariance under general coord. transformations.

(ii) " " local Lorentz " .

(iii) Treat field variables in the theory

ψ, $\bar\psi$

$e^a{}_\mu$. vierbein

$\omega^{ab}{}_\mu$. Lorentz connection, spin-connection

as <u>independent</u>. They are determined by the resulting field equations.

H.-T. Nieh and M.L. Yan, Ann. Phys. $\underline{138}$, 237 (1982).

Field Equations

(1) $\delta\psi$, $\delta\bar\psi$: Dirac equation

$$\delta\bar\psi : \quad \frac{1}{2} h i \gamma^a e_a{}^\mu D_\mu \psi + \frac{1}{2}\left(\partial_\mu - \frac{i}{4}\sigma_{cd}\omega^{cd}{}_\mu\right)(h i \gamma^a e_a{}^\mu \psi) - h m \psi =$$

$$\begin{aligned}
\partial_\mu (h e_a{}^\mu) &= h_{,\mu} e_a{}^\mu + h e_a{}^\mu{}_{,\mu} \\
&= \left[h\, \partial_\mu \ln(\det e^a{}_\lambda)\right] e_a{}^\mu + h e_a{}^\mu{}_{,\mu} \\
&= (h e_b{}^\lambda e^b{}_{\lambda,\mu}) e_a{}^\mu + h e_a{}^\mu{}_{,\mu} \\
&= h \Gamma^\lambda{}_{\lambda\mu} e_a{}^\mu + h e_a{}^\mu{}_{,\mu} \\
&= h (\Gamma^\lambda{}_{\mu\lambda} + C^\lambda{}_{\lambda\mu}) e_a{}^\mu + h e_a{}^\mu{}_{,\mu} \\
&= h \left(C^\lambda{}_{\lambda\mu} e_a{}^\mu + e_a{}^\mu{}_{;\mu} + \omega^b{}_{a\mu} e_b{}^\mu\right) \\
&= h \left(C^\lambda{}_{\lambda\mu} e_a{}^\mu + \omega^b{}_{a\mu} e_b{}^\mu\right)
\end{aligned}$$

$$[\sigma_{cd}\omega^{cd}{}_\mu, \gamma_a e^{a\mu}] = 4 i \omega^{ab}{}_\mu \gamma_a e_b{}^\mu$$

$$\boxed{\begin{gathered} i\gamma^a e_a{}^\mu \left(D_\mu + \frac{1}{2} C^\lambda{}_{\lambda\mu}\right)\psi - m\psi = 0 \\ D_\mu = \partial_\mu - \frac{i}{4}\sigma_{bc}\omega^{bc}{}_\mu \end{gathered}}$$

Dirac equation

(2) $\delta e_a^{\ \mu}$: Einstein equation

$$-\frac{1}{16\pi G}\left[-h\, e^a_{\ \mu} R + 2h\, e_b^{\ \nu} R^{ab}_{\ \ \mu\nu}\right]$$

$$+ \cdots\cdots \qquad = 0$$

$$e_{a\nu}\cdot \ \Big\Downarrow \ \text{Dirac eq.}$$

$$\boxed{\begin{gathered} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G\, T_{\nu\mu} \\ T_{\nu\mu} = \frac{1}{2}\left(\bar{\psi} i\gamma_\nu D_\mu \psi - \bar{\psi} \overleftarrow{D}_\mu i\gamma_\nu \psi\right) \end{gathered}} \qquad \gamma_\nu = e_{a\nu}\gamma^a$$

Note : $\left\{ \begin{array}{l} R_{\mu\nu} \neq R_{\nu\mu} \\ T_{\nu\mu} \neq T_{\mu\nu} \end{array} \right.$ when torsion $\neq 0$.

(3) $\delta\omega^{ab}{}_\mu$: equation for Lorentz connection $\omega^{ab}{}_\mu$.

$$\frac{1}{8\pi G}\Big\{\left[h(e_a{}^\mu e_b{}^\nu - e_a{}^\nu e_b{}^\mu)\right]_{,\nu} - \omega^d{}_{a\nu}\, h\,(e_d{}^\mu e_b{}^\nu - e_d{}^\nu e_b{}^\mu)$$
$$- \omega^d{}_{b\nu}\, h\,(e_a{}^\mu e_d{}^\nu - e_a{}^\nu e_d{}^\mu)\Big\}$$
$$+ h\frac{1}{4c}\bar\psi(\gamma^\mu \sigma_{ab} + \sigma_{ab}\gamma^\mu)\psi = 0.$$

Let $\left\{\left[h(e_a{}^\mu e_b{}^\nu - e_a{}^\nu e_b{}^\mu)\right]_{,\nu} - \cdots\cdots - \cdots\cdots\right\} \equiv h\, S_{ab}{}^\mu$.

$$\gamma^\mu \sigma_{ab} + \sigma_{ab}\gamma^\mu = 2e^{c\mu}\eta_{abcd}\gamma_5\gamma^d. \quad (\eta_{0123} = -1)$$

$$e_{c\mu} S_{ab}{}^\mu \equiv S_{abc}$$

$$\boxed{\therefore\ S_{abc} = -4\pi G\, \eta_{abcd}\, \bar\psi\gamma_5\gamma^d\psi}$$

On the other hand,

$$h\,(S_{abc} - S_{bca} - S_{cab} - \eta_{ac}S^d{}_{bd} - \eta_{bc}S_a{}^d{}_d)$$

$$= \cdots\cdots\cdots \qquad \boxed{e_a{}^\lambda e^a{}_{\lambda,\mu} = \frac{\partial}{\partial x^\mu}\ln\det e_a{}^\lambda = h^{-1}h_{,\mu}}$$

$$= 2h\,\omega_{abc}$$
$$+ h\Big[-(e_a{}^\mu e_b{}^\nu - e_a{}^\nu e_b{}^\mu)\, e_{c\mu,\nu}$$
$$+ (e_b{}^\mu e_c{}^\nu - e_b{}^\nu e_c{}^\mu)\, e_{a\mu,\nu}$$
$$+ (e_c{}^\mu e_a{}^\nu - e_c{}^\nu e_a{}^\mu)\, e_{b\mu,\nu}\Big]$$

$$\boxed{\begin{aligned} \omega_{abc} &= \tfrac{1}{2}\left(c_{abc} - c_{bca} - c_{cab}\right) \\ &\quad + \tfrac{1}{2}\left(S_{abc} - S_{bca} - S_{cab} - \eta_{ac} S^{d}{}_{bd} - \eta_{bc} S_{a}{}^{d}{}_{d}\right) \\ c_{abc} &\equiv \left(e_a{}^{\mu} e_b{}^{\nu} - e_a{}^{\nu} e_b{}^{\mu}\right) e_{c\mu,\nu} \\ S_{abc} &= - \quad \eta_{abcd} \bar{\psi} \gamma_5 \gamma^d \psi \end{aligned}}$$

$$\boxed{\begin{aligned} \omega_{abc} &= \overset{\circ}{\omega}_{abc} + \frac{2\pi G}{c^3} \eta_{abcd} \bar{\psi} \gamma_5 \gamma^d \psi \\ \overset{\circ}{\omega}_{abc} &= \tfrac{1}{2}\left(c_{abc} - c_{bca} - c_{cab}\right) \end{aligned}}$$

Recall : $\Gamma^{\lambda}{}_{\mu\nu} = e_a{}^{\lambda}\left(e^a{}_{\mu,\nu} + \omega^a{}_{b\nu} e^b{}_{\mu}\right)$

$$\begin{aligned} \overset{\circ}{\Gamma}{}^{\lambda}{}_{\mu\nu} &= e_a{}^{\lambda}\left(e^a{}_{\mu,\nu} + \overset{\circ}{\omega}{}^a{}_{b\nu} e^b{}_{\mu}\right) \\ &= e_a{}^{\lambda}\left(e^a{}_{\mu,\nu} + \overset{\circ}{\omega}{}^a{}_{bc} e^c{}_{\nu} e^b{}_{\mu}\right) \\ &= \cdots\cdots \\ &= \tfrac{1}{2} g^{\lambda\rho}\left(g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho}\right) = \left\{ {\lambda \atop \mu\nu} \right\} \end{aligned}$$

Christoffel

$$\boxed{\Gamma^{\lambda}{}_{\mu\nu} = \overset{\circ}{\Gamma}{}^{\lambda}{}_{\mu\nu} + \frac{2\pi G}{c^4} \varepsilon^{\lambda}{}_{\mu\nu\rho} \bar{\psi} \gamma_5 \gamma^{\rho} \psi}$$

$$C^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\nu\mu} = 4\pi G \; \varepsilon^{\lambda}{}_{\mu\nu\rho} \bar{\psi} \gamma_5 \gamma^{\rho} \psi$$

$$R = g_\lambda{}^\mu g^{\rho\nu} R^\lambda{}_{\rho\mu\nu}$$

$$= g_\lambda{}^\mu g^{\rho\nu}\left[\Gamma^\lambda{}_{\rho\mu,\nu} - \Gamma^\lambda{}_{\rho\nu,\mu} - \Gamma^\lambda{}_{\sigma\mu}\Gamma^\sigma{}_{\rho\nu} + \Gamma^\lambda{}_{\sigma\nu}\Gamma^\sigma{}_{\rho\mu}\right]$$

$$\Gamma^\lambda{}_{\mu\nu} = \mathring{\Gamma}^\lambda{}_{\mu\nu} - \frac{1}{2}S^\lambda{}_{\mu\nu}, \qquad \mathring{\Gamma}^\lambda{}_{\mu\nu} = \left\{ {\lambda \atop \mu\nu} \right\}$$

$$R = g_\lambda{}^\mu g^{\rho\nu}\left[(\mathring{\Gamma}^\lambda{}_{\rho\mu} - \frac{1}{2}S^\lambda{}_{\rho\mu})_{,\nu} - (\mathring{\Gamma}^\lambda{}_{\rho\nu} - \frac{1}{2}S^\lambda{}_{\rho\nu})_{,\mu} - \cdots\cdots - \cdots\right]$$

$$= \cdots\cdots$$

$$= \mathring{R} + \frac{1}{4}S^{\lambda\mu\nu}S_{\lambda\mu\nu}, \qquad S_{\lambda\mu\nu} = -4\pi G\,\varepsilon_{\lambda\mu\nu\rho}\bar{\Psi}\gamma_5\gamma^\rho\psi$$

$$= \mathring{R} + \frac{1}{4}(-4\pi G)^2(-6)(\bar{\Psi}\gamma_5\gamma^a\psi)(\bar{\Psi}\gamma_5\gamma_a\psi)$$

$$\frac{1}{2}\left(\bar{\Psi}i\gamma^a e_a{}^\mu D_\mu\psi - \bar{\Psi}\bar{D}_\mu i\gamma^a e_a{}^\mu\psi\right)$$

$$D_\mu = \partial_\mu - \frac{i}{4}\omega^{ab}{}_\mu\sigma_{ab}$$

$$= \partial_\mu - \frac{i}{4}(\mathring{\omega}^{ab}{}_\mu - \frac{1}{2}S^{ab}{}_\mu)\sigma_{ab} = \mathring{D}_\mu + \frac{i}{8}S^{ab}{}_\mu\sigma_{ab}$$

$$\bar{D}_\mu = \cdots\cdots = \bar{\mathring{D}}_\mu - \frac{i}{8}S^{ab}{}_\mu\sigma_{ab}$$

$$\therefore \quad \frac{1}{2}\left(-\frac{1}{8}\right)e_a{}^\mu S_{bc\mu}(\gamma^a\sigma^{bc} + \sigma^{bc}\gamma^a)$$

$$= \left(-\frac{1}{8}\right)(-4\pi G)(-6)(\bar{\Psi}\gamma_5\gamma^a\psi)(\bar{\Psi}\gamma_5\gamma_a\psi)$$

$$W = -\frac{1}{16\pi G}\int d^4x \, h \; R$$

$$+ \int d^4x \, h \left[\frac{1}{2}\left(\bar{\psi} i\gamma^a e_a{}^\mu D_\mu \psi - \bar{\psi} \bar{D}_\mu i\gamma^a e_a{}^\mu \psi \right) - m\bar{\psi}\psi \right]$$

$$= -\frac{1}{16\pi G}\int d^4x \sqrt{-g} \left[\mathring{R} - \frac{3}{2}(-4\pi G)^2 (\bar{\psi}\gamma_5\gamma^a\psi)(\bar{\psi}\gamma_5\gamma_a\psi) \right]$$

$$+ \int d^4x \sqrt{-g} \left[\frac{1}{2}\left(\bar{\psi} i\gamma^a e_a{}^\mu \mathring{D}_\mu \psi - \bar{\psi} \mathring{\bar{D}}_\mu i\gamma^\mu e_a{}^\mu \psi \right) - m\bar{\psi}\psi \right.$$

$$\left. - 3\pi G (\bar{\psi}\gamma_5\gamma^a\psi)(\bar{\psi}\gamma_5\gamma_a\psi) \right]$$

$$\boxed{\begin{aligned} W = &-\frac{1}{16\pi G}\int d^4x \sqrt{-g}\, \mathring{R} \\ &+ \int d^4x \sqrt{-g} \left[\frac{1}{2}\left(\bar{\psi} i\gamma^a e_a{}^\mu \mathring{D}_\mu \psi - \bar{\psi} \mathring{\bar{D}}_\mu i\gamma^a e_a{}^\mu \psi \right) - m\bar{\psi}\psi \right] \\ &- \frac{3}{2}\pi G \int d^4x \sqrt{-g}\, (\bar{\psi}\gamma_5\gamma^a\psi)(\bar{\psi}\gamma_5\gamma_a\psi) \end{aligned}}$$