Non-Uniform Black ring/branes with Schwarzchild-(A)dS Foliation

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1. Introduction

- Subject: higher dimensional spacetime solutions with event horizon(s),e.g. black hole and black p-brane (5D cases: black hole, black ring, black string)
- Why higher dimensions: There are several reasons.
  - String theories require many extra dimensions (total of 26 for bosonic string and 10 for supersymmetric, and 11 for the non-perturbative unification of all super strings, i.e. M-theory);
  - Holography principle: inspired by the founding of black hole entropy (proportional to the *area*, rather than *volumn* of the horizon). There might be a codimension 1 system bearing exactly the same information. Concrete example: AdS/CFT in 5D.
  - Braneworld scenario and Kaluza-Klein theory: the world might indeed be higher dimensional, with invisible dimensions being warpped or compactified. Endeavors in searching TeV quantum gravity effects.
  - Uniqueness and stability issues: 4D black hole solutions are unique under proper energy condition if the charges are fixed. In particular, all horizons have spherical topology. In higher dimensions, uniqueness does not hold, and the physics behind is very rich.

2. What are black rings and black strings?

2.1. Naïve view: Hole – String – Ring

- Take a 4-dim black hole spacetime (Horizon topology: S^2);
- Make a 1-dim distribution of 4-dim black holes (along a fifth dimension) $\Rightarrow$ "black string" (Horizon topology: $S^2 \times U, U \subset \mathbb{R}$);
- Bend the string to form a closed circle $\Rightarrow$ "black ring" (Horizon topology: $S^2 \times S^1$)!

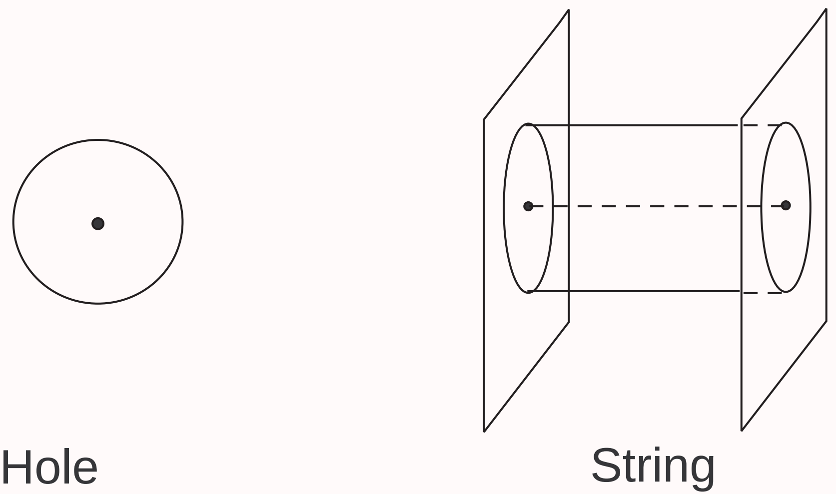

Figure: 4d black hole and 5d black string

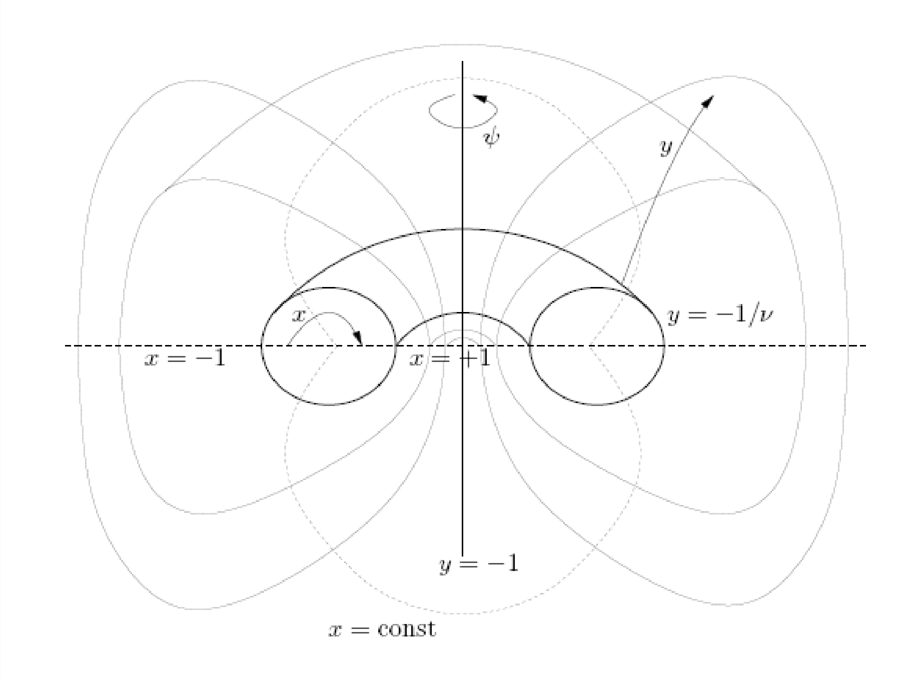

Figure: A slice of a 5d black ring

Black rings and strings are exceptional black objects in that

- they are purely higher dimensional objects and have no 4-dimensional analogues;
- they have horizons of non-spherical topology which can be regarded as sort of new "hair" and thus provide the first explicit examples for the violation of black hole uniqueness;
- can be identified as intersecting brane configurations in string theoretic settings, etc.

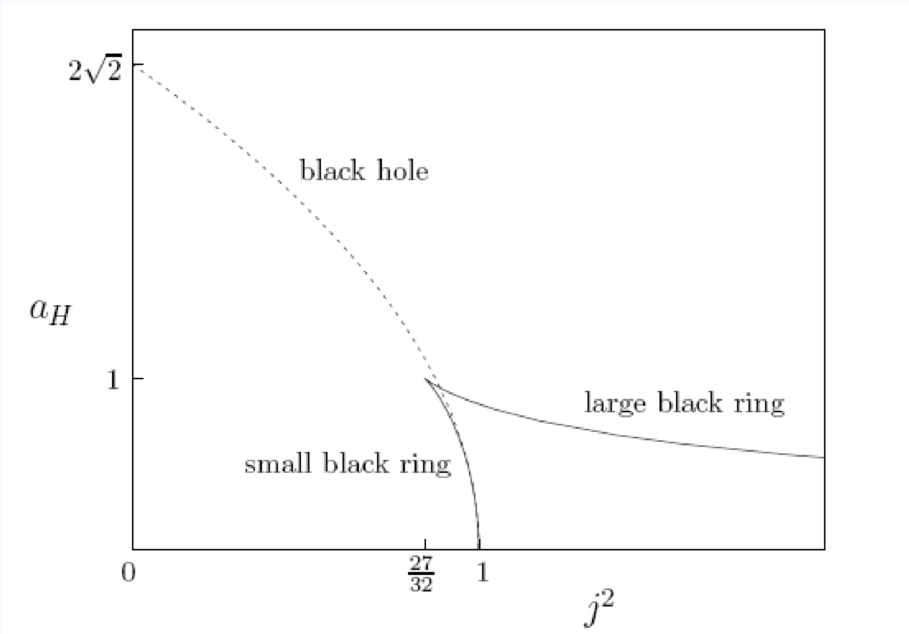

Figure: Non-uniqueness of 5d neutral black hole/rings

$$a_H = \frac{3}{16}\sqrt{\frac{3}{\pi}}\frac{\mathcal{A}_{\mathcal{H}}}{(GM)^{3/2}},\ j^2 = \frac{27\pi}{32G}\frac{J^2}{M^3}$$

(from Emparan & Reall 2001)

2.2. A closer look at black string: Are they stable?

The simplest black string is the Schwarzchild black string with metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2 + dz^2.$$

It is uniform along the z-direction. It is known that this string is classically unstable. The simplest way to understand this instability is to compare the entropy thereof with that of a black hole of the same mass. It turns out that

$$S_{B.S} \propto M^2/L, \qquad S_{B.H.} \propto M^{3/2}.$$

Thus for long enough black strings the entropy contained by its event horizon is much less than that contained in the event horizon of a black hole.

Warning: This argument is too naive and is replaced by the Gusber-Mitra conjecture: a black brane with a non-compact translational symmetry is classically stable if and only if it is locally thermodynamically stable.

2.3. Uniform Black rings

- Known solutions all written in terms of C-metric like coordinates, since the first known black ring was in fact obtained from a double Wick rotation of the C-metric;
- What hold them from collapse? In order for uniform black rings to exist and classically stable, the metric must be either
  - stationary but non-static, i.e. having rotational symmetry and carrying angular momentum;
  - or, carrying electronic charge;
  - or, carrying other kinds of charges (e.g. local magnetic charge or "dipole charge").

Note: In 5 or higher dimensions, charges can be electromagnetic or brane charges, but I shall be talking only about electric charges. (Consider only black ring solutions to Einstein-Maxwell-dilaton (EMD) theory.)

$$S = \frac{1}{16\pi G}\int d^5x\sqrt{g}\left(R - \frac{1}{2}(\partial\Phi)^2 - \frac{1}{4}e^{-\alpha\Phi}F^2\right).$$

- Problems with black string/branes:
  - Dynamical (Gregory-Laflamme) instability (1993): Uniform black string/branes are classically dynamically unstable against small massive perturbations – massive s-wave instability $\Rightarrow$ black string/branes might segregate into an array of black holes when final stable state is achieved;
  - Horowitz-Maeda (2001): Black string horizon cannot not pinch off $\Rightarrow$ final fate of black string instability cannot be an array of black holes and remains to be found;
  - Hirayama-Kang (2001): Black string in 5D AdS bulk can be stable if the length of the string exceeds the AdS radius.
- Question: What about black strings in flat background?
  - more interested in non-uniform solutions;
  - numerical studies persued in many groups;
  - better to find explicit analytic solutions.

3. A black string in 5D pure gravity: exact, non-uniform with (A)dS foliation

3.1. The solution

We take the metric ansatz

$$ds^2 = e^{B(z)}\left(-f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right) + e^{-C(z)}dz^2.$$

Direct calculations show that the only non-vanishing components of the Ricci tensor are

$$\begin{aligned}
\hat{R}_{00} &= \frac{f(r)}{4r}\left[2rf''(r) + 4f'(r) + r\left(2B''(z) + 4B'(z)^2 + B'(z)C'(z)\right)e^{B(z)}e^{C(z)}\right],\\
\hat{R}_{11} &= -\frac{1}{f(r)^2}\hat{R}_{00},\\
\hat{R}_{22} &= -rf'(r) + 1 - f(r) - \frac{1}{4}r^2\left(2B''(z) + 4B'(z)^2 + B'(z)C'(z)\right)e^{B(z)}e^{C(z)},\\
\hat{R}_{33} &= \left(\sin^2\theta\right)\hat{R}_{22},\\
\hat{R}_{44} &= -\left(2B''(z) + B'(z)^2 + B'(z)C'(z)\right).
\end{aligned}$$

The solution is

$$
\begin{aligned}
f(r) &= 1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}, \\
C(z) &= -B(z) - 2\log B'(z) + \log\left(\frac{4\Lambda}{3}\right),
\end{aligned}
$$

so the 4D slices of the black string are Schwarzchild-(Anti) de Sitter black holes. One of the functions $B(z)$ and $C(z)$ can be solved for from the last equation by manually specifying the other. The freedom in choice of one of the two metric functions is the consequence of general coordinate invariance for the variable z.

- Choosing $C(z) = B(z) + iK\pi$, we have

$$
\begin{aligned}
ds^2 &= \left(c + e^{iK\pi/2}\sqrt{\frac{4\Lambda}{3}}z\right)(\gamma_{\mu\nu}dx^\mu dx^\nu) + \left(c + e^{iK\pi/2}\sqrt{\frac{4\Lambda}{3}}z\right)^{-1} dz^2, \\
\gamma_{\mu\nu}dx^\mu dx^\nu &= -f(r)dt^2 + f(r)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right).
\end{aligned}
$$

  Here c is an integration constant, $z = -\frac{c}{e^{iK\pi/2}\sqrt{\frac{4\Lambda}{3}}}$ is a coordinate singularity.

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- Choosing $C(z) = 0 + iK\pi$, we get

$$ds^2 = e^{-iK\pi} \left(\sqrt{\frac{\Lambda}{3}} z \right)^2 \gamma_{\mu\nu} dx^\mu dx^\nu + dz^2.$$

  Here z needs not to be constrained to be positive but $z = 0$ is still somewhat singular.

- Choosing $C(z) = -B(z) - iK\pi$, we get

$$B(z) = e^{iK\pi/2} \sqrt{\frac{4\Lambda}{3}} z,$$

$$ds^2 = e^{B(z)} \left(\gamma_{\mu\nu} dx^\mu dx^\nu + dz^2 \right).$$

  Here z can take any real value and the metric is non-singular for any finite z.

3.2. Analysis of the solution

- The $C(z) = B(z) + iK\pi$ case contains coordinate singularity and thus discarded;
- The $C(z) = 0 + iK\pi$ case has some interesting features. For any non-zero constant value of z, the metric is that of a Schwarzchild-de Sitter black hole, and the radius of the event horizon depends on the value of z. So this is a non-uniform black string solution. Moreover, we set $M = 0, \Lambda = 3k$ and make the double Wick rotation

$$z \to i\tau, r \to ir,$$

we get (by renaming t as y)

$$d\tilde{s}^2 = -d\tau^2 + \tau^2 \left[\left(1 - kr^2\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right) + \left(1 - kr^2\right) dy^2 \right].$$

The constant y slice is a time-dependent FRW-type solution with spacial curvature k.

- The case $C(z) = -B(z) + iK\pi$ $(K = 0, 1)$ looks more interesting, because there is no coordinate singularity along the z-axis. So we would like to study its stability against a small perturbation.

  Let the metric be perturbed in such a way that

$$\begin{aligned} g_{MN} &\rightarrow g_{MN} + \delta g_{MN}, \\ ds^2 &\rightarrow ds'^2 = e^{a(z)}\left(\left(\gamma_{\mu\nu} + e^{-a(z)} h_{\mu\nu}(x,z)\right) dx^\mu dx^\nu + dz^2\right), \end{aligned}$$

  where $a(z) = 2z/\ell$, $\ell^2 \equiv 3/\Lambda$. To linear order in $h_{\mu\nu}(x,z)$, the perturbation equation reads

$$\delta R_{MN}(g) = \frac{1}{2} g^{KL} \left(h_{KL;M;N} - h_{KM;N;L} - h_{KN;M;L} + h_{MN;K;L}\right) = 0. \tag{1}$$

  Under the gauge conditions

$$\begin{aligned} h &= h^\mu{}_\mu = \gamma^{\mu\nu} h_{\mu\nu} = 0, \\ h^{\mu\nu}{}_{;\mu} &= 0, \end{aligned}$$

  the perturbation equation becomes

$$\Box^{(\gamma)} h_{\mu\nu} + 2R_{\mu\rho\nu\lambda}(\gamma) h^{\rho\lambda} + \left[\partial_z^2 - \frac{1}{2} a'(z)\partial_z - a''(z) - \frac{1}{2} a'(z)^2\right] h_{\mu\nu} = 0. \tag{2}$$

Separation of variables:

$$h_{\mu\nu}(x,z) = h_{\mu\nu}(x)e^{a(z)/4}\xi(z)$$

yields

$$\left[\Box^{(\gamma)}h_{\mu\nu}(x) + 2R_{\mu\rho\nu\lambda}(\gamma)h^{\rho\lambda}(x)\right] = m^2 h_{\mu\nu}(x), \tag{3}$$

$$\left[-\partial_z^2 + \frac{3}{4}a''(z) + \frac{9}{16}a'(z)^2\right]\xi(z) = m^2\xi(z). \tag{4}$$

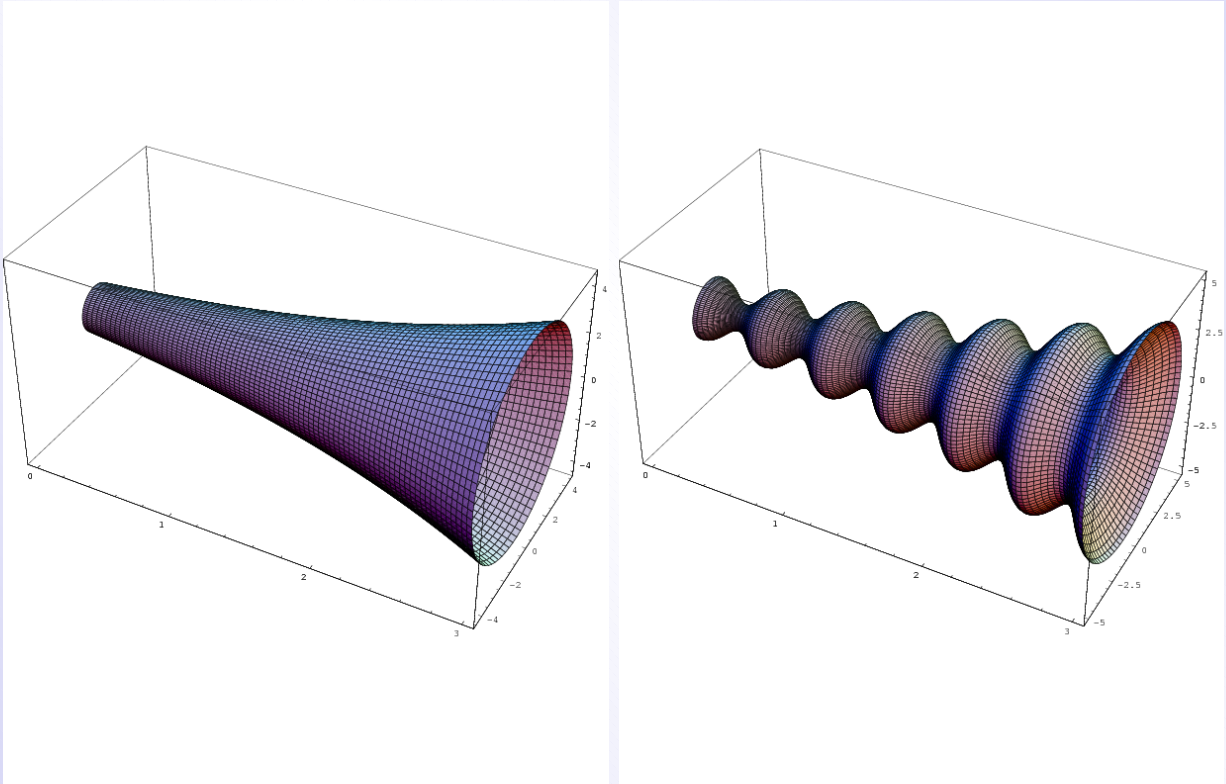
Inserting $a(z) = 2z/\ell$, the equation (4) turns out to be a Schrodinger equation with constant potential $V(z) = \frac{9}{4\ell^2}$. So the solutions are plane waves with continuous eigenvalues

$$m > \frac{3}{2\ell}.$$

We stress that the minimum of the eigenvalues $m_{min} = \frac{3}{2\ell}$ is non-zero.
The equation (3) is a 4D Lichnerowicz equation with 4d Schwarzchild-(AdS) background. This is exactly the equationa arised in the study of Hirayama-Kang for classical stability of Black string in 5D AdS bulk. So without repeating the procedure for solving the equation, we can just quote the conclusion: our non-uniform black string solution is stable if $\Lambda < 0$ and the length of the string exceeds the AdS redius, but otherwise is unstable if $\Lambda > 0$.

3.3. Horizon geometry

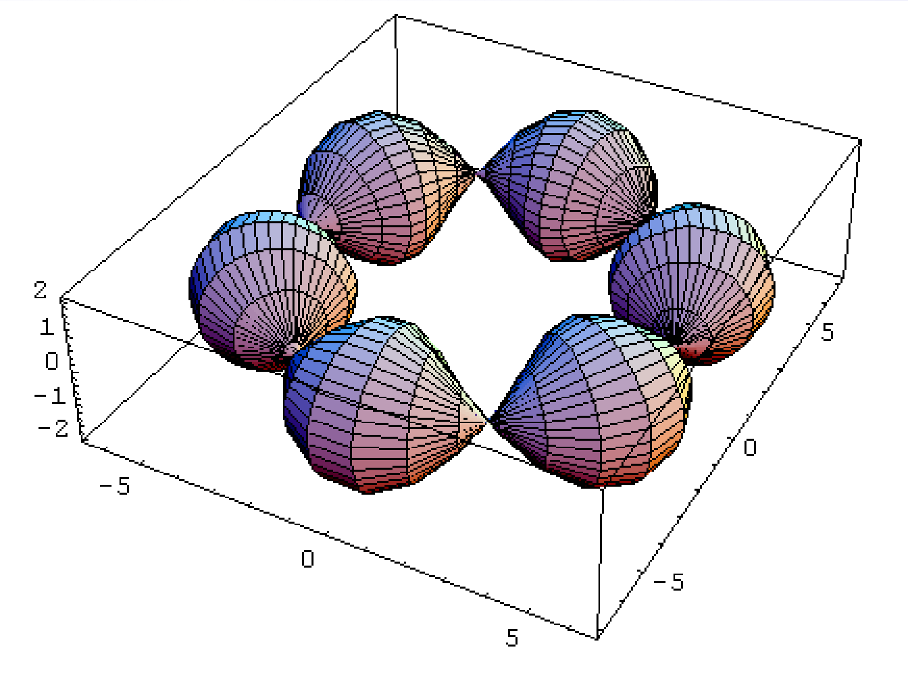
- $\Lambda > 0, K = 0$:



(a) without perturbation, (b) with a single s-wave mode perturbation

- $\Lambda < 0, K = 1$:

  In this case the metric function $e^{B(z)} = \exp\left(i\sqrt{\frac{4|\Lambda|}{3}}z\right)$, which is periodic in z. So, effectively our solution becomes a non-uniform black ring. Schematically, the horizon may look like this (looking only at the real part)



So the stability in our case agrees with the general argument for the classical stability of black rings.

4. Black branes

The solution described above can easily be generalized to the case with more than one longitudinal dimensions.

- Example: 6D vacuum black brane solution: The metric ansatz reads

$$ds^2 = B(y,z)\left(-f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2\right) + C_1(y)dy^2 + C_2(z)dz^2.$$

Then Ricci flat condition has the following solution for the metric functions:

$$\begin{aligned} (B_{,y})^2 &= \Lambda_1 B C_1, \\ (B_{,z})^2 &= \Lambda_2 B C_2, \\ f(r) &= 1 - \frac{2M}{r} - \frac{1}{4}\left(\Lambda_1 + \Lambda_2\right) r^2. \end{aligned}$$

More general cases can be easily treated following the same spirit: assume that the bulk is $(d+n)$-dimensional, where d is the dimension of the black hole slices and n is the number of noncompact dimensions. Then the metric should take the form

$$ds^2 = B(\{z\}) \left(-f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2_{d-2} \right) + \sum_{i=1}^{n} C_i(z_i) dz_i^2.$$

Ricci flatness then imply the following relations:

$$(B_{,z_i})^2 = \Lambda_i B C_i,$$
$$f(r) = 1 - \frac{2M}{r^{d-3}} - \frac{1}{4} \sum_{i=1}^{n} \Lambda_i r^2.$$

The black hole slices are Tangerlini-Schwarzchild-(A)dS holes in d-spacetime dimensions.
For $C_i(z_i) = 1$ for all i,

$$B(\{z\}) \propto \left(\sum_i z_i \right)^2.$$

5. Charged dilitonic black strings

Back to 5-dimensions. Our black string solution can be used to generate solutions of Einstein-Maxwell-dilaton theory. This is accomplished by the following three-step process:

- add one extra flat direction y to our original 5D vacuum solution to form a 6D vacuum solution:

$$ds^2 = e^{B(z)}\left[-f(r)dt^2 + f(r)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2(\theta)d\varphi^2\right) + dz^2\right] + dy^2.$$

Then perform a boost in the (t, y) plane:

$$\begin{aligned} t \to T &= t\cosh\psi - y\sinh\psi, \\ y \to Y &= -t\sinh\psi + y\cosh\psi. \end{aligned}$$

The 6D metric becomes

$$\begin{aligned} \hat{g}_{00} &= \cosh^2\psi\left[1 - e^{B(z)}f(r)\right] - 1, \\ \hat{g}_{05} &= \sinh\psi\cosh\psi\left[1 - e^{B(z)}f(r)\right], \\ \hat{g}_{55} &= \cosh^2\psi\left[1 - e^{B(z)}f(r)\right] + e^{B(z)}f(r), \end{aligned}$$

with the rest components unchanged.

- Next, make a Kaluza-Klein reduction along the 6-th coordinate dimsneion, yielding

$$
\begin{aligned}
d\tilde{s}^2 &= V(r,z)^{1/3} e^{B(z)} \\
&\quad \times \left(-V(r,z)^{-1} f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\theta^2 + r^2 \sin\theta d\varphi^2 + dz^2 \right), \\
A &= V(r,z)^{-1} \frac{k\left[1 - e^{B(z)} f(r)\right]}{1-k^2} dt, \\
e^{-\alpha\phi} &= V(r,z)^{4/3},
\end{aligned}
$$

where

$$
V(r,z) = \frac{1 - e^{B(z)} f(r) k^2}{1-k^2}, \qquad \alpha = \sqrt{8/3},
$$

and

$$
k = \tanh\psi.
$$

This is a solution of EMD theory with dilaton coupling taking the Kaluza-Klein value.

- Introducing

$$N = 4/(\alpha^2 + 4/3),$$

we can extend the last solution to the case of arbitrary dilaton coupling α:

$$\begin{aligned} d\tilde{s}^2 &= V(r,z)^{N/3} e^{B(z)} \\ &\quad \times \left(-V(r,z)^{-N} f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\theta^2 + r^2 \sin\theta d\varphi^2 + dz^2 \right), \\ A &= \sqrt{N} V(r,z)^{-1} \frac{k\left[1 - e^{B(z)} f(r)\right]}{1-k^2} dt, \\ e^{-\alpha\phi} &= V(r,z)^{4N/3}. \end{aligned}$$

Future works:

- Adding rotation parameter to the non-uniform black string/ring;
- Detailed analysis for the stability of each case;
- Non-uniform black string/ring/branes in supersymmetric models?

Thank you!

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