西北大学

现代物理研究所

$ADS_2 \times S^2$ string and integrability

Ruihong Yue

Institute of Modern Physics, Northwest University

rhyue@nwu.edu.cn

$ADS_2 \times S^2$ string and integrability

- The hybrid superstring in an $AdS_2 \times S^2$ background
- The twisted duality and Lax connection
- TsT transformation on $AdS_2 \times S^2$
- Integrability in γ-deformed backgrounds

1 The hybrid superstring in an $AdS_2 \times S^2$ background

Superstring $AdS_2 \times S^2$ can be considered as the non-linear sigma model with a Wess-Zumino-Witten (WZW) term :

$$\frac{PSU(1,1|2)}{U(1) \times U(1)}.$$

The element of $psu(1,1|2)$

$$G = \begin{pmatrix} A & X \\ Y & B \end{pmatrix}, \qquad \begin{matrix} strM = trA - trB = 0 \\ HG + G^\dagger H = 0 \end{matrix}, \qquad H = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\mathbb{I} \end{pmatrix}. \tag{1}$$

The $\mathbb{Z}_4$ automorphism Ω:

$$\Omega(G) = \begin{pmatrix} JA^TJ & -JY^TJ \\ JX^TJ & JB^TJ \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2}$$

$$\mathcal{G} = \mathcal{H}^0 \oplus \mathcal{H}^1 \oplus \mathcal{H}^2 \oplus \mathcal{H}^3, \qquad \Omega(H^p) = i^p H^p. \tag{3}$$

Here $\mathcal{H}^0 = u(1) \times u(1)$, $\mathcal{H}^2$ even, $\mathcal{H}^1, \mathcal{H}^3$ odd.

$$[H^p, H^q] \in \mathcal{H}^{p+q} \pmod 4\,. \tag{4}$$

The action

$$S_{AdS} = S_{GS} + \int d^2z\, (d_\alpha J_-^\alpha + \hat{d}_{\hat{\alpha}} J_+^{\hat{\alpha}} - \frac{1}{2} d_\alpha \hat{d}_{\hat{\beta}} F^{\alpha\hat{\beta}}) + S_{com} + S_{ghost} \quad (5)$$

$$= \int d^2z \left[\frac{1}{2}\eta_{cd} J_+^c J_-^d - \frac{1}{4}\delta_{\alpha\hat{\beta}}(J_+^\alpha J_-^{\hat{\beta}} - J_-^\alpha J_+^{\hat{\beta}})\right]$$
$$+ \int d^2z (d_\alpha J_-^\alpha + \hat{d}_{\hat{\alpha}} J_+^{\hat{\alpha}} - \frac{1}{2} d_\alpha \hat{d}_{\hat{\beta}} F^{\alpha\hat{\beta}}) +$$
$$+ S_{com} + S_{ghost}. \quad (6)$$

where $J_\pm^A = (g^{-1}\partial_\pm g)^A$, $g \in \frac{PSU(1,1|2)}{U(1)\times U(1)}$, $A = (cd, c, \alpha, \hat{\alpha})$
$F^{\alpha\hat{\beta}}$ is constant RR flux $F^{\alpha\hat{\beta}} = -2\delta^{\alpha\hat{\beta}}$. d_α and $d_{\hat{\alpha}}$ are auxiliary fields.

Integrated out the auxiliary fields

$$d_\alpha = \delta_{\alpha\hat{\beta}} J_+^{\hat{\beta}}, \quad \hat{d}_{\hat{\beta}} = -\delta_{\alpha\hat{\beta}} J_-^{\alpha}.$$

$$\begin{aligned} S_{AdS} &= \int d^2z\, [\frac{1}{2}\eta_{cd} J_+^c J_-^d - \frac{1}{4}\delta_{\alpha\hat{\beta}}(J_+^\alpha J_-^{\hat{\beta}} + 3J_-^\alpha J_+^{\hat{\beta}})] + S_{com} + S_{ghost}, \\ &= \int d^2z\, [\frac{1}{2} str(J_+^2 J_-^2) + \frac{1}{4} str(J_-^3 J_+^1) + \frac{3}{4} str(J_+^3, J_-^1) + S_{com} + S_{ghost}. \end{aligned} \tag{7}$$

Where the currents $J_\pm^p, (p = 0, 1, 2, 3)$ is given by

$$\begin{aligned} J_\pm^0 &= J_\pm^{cd} T_{cd}, \quad J_\pm^1 = J_\pm^\alpha T_\alpha, \\ J_\pm^2 &= J_\pm^c T_c, \quad\ \ J_\pm^3 = J_\pm^{\hat{\alpha}} T_{\hat{\alpha}}, \end{aligned} \tag{8}$$

Considering the variation of the group element g as $\delta g^{-1} g = \delta X = \delta X^{(0)} + \delta X^{(1)} + \delta X^{(2)} + \delta X^{(3)}$, the variation of the current $J = g^{-1} dg$ reads

$$\begin{aligned} \delta J_\pm^0 &= \partial_\pm \delta X^0 + [J_\pm^0, \delta X^0] + [J_\pm^1, \delta X^3] + [J_\pm^2, \delta X^2] + [J_\pm^3, \delta X^1], \\ \delta J_\pm^1 &= \partial_\pm \delta X^1 + [J_\pm^0, \delta X^1] + [J_\pm^1, \delta X^0] + [J_\pm^2, \delta X^3] + [J_\pm^3, \delta X^2], \\ \delta J_\pm^2 &= \partial_\pm \delta X^2 + [J_\pm^0, \delta X^2] + [J_\pm^1, \delta X^1] + [J_\pm^2, \delta X^0] + [J_\pm^3, \delta X^3], \\ \delta J_\pm^3 &= \partial_\pm \delta X^3 + [J_\pm^0, \delta X^3] + [J_\pm^1, \delta X^2] + [J_\pm^2, \delta X^1] + [J_\pm^3, \delta X^0]. \end{aligned} \tag{9}$$

The equations of motion

$$\frac{1}{2}D_+J^3_- + \frac{3}{2}D_-J^3_+ = -\frac{1}{2}[J^2_+, J^1_-] - \frac{1}{2}[J^1_+, J^2_-], \quad (10)$$

$$\frac{3}{2}D_+J^1_- + \frac{1}{2}D_-J^1_+ = \frac{1}{2}[J^2_+, J^3_-] + \frac{1}{2}[J^3_+, J^2_-], \quad (11)$$

$$D_+J^2_- + D_-J^2_+ = -[J^1_+, J^1_-] + [J^3_+, J^3_-], \quad (12)$$

where $D_\pm = \partial_\pm + [J^0_\pm, \ ]$.

The Maurer-Cartan equations $\partial_+J_- - \partial_-J_+ + [J_+, J_-] = 0$ can be rewritten as

$$\partial_+J^0_- - \partial_-J^0_+ = -[J^0_+, J^0_-] - [J^1_+, J^3_-] - [J^2_+, J^2_-] - [J^3_+, J^1_-], \quad (13)$$

$$D_+J^3_- - D_-J^3_+ = -[J^1_+, J^2_-] - [J^2_+, J^1_-], \quad (14)$$

$$D_+J^1_- - D_-J^1_+ = -[J^2_+, J^3_-] - [J^3_+, J^2_-], \quad (15)$$

$$D_+J^2_- - D_-J^2_+ = -[J^1_+, J^1_-] - [J^3_+, J^3_-]. \quad (16)$$

Using the Maurer-Cartan equations the equations of motion can be rewritten as

$$D_+ J_-^3 + D_- J_+^3 = -[J_+^1, J_-^2] - [J_+^2, J_-^1], \tag{17}$$

$$D_+ J_-^1 + D_- J_+^1 = [J_+^2, J_-^3] + [J_+^3, J_-^2], \tag{18}$$

$$D_+ J_-^2 + D_- J_+^2 = -[J_+^1, J_-^1] + [J_+^3, J_-^3]. \tag{19}$$

2 The twisted duality and Lax connection

Hodge star of the bosonic currents

$$\star\, J_i = g_{ik}\epsilon^{kj} J_j, \tag{20}$$

or

$$\begin{aligned} \star J_+ &= \frac{1}{2}(\star J_\tau + \star J_\sigma) = \quad J_+, \\ \star J_- &= \frac{1}{2}(\star J_\tau - \star J_\sigma) = -J_-. \end{aligned} \tag{21}$$

The currents J^0 is invariant at the Hodge star. The Hodge star of the fermionic currents

$$\star\, J_i^\alpha = g_{ik}\epsilon^{kj} J_j^\alpha, \tag{22}$$

or

$$\star\, J_\pm^{1,3} = \pm J_\pm^{1,3}, \tag{23}$$

Define a duality transformation

$$J^a_\pm \to i^{\frac{a}{2}} \star J^a_\pm, \qquad a = 1, 2, 3 \tag{24}$$

The MCE's will be derived easily from the equations of motion EOM's, moreover, the reverse is also true by the same transformation.

Introduce twisted duality transformations

$$\begin{aligned}
\mathcal{J}^0_i(\lambda) &= J^0_i, \\
\mathcal{J}^2_i(\lambda) &= \frac{1}{2}(\lambda^2 + \lambda^{-2}) J^2_i + \frac{1}{2}(\lambda^2 - \lambda^{-2}) \star J^2_i, \\
\mathcal{J}^1_i(\lambda) &= \frac{1}{2}(\lambda + \lambda^{-3}) J^1_i + \frac{1}{2}(\lambda - \lambda^{-3}) \star J^1_i, \\
\mathcal{J}^3_i(\lambda) &= \frac{1}{2}(\lambda^3 + \lambda^{-1}) J^3_i + \frac{1}{2}(\lambda^3 - \lambda^{-1}) \star J^3_i,
\end{aligned} \tag{25}$$

satisfying MCE, and recovering the above duality transformation ($\lambda^2 = i$).

The Lax connection $A_i(\lambda)$ with the spectral parameter λ

$$\begin{aligned} A_i &= \mathcal{J}_i^0 + \mathcal{J}_i^2 + \mathcal{J}_i^1 + \mathcal{J}_i^3 \\ &= J_i^0 + \frac{1}{2}(\lambda^2 + \lambda^{-2}) J_i^2 + \frac{1}{2}(\lambda^2 - \lambda^{-2}) \star J_i^2 + \\ &\quad + \frac{1}{2}(\lambda + \lambda^{-3}) J_i^1 + \frac{1}{2}(\lambda - \lambda^{-3}) \star J_i^1 + \\ &\quad + \frac{1}{2}(\lambda^3 + \lambda^{-1}) J_i^3 + \frac{1}{2}(\lambda^3 - \lambda^{-1}) \star J_i^3, \end{aligned} \tag{26}$$

satisfies the compatibility condition

$$\partial_+ A_- - \partial_- A_+ + [A_+, A_-] = 0. \tag{27}$$

The monodromy matrix

$$T(\lambda) = Pexp \int_0^{2\pi} A_\sigma(\lambda) dx. \tag{28}$$

3 TsT transformation on $AdS_2 \times S^2$

3.1. TsT transformation on S^2

The S^2

$$g = \begin{pmatrix} r_1 e^{i\phi_1''} & r_2 e^{i\phi_2''} \\ -r_2 e^{-i\phi_2''} & r_1 e^{-i\phi_1''} \end{pmatrix}, \qquad r_1^2 + r_2^2 = 1, \tag{29}$$

The string action on S_2

$$S'' = -\frac{\sqrt{\lambda}}{2} \int d\tau \frac{d\sigma}{2\pi} [\gamma^{\alpha\beta}(\partial_\alpha r_i \partial_\beta r_i + g_{ij} \partial_\alpha \phi_i'' \partial_\beta \phi_i'') + \Lambda(r_1^2 + r_2^2 - 1)], \tag{30}$$

$\sqrt{\lambda} = R^2/\alpha'$, R the radius of S^2, $\gamma^{\alpha\beta} = \mathrm{diag}(-1, 1)$. $g_{ij} = \mathrm{diag}(r_1, r_2)$

First T-duality transformation

$$g'_{ij} = (r_1^{-2}, r_2^2), \quad \partial^\alpha \phi''_1 = \epsilon^{\alpha\beta}\partial_\beta \phi'_1 g'_{11}, \quad \phi''_2 = \phi'_2. \tag{31}$$

$$S' = -\frac{\sqrt{\lambda}}{2}\int d\tau \frac{d\sigma}{2\pi}[\gamma^{\alpha\beta}(\partial_\alpha r_i \partial_\beta r_i + g'_{ij}\partial_\alpha \phi'_i \partial_\beta \phi'_i) + \Lambda(r_1^2 + r_2^2 - 1)], \tag{32}$$

Secondly, angle shift

$$\phi_2' \to \phi_2' + \hat{\gamma}\phi_1', \qquad G_{11}' = r_1^{-2}(1 + \hat{\gamma}^2 r_1^2 r_2^2),$$
$$G_{12}' = \hat{\gamma} r_2^2, \qquad G_{22}' = r_2^2, \tag{33}$$

another T transformation

$$\begin{aligned} \partial_\alpha \phi_1' &= \gamma_{\alpha\beta}\epsilon^{\beta\gamma}\partial_\gamma \phi_1 G r_1^2 - \partial_\alpha \phi_2 \hat{\gamma} G r_1^2 r_2^2, \\ \phi_2' &= \phi_2, \quad G^{-1} = 1 + \hat{\gamma}^2 r_1^2 r_2^2. \end{aligned} \tag{34}$$

Action on the γ-deformed background

$$S=-\frac{\sqrt{\lambda}}{2}\int d\tau\frac{d\sigma}{2\pi}[\gamma^{\alpha\beta}(\partial_\alpha r_i\partial_\beta r_i+Gr_i^2\partial_\alpha\phi_i\partial_\beta\phi_i)-2\hat{\gamma}Gr_1^2r_2^2\epsilon^{\alpha\beta}\partial_\alpha\phi_1\partial_\beta\phi_2+\Lambda(r_i^2-1)] \quad (35)$$

The corresponding relation

$$\begin{aligned}\partial_\alpha\phi_1'' &= G(\partial_\alpha\phi_1-\hat{\gamma}r_2^2\gamma_{\alpha\beta}\epsilon^{\beta\gamma}\partial_\gamma\phi_2),\\ \partial_\alpha\phi_2'' &= G(\partial_\alpha\phi_2+\hat{\gamma}r_1^2\gamma_{\alpha\beta}\epsilon^{\beta\gamma}\partial_\gamma\phi_1),\\ G^{-1} &= 1+\hat{\gamma}^2r_1^2r_2^2.\end{aligned} \quad (36)$$

What is the meaning of the above relations?

$$
\begin{aligned}
J_i''^{\alpha} &= -\sqrt{\lambda} r_i^2 \gamma^{\alpha\beta} \partial_\beta \phi_i'', \\
J_1^{\alpha} &= -\sqrt{\lambda} r_1^2 \gamma^{\alpha\beta} G(\partial_\alpha \phi_1 - \hat{\gamma} r_2^2 \gamma_{\alpha\beta} \epsilon^{\beta\gamma} \partial_\gamma \phi_2), \\
J_2^{\alpha} &= -\sqrt{\lambda} r_2^2 \gamma^{\alpha\beta} G(\partial_\alpha \phi_2 + \hat{\gamma} r_1^2 \gamma_{\alpha\beta} \epsilon^{\beta\gamma} \partial_\gamma \phi_1).
\end{aligned} \tag{37}
$$

The $U(1)$ currents of strings on S^2 are equal to those on the γ-deformed background. i.e.

$$
J_i''^{\alpha} = J_i^{\alpha}. \tag{38}
$$

3.2. TsT transformation on AdS^2

$$g = \begin{pmatrix} r_1 e^{i\phi_1''} & r_2 e^{i\phi_2''} \\ r_2 e^{-i\phi_2''} & r_1 e^{-i\phi_1''} \end{pmatrix}, \qquad r_1^2 - r_2^2 = 1, \qquad g_{ij} = \mathrm{diag}(r_1^2, -r_2^2) \tag{39}$$

The string action on AdS^2

$$S'' = -\frac{\sqrt{\lambda}}{2} \int d\tau \frac{d\sigma}{2\pi} [\gamma^{\alpha\beta} (\partial_\alpha r_1 \partial_\beta r_1 - \partial_\alpha r_2 \partial_\beta r_2 + g_{ij} \partial_\alpha \phi_i'' \partial_\beta \phi_j'') + \Lambda (r_1^2 - r_2^2 - 1)] \tag{40}$$

Here, λ is same as in S^2 case excepting the radius R of AdS_2.

Similarly one can make the TsT transformation

$$\begin{aligned}
\partial_\alpha \phi_1'' &= -G(\partial_\alpha \phi_1 - \hat{\gamma} r_2^2 \gamma_{\alpha\beta} \epsilon^{\beta\gamma} \partial_\gamma \phi_2), \\
\partial_\alpha \phi_2'' &= -G(\partial_\alpha \phi_2 - \hat{\gamma} r_1^2 \gamma_{\alpha\beta} \epsilon^{\beta\gamma} \partial_\gamma \phi_1), \\
G^{-1} &= -1 + \hat{\gamma}^2 r_1^2 r_2^2.
\end{aligned} \tag{41}$$

The string action in γ-deformed background about AdS_2,

$$\begin{aligned}
S &= -\frac{\sqrt{\lambda}}{2} \int d\tau \frac{d\sigma}{2\pi} [\gamma^{\alpha\beta} (\partial_\alpha r_1 \partial_\beta r_1 - \partial_\alpha r_2 \partial_\beta r_2 - G r_1^2 \partial_\alpha \phi_1 \partial_\beta \phi_1 + G r_2^2 \partial_\alpha \phi_2 \partial_\beta \phi_2) + \\
&\quad + 2\hat{\gamma} G r_1^2 r_2^2 \epsilon^{\alpha\beta} \partial_\alpha \phi_1 \partial_\beta \phi_2 + \Lambda (r_1^2 - r_2^2 - 1)],
\end{aligned} \tag{42}$$

Compute the conserved $U(1)$ isometry currents

$$\begin{aligned} J_1''^{\alpha} &= -\sqrt{\lambda} r_1^2 \gamma^{\alpha\beta} \partial_\beta \phi_1'', \\ J_2''^{\alpha} &= \sqrt{\lambda} r_2^2 \gamma^{\alpha\beta} \partial_\beta \phi_2'', \\ J_1^{\alpha} &= -\sqrt{\lambda} r_1^2 \gamma^{\alpha\beta} (-G)(\partial_\alpha \phi_1 - \hat{\gamma} r_2^2 \gamma_{\alpha\beta} \epsilon^{\beta\gamma} \partial_\gamma \phi_2), \\ J_2^{\alpha} &= \sqrt{\lambda} r_2^2 \gamma^{\alpha\beta} (-G)(\partial_\alpha \phi_2 - \hat{\gamma} r_1^2 \gamma_{\alpha\beta} \epsilon^{\beta\gamma} \partial_\gamma \phi_1). \end{aligned} \tag{43}$$

The $U(1)$ currents of strings on AdS^2 are equal to those on the γ-deformed background $J_i''^{\alpha} = J_i^{\alpha}$.

4 Integrability in γ-deformed backgrounds

The sigma model on S_2 (or AdS^2) follows from the usual action for the principal chiral field:

$$S = \int dz_+ dz_- tr(J_+ J_-), \qquad J_\pm = g^{-1}\partial_\pm g \tag{44}$$

EOM and MCE are

$$\partial_+ J_- + \partial_- J_+ = 0, \qquad \partial_+ J_- - \partial_- J_+ + [J_+, J_-] = 0. \tag{45}$$

Now the Lax connection $A_i(l)$ with the spectral parameter λ is constructed,

$$A_\pm = a_\pm(l) J_\pm = \frac{\mp l}{1 \mp l} J_\pm, \qquad [D_+, D_-] = 0. \tag{46}$$

How to define the Lax connection on the γ-deformed background?

$$g(r_i, \phi_i'') = M_+(\phi_i'')\hat{g}(r_i)\, M_-(\phi_i''), \tag{47}$$

$$M_\pm = \begin{pmatrix} 0 & e^{\frac{i}{2}(\phi_2''\pm\phi_1'')} \\ e^{-\frac{i}{2}(\phi_2''\pm\phi_1'')} & 0 \end{pmatrix}, \qquad M' = \begin{pmatrix} 0 & e^{\frac{i}{2}(\phi_1''+\phi_2'')} \\ e^{-\frac{i}{2}(\phi_1''+\phi_2'')} & 0 \end{pmatrix} \tag{48}$$

we get

$$\begin{aligned} J_\pm(r_i, \phi''^{i}) &= M^{-1}\hat{J}_\pm(r_i, \partial\phi_i'')M, \\ \hat{J}_\pm &= \frac{i}{2}(\partial_\pm\phi_2'' - \partial_\pm\phi_1'')\sigma_3 + \hat{g}^{-1}\partial_\pm\hat{g} - \frac{i}{2}(\partial_\pm\phi_1'' + \partial_\pm\phi_2'')\hat{g}^{-1}\sigma_3\hat{g}. \end{aligned} \tag{49}$$

Making a gauge transformation,

$$D_\pm \to M D_\pm M^{-1} = \partial_\pm + A'_\pm,$$
$$A'_\pm = M A_\pm M^{-1} + M\partial_\pm M^{-1} = a_\pm \hat{J}_\pm + \frac{i}{2}(\partial_\pm \phi''_2 - \partial_\pm \phi''_1)\sigma_3. \quad (50)$$

The monodromy matrix $T(l)$,

$$T(l) = -\mathcal{P}exp\int_0^{2\pi} A'_\sigma(l)d\sigma. \quad (51)$$

Thank You

Adress: Institute of Modern Physics
Northwest University
No. 229, TaiBaiBeiLu, Xian
Shaanxi 710069, China
Tel: 0086-29-88303491
Email:rhyue@nwu.edu.cn