Boundary contribution of on-shell recursion relation

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based on papers arXiv:1411.0452, arXiv:1412.8170 and on-going work

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- Background and motivation
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Part I: Background and motivation

- The amazing development from 2003 to 2014 tells us that although the Feynman diagram method is the most standard way to calculate scattering amplitudes, it is not the most efficient way.
- For tree-level amplitude, the one-shell recursion relation becomes one of most useful new efficient method. Let us review its derivation.

- **First fact:** Tree-level amplitude is the rational function of external momenta and external wave functions.
- **Deformation:** We can consider following deformation: shifting two chosen external momenta, for example p_1, p_2, by an auxiliary momentum q as

$$p_1(z) = p_1 + zq, \qquad p_2(z) = p_2 - zq$$

With the deformation, the original tree-level amplitude $\mathcal{M}$ becomes the function of z, q, i.e., $\mathcal{M}(z, q)$. Furthermore, the momentum conservation is still hold.

- **On-shell conditions:** Asking $p_1^2 = p_1(z)^2$, $p_2^2 = p_2(z)^2$ for all z-values leads to

$$q^2 = q \cdot p_1 = q \cdot p_2 = 0$$

- **Structure:** From Feynman diagrams, tree-level amplitude can become singular when propagators are on-shell.
- Under the deformation, either propagator is not affected, or $(P + zq)^2 = P^2 + z(2P \cdot q)$, i.e., the single pole structure of complex variable z.
- **Fact:** For rational function of single complex variable z, one can use the information of its pole locations and residues to determine it. How to do it?

- Let us consider the contour integration $I = \oint dz A(z)/z$. One can evaluate by two ways:
  - Doing it along the point $z = \infty$, we get the **"boundary contribution"** which will denote as $\mathcal{B}$.
  - Doing it for big cycle around $z = 0$, we have $I = A(0) + \sum_\alpha \mathrm{Res}(A(z)/z)|_{z_\alpha}$.
- Combining above we have

$$A(z=0) = B - \sum_{poles\ z_\alpha} \mathrm{Res}\left(\frac{A(z)}{z}\right)_{z=z_\alpha}$$

[Britto, Cachazo, Feng , 2004] [Britto, Cachazo, Feng , Witten, 2004]

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Residue of finite pole z_α:

- **Location:** It can be found by solving $(P+zq)^2 = P^2 + z_\alpha(2P\cdot q)$.
- **Residue:** there is an important Factorization property: when one propagator goes to on-shell, i.e., $P^2 - m^2 \to 0$, we have

$$A^{tree}(1,..,n) \;\; \to \;\; \sum_\lambda A_{m+1}(1,..,m,P^\lambda)\frac{1}{P_{1m}^2 - m^2} A_{n-m+1}(-P^{-\lambda}, m+1,...,n)$$

Using it we get

$$\left(\frac{A(z)}{z}\right)_{z=z_\alpha} = \sum_\lambda A^L_{m+1}(1,..,m,P^\lambda(z_\alpha)) \frac{1}{P^2} A^R_{n-m+1}(-P^{-\lambda}(z_\alpha), m+1,...,n)$$

- How about the boundary contribution? It has following three cases:
  - When $z \to \infty$, $A(z) \to \sum_{i=0}^{k} c_i z^i + \mathcal{O}(1/z)$ with $c_0 \neq 0 \Longrightarrow$ nonzero boundary contribution
  - When $z \to \infty$, $A(z) \sim \frac{1}{z} \Longrightarrow$ zero boundary contribution
  - When $z \to \infty$, $A(z) \sim \frac{1}{z^k}, \quad k \geq 2 \Longrightarrow$ zero boundary contribution and bonus relations
- But how to determine which case it will belong to for a given theory ? A nice method is the background field method. [Arkani-Hamed, Kaplan 2008]

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Current situations:

- Not all theories have $\mathcal{B} = 0$.
- Fortunately, for theory involving gluons and gravitons, in many cases, there is a deformation to make $\mathcal{B} = 0$.
- However, for standard model with scalar and fermions, in general boundary contribution is unavoidable.
- Thus **determining $\mathcal{B}$ is an important problem for applications**.

There are three ways to deal with boundary contributions:

- Using auxiliary fields to make contributions in new QFT zero.

  [Benincasa, Cachazo, 2007; Boels, 2010]

- Analyze Feynman diagrams directly

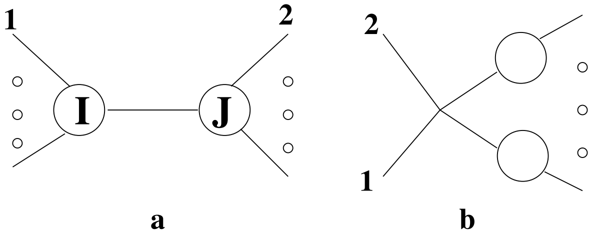
  [Feng, Wang, Wang, Zhang, 2009; Feng, Liu, 2010; Feng, Zhang, 2011]

- Transfer to the discussion of roots of amplitude

  [Benincasa, Conde, 2011; Feng, Jia, Luo, Luo, 2011]

Feynman diagram for $\lambda\phi^4$ theory

- With $(1,2)$-pair deformation, Feynman diagrams will be following two types:



- Boundary contribution is

$$A_b = (-i\lambda) \sum_{\mathcal{I}' \bigcup \mathcal{J}' = \{n\} \backslash \{i,j\}} A_{\mathcal{I}'}(\{K_{\mathcal{I}'}\}) \frac{1}{p^2_{\mathcal{I}'}} \frac{1}{p^2_{\mathcal{J}'}} A_{\mathcal{J}'}(\{K_{\mathcal{J}'}\})$$

Part II: Proposal for boundary contribution

[Bo Feng, Kang Zhou, Chenkai Qiao, Junjie Rao, arXiv:1411.0452]

Let us consider more carefully the derivation of BCFW recursion relation under the deformation $\underline{0} \equiv \langle i_0 | j_0]$:

- There are **physical poles** and **spurious poles** and denote the set of them as $\mathcal{S}^{\underline{0}}$.
- Classification: **detectable propagators** which depend on $z_{\underline{0}}$ and the **undetectable propagators** which are $z_{\underline{0}}$-independent.

- The expansion

$$-A_n^{\underline{0}}(z_{\underline{0}}) \;=\; \frac{N(z_{\underline{0}})}{\prod P_t^2(z_{\underline{0}})} = \mathcal{R}^{\underline{0}}(z_{\underline{0}}) + \mathcal{B}^{\underline{0}}(z_{\underline{0}}).$$

with **recursive part** as

$$\mathcal{R}^{\underline{0}}(z_{\underline{0}}) = \sum_{P_t \in \mathcal{D}^{\underline{0}}} \frac{A_{t;L}(\widehat{z}_{\underline{0},t})A_{t;R}(\widehat{z}_{\underline{0},t})}{P_t^2(z_{\underline{0}})},$$

and the **regular part** as

$$\mathcal{B}^{\underline{0}}(z_{\underline{0}}) = C_0^{\underline{0}} + \sum C_i^{\underline{0}} z_{\underline{0}}^i \;.$$

- **Key observation:** the poles $P_t \in \mathcal{D}^{\underline{0}}$ will appear once and only once with power one in $\mathcal{R}^{\underline{0}}$, i.e., they cannot be the poles of coefficients $\mathcal{B}^{\underline{0}}$
- **Pole structure of boundary:** (I) It belongs to $\mathcal{U}^{\underline{0}}$ or $\mathcal{S}^{\underline{0}}$; (II) The powers of spurious poles in $\mathcal{B}^{\underline{0}}$ may be larger than one.
- **Fact:** The part $\mathcal{R}^{\underline{0}}$ is **known** by recursion relation, while the part $\mathcal{B}^{\underline{0}}$ is not known.

Let us consider the second deformation $\underline{1} \equiv \langle i_1 | j_1]$:

- The full amplitude can be calculated by two ways:
  - **The first way:** Using the recursion relation

$$-A_n^{\underline{1}}(z_{\underline{1}}) = \mathcal{R}^{\underline{1}}(z_{\underline{1}}) + \mathcal{B}^{\underline{1}}(z_{\underline{1}})$$

  - **The second way:** Using expression $-A_n^{\underline{0}}(z_{\underline{0}} = 0)$ to make the deformation and the expansion

$$\begin{aligned} \mathcal{R}^{\underline{0}}(z_{\underline{1}}) &= \mathcal{R}\mathcal{R}^{\underline{0},\underline{1}}(z_{\underline{1}}) + \mathcal{R}\mathcal{B}^{\underline{0},\underline{1}}(z_{\underline{1}}) \\ \mathcal{B}^{\underline{0}}(z_{\underline{1}}) &= \mathcal{B}\mathcal{R}^{\underline{0},\underline{1}}(z_{\underline{1}}) + \mathcal{B}^{\underline{01}}(z_{\underline{1}}), \end{aligned}$$

- **Key observation:** Identifying two ways,

$$\mathcal{R}^{\underline{1}}(z_{\underline{1}}) = \mathcal{R}\mathcal{R}^{\underline{0},\underline{1}}(z_{\underline{1}}) + \mathcal{B}\mathcal{R}^{\underline{0},\underline{1}}(z_{\underline{1}}).$$

Brief summary:

- Using two deformations, we can find part of unknown boundary $\mathcal{B}^{\underline{01}}$, which depends on poles $P_t \in \mathcal{U}^{\underline{0}} \bigcap \mathcal{D}^{\underline{1}}$.
- It is easy to see our strategy: using enough deformations to detect all possible poles of unknown boundary $\mathcal{B}^{\underline{01}}$, thus we can determine it **up to polynomial part.**

How many steps do we need to take? When we can stop? A judicious criteria are following:

- All spurious poles must be canceled out.
- The power of any physical pole must be at most one.
- It must have correct factorization limits for all physical poles.
- If the result satisfies above three conditions, it is very likely to be correct (up to possible polynomial terms).

Example: $A(1^+, 2, 3, 4, 5^+)$ of the color ordered Yukawa Theory

- Possible dependence of physical poles $\{\langle 1|2\rangle, \langle 4|5\rangle, \langle 5|1\rangle\}$
- With $\underline{0} = \langle 1|5]$

$$-A^{\underline{0}} \quad = \quad g^3 \frac{\langle 2|4\rangle}{\langle 2|1\rangle \langle 5|4\rangle} + \mathcal{B}^{\underline{0}}$$

with sets $\mathcal{D}^{\underline{0}} = \{\langle 1|2\rangle\}, \mathcal{U}^{\underline{0}} = \{\langle 4|5\rangle, \langle 5|1\rangle\}, \mathcal{S}^{\underline{0}} = \emptyset$

- With $\underline{1} = \langle 5|4]$,

$$\mathcal{R}^{\underline{1}}(z_{\underline{1}}) = g\lambda \frac{1}{\langle 1|5\rangle - z_{\underline{1}} \langle 1|4\rangle},$$

so we get

$$\mathcal{BR}^{\underline{0},\underline{1}} = g\lambda \frac{1}{\langle 1|5\rangle}.$$

- After the deformation $\underline{1}$ we get

$$-A_5 = g^3 \frac{\langle 2|4\rangle}{\langle 2|1\rangle \langle 5|4\rangle} - g\lambda \frac{1}{\langle 5|1\rangle} + \mathcal{B}^{0\underline{1}},$$

  with the corresponding sets

$$\mathcal{D}^{0\underline{1}} = \{\langle 1|2\rangle, \langle 5|1\rangle\}, \quad \mathcal{U}^{0\underline{1}} = \{\langle 4|5\rangle\}, \quad \mathcal{S}^{0\underline{1}} = \emptyset.$$

- To continue, we need to perform another deformation, *e.g.*, $\underline{2} = \langle 5|1]$ to detect $\langle 4|5\rangle$. However, it can be checked that under $\underline{2}$ the pole part of $\mathcal{B}^{0\underline{1}}$ is zero. Since all physical poles have been detected, we can conclude that $\mathcal{B}^{0\underline{1}} = 0$, and the correct answer is

$$-A_5 = g^3 \frac{\langle 2|4\rangle}{\langle 2|1\rangle \langle 5|4\rangle} - g\lambda \frac{1}{\langle 5|1\rangle},$$

Our algorithm is very general, but there are a lot of unanswered questions:

- How to choose deformations in consequence to make the calculation most efficient?
- Does the algorithm terminate eventually? How to judge it after several steps?
- Which theory it can be applied and which theory it can not be applied?
- Could the idea to be generalized to other places?

Part III: Recursion relation for boundary contribution

[Qingjun Jin, Bo Feng, arXiv:1412.8170]

Let us consider a special type of deformation $\langle i|n]$ with $i = 2, ..., n-1$

- Poles of boundary can be easily read out by large z-expansion

$$\frac{1}{(P_J + p_1 - z\lambda_n\widetilde{\lambda}_1)^2} = \frac{1}{-z\langle n|P_J + p_1|1]}\sum_{i=0}\left(\frac{(P_J + p_1)^2}{z\langle n|P_J + p_1|1]}\right)$$

- Thus all spurious poles $\langle n|P_{J\subset\mathcal{T}}|i]$ are invariant under deformations.
- The most important thing is that we can establish corresponding **on-shell recursion relation for boundary contribution**

Derivation:

- First, the boundary is defined as

$$B_0^1(\{\lambda_1,\widetilde{\lambda}_1\},p_2,...,p_{n-1},\{\lambda_n,\widetilde{\lambda}_n\})$$
$$=\oint_{w=\infty}\frac{dw}{w}A_n(\{\lambda_1-w\lambda_n,\widetilde{\lambda}_1\},p_2,...,p_{n-1},\{\lambda_n,\widetilde{\lambda}_n+w\widetilde{\lambda}_1\})$$

- Now using the contour integration $\oint_{|z|=R\to\infty} dz\frac{B_0^1(z)}{z}$ we arrive

$$B_0^1=B_0^{12}-\sum_{z_{\mathcal{I}}}\mathrm{Res}\left(\frac{B_0^1}{z}\right)_{z=z_{\mathcal{I}}}$$

where the second deformation is $\langle 2|n]$ and $z_{\mathcal{I}}=\frac{(p_2+P_{\mathcal{I}})^2}{\langle n|P_{\mathcal{I}}|2]}$ and $\mathcal{I}\bigcup\overline{\mathcal{I}}=\{3,4,...,n-1\}$.

- Evaluation of residue part is given by

$$
\begin{aligned}
&\text{Res}\left(\frac{B_0^1}{z}\right)_{z=z_{\mathcal{I}}} \\
= &\oint_{z=z_{\mathcal{I}}} \frac{dz}{z} B_0^1(\{\lambda_1, \widetilde{\lambda}_1\}, \{\lambda_2 - z\lambda_n, \widetilde{\lambda}_2\}, ..., p_{n-1}, \{\lambda_n, \widetilde{\lambda}_n + z\widetilde{\lambda}_2\}) \\
= &\oint_{z_{\mathcal{I}}} \frac{dz}{z} \oint_{\infty} \frac{dw}{w} A_n(\{\lambda_1 - w\lambda_n, \widetilde{\lambda}_1\}, \{\lambda_2 - z\lambda_n, \widetilde{\lambda}_2\}, \\
&p_3, ..., p_{n-1}, \{\lambda_n, \widetilde{\lambda}_n + z\widetilde{\lambda}_2 + w\widetilde{\lambda}_1\})
\end{aligned}
$$

- The key is then to use the **Fubini-Tonelli theorem** to exchange the ordering of two integrations

- Now we have

$$
\begin{aligned}
& \oint_{w=\infty} \frac{dw}{w} \oint_{z=z_{\mathcal{I}}} \frac{dz}{z} A_n(\{\lambda_1 - w\lambda_n, \widetilde{\lambda}_1\}, \{\lambda_2 - z\lambda_n, \widetilde{\lambda}_2\}, p_3, \ldots, p_{n-1}, \{\lambda_n, \widetilde{\lambda}_n + z\widetilde{\lambda}_2 + w\widetilde{\lambda}_1\}) \\
= & \oint_{w=\infty} \frac{dw}{w} \sum_h A_L(\widehat{p}_2(z_{\mathcal{I}}), \mathcal{I}, -P^h(z_{\mathcal{I}})) \frac{-1}{(p_2 + P_{\mathcal{I}})^2} A_R(\{\lambda_1 - w\lambda_n, \widetilde{\lambda}_1\}, \overline{\mathcal{I}}, \{\lambda_n, \widetilde{\lambda}_n + z_{\mathcal{I}}\widetilde{\lambda}_2 + w\widetilde{\lambda}_1 \\
= & \sum_h A_L(\widehat{p}_2(z_{\mathcal{I}}), \mathcal{I}, -P^h(z_{\mathcal{I}})) \frac{-1}{(p_2 + P_{\mathcal{I}})^2} \\
& \oint_{w=\infty} \frac{dw}{w} A_R(\{\lambda_1 - w\lambda_n, \widetilde{\lambda}_1\}, \overline{\mathcal{I}}, \{\lambda_n, \widetilde{\lambda}_n + z_{\mathcal{I}}\widetilde{\lambda}_2 + w\widetilde{\lambda}_1\}, P^{-h}(z_{\mathcal{I}})) \\
= & \sum_h A_L(\widehat{p}_2(z_{\mathcal{I}}), \mathcal{I}, -P^h(z_{\mathcal{I}})) \frac{-1}{(p_2 + P_{\mathcal{I}})^2} B_0^1(p_1, \widehat{p}_n(z_{\mathcal{I}}), \overline{\mathcal{I}}, P^{-h}(z_{\mathcal{I}}))
\end{aligned}
$$

Remarks:

- In the derivation, the commutativity of two integrations is crucial. In general with arbitrary pair of deformations, it is not true, but with our special choice of the type $\langle i|n]$, it is true.
- For it to be useful, one should show by other ways that after finite steps, there is no boundary left anymore. We have made the analysis for **standard like model**, i.e., similar matter contents and similar interaction except all particles are massless.

Example II: Six scalars in scalar-Yang-Mills theory

$$L = \mathrm{Tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - D_\mu\bar{\Phi}D^\mu\Phi - \frac{g^2}{2}[\Phi,\bar{\Phi}]^2\right)$$

- Step 1: With deformation $\underline{0} = \langle 1|6]$, the recursive part is given by

$$R_6^{\underline{0}} = -\frac{\langle 16\rangle[35]^2[4|1+6|2\rangle^2}{\tau_{612}\langle 12\rangle[34][45][5|1+6|2\rangle[3|1+2|6\rangle} + \frac{[13]^2\langle 46\rangle^2[1|2+3|5\rangle^2[2|1+3|6\rangle^2}{\tau_{123}[12][23]\langle 45\rangle\langle 56\rangle[1|2+3|4\rangle[3|1+2|6\rangle[1|2+3|6\rangle^2} + \frac{[16]\langle 24\rangle^2[5|1+6|3\rangle^2}{\tau_{234}[56]\langle 23\rangle\langle 34\rangle[1|2+3|4\rangle[5|1+6|2\rangle}$$

- Next under the deformation $\underline{1} = \langle 2|6]$ and using

$$\mathcal{B}^{\underline{0}}(g^-(k_7),4,5,6,1) = \frac{[14]\Big(-2[15][46]+[14][56]\Big)}{[16][17][45][57]}$$

$$\mathcal{B}^{\underline{0}}(\bar{\Phi}(k_7),5,6,1) = \frac{-\langle 17\rangle\langle 56\rangle - 2\langle 15\rangle\langle 67\rangle}{\langle 16\rangle\langle 57\rangle}$$

$$\mathcal{B}^{\underline{0}}(g^-(k_7),6,1) = \frac{\langle 17\rangle\langle 67\rangle}{\langle 16\rangle}$$

we find

$$\mathcal{B}\mathcal{R}^{\underline{01}} = A(\hat{2},3,-\hat{p}_{23})\frac{1}{p_{23}^2}\mathcal{B}^{\underline{0}}(\hat{p}_{23},4,5,\hat{6},1)$$
$$+A(\hat{2},3,4,-\hat{p}_{234})\frac{1}{p_{234}^2}\mathcal{B}^{\underline{0}}(\hat{p}_{234},5,\hat{6},1)$$
$$+A(\hat{2},3,4,5,\hat{p}_{16})\frac{1}{p_{16}^2}\mathcal{B}^{\underline{0}}(-\hat{p}_{16},\hat{6},1)$$

- At the third step, using the deformation $\underline{2} = \langle 3|6]$ and

$$\mathcal{B}^{\underline{01}}(g^-(k_7), 5, 6, 1, 2) = \frac{[25]}{[27][57]}, \quad \mathcal{B}^{\underline{01}}(\Phi(k_7), 6, 1, 2) = -1$$

we find

$$\begin{aligned} \mathcal{BR}^{\underline{012}} &= A(\hat{3}, 4, -\hat{p}_{34})\frac{1}{p_{34}^2}\mathcal{B}^{\underline{01}}(\hat{p}_{34}, 5, \hat{6}, 1, 2) \\ &+ A(\hat{3}, 4, 5, -\hat{p}_{345})\frac{1}{p_{345}^2}\mathcal{B}^{\underline{01}}(\hat{p}_{345}, \hat{6}, 1, 2) \end{aligned}$$

- At the fourth step with deformation $\underline{3} = \langle 4|6]$, using

$$\mathcal{B}^{\underline{012}}(g^-(k_7), 6, 1, 2, 3) = \frac{[13]^2[27]}{[23][37][17]^2}$$

we find

$$\mathcal{BR}^{\underline{0123}} = A(\hat{4}, 5, -\hat{p}_{45})\frac{1}{p_{45}^2}\mathcal{B}^{\underline{012}}(\hat{p}_{45}, \hat{6}, 1, 2, 3)$$

- Finally $\mathcal{R}_6^0 + \mathcal{BR}^{\underline{01}} + \mathcal{BR}^{\underline{012}} + \mathcal{BR}^{\underline{0123}}$ is equal to the total amplitude.

Part IV: The boundary Lagrangian

[work going-on]

- Our algorithm is reduced to the study of boundary of boundary. How to study it? Could we have an understanding like the Feynman diagrams? Could we have the corresponding Lagrangian?
- The key observation is that the boundary comes from the **large z-limit** of deformation parameter. Thus two momenta $p_i + zq, p_j - zq$ become **infinity**, i.e., we have **two very heavy particles**.

We can view it from two different aspects:

- The first one is **background field method**, i.e., the two heavy particles can be taken as classical background, while other fields as soft (quantum) fluctuation. Thus we can use Wilson's idea to integrate them out.
- The second one is using OPE method to replace the product of two quantum fields by a **boundary operator**,

$$\mathcal{O}_I(k_L + zq)\mathcal{O}_J(k_R - zq) = \sum_K C_{IJ}^K(k_L + zq)\mathcal{O}_K(k_L + k_R)$$

Expanding the coefficient around $z = \infty$

$$C_{IJ}^K(k_L + zq) = \sum_i C_{IJ,i}^K z^i$$

we get the boundary operator

$$\mathcal{F} = \sum_K C_{IJ,0}^K \mathcal{O}_K(k_1 + k_n)$$

A simple calculation for theory

$$L = -\frac{1}{2}(\partial\phi)^2 + \frac{\lambda}{m!}\phi^m$$

- The n-point amplitude can be calculated from

$$\left\langle 0|\mathcal{T}\left(\exp[-i\int d^4x \frac{\lambda}{m!}\phi^m]\right)|\phi(p_1)\phi(p_2)...\phi(p_n)\right\rangle$$

$$= \langle 0|[-i\frac{\lambda}{(m-2)!}\phi^{m-2}]\mathcal{T}\left(\exp[-i\int d^4x \frac{\lambda}{m!}\phi^m]\right) |\phi(p_2)...\phi(p_{n-1})\rangle$$

where we have contracted the $p_1 - zq, p_n + zq$.

- Above result can be reproduced by

$$\langle 0|\mathcal{T}\left(\exp[-i\int d^4x\frac{\lambda}{m!}\phi^m - i\mathcal{F}^{\langle 1|n]}\frac{\lambda}{(m-2)!}\phi^{m-2}]\right)$$
$$|\mathcal{F}^{\langle 1|n]}(p_1+p_n)\phi(p_2)...\phi(p_{n-1})\rangle$$

Thus we have derived the **boundary Lagrangian** as

$$L_{\mathcal{B}^{\langle 1|n]}} = -\frac{1}{2}(\partial\phi)^2 + \frac{\lambda}{m!}\phi^m + \mathcal{F}^{\langle 1|n]}\frac{\lambda}{(m-2)!}\phi^{m-2}$$

where ϕ is the soft fields.

Path integration approach:

- **Two key observations**: (1) along the hard line, each vertex has two and only two hard fields; (2) hard particles are contracted as inner propagator.
- Thus interaction vertex should be changed to

$$\frac{\lambda_m}{m!}\phi^m \to \frac{\lambda_m}{m!}\phi^{m-2}H^2 C_m^2 = \frac{\lambda_m}{2(m-2)!}\phi^{m-2}H^2$$

- Thus the evaluation of amplitudes can be divided as following

$$
\begin{aligned}
& \int [D\phi][DH] e^{iS[\phi]+i(\frac{1}{2}(\partial H)^2-\frac{\lambda m}{2(m-2)!}\phi^{m-2}H^2)} \\
& H(p_1-zq)H(p_n+zq)\phi(p_2)...\phi(p_{n-1}) \\
= & \int [D\phi] e^{iS[\phi]}\phi(p_2)...\phi(p_{n-1}) \\
& \left\{\int [DH] e^{i\int d^4x(\frac{1}{2}(\partial H)^2-\frac{\lambda m}{2(m-2)!}\phi^{m-2}H^2)} H(p_1-zq)H(p_n+zq)\right\}
\end{aligned}
$$

where the $\frac{1}{2}(\partial H)^2$ is needed for contraction to get propagator.

- The hard part is Guassian and can be evaluated using the generating function

$$Z[J] = \int [DH] e^{i \int d^4x \left(\frac{1}{2}(\partial H)^2 - \frac{\lambda_m}{2(m-2)!}\phi^{m-2}H^2 + JH\right)}$$

- Let us define the operator $\mathcal{D} = \partial^2 + \frac{\lambda_m}{(m-2)!}\phi^{m-2}$, thus the two-point correlation function is given by $\mathcal{D}^{-1}$.
- The inverse can be expanded as following

$$\mathcal{D}^{-1} = \left\{ \left(1 + \frac{\lambda_m}{(m-2)!}\phi^{m-2}\frac{1}{\partial^2} \right) \partial^2 \right\}^{-1}$$

$$= \sum_{k=0}^{\infty} (-)^k \frac{1}{\partial^2} \left(\frac{\lambda_m}{(m-2)!}\phi^{m-2}\frac{1}{\partial^2} \right)^k$$

- To get the scattering amplitude, we need to use the **LSZ reduction**, i.e., multiplying ∂^2 for each field on correlation function and contracting the off-shell quantities with wave (polarization) function.

$$\epsilon(p_1 - zq)\partial^2 \left\{ \sum_{k=0}^{\infty} (-)^k \frac{1}{\partial^2} \left(\frac{\lambda_m}{(m-2)!} \phi^{m-2} \frac{1}{\partial^2} \right)^k \right\} \partial^2 \epsilon(p_n + zq)$$

$$= \sum_{k=1}^{\infty} \left(\frac{-\lambda_m}{(m-2)!} \phi^{m-2} \frac{1}{\partial^2} \right)^{k-1} \frac{-\lambda_m}{(m-2)!} \phi^{m-2} \epsilon(p_1 - zq)\epsilon(p_n + zq)$$

- For scalar $\epsilon(p_1 - zq) = \epsilon(p_n + zq) = 1$, and each $\frac{1}{\partial^2} \sim \frac{1}{z}$, only the term $k = 1$ contribute and we arrive

$$L_{\mathcal{B}\langle 1|n]} = \frac{1}{2}(\partial\phi)^2 - \frac{\lambda_m}{m!}\phi^m - \mathcal{F}^{\langle 1|n]} \frac{\lambda_m}{(m-2)!}\phi^{m-2}$$

Part V: Conclusion

There are a lot of unsolved problems for our approach:

- How to choose deformations in consequence to make the calculation most efficient?
- Does the algorithm terminate eventually? How to judge it after several steps?
- Which theory it can be applied and which theory it can not be applied?
- Could the idea to be generalized to other places?
- What is the relation between boundary and the zero of amplitudes?
- To loop level?

Thanks a lot for listening!!!