Modular symmetry at level 6 and a new route towards finite modular groups

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In collaboration with

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Outline

1. Motivation
2. Modular symmetry
3. Modular forms and gCP
4. A new route towards
5. Model building and predictions
6. Conclusions

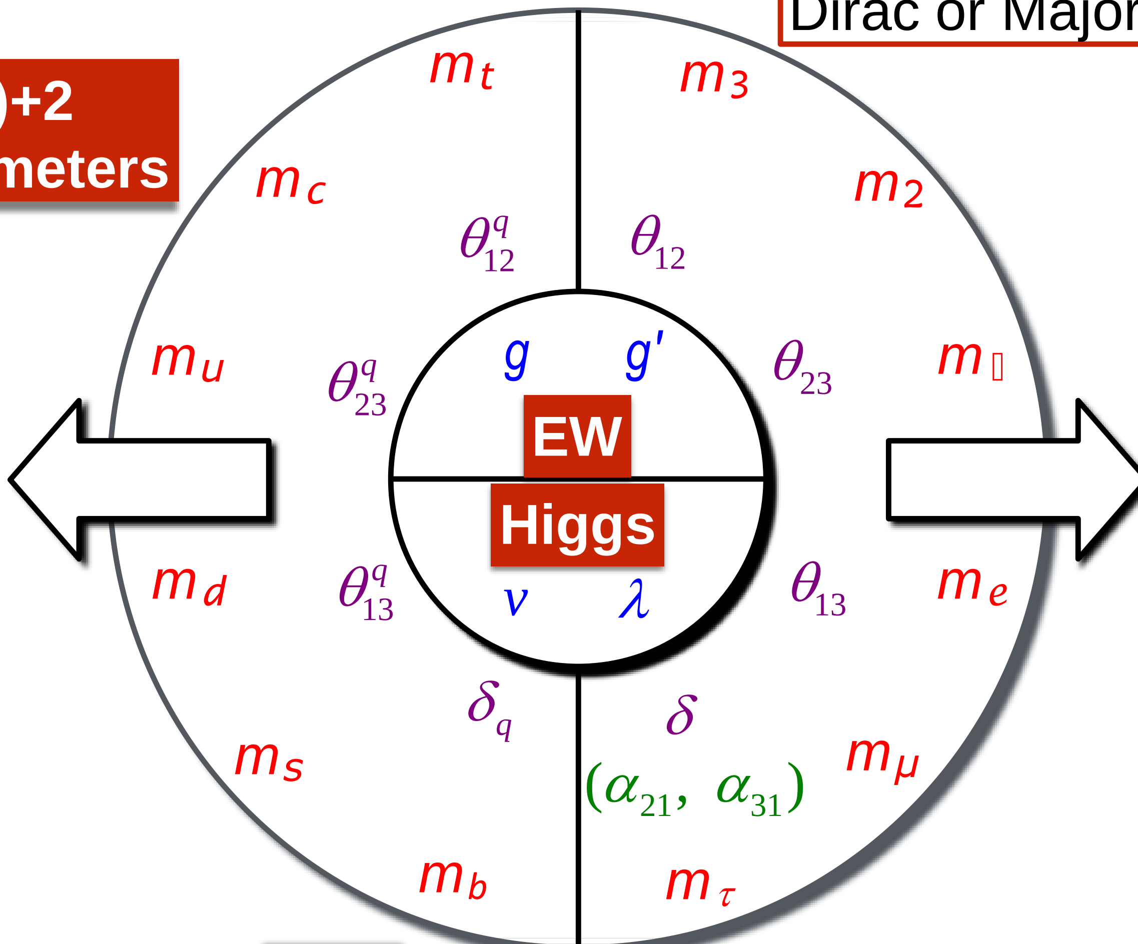
1. Motivation

SM + massive neutrinos

Neutrino may take Dirac or Majorana masses

24(26)+2 free parameters

Quark masses and mixing



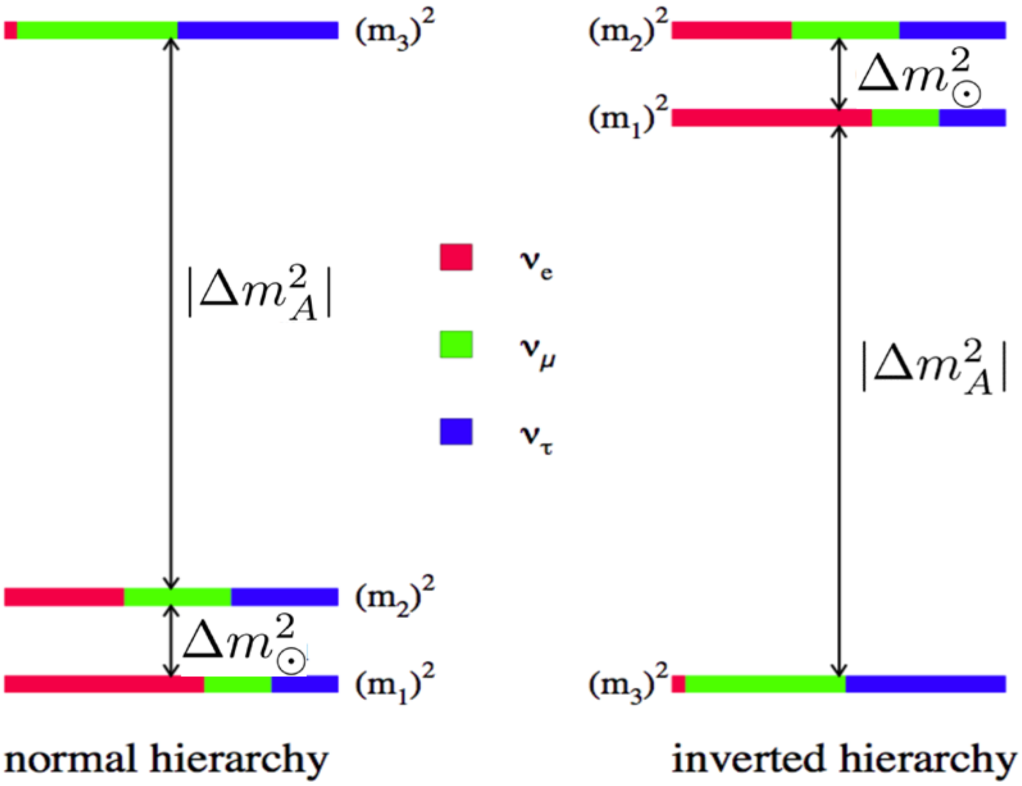
Lepton masses and mixing

QCD g_s $\bar{\theta}$ **Strong CP**

3ν flavour paradigm

Masses: ordering

$$\frac{\Delta m^2_\odot}{|\Delta m^2_A|} \sim \frac{1}{30}$$



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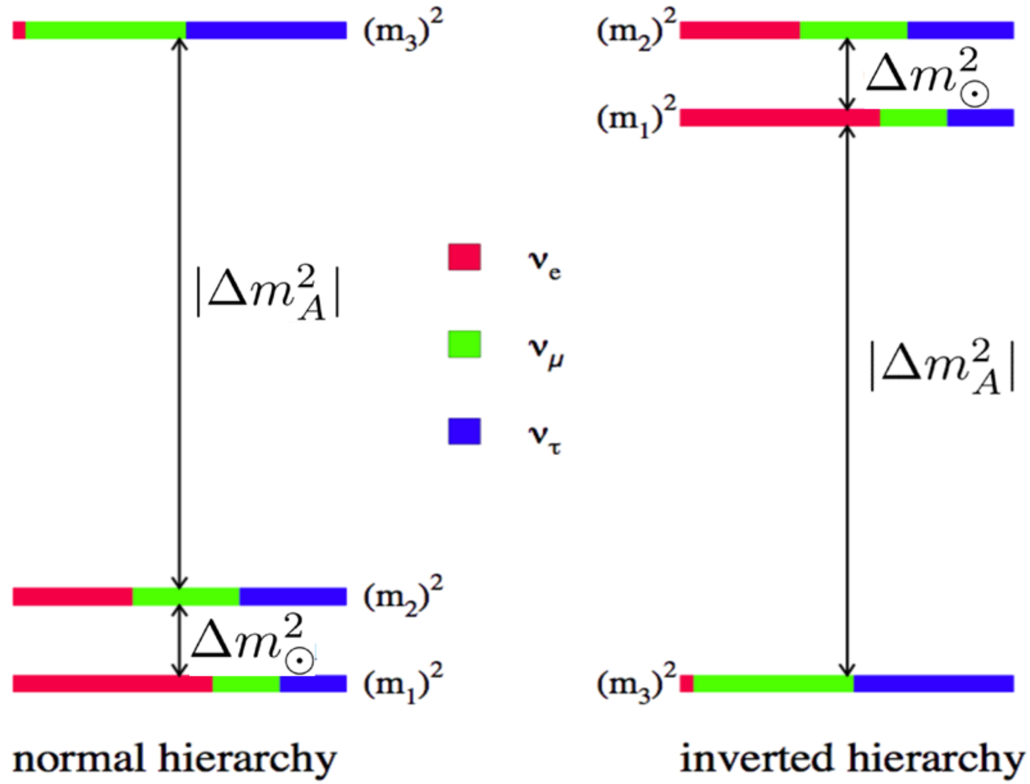


Mixing: parameterisation

$$U_{PMNS} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{CP}} \\ 0 & 1 & 0 \\ -s_{13}e^{-i\delta_{CP}} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha_{21}/2} & 0 \\ 0 & 0 & e^{i\alpha_{31}/2} \end{pmatrix}$$

Atmospheric mixing

Reactor mixing & Dirac CP phase

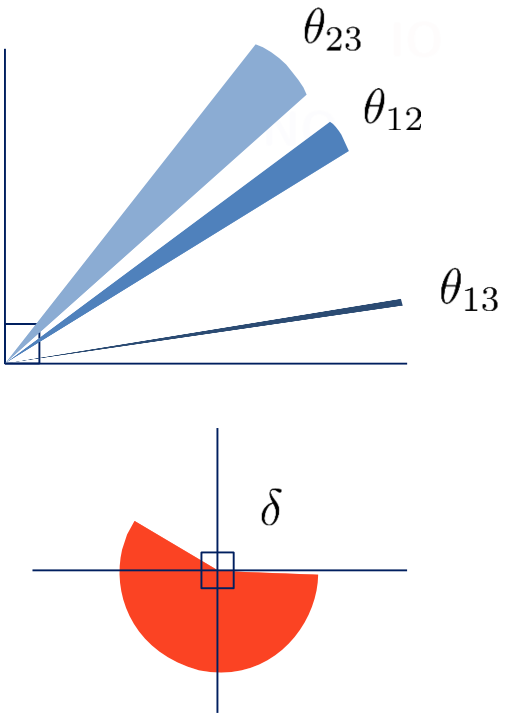
Solar mixing

Majorana CP phases

NuFit 5.1

For a spectrum with NO:

Parameter	Best-fit value
$\Delta m^2_{\odot}$	7.42×10^{-5} eV^2
$\lvert\Delta m^2_{A}\rvert$	2.510×10^{-3} eV^2
$\sin^2\theta_{12}$	0.304
$\sin^2\theta_{13}$	0.2246
$\sin^2\theta_{23}$	0.450
δ	$1.278\ \pi$



$\sim 3\sigma$

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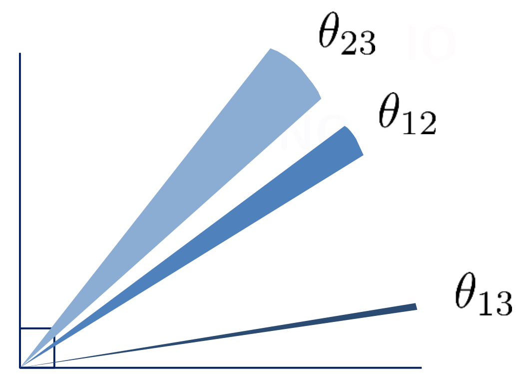

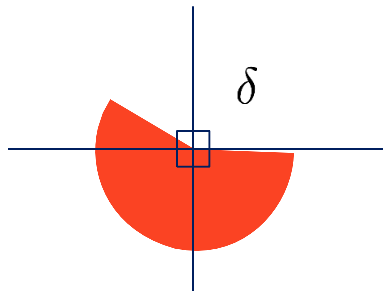

$\sim 3\sigma$

Is there an organizing principle behind this?

Flavour symmetries

For the lepton sector, at low energy and in some flavour basis:

$$\mathcal{L} = -\overline{l_L} m_l l_R - \frac{1}{2}\nu_L^T C m_\nu \nu_L + \frac{g}{\sqrt{2}}\overline{l_L}\gamma^\mu \nu_L W_\mu^- + h.c.$$

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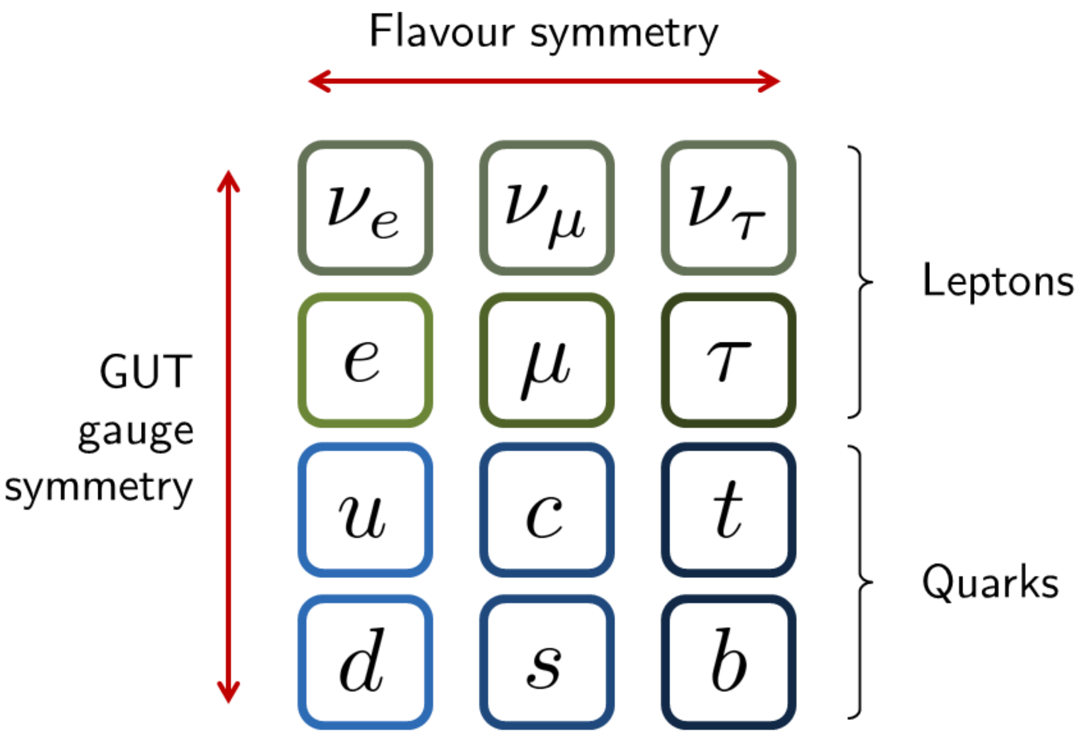

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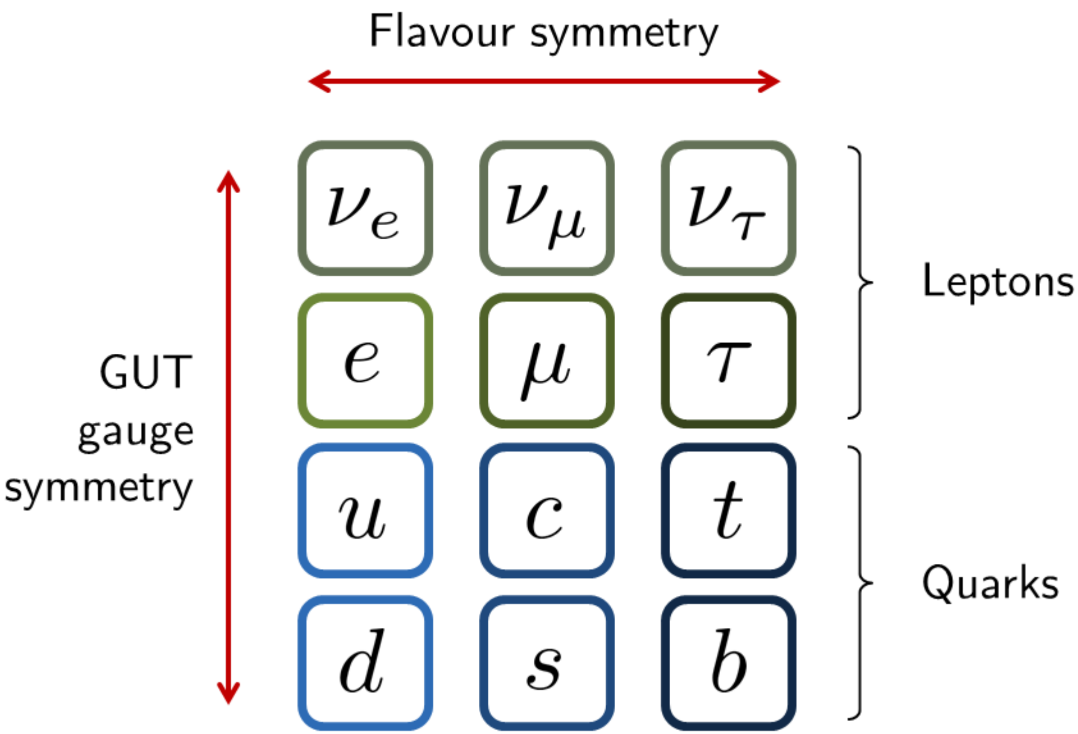
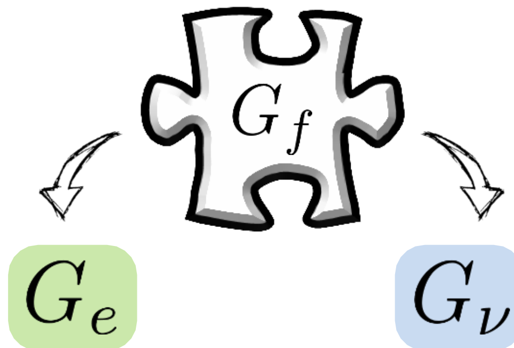


Non-Abelian discrete flavour symmetries



constrain mixing and Dirac phase

For reviews, see: Altarelli and Feruglio (2010), Ishimori et al. (2010),

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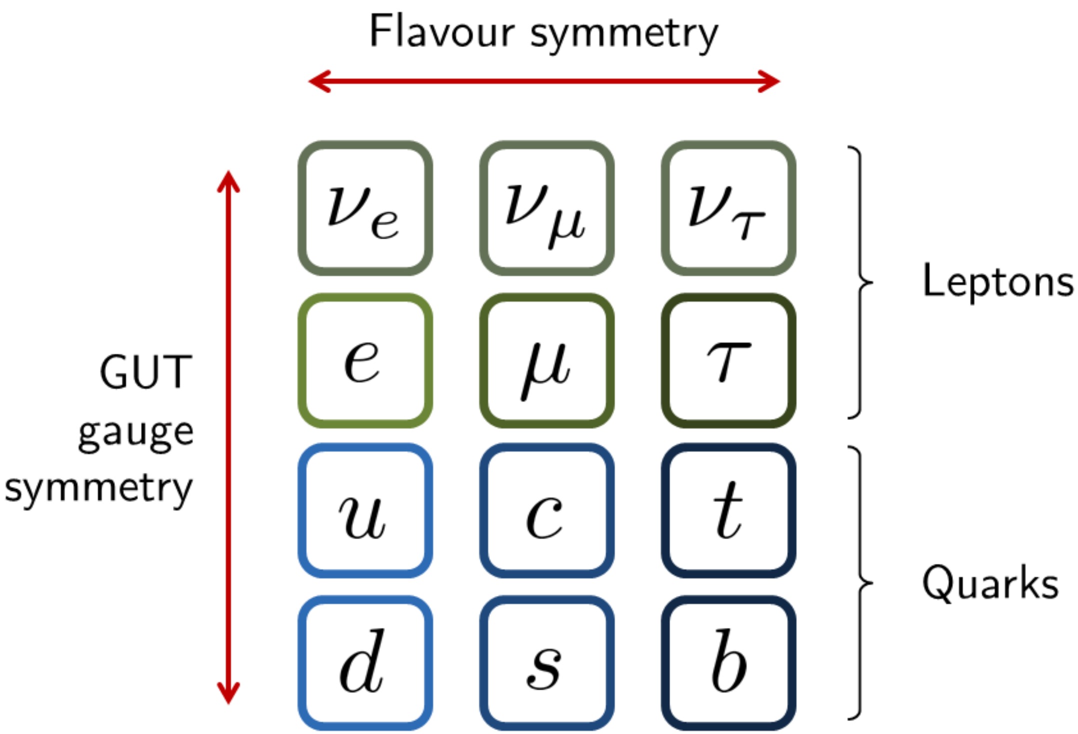


Tri-bimaximal:

$$U_{TB} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\theta_{12}^{TB} = 35.26^\circ, \theta_{23}^{TB} = 45^\circ, \theta_{13}^{TB} = 0^\circ$$

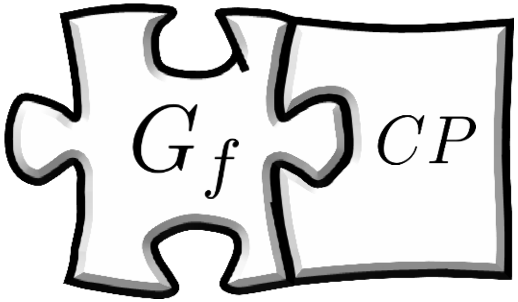
For reviews, see: Altarelli and Feruglio (2010), Ishimori et al. (2010),

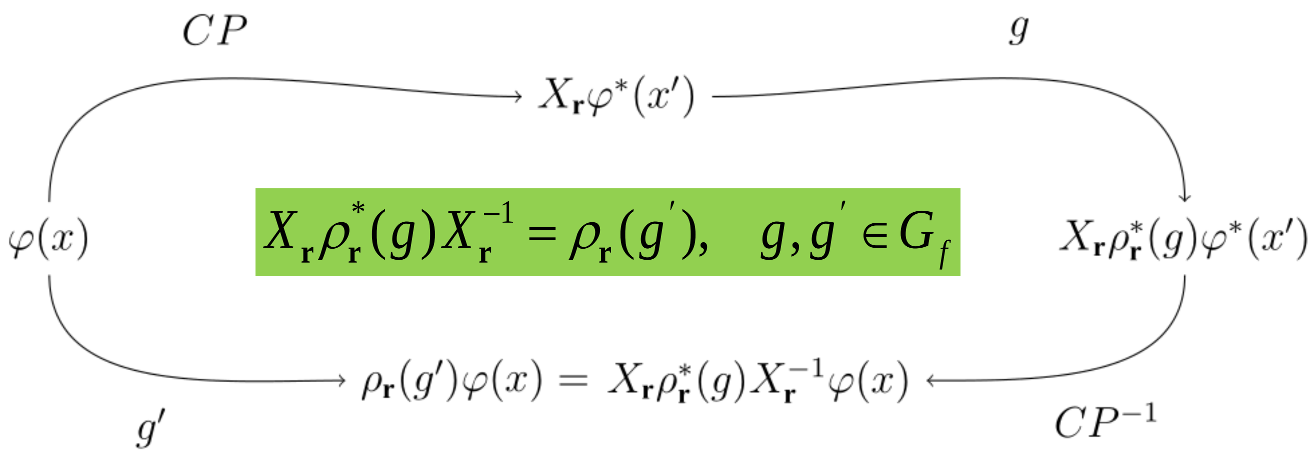
Flavour symmetries + gCP

$$\psi(x) \to \rho_{\mathbf{r}}(g)\,\psi(x) \qquad\qquad \psi(x) \to X_{\mathbf{r}}^{\mathrm{CP}}\,\overline{\psi}(x_{\mathrm{P}})$$

Branco, Lavoura, Rebelo (1986), Harrison, Scott (2002),
Grimus, Lavoura (2003), Farzan, Smirnov (2006),
Ferreira, Grimus, Lavoura, Ludl (2012) , ...

Flavour symmetries + gCP

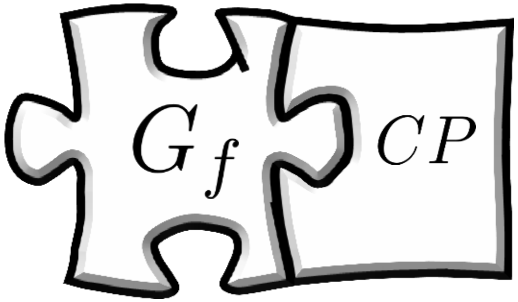




Consistency condition [Feruglio, et al., Holthausen et al. (2012)]

Class-inverting outer automorphism [Chen et al. (2014)]

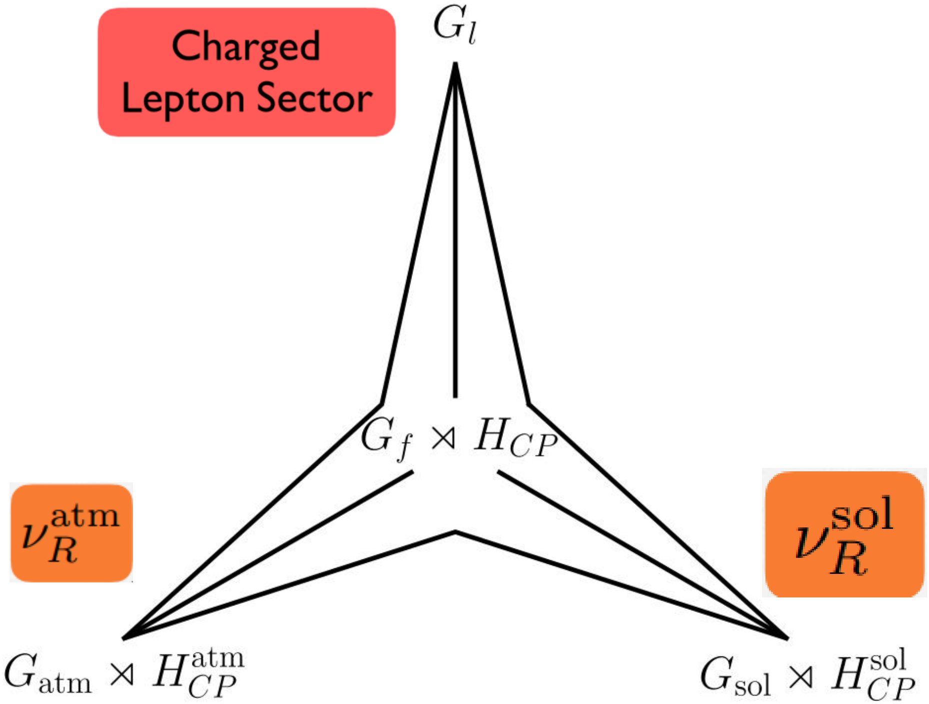
Flavour symmetries + gCP

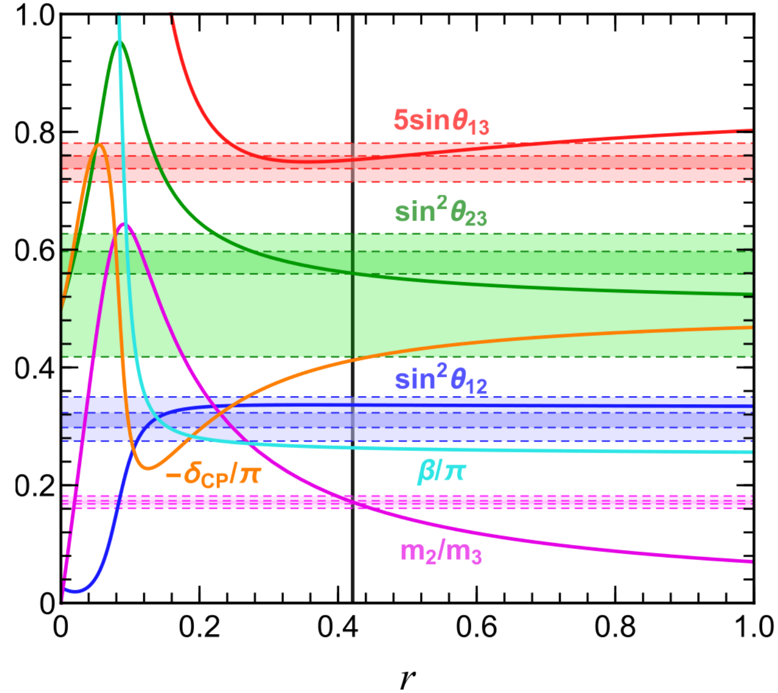


constrain mixing, Dirac and Majorana phases

Feruglio, Hagedorn, Ziegler (2012),
Holthausen, Lindner, Schmidth (2013),
Ding, King, Luhn, Stuart (2013)
Chen, Fallbacher, Mahanthappa, Ratz, Trautner (2014), ...

Tri-Direct CP approach

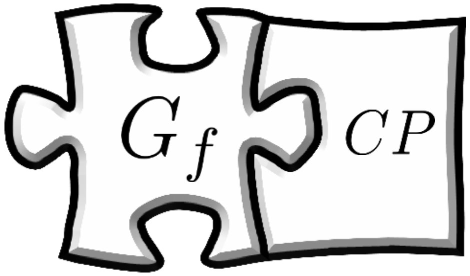




G.J.Ding, S.F.King and C.C.Li, JHEP 12 (2018) 003

Constrain mixing angles, Dirac phase, Majorana phase and neutrino masses

Problems with the usual approach



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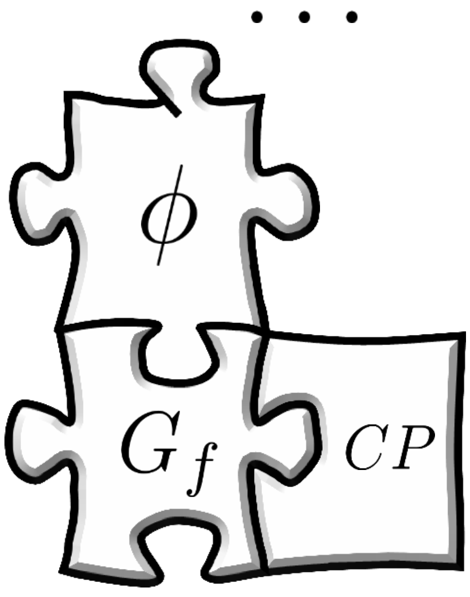

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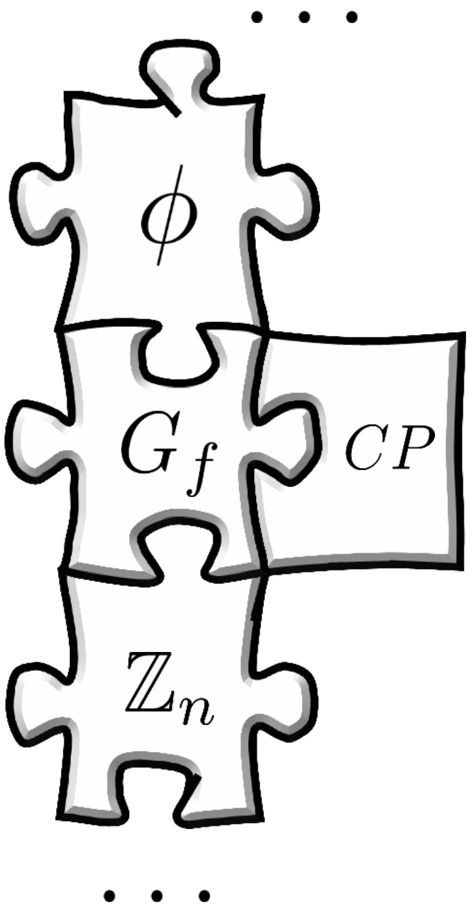

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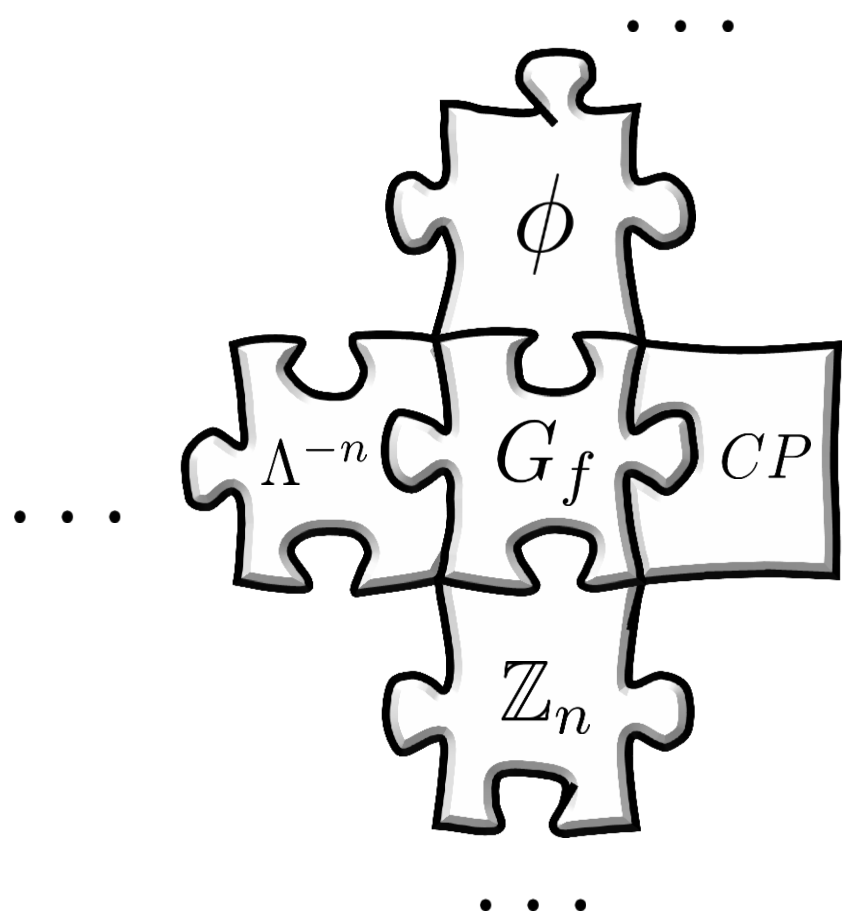
J.T. Penedo FLASY 2019



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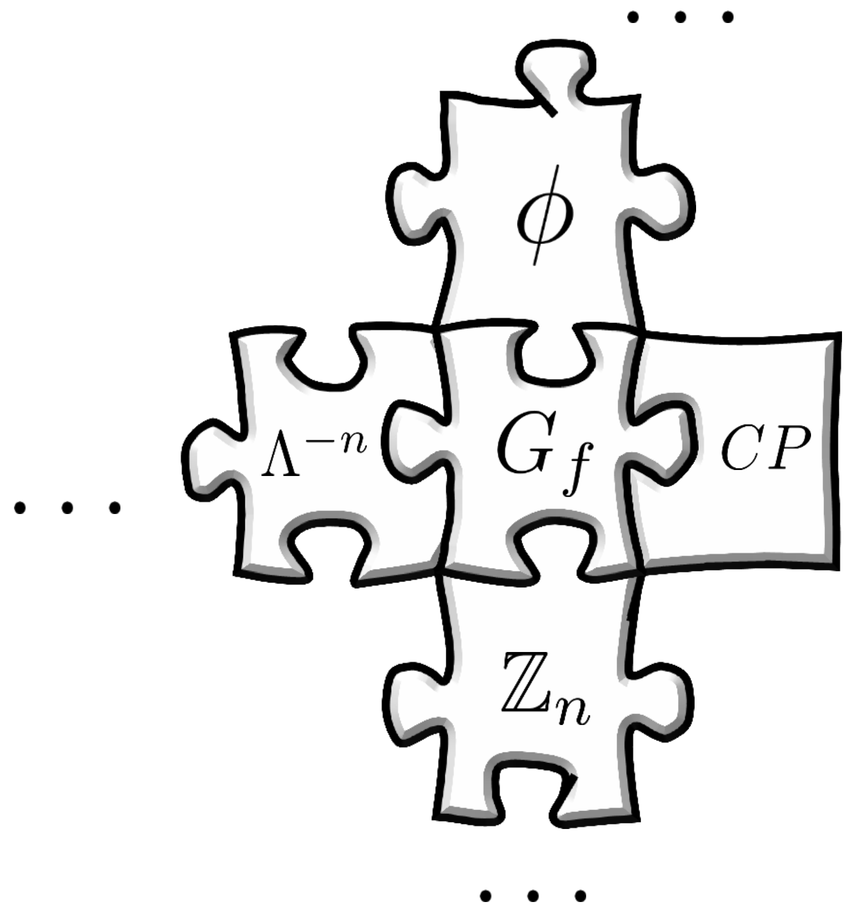
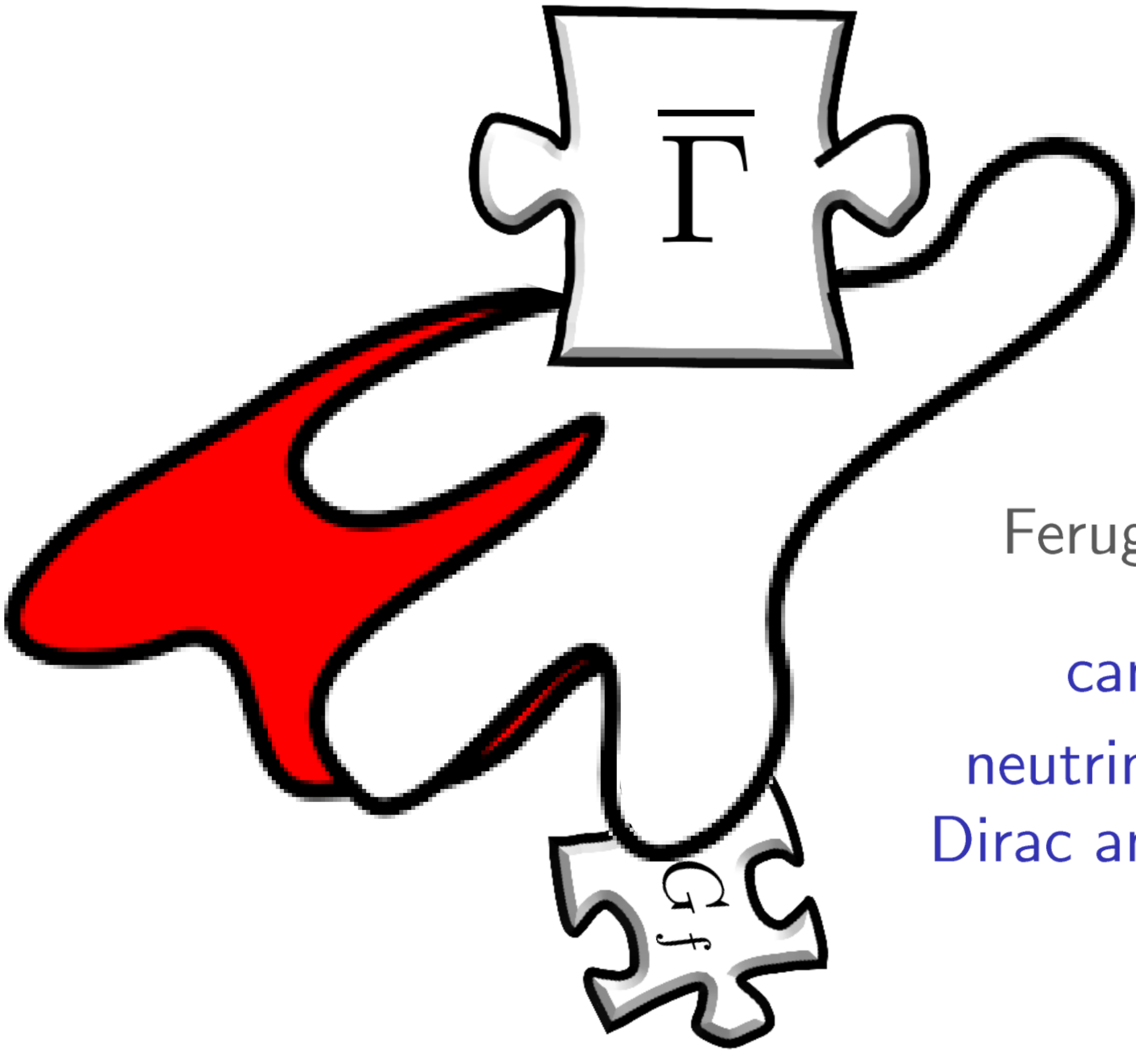


What is the origin of finite groups ?



J.T. Penedo
FLASY 2019

Feruglio, 1706.08749

can constrain all:

neutrino masses, mixing,
Dirac and Majorana phases

2. Modular symmetry

The homogeneous modular group $\Gamma \cong SL(2,\mathbb{Z})$

$$SL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, ad - bc = 1, \quad a,b,c,d \in \mathbb{Z} \right\}.$$

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Generators S and T

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \boxed{S^2 = -1, \ (ST)^3 = 1, \ S^2 T = TS^2.}$$

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The modular group $\bar{\Gamma} \cong PSL(2,\mathbb{Z}) \cong SL(2,\mathbb{Z}) / \{I, -I\}$

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$$\tau \mapsto \gamma\tau \equiv \frac{a\tau + b}{c\tau + d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \overline{\Gamma}, \qquad \boxed{S : \tau \mapsto -\frac{1}{\tau}, \qquad T : \tau \mapsto \tau + 1}$$

Principal congruence subgroups of $SL(2,\mathbb{Z})$, $N=2,3,4...$

$$\Gamma(N)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL(2,\mathbb{Z}), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N}\right\}$$

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$$\Rightarrow \boxed{T^N \in \Gamma(N)}$$

Projective principal congruence subgroups

$$\overline{\Gamma}(2) \equiv \Gamma(2)/\{I, -I\} \qquad \overline{\Gamma}(N) \equiv \Gamma(N), \quad N > 2$$

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$$S^2 = (ST)^3 = T^N = 1, \quad \text{for} \quad N \leq 5$$

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$\Gamma_2 \cong S_3$, $\Gamma_3 \cong A_4$, $\Gamma_4 \cong S_4$, $\Gamma_5 \cong A_5$

Principal congruence subgroups of $SL(2,\mathbb{Z})$, $N = 2,3,4...$

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$\Gamma_2 \simeq S_3$;

Kobayashi et al., 1803.10391 (+A_4)
Kobayashi et al., 1812.11072 (+A_4)
Kobayashi et al., 1906.10341
Okada, Orikasa, 1907.04716

$\Gamma_3 \simeq A_4$;

Feruglio, 1706.08749
Feruglio, Criado, 1807.01125
Kobayashi et al., 1808.03012
Okada, Tanimoto, 1812.09677
Novichkov et al., 1812.11289
Nomura, Okada, 1904.03937
Okada, Tanimoto, 1905.13421
Nomura, Okada, 1906.03927
Nomura, Okada 1908.07457
Ding, King 1910.03460
Wang 1912.13284
Ding, Feruglio 2003.13448
Okada, Tanimoto 2005.00775
Asaka, Heo 2009.12120
Yao, Lu, 2012.13390
Okada, Shimizu 2105.14292
Novichkov, Penedo 2102.07488
Feruglio, Gherardi 2101.08718

$\Gamma_4 \simeq S_4$;

JP, Petcov, 1806.11040
Novichkov et al., 1811.04933
Kobayashi et al., 1907.09141
King, Zhou 1908.02770
Wang, Zhou 1910.09473
Wang 2007.05913
Ding, King 2103.16311
Qu, Liu 2106.11659

$\Gamma_5 \simeq A_5$

Novichkov et al., 1812.02158
Ding et al., 1903.12588
Wang, Yu 2010.10159
Yao, Liu 2011.03501
Wang, Zhou 2102.04358

Principal congruence subgroups of $SL(2,\mathbb{Z})$, $N=2,3,4\ldots$

$$\Gamma(N)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL(2,\mathbb{Z}),\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\ (\mathrm{mod}\ N)\right\}$$

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Projective principal congruence subgroups

$$\overline{\Gamma}(2)\equiv\Gamma(2)/\{I,-I\}\qquad \overline{\Gamma}(N)\equiv\Gamma(N),\quad N>2$$

Finite modular groups: $\boxed{\Gamma_N\equiv\overline{\Gamma}/\overline{\Gamma}(N)}$

$$S^2=(ST)^3=T^N=1,\ \ \text{for}\ \ N\le 5$$

$$\Gamma_2\cong S_3,\qquad \Gamma_3\cong A_4,\qquad \Gamma_4\cong S_4,\qquad \Gamma_5\cong A_5$$

Homogeneous FMGs: $\boxed{\Gamma'_N\equiv\Gamma/\Gamma(N)\cong SL(2,\mathbb{Z}/N\mathbb{Z})}$

$$S^4=(ST)^3=T^N=1,\ S^2T=TS^2\ \ \text{for}\ \ N\le 5$$

X.G.Liu and G.J.Ding, JHEP 08 (2019) 134

Principal congruence subgroups of $SL(2,\mathbb{Z})$, $N=2,3,4\ldots$

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X.G.Liu and G.J.Ding, JHEP 08 (2019) 134

$$\Gamma'_2 \cong S_3,\qquad \Gamma'_3 \cong T',\qquad \Gamma'_4 \cong S'_4,\qquad \Gamma'_5 \cong A'_5$$

Principal congruence subgroups of $SL(2,\mathbb{Z})$, $N=2,3,4\ldots$

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X.G.Liu and G.J.Ding, JHEP (2019)
Jun-Nan Lu, Xiang-Gan Liu, Gui-Jun Ding, Phys.Rev.D (2020)

X.G.Liu, C.Y. Yao and G.J.Ding, Phys.Rev.D (2021)
B.Y. Qu, X.G.Liu, P.T. Chen and G.J.Ding, Phys.Rev.D (2021) 7
P. P. Novichkov, J. T. Penedo, and S. T. Petcov, Nucl.Phys.B (2021)

X. Wang, B. Yu, and S. Zhou, Phys.Rev.D (2021)
C.Y. Yao , X.G.Liu and G.J.Ding, Phys.Rev.D (2021)

Multiplication rules of $\Gamma_7 \simeq \Sigma(168)$

G.J.Ding, S.F.King, C.C.Li and Y.L.Zhou, JHEP 08 (2020) 164

$$S^2 = (ST)^3 = T^7 = (ST^3)^4 = 1$$

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G.J.Ding, S.F.King, C.C.Li and Y.L.Zhou, JHEP 08 (2020) 164

$$S^2 = (ST)^3 = T^7 = (ST^3)^4 = 1$$

	Conjugacy Classes					
	$1C_1$	$21C_2$	$56C_3$	$42C_4$	$24C_7$	$24C'_7$
1	1	1	1	1	1	1
3	3	-1	0	1	b_7	$\bar{b}_7$
$\bar{\mathbf{3}}$	3	-1	0	1	$\bar{b}_7$	b_7
6	6	2	0	0	-1	-1
7	7	-1	1	-1	0	0
8	8	0	-1	0	1	1

The character table of the Γ_7 group with $b_7 = (-1 + i\sqrt{7})/2$ and $\bar{b}_7 = b_7^* = -(1 + i\sqrt{7})/2$.

3. Modular forms and gCP

Holomorphic functions transforming under

$$f\left(\gamma\tau\right)=(c\tau+d)^{k}\,f(\tau),\qquad \gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma(N)$$

3. Modular forms and gCP

Holomorphic functions transforming under

X.G.Liu and G.J.Ding, JHEP 08 (2019) 134

weight (non-negative, integer)

$$f\left(\gamma\tau\right)=(c\tau+d)^{k} f(\tau), \quad \gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N)$$

level (natural)

3. Modular forms and gCP

C.Y Yao, X.G.Liu and G.J.Ding, Phys.Rev.D 103 (2021) 9, 095013

Holomorphic functions transforming under

weight (non-negative, fractional)

$$f\left(\gamma\tau\right)=(c\tau+d)^{k}f(\tau),\qquad \gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma(N)$$

level (natural)

3. Modular forms and gCP

Holomorphic functions transforming under

weight (non-negative, integer)

$$f(\gamma\tau) = (c\tau + d)^k f(\tau), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N)$$

level (natural)

Dimension of modular forms of
level N and weight k

$$\mathrm{dim}\mathcal{M}_k(\Gamma(N)) = \frac{(k-1)N+6}{24} N^2 \prod_{p|N}(1-\frac{1}{p^2}), \quad N > 2, k \geq 1$$

F. Diamond and J.M. Shurman, Springer (2005)

3. Modular forms and gCP

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$N\backslash k$	k	0	1	2	3	4
2	$k/2+1$	1	-	2	-	3
3	$k+1$	1	2	3	4	5
4	$2k+1$	1	3	5	7	9
5	$5k+1$	1	6	11	16	21
6	6k	1	6	12	18	24
7	$14k-2$	1	12	26	40	54

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$$\dim \mathcal{M}_k(\Gamma(N)) = \frac{(k-1)N+6}{24} N^2 \prod_{p|N} (1 - \frac{1}{p^2}),$$

There exists a basis in this space

$$f_i(\gamma\tau) = (c\tau + d)^k (\rho_{\mathbf{r}}(\gamma))_{ij} f_j(\tau), \qquad \gamma \in \Gamma,$$

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representation matrix of Γ_N

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2	$k/2+1$	1	-	2	-	3
3	$k+1$	1	2	3	4	5
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An example: Modular forms of level 7

G.J.Ding, S.F.King, C.C.Li and Y.L.Zhou, JHEP 08 (2020) 164

Jacobi theta functions

$$\theta_3(u,\tau)=\prod_{n=1}^{\infty}(1-q^n)(1+q^{n-1/2}e^{2\pi iu})(1+q^{n-1/2}e^{-2\pi iu}),$$

$q=e^{2\pi i\tau}$

"seed" functions

$$\alpha_{1,-1}(\tau)=\theta_3\left(\frac{\tau+1}{2},7\tau\right),\ \alpha_{2,-1}(\tau)=e^{\frac{2\pi i\tau}{7}}\theta_3\left(\frac{3\tau+1}{2},7\tau\right),\ \alpha_{3,-1}(\tau)=e^{\frac{6\pi i\tau}{7}}\theta_3\left(\frac{5\tau+1}{2},7\tau\right),$$

$$\alpha_{1,0}(\tau)=\theta_3\left(\frac{\tau+1}{14},\frac{\tau}{7}\right),\quad \alpha_{2,0}(\tau)=\theta_3\left(\frac{\tau+3}{14},\frac{\tau}{7}\right),\quad \alpha_{3,0}(\tau)=\theta_3\left(\frac{\tau+5}{14},\frac{\tau}{7}\right),$$

$$\alpha_{1,1}(\tau)=\theta_3\left(\frac{\tau+2}{14},\frac{\tau+1}{7}\right),\quad \alpha_{2,1}(\tau)=\theta_3\left(\frac{\tau+4}{14},\frac{\tau+1}{7}\right),\quad \alpha_{3,1}(\tau)=\theta_3\left(\frac{\tau+6}{14},\frac{\tau+1}{7}\right),$$

$$\alpha_{1,2}(\tau)=\theta_3\left(\frac{\tau+3}{14},\frac{\tau+2}{7}\right),\quad \alpha_{2,2}(\tau)=\theta_3\left(\frac{\tau+5}{14},\frac{\tau+2}{7}\right),\quad \alpha_{3,2}(\tau)=\theta_3\left(\frac{\tau+7}{14},\frac{\tau+2}{7}\right),$$

$$\alpha_{1,3}(\tau)=\theta_3\left(\frac{\tau+4}{14},\frac{\tau+3}{7}\right),\quad \alpha_{2,3}(\tau)=\theta_3\left(\frac{\tau+6}{14},\frac{\tau+3}{7}\right),\quad \alpha_{3,3}(\tau)=\theta_3\left(\frac{\tau+8}{14},\frac{\tau+3}{7}\right),$$

$$\alpha_{1,4}(\tau)=\theta_3\left(\frac{\tau+5}{14},\frac{\tau+4}{7}\right),\quad \alpha_{2,4}(\tau)=\theta_3\left(\frac{\tau+7}{14},\frac{\tau+4}{7}\right),\quad \alpha_{3,4}(\tau)=\theta_3\left(\frac{\tau+9}{14},\frac{\tau+4}{7}\right),$$

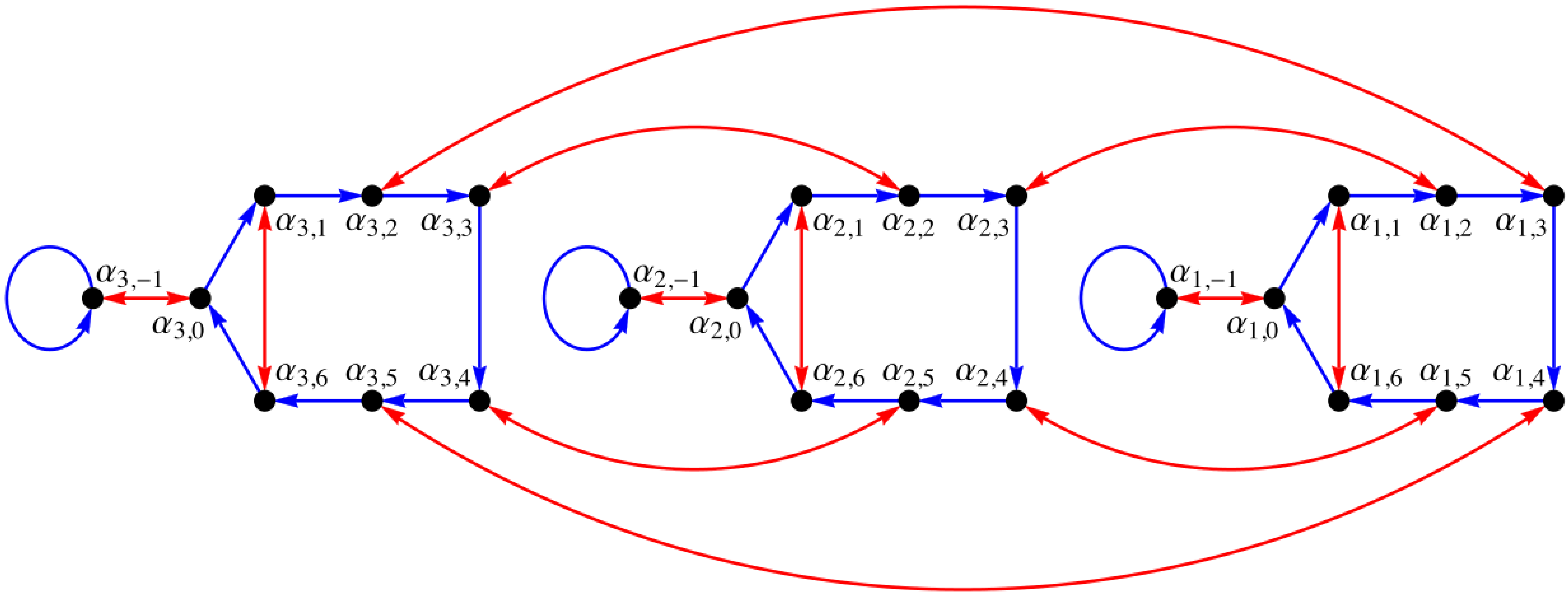
$$\alpha_{1,5}(\tau)=\theta_3\left(\frac{\tau+6}{14},\frac{\tau+5}{7}\right),\quad \alpha_{2,5}(\tau)=\theta_3\left(\frac{\tau+8}{14},\frac{\tau+5}{7}\right),\quad \alpha_{3,5}(\tau)=\theta_3\left(\frac{\tau+10}{14},\frac{\tau+5}{7}\right),$$

$$\alpha_{1,6}(\tau)=\theta_3\left(\frac{\tau+7}{14},\frac{\tau+6}{7}\right),\quad \alpha_{2,6}(\tau)=\theta_3\left(\frac{\tau+9}{14},\frac{\tau+6}{7}\right),\quad \alpha_{3,6}(\tau)=\theta_3\left(\frac{\tau+11}{14},\frac{\tau+6}{7}\right).$$

An example: Modular forms of level 7

G.J.Ding, S.F.King, C.C.Li and Y.L.Zhou, JHEP 08 (2020) 164

The transformations of the set of seed functions



The modular functions

$$Y(x_{1,-1},\ldots,x_{1,6};x_{2,-1},\ldots,x_{2,6};x_{3,-1},\ldots,x_{3,6}\,|\,\tau)\equiv\sum_{i,j}x_{i,j}\frac{d}{d\tau}\log\alpha_{i,j}(\tau),\quad \text{with}\sum_{i,j}x_{i,j}=0$$

An example: Modular forms of level 7

G.J.Ding, S.F.King, C.C.Li and Y.L.Zhou, JHEP 08 (2020) 164

Modular forms of weight 2 and level 7

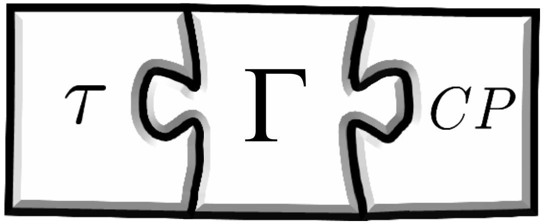
$$Y_{\mathbf{3}}^{(2)}(\tau)=\begin{pmatrix} q^{1/7}(1-3q+4q^3+2q^4)+\dots \\ q^{2/7}(1-3q-q^2+8q^3)+\dots \\ -q^{4/7}(1-4q+3q^2+5q^3-5q^4)+\dots \end{pmatrix},$$

$$Y_{\mathbf{7}}^{(2)}(\tau)=\begin{pmatrix} 1+4q+12q^2+16q^3+28q^4+24q^5+\dots \\ -\sqrt{2}q^{1/7}(1+15q+24q^2+36q^3+30q^4)+\dots \\ -\sqrt{2}q^{2/7}(3+13q+31q^2+24q^3+72q^4)+\dots \\ -2\sqrt{2}q^{3/7}(2+9q+9q^2+30q^3+16q^4)+\dots \\ -\sqrt{2}q^{4/7}(7+12q+39q^2+31q^3+63q^4)+\dots \\ -2\sqrt{2}q^{5/7}(3+14q+10q^2+21q^3+24q^4)+\dots \\ -2\sqrt{2}q^{6/7}(6+7q+21q^2+20q^3+27q^4)+\dots \end{pmatrix},$$

$$Y_{\mathbf{8}a}^{(2)}(\tau)=\begin{pmatrix} 2+2q-24q^2+56q^3-q^4+\dots \\ -\sqrt{3}q(22+30q+56q^2+59q^3+84q^4)+\dots \\ -\sqrt{2}q^{1/7}(4+30q+42q^2+93q^3+120q^4)+\dots \\ \sqrt{2}q^{2/7}(3-16q+11q^2-18q^3+21q^4)+\dots \\ 7\sqrt{2}q^{3/7}(2+3q+12q^2+12q^3+22q^4)+\dots \\ -\sqrt{2}q^{4/7}(11+54q+54q^2+104q^3+99q^4)+\dots \\ -7\sqrt{2}q^{12/7}(-1-2q+6q^2-12q^3+6q^4)+\dots \\ 7\sqrt{2}q^{6/7}(3+8q+9q^2+10q^3+21q^4)+\dots \end{pmatrix},$$

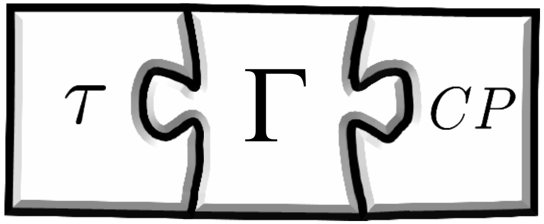
$$Y_{\mathbf{8}b}^{(2)}(\tau)=\begin{pmatrix} 3q(20+40q+42q^2+53q^3+112q^4)+\dots \\ \sqrt{3}(-2+12q-18q^2+42q^3-27q^4)+\dots \\ 3\sqrt{2}q^{1/7}(2+8q+28q^2+15q^3+60q^4)+\dots \\ -3\sqrt{2}q^{2/7}(5+20q+37q^2+68q^3+63q^4)+\dots \\ 21\sqrt{2}q^{10/7}(3+2q+2q^2+4q^3)+\dots \\ -3\sqrt{2}q^{4/7}(5+8q+22q^2+32q^3+45q^4)+\dots \\ 21\sqrt{2}q^{5/7}(2+3q+8q^2+10q^3+10q^4)+\dots \\ -21\sqrt{2}q^{6/7}(1+4q+q^2+8q^3+3q^4)+\dots \end{pmatrix}.$$

Modular symmetry + gCP



$$\tau \xrightarrow{\text{CP}} ? \qquad \psi \xrightarrow{\text{CP}} ? \qquad Y(\tau) \xrightarrow{\text{CP}} ?$$

Modular symmetry + gCP



$\tau \xrightarrow{\text{CP}} ?$

$\psi(x) \xrightarrow{\text{CP}} X_{\mathbf{r}}^{\text{CP}}\,\overline{\psi}(x_P)$

$Y(\tau) \xrightarrow{\text{CP}} ?$

Modular symmetry + gCP: the modulus

P. P. Novichkov, J. T. Penedo, S. T. Petcov and A. V. Titov, JHEP 07 (2019) 165

$$\underbrace{CP \to \gamma \in \Gamma \to CP^{-1}}_{\gamma' \in \Gamma}$$

$$\Rightarrow \quad \boxed{\tau_{\mathrm{CP}} = -\tau^*}$$

$$\left(\tau_{\mathrm{CP}^2} = \tau\right)$$

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$$\underbrace{CP \to \gamma \in \Gamma \to CP^{-1}}_{\gamma' \in \Gamma}$$

$$\tau_{\mathrm{CP}} = -\tau^*$$

$$\left(\tau_{\mathrm{CP}^2} = \tau\right)$$

Extended modular group

$$\tau \xrightarrow{\mathrm{CP}} -\tau^* \xrightarrow{\gamma} -\frac{a\tau^* + b}{c\tau^* + d} \xrightarrow{\mathrm{CP}^{-1}} \frac{a\tau - b}{-c\tau + d}$$

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$$u(\gamma) \equiv {\rm CP}\,\gamma\,{\rm CP}^{-1} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

$${\rm CP} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Modular symmetry + gCP: the modulus

P. P. Novichkov, J. T. Penedo, S. T. Petcov and A. V. Titov, JHEP 07 (2019) 165

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$$\left(\tau_{\mathrm{CP}^2} = \tau\right)$$

Extended modular group

$$\Gamma^* = \left\{ \tau \xrightarrow{S} -1/\tau,\ \tau \xrightarrow{T} \tau+1,\ \tau \xrightarrow{\mathcal{CP}} -\tau^* \right\} \simeq GL(2,\mathbb{Z})$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma^* : \begin{cases} \tau \to \dfrac{a\tau+b}{c\tau+d} & \text{for} \quad ad-bc=1, \\ \tau \to \dfrac{a\tau^*+b}{c\tau^*+d} & \text{for} \quad ad-bc=-1. \end{cases}$$

Modular symmetry + gCP: consistency

$$X_{\mathbf{r}}^{\mathrm{CP}}\,\rho_{\mathbf{r}}^{*}(\gamma)\,\left(X_{\mathbf{r}}^{\mathrm{CP}}\right)^{-1}=\rho_{\mathbf{r}}(u(\gamma))$$

but now there is a unique automorphism

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$$u(\gamma) \equiv \mathrm{CP}\,\gamma\,\mathrm{CP}^{-1} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \quad \Rightarrow \quad X_{\mathbf{r}}^{\mathrm{CP}} \text{ uniquely determined}$$

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P. P. Novichkov, J. T. Penedo, S. T. Petcov and A. V. Titov, JHEP 07 (2019) 165

In a symmetric basis: $\rho_{\mathbf{r}}^{*}(\gamma)=\rho_{\mathbf{r}}(u(\gamma))$

$$X_{\mathbf{r}}^{\mathrm{CP}}=\mathbb{1}_{\mathbf{r}}$$

Modular symmetry + gCP: the modular forms

$$Y(\tau) \xrightarrow{\mathrm{CP}} ?$$

$$Y(\tau) \xrightarrow{\mathrm{CP}} Y(\tau_{\mathrm{CP}}) = Y(-\tau^*)$$

$$Y(\tau) \sim \psi$$

under the modular group

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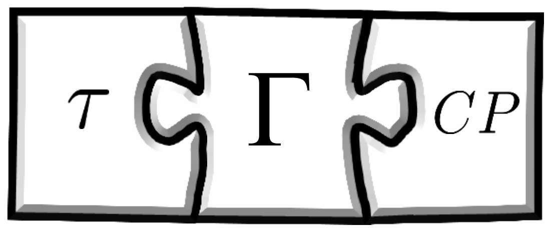
$$Y(-\tau^*) = X_{\mathbf{r}}^{\text{CP}}\, Y^*(\tau)$$

P. P. Novichkov, J. T. Penedo, S. T. Petcov
and A. V. Titov, JHEP 07 (2019) 165

Modular symmetry + gCP



$\tau \xrightarrow{\text{CP}} -\tau^*$ $\quad$ $\psi(x) \xrightarrow{\text{CP}} X_{\mathbf{r}}^{\text{CP}}\,\overline{\psi}(x_{\text{P}})$ $\quad$ $Y(\tau) \xrightarrow{\text{CP}} X_{\mathbf{r}}^{\text{CP}}\,Y^*(\tau)$

P. P. Novichkov, J. T. Penedo, S. T. Petcov and A. V. Titov, JHEP 07 (2019) 165;
G.J. Ding, F. Feruglio and X.G. Liu, SciPost Phys. 10 (2021) 6, 133 ...

4. A new route towards

Chinese remainder theorem gives the isomorphism of rings

$$\mathbb{Z}/N\mathbb{Z} \cong \left(\mathbb{Z}/p_1^{n_1}\mathbb{Z}\right)\times\ldots\times\left(\mathbb{Z}/p_k^{n_k}\mathbb{Z}\right), \qquad N = p_1^{n_1}\ldots p_k^{n_k}$$

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Then this gives isomorphism

$$SL(2,\mathbb{Z}/N\mathbb{Z}) \cong SL(2,\mathbb{Z}/p_1^{n_1}\mathbb{Z})\times\ldots\times SL(2,\mathbb{Z}/p_k^{n_k}\mathbb{Z}).$$

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$$\Gamma(1)/\Gamma(N) \cong (\Gamma(1)/\Gamma(p_1^{n_1}))\times\ldots\times(\Gamma(1)/\Gamma(p_k^{n_k})),$$

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N	1	2	3	4	5	6	7	8	9	10
	1	2	3	2^2	5	$2\cdot 3$	7	2^3	3^2	$2\cdot 5$
N	11	12	13	14	15	16	17	18	19	20
	11	$2^2\cdot 3$	13	$2\cdot 7$	$3\cdot 5$	2^4	17	$2\cdot 3^2$	19	$2^2\cdot 5$

Table 1: The prime factorization of positive integer $0 < N < 21$.

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$$\Gamma'_2 = \Gamma(1)/\Gamma(2) \cong \Gamma(3)/\Gamma(6) \cong \Gamma(5)/\Gamma(10),$$

$$\Gamma'_3 = \Gamma(1)/\Gamma(3) \cong \Gamma(2)/\Gamma(6) \cong \Gamma(4)/\Gamma(12),$$

$$\Gamma'_4 = \Gamma(1)/\Gamma(4) \cong \Gamma(3)/\Gamma(12) \cong \Gamma(5)/\Gamma(20),$$

$$\Gamma'_5 = \Gamma(1)/\Gamma(5) \cong \Gamma(2)/\Gamma(10) \cong \Gamma(3)/\Gamma(15).$$

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$$\Gamma'_N = \Gamma(1)/\Gamma(N) \quad \Rightarrow \quad \Gamma'_N = \Gamma(N')/\Gamma(N'')$$

Weight 1 modular forms of level 6

$N\backslash k$	k	0	1	2	3	4
6	6k	1	6	12	18	24

The Dedekind eta function

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}\left(1-q^n\right), \qquad q = e^{2\pi i\tau}.$$

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The eta function transforms

$$\eta(\tau)\xrightarrow{S}\eta(-1/\tau)=\sqrt{-i\tau}\,\eta(\tau),\qquad \eta(\tau)\xrightarrow{T}\eta(\tau+1)=e^{i\pi/12}\eta(\tau).$$

Weight 1 modular forms of level 6

$N\backslash k$	k	0	1	2	3	4
6	6k	1	6	12	18	24

The Dedekind eta function

$$\eta(\tau)=q^{1/24}\prod_{n=1}^{\infty}\left(1-q^{n}\right),\qquad q=e^{2\pi i\tau}.$$

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The modular forms space of level N=3 and weight k=1

$$\mathcal{M}_1(\Gamma(3))=\left\{\frac{\eta^3(3\tau)}{\eta(\tau)},\quad \frac{\eta^3(\tau/3)}{\eta(\tau)}\right\},$$

X.G.Liu and G.J.Ding, JHEP 08 (2019) 134

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If $f(\tau)$ is the modular form of $\Gamma(3)$

$$f(\tau)\in\mathcal{M}_1(\Gamma(3)), \quad \Rightarrow \quad f(2\tau)\in\mathcal{M}_1(\Gamma(6))$$

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Four weight 1 modular forms of $\Gamma(6)$ are obtained

$$\frac{\eta^3(3\tau)}{\eta(\tau)}, \frac{\eta^3(\tau/3)}{\eta(\tau)}, \frac{\eta^3(6\tau)}{\eta(2\tau)}, \frac{\eta^3(2\tau/3)}{\eta(2\tau)}.$$

Weight 1 modular forms of level 6

$N\backslash k$	k	0	1	2	3	4
6	6k	1	6	12	18	24

The modular form space $M_{\mathbf{k}}(\Gamma(6))$ must be closed up

$$\frac{\eta^3(6\tau)}{\eta(2\tau)} \xrightarrow{S} \frac{\eta^3(-6/\tau)}{\eta(-2/\tau)} = -\tau \frac{i}{6\sqrt{3}} \frac{\eta^3(\tau/6)}{\eta(\tau/2)},$$

$$\frac{\eta^3(2\tau/3)}{\eta(2\tau)} \xrightarrow{S} \frac{\eta^3(-2/(3\tau))}{\eta(-2/\tau)} = -\tau \frac{3\sqrt{3}i}{2} \frac{\eta^3(3\tau/2)}{\eta(\tau/2)}.$$

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The six independent basis vectors of the linear space $M_1(\Gamma(6))$

$$e_1(\tau) = \frac{\eta^3(3\tau)}{\eta(\tau)}, \quad e_2(\tau) = \frac{\eta^3(\tau/3)}{\eta(\tau)}, \quad e_3(\tau) = \frac{\eta^3(6\tau)}{\eta(2\tau)},$$

$$e_4(\tau) = \frac{\eta^3(\tau/6)}{\eta(\tau/2)}, \quad e_5(\tau) = \frac{\eta^3(2\tau/3)}{\eta(2\tau)}, \quad e_6(\tau) = \frac{\eta^3(3\tau/2)}{\eta(\tau/2)}.$$

Modular forms of level 6 arrange into: N"=6 and N'=1

Multiplication rules of $\Gamma'_6 = \Gamma(1)/\Gamma(6) \cong \Gamma(1)/\Gamma(2) \times \Gamma(1)/\Gamma(3) \cong S_3 \times T'$

$$S^4 = T^6 = (ST)^3 = ST^2ST^3ST^4ST^3 = 1, \qquad S^2T = TS^2.$$

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$$S^4 = T^6 = (ST)^3 = ST^2ST^3ST^4ST^3 = 1, \qquad S^2T = TS^2.$$

or

$$\begin{aligned} &\tilde{a}^4 = \tilde{b}^3 = (\tilde{a}\tilde{b})^3 = 1, \quad \tilde{a}^2\tilde{b} = \tilde{b}\tilde{a}^2, \\ &\tilde{c}^2 = \tilde{d}^2 = (\tilde{c}\tilde{d})^3 = 1, \\ &\tilde{a}\tilde{c} = \tilde{c}\tilde{a}, \quad \tilde{a}\tilde{d} = \tilde{d}\tilde{a}, \quad \tilde{b}\tilde{c} = \tilde{c}\tilde{b}, \quad \tilde{b}\tilde{d} = \tilde{d}\tilde{b}. \end{aligned}$$

Modular forms of level 6 arrange into: N"=6 and N'=1

Multiplication rules of $\Gamma_6' = \Gamma(1)/\Gamma(6) \cong \Gamma(1)/\Gamma(2)\times\Gamma(1)/\Gamma(3) \cong S_3 \times T'$

$$S^4 = T^6 = (ST)^3 = ST^2ST^3ST^4ST^3 = 1, \qquad S^2T = TS^2.$$

or

$$\tilde{a}^4 = \tilde{b}^3 = (\tilde{a}\tilde{b})^3 = 1, \quad \tilde{a}^2\tilde{b} = \tilde{b}\tilde{a}^2,$$

$$\tilde{c}^2 = \tilde{d}^2 = (\tilde{c}\tilde{d})^3 = 1,$$

$$\tilde{a}\tilde{c} = \tilde{c}\tilde{a}, \quad \tilde{a}\tilde{d} = \tilde{d}\tilde{a}, \quad \tilde{b}\tilde{c} = \tilde{c}\tilde{b}, \quad \tilde{b}\tilde{d} = \tilde{d}\tilde{b}.$$

The two bases are fulfilled

$$\tilde{a} = TST^4S^3T, \qquad \tilde{b} = T^4, \qquad \tilde{c} = S^3T^3S, \qquad \tilde{d} = T^3,$$

Modular forms of level 6 arrange into: N"=6 and N'=1

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$$\tilde{a}^4 = \tilde{b}^3 = (\tilde{a}\tilde{b})^3 = 1, \quad \tilde{a}^2\tilde{b} = \tilde{b}\tilde{a}^2,$$
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$$\tilde{a}\tilde{c} = \tilde{c}\tilde{a}, \quad \tilde{a}\tilde{d} = \tilde{d}\tilde{a}, \quad \tilde{b}\tilde{c} = \tilde{c}\tilde{b}, \quad \tilde{b}\tilde{d} = \tilde{d}\tilde{b}.$$

The two bases are fulfilled

$$\tilde{a} = TST^4S^3T, \qquad \tilde{b} = T^4, \qquad \tilde{c} = S^3T^3S, \qquad \tilde{d} = T^3,$$

$$S = \tilde{a}\tilde{c}\tilde{d}\tilde{c}, \qquad T = \tilde{b}\tilde{d}.$$

T'			S_3		
	$\tilde{a}$	$\tilde{b}$		$\tilde{c}$	$\tilde{d}$
$\mathbf{1}_k^0$	1	ω^k	$\mathbf{1}_0^r$	$(-1)^r$	$(-1)^r$
$\mathbf{2}_k^0$	$\boldsymbol{a_2}$	$\omega^{k+1}\boldsymbol{b_2}$	$\mathbf{2}_0$	$-\frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$\mathbf{3}^0$	$\boldsymbol{a_3}$	$\boldsymbol{b_3}$	–	–	–

$S_3 \times T'$				
	$\tilde{a}$	$\tilde{b}$	$\tilde{c}$	$\tilde{d}$
$\mathbf{1}_k^r = \mathbf{1}_0^r \times \mathbf{1}_k^0$	1	ω^k	$(-1)^r$	$(-1)^r$
$\mathbf{2}_k = \mathbf{2}_0 \times \mathbf{1}_k^0$	$\mathbb{1}_2$	$\omega^k \mathbb{1}_2$	$-\frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$\mathbf{2}_k^r = \mathbf{1}_0^r \times \mathbf{2}_k^0$	$\boldsymbol{a_2}$	$\omega^{k+1}\boldsymbol{b_2}$	$(-1)^r \mathbb{1}_2$	$(-1)^r \mathbb{1}_2$
$\mathbf{3}^r = \mathbf{1}_0^r \times \mathbf{3}^0$	$\boldsymbol{a_3}$	$\boldsymbol{b_3}$	$(-1)^r \mathbb{1}_3$	$(-1)^r \mathbb{1}_3$
$\mathbf{4}_k = \mathbf{2}_0 \times \mathbf{2}_k^0$	$\begin{pmatrix} \boldsymbol{a_2} & \mathbb{0}_2 \\ \mathbb{0}_2 & \boldsymbol{a_2} \end{pmatrix}$	$\omega^{k+1}\begin{pmatrix} \boldsymbol{b_2} & \mathbb{0}_2 \\ \mathbb{0}_2 & \boldsymbol{b_2} \end{pmatrix}$	$-\frac{1}{2}\begin{pmatrix} \mathbb{1}_2 & \sqrt{3}\mathbb{1}_2 \\ \sqrt{3}\mathbb{1}_2 & -\mathbb{1}_2 \end{pmatrix}$	$\begin{pmatrix} \mathbb{1}_2 & \mathbb{0}_2 \\ \mathbb{0}_2 & -\mathbb{1}_2 \end{pmatrix}$
$\mathbf{6} = \mathbf{2}_0 \times \mathbf{3}^0$	$\begin{pmatrix} \boldsymbol{a_3} & \mathbb{0}_3 \\ \mathbb{0}_3 & \boldsymbol{a_3} \end{pmatrix}$	$\begin{pmatrix} \boldsymbol{b_3} & \mathbb{0}_3 \\ \mathbb{0}_3 & \boldsymbol{b_3} \end{pmatrix}$	$-\frac{1}{2}\begin{pmatrix} \mathbb{1}_3 & \sqrt{3}\mathbb{1}_3 \\ \sqrt{3}\mathbb{1}_3 & -\mathbb{1}_3 \end{pmatrix}$	$\begin{pmatrix} \mathbb{1}_3 & \mathbb{0}_3 \\ \mathbb{0}_3 & -\mathbb{1}_3 \end{pmatrix}$

T'			S_3		
	$\tilde{a}$	$\tilde{b}$		$\tilde{c}$	$\tilde{d}$
$\mathbf{1}_k^0$	1	ω^k	$\mathbf{1}_0^r$	$(-1)^r$	$(-1)^r$
$\mathbf{2}_k^0$	$\boldsymbol{a_2}$	$\omega^{k+1}\boldsymbol{b_2}$	$\mathbf{2}_0$	$-\frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$\mathbf{3}^0$	$\boldsymbol{a_3}$	$\boldsymbol{b_3}$	–	–	–

$S_3 \times T'$				
	$\tilde{a}$	$\tilde{b}$	$\tilde{c}$	$\tilde{d}$
$\mathbf{1}_k^r = \mathbf{1}_0^r \times \mathbf{1}_k^0$	1	ω^k	$(-1)^r$	$(-1)^r$
$\mathbf{2}_k = \mathbf{2}_0 \times \mathbf{1}_k^0$	$\mathbb{1}_2$	$\omega^k\mathbb{1}_2$	$-\frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$\mathbf{2}_k^r = \mathbf{1}_0^r \times \mathbf{2}_k^0$	$\boldsymbol{a_2}$	$\omega^{k+1}\boldsymbol{b_2}$	$(-1)^r\mathbb{1}_2$	$(-1)^r\mathbb{1}_2$
$\mathbf{3}^r = \mathbf{1}_0^r \times \mathbf{3}^0$	$\boldsymbol{a_3}$	$\boldsymbol{b_3}$	$(-1)^r\mathbb{1}_3$	$(-1)^r\mathbb{1}_3$
$\mathbf{4}_k = \mathbf{2}_0 \times \mathbf{2}_k^0$	$\begin{pmatrix} \boldsymbol{a_2} & \mathbb{0}_2 \\ \mathbb{0}_2 & \boldsymbol{a_2} \end{pmatrix}$	$\omega^{k+1}\begin{pmatrix} \boldsymbol{b_2} & \mathbb{0}_2 \\ \mathbb{0}_2 & \boldsymbol{b_2} \end{pmatrix}$	$-\frac{1}{2}\begin{pmatrix} \mathbb{1}_2 & \sqrt{3}\mathbb{1}_2 \\ \sqrt{3}\mathbb{1}_2 & -\mathbb{1}_2 \end{pmatrix}$	$\begin{pmatrix} \mathbb{1}_2 & \mathbb{0}_2 \\ \mathbb{0}_2 & -\mathbb{1}_2 \end{pmatrix}$
$\mathbf{6} = \mathbf{2}_0 \times \mathbf{3}^0$	$\begin{pmatrix} \boldsymbol{a_3} & \mathbb{0}_3 \\ \mathbb{0}_3 & \boldsymbol{a_3} \end{pmatrix}$	$\begin{pmatrix} \boldsymbol{b_3} & \mathbb{0}_3 \\ \mathbb{0}_3 & \boldsymbol{b_3} \end{pmatrix}$	$-\frac{1}{2}\begin{pmatrix} \mathbb{1}_3 & \sqrt{3}\mathbb{1}_3 \\ \sqrt{3}\mathbb{1}_3 & -\mathbb{1}_3 \end{pmatrix}$	$\begin{pmatrix} \mathbb{1}_3 & \mathbb{0}_3 \\ \mathbb{0}_3 & -\mathbb{1}_3 \end{pmatrix}$

$$a_2 = \frac{i}{\sqrt{3}}\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}, \quad a_3 = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad b_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}.$$

These six modular forms $e_{\mathrm{i}}(\tau)$ can be arranged into:

$$Y^{(1)}_{\mathbf{2}^0_2}(\tau) \equiv \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 3e_1(\tau) + e_2(\tau) \\ 3\sqrt{2}e_1(\tau) \end{pmatrix},$$

$$Y^{(1)}_{\mathbf{4}_1}(\tau) \equiv \begin{pmatrix} Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \end{pmatrix} = \begin{pmatrix} 3\sqrt{2}e_3(\tau) \\ -3e_3(\tau) - e_5(\tau) \\ \sqrt{6}(e_3(\tau) - e_6(\tau)) \\ -\sqrt{3}e_3(\tau) + \frac{1}{\sqrt{3}}e_4(\tau) - \frac{1}{\sqrt{3}}e_5(\tau) + \sqrt{3}e_6(\tau) \end{pmatrix}.$$

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$$Y^{(1)}_{\mathbf{2}^0_2}(\tau) \equiv \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 3e_1(\tau) + e_2(\tau) \\ 3\sqrt{2}e_1(\tau) \end{pmatrix},$$

$$Y^{(1)}_{\mathbf{4}_1}(\tau) \equiv \begin{pmatrix} Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \end{pmatrix} = \begin{pmatrix} 3\sqrt{2}e_3(\tau) \\ -3e_3(\tau) - e_5(\tau) \\ \sqrt{6}(e_3(\tau) - e_6(\tau)) \\ -\sqrt{3}e_3(\tau) + \frac{1}{\sqrt{3}}e_4(\tau) - \frac{1}{\sqrt{3}}e_5(\tau) + \sqrt{3}e_6(\tau) \end{pmatrix}.$$

The product of two modular forms of level N and weights k, k' is a modular form of level N and weight $k+k'$

	Modular form $Y_{\boldsymbol{r}}^{(k)}$
$k=1$	$Y^{(1)}_{\mathbf{2^0_2}},\ Y^{(1)}_{\mathbf{4_1}}$
$k=2$	$Y^{(2)}_{\mathbf{1^1_2}},\ Y^{(2)}_{\mathbf{2_0}},\ Y^{(2)}_{\mathbf{3^0}},\ Y^{(2)}_{\mathbf{6}}$
$k=3$	$Y^{(3)}_{\mathbf{2^0_1}},\ Y^{(3)}_{\mathbf{2^0_2}},\ Y^{(3)}_{\mathbf{2^1_1}},\ Y^{(3)}_{\mathbf{4_0}},\ Y^{(3)}_{\mathbf{4_1}},\ Y^{(3)}_{\mathbf{4_2}}$
$k=4$	$Y^{(4)}_{\mathbf{1^0_0}},\ Y^{(4)}_{\mathbf{1^0_1}},\ Y^{(4)}_{\mathbf{2_0}},\ Y^{(4)}_{\mathbf{2_2}},\ Y^{(4)}_{\mathbf{3^0}},\ Y^{(4)}_{\mathbf{3^1}},\ Y^{(4)}_{\mathbf{6}i},\ Y^{(4)}_{\mathbf{6}ii}$
$k=5$	$Y^{(5)}_{\mathbf{2^0_0}},\ Y^{(5)}_{\mathbf{2^0_1}},\ Y^{(5)}_{\mathbf{2^0_2}},\ Y^{(5)}_{\mathbf{2^1_0}},\ Y^{(5)}_{\mathbf{2^1_1}},\ Y^{(5)}_{\mathbf{4_0}},\ Y^{(5)}_{\mathbf{4_1}i},\ Y^{(5)}_{\mathbf{4_1}ii},\ Y^{(5)}_{\mathbf{4_2}i},\ Y^{(5)}_{\mathbf{4_2}ii}$
$k=6$	$Y^{(6)}_{\mathbf{1^0_0}},\ Y^{(6)}_{\mathbf{1^1_0}},\ Y^{(6)}_{\mathbf{1^1_2}},\ Y^{(6)}_{\mathbf{2_0}},\ Y^{(6)}_{\mathbf{2_1}},\ Y^{(6)}_{\mathbf{2_2}},\ Y^{(6)}_{\mathbf{3^0}i},\ Y^{(6)}_{\mathbf{3^0}ii},\ Y^{(6)}_{\mathbf{3^1}},\ Y^{(6)}_{\mathbf{6}i},\ Y^{(6)}_{\mathbf{6}ii},\ Y^{(6)}_{\mathbf{6}iii}$

Integral weight modular multiplets of level 6 up to weight 6

Decomposing modular forms of level N=6: N''=6, N'=2 and 3

The $\Gamma(6)$ is a normal subgroup of $\Gamma(2)$

$$\Gamma(2)/\Gamma(6) = \Gamma'_3 \cong T' = \langle \tilde{a}, \tilde{b} \,|\, \tilde{a}^4 = (\tilde{a}\tilde{b})^3 = \tilde{b}^3 = 1, \tilde{a}^2\tilde{b} = \tilde{b}\tilde{a}^2 \rangle$$

Decomposing modular forms of level N=6:N''=6, N'=2 and 3

The $\Gamma(6)$ is a normal subgroup of $\Gamma(2)$

$$\Gamma(2)/\Gamma(6)=\Gamma'_3\cong T'=\langle \tilde{a},\tilde{b}\,|\,\tilde{a}^4=(\tilde{a}\tilde{b})^3=\tilde{b}^3=1,\tilde{a}^2\tilde{b}=\tilde{b}\tilde{a}^2\rangle$$

Arrange into three doublets of T'

$$Y'^{(1)}_{\mathbf{2}_2^0}(\tau)=\begin{pmatrix}Y_1\\Y_2\end{pmatrix},\qquad Y'^{(1)}_{2_1^0 i}(\tau)=\begin{pmatrix}Y_3\\Y_4\end{pmatrix},\qquad Y'^{(1)}_{2_1^0 ii}(\tau)=\begin{pmatrix}Y_5\\Y_6\end{pmatrix},$$

Decomposing modular forms of level N=6:N''=6, N'=2 and 3

The $\Gamma(6)$ is a normal subgroup of $\Gamma(2)$

$$\Gamma(2)/\Gamma(6)=\Gamma'_3\cong T'=\langle\tilde{a},\tilde{b}\,|\,\tilde{a}^4=(\tilde{a}\tilde{b})^3=\tilde{b}^3=1,\tilde{a}^2\tilde{b}=\tilde{b}\tilde{a}^2\rangle$$

Arrange into three doublets of T'

$$Y'^{(1)}_{\mathbf{2}^0_2}(\tau)=\begin{pmatrix}Y_1\\Y_2\end{pmatrix},\qquad Y'^{(1)}_{2^0_1 i}(\tau)=\begin{pmatrix}Y_3\\Y_4\end{pmatrix},\qquad Y'^{(1)}_{2^0_1 ii}(\tau)=\begin{pmatrix}Y_5\\Y_6\end{pmatrix},$$

The $\Gamma(6)$ is a normal subgroup of $\Gamma(3)$

$$\Gamma(3)/\Gamma(6)\cong S_3=\langle\tilde{c},\tilde{d}\,|\,\tilde{c}^2=(\tilde{c}\tilde{d})^3=\tilde{d}^2=1\rangle,$$

Decomposing modular forms of level N=6:N''=6, N'=2 and 3

The $\Gamma(6)$ is a normal subgroup of $\Gamma(2)$

$$\Gamma(2)/\Gamma(6)=\Gamma'_3\cong T'=\langle\tilde{a},\tilde{b}\,|\,\tilde{a}^4=(\tilde{a}\tilde{b})^3=\tilde{b}^3=1,\tilde{a}^2\tilde{b}=\tilde{b}\tilde{a}^2\rangle$$

Arrange into three doublets of T'

$$Y'^{(1)}_{\mathbf{2}^0_2}(\tau)=\begin{pmatrix}Y_1\\Y_2\end{pmatrix},\qquad Y'^{(1)}_{2^0_1 i}(\tau)=\begin{pmatrix}Y_3\\Y_4\end{pmatrix},\qquad Y'^{(1)}_{2^0_1 ii}(\tau)=\begin{pmatrix}Y_5\\Y_6\end{pmatrix},$$

The $\Gamma(6)$ is a normal subgroup of $\Gamma(3)$

$$\Gamma(3)/\Gamma(6)\cong S_3=\langle\tilde{c},\tilde{d}\,|\,\tilde{c}^2=(\tilde{c}\tilde{d})^3=\tilde{d}^2=1\rangle,$$

Arranged into two singlets and two doublets of S_3

$$Y''^{(1)}_{1^0_0 i}(\tau)=Y_1,\qquad Y''^{(1)}_{1^0_0 ii}(\tau)=Y_2,\qquad Y''^{(1)}_{2_0 i}(\tau)=\begin{pmatrix}Y_3\\Y_5\end{pmatrix},\qquad Y''^{(1)}_{2_0 ii}(\tau)=\begin{pmatrix}Y_4\\Y_6\end{pmatrix}.$$

Summary of modular forms of level N = 6 up to weight 6 in the irreducible multiplets of finite group T'

	Modular form $Y_{\boldsymbol{r}}^{(k)}$
$k=1$	$Y'^{(1)}_{\mathbf{2}^0_2}(\tau)$, $Y'^{(1)}_{\mathbf{2}^0_1 i}(\tau)$, $Y'^{(1)}_{\mathbf{2}^0_1 ii}(\tau)$
$k=2$	$Y'^{(2)}_{\mathbf{1}^0_0 i}$, $Y'^{(2)}_{\mathbf{1}^0_0 ii}$, $Y'^{(2)}_{\mathbf{1}^0_2}$, $Y'^{(2)}_{\mathbf{3}^0 i}$, $Y'^{(2)}_{\mathbf{3}^0 ii}$, $Y'^{(2)}_{\mathbf{3}^0 iii}$
$k=3$	$Y'^{(3)}_{\mathbf{2}^0_0 i}$, $Y'^{(3)}_{\mathbf{2}^0_0 ii}$, $Y'^{(3)}_{\mathbf{2}^0_1 i}$, $Y'^{(3)}_{\mathbf{2}^0_1 ii}$, $Y'^{(3)}_{\mathbf{2}^0_1 iii}$, $Y'^{(3)}_{\mathbf{2}^0_1 iv}$, $Y'^{(3)}_{\mathbf{2}^0_2 i}$, $Y'^{(3)}_{\mathbf{2}^0_2 ii}$, $Y'^{(3)}_{\mathbf{2}^0_2 iii}$
$k=4$	$Y'^{(4)}_{\mathbf{1}^0_0 i}$, $Y'^{(4)}_{\mathbf{1}^0_0 ii}$, $Y'^{(4)}_{\mathbf{1}^0_0 iii}$, $Y'^{(4)}_{\mathbf{1}^0_1}$, $Y'^{(4)}_{\mathbf{1}^0_2 i}$, $Y'^{(4)}_{\mathbf{1}^0_2 ii}$, $Y'^{(4)}_{\mathbf{3}^0 i}$, $Y'^{(4)}_{\mathbf{3}^0 ii}$, $Y'^{(4)}_{\mathbf{3}^0 iii}$, $Y'^{(4)}_{\mathbf{3}^0 iv}$, $Y'^{(4)}_{\mathbf{3}^0 v}$, $Y'^{(4)}_{\mathbf{3}^0 vi}$
$k=5$	$Y'^{(5)}_{\mathbf{2}^0_0 i}$, $Y'^{(5)}_{\mathbf{2}^0_0 ii}$, $Y'^{(5)}_{\mathbf{2}^0_0 iii}$, $Y'^{(5)}_{\mathbf{2}^0_0 iv}$, $Y'^{(5)}_{\mathbf{2}^0_1 i}$, $Y'^{(5)}_{\mathbf{2}^0_1 ii}$, $Y'^{(5)}_{\mathbf{2}^0_1 iii}$, $Y'^{(5)}_{\mathbf{2}^0_1 iv}$, $Y'^{(5)}_{\mathbf{2}^0_1 v}$, $Y'^{(5)}_{\mathbf{2}^0_1 vi}$, $Y'^{(5)}_{\mathbf{2}^0_2 i}$, $Y'^{(5)}_{\mathbf{2}^0_2 ii}$, $Y'^{(5)}_{\mathbf{2}^0_2 iii}$, $Y'^{(5)}_{\mathbf{2}^0_2 iv}$, $Y'^{(5)}_{\mathbf{2}^0_2 v}$
$k=6$	$Y'^{(6)}_{\mathbf{1}^0_0 i}$, $Y'^{(6)}_{\mathbf{1}^0_0 ii}$, $Y'^{(6)}_{\mathbf{1}^0_0 iii}$, $Y'^{(6)}_{\mathbf{1}^0_0 iv}$, $Y'^{(6)}_{\mathbf{1}^0_1 i}$, $Y'^{(6)}_{\mathbf{1}^0_1 ii}$, $Y'^{(6)}_{\mathbf{1}^0_2 i}$, $Y'^{(6)}_{\mathbf{1}^0_2 ii}$, $Y'^{(6)}_{\mathbf{1}^0_2 iii}$, $Y'^{(6)}_{\mathbf{3}^0 i}$, $Y'^{(6)}_{\mathbf{3}^0 ii}$, $Y'^{(6)}_{\mathbf{3}^0 iii}$, $Y'^{(6)}_{\mathbf{3}^0 iv}$, $Y'^{(6)}_{\mathbf{3}^0 v}$, $Y'^{(6)}_{\mathbf{3}^0 vi}$, $Y'^{(6)}_{\mathbf{3}^0 vii}$, $Y'^{(6)}_{\mathbf{3}^0 viii}$, $Y'^{(6)}_{\mathbf{3}^0 ix}$

5. Model building and predictions

Modular-invariant SUSY actions

$$W(\psi;\tau)=\sum_{n}\sum_{\{i_1,\ldots,i_n\}}\sum_{s} g_{i_1\ldots i_n,s}\left(Y_{i_1\ldots i_n,s}(\tau)\,\psi_{i_1}\ldots\psi_{i_n}\right)_{\mathbf{1},s}$$

Modular-invariant SUSY actions

$$W(\psi;\tau)=\sum_n \sum_{\{i_1,\ldots,i_n\}} \sum_s g_{i_1\ldots i_n,s}\left(Y_{i_1\ldots i_n,s}(\tau)\,\psi_{i_1}\ldots\psi_{i_n}\right)_{\mathbf{1},s}$$

$$\begin{cases} \tau \;\to\; \gamma\tau = \dfrac{a\tau+b}{c\tau+d} \\ \psi_i \;\to\; (c\tau+d)^{-k_i}\,\rho_i(\gamma)\,\psi_i \\ Y(\tau) \;\to\; Y(\gamma\tau)=(c\tau+d)^{k_Y}\,\rho_Y(\gamma)\,Y(\tau) \end{cases}$$

$$\text{with } \gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

5. Model building and predictions

Modular-invariant SUSY actions

$$W(\psi;\tau)=\sum_{n}\sum_{\{i_1,\dots,i_n\}}\sum_{s} g_{i_1\,\dots\,i_n,s}\left(Y_{i_1\,\dots\,i_n,s}(\tau)\,\psi_{i_1}\,\dots\,\psi_{i_n}\right)_{\mathbf{1},s}$$

$$\begin{cases}\tau\ \to\ \gamma\tau=\dfrac{a\tau+b}{c\tau+d}\\[2ex] \psi_i\ \to\ (c\tau+d)^{-k_i}\rho_i(\gamma)\,\psi_i\\[2ex] Y(\tau)\ \to\ Y(\gamma\tau)=(c\tau+d)^{k_Y}\rho_Y(\gamma)\,Y(\tau)\end{cases}\qquad \text{with } \gamma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$$

weights (k_i, k_Y)

5. Model building and predictions

Modular-invariant SUSY actions

$$W(\psi;\tau) = \sum_n \sum_{\{i_1,\ldots,i_n\}} \sum_s g_{i_1\ldots i_n,s} \left(Y_{i_1\ldots i_n,s}(\tau)\, \psi_{i_1}\ldots\psi_{i_n}\right)_{\mathbf{1},s}$$

$$\begin{cases} \tau \to \gamma\tau = \dfrac{a\tau+b}{c\tau+d} \\ \psi_i \to (c\tau+d)^{-k_i}\rho_i(\gamma)\,\psi_i \\ Y(\tau) \to Y(\gamma\tau) = (c\tau+d)^{k_Y}\rho_Y(\gamma)\,Y(\tau) \end{cases} \quad \text{with } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

weights

Feruglio, 1706.08749

$Y(\tau)$ are **modular forms** obeying $\begin{cases} k_Y = k_{i_1} + \ldots + k_{i_n} \\ \rho_Y \otimes \rho_{i_1} \otimes \ldots \otimes \rho_{i_1} \supset \mathbf{1} \end{cases}$

Unbroken gCP: couplings

$$W \overset{\text{CP}}{\longleftrightarrow} \overline{W}$$

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In a symmetric basis:

$$\psi(x) \xrightarrow{\mathrm{CP}} \overline{\psi}(x_{\mathrm{P}})$$

$$Y(\tau) \xrightarrow{\mathrm{CP}} Y^*(\tau)$$

$$g(Y^*\overline{\psi}\ldots\overline{\psi})_{\mathbf{1}}$$

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$$g \in \mathbb{R}$$

Guidelines for model building

Using minimality as a guiding principle …

- **No flavons** are introduced,
- Higgs multiplets transform trivially,
- Lepton doublets transform as an Γ_6' **triplet or singlet+doublet**
- Lepton singlets transform as Γ_6' **singlets**, **doublet or triplet**
- Lowest possible weights are chosen such that all charged leptons are massive

An Γ_6' example

Fields \ Models	Type I	Type II	Type III	Type IV	Type V
L	triplet	triplet	triplet	singlet+doublet	singlet+doublet
E^c	three singlets	three singlets	singlet+doublet	singlet+doublet	triplet
N^c	triplet	singlet+doublet	triplet	—	doublet

Table 2. The representation assignments of lepton matter fields L, E^c and N^c under the finite modular group Γ_6' in the five types of models.

An Γ_6' example

$\Gamma_3' \simeq T'$

C.Y Yao, J.N Lu and G.J.Ding, JHEP 05 (2021) 102

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	L_1	$L_d=(L_2,L_3)$	E_1	$E_d=(E_2,E_3)$	H_u, H_d
Weight	3	2	-3	3	0
Γ_6'	$\mathbf{1}_2^0$	$\mathbf{2}_1^1$	$\mathbf{1}_1^0$	$\mathbf{2}_2$	$\mathbf{1}$

An Γ_6' example

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Γ_6'	$\mathbf{1}_2^0$	$\mathbf{2}_1^1$	$\mathbf{1}_1^0$	$\mathbf{2}_2$	$\mathbf{1}$

The superpotential

$$\mathcal{W} = \alpha E_1^c L_1 H_d + \beta L_1 \left(Y_{\mathbf{2}_2}^{(6)} E_d^c\right)_{\mathbf{1}_1^0} H_d + \gamma \left(Y_{\mathbf{4}_0}^{(5)} L_d E_d^c\right)_{\mathbf{1}_0^0} H_d$$

$$+ \frac{g_1}{2\Lambda} L_1 \left(Y_{\mathbf{2}_0^1}^{(5)} L_d\right)_{\mathbf{1}_1^0} H_u H_u + \frac{g_2}{2\Lambda} \left(Y_{\mathbf{3}^0}^{(4)} L_d L_d\right)_{\mathbf{1}_0^0} H_u H_u .$$

Lepton mass matrices

$$m_e = \begin{pmatrix} \alpha & 0 & 0 \\ \beta Y^{(6)}_{\mathbf{2_2},1} & \gamma Y^{(5)}_{\mathbf{4_0},4} & -\gamma Y^{(5)}_{\mathbf{4_0},3} \\ \beta Y^{(6)}_{\mathbf{2_2},2} & -\gamma Y^{(5)}_{\mathbf{4_0},2} & \gamma Y^{(5)}_{\mathbf{4_0},1} \end{pmatrix} v_d, \qquad m_\nu = \frac{v_u^2}{2\Lambda} \begin{pmatrix} 0 & -g_1 Y^{(5)}_{\mathbf{2^1_0},2} & g_1 Y^{(5)}_{\mathbf{2^1_0},1} \\ -g_1 Y^{(5)}_{\mathbf{2^1_0},2} & -2\sqrt{2} g_2 Y^{(4)}_{\mathbf{3^0},3} & 2 g_2 Y^{(4)}_{\mathbf{3^0},2} \\ g_1 Y^{(5)}_{\mathbf{2^1_0},1} & 2 g_2 Y^{(4)}_{\mathbf{3^0},2} & 2\sqrt{2} g_2 Y^{(4)}_{\mathbf{3^0},1} \end{pmatrix}.$$

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Best fit values of the free parameters

$$\Re\langle\tau\rangle = -0.329, \qquad \Im\langle\tau\rangle = 1.080, \qquad \beta/\alpha = 105.467, \qquad \gamma/\alpha = 12.954,$$

$$|g_2/g_1| = 0.816, \qquad \arg(g_2/g_1) = 0.958\pi, \qquad g_1 v_u^2/\Lambda = 29.321\text{meV}.$$

Lepton mass matrices

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$$\chi^2_{\min} = 2.871$$

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The minimum of χ^2

$$\chi^2_{\min} = 2.871$$

The predictions for various observable quantities

$$\sin^2\theta_{13} = 0.02217, \quad \sin^2\theta_{12} = 0.304, \quad \sin^2\theta_{23} = 0.570, \quad \delta_{CP} = 1.347\pi,$$

$$\alpha_{21} = 1.942\pi, \quad \alpha_{31} = 0.953\pi, \quad m_1 = 37.424\text{meV}, \quad m_2 = 38.399\text{meV},$$

$$m_3 = 62.588\text{meV}, \quad \sum_i m_i = 138.411\text{meV}, \quad m_{\beta\beta} = 37.661\text{meV},$$

$$\text{gCP} \quad \Rightarrow \quad g_1, g_2 \in \mathbb{R}$$

Best fit values of the free parameters when gCP is considered

$$\Re\langle\tau\rangle = -0.334, \quad \Im\langle\tau\rangle = 1.092, \quad \beta/\alpha = 105.717, \quad \gamma/\alpha = 12.696,$$

$$g_2/g_1 = -0.810, \quad g_1 v_u^2/\Lambda = 30.315\text{meV}.$$

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The minimum of χ^2

$$\chi^2_{\min} = 2.914$$

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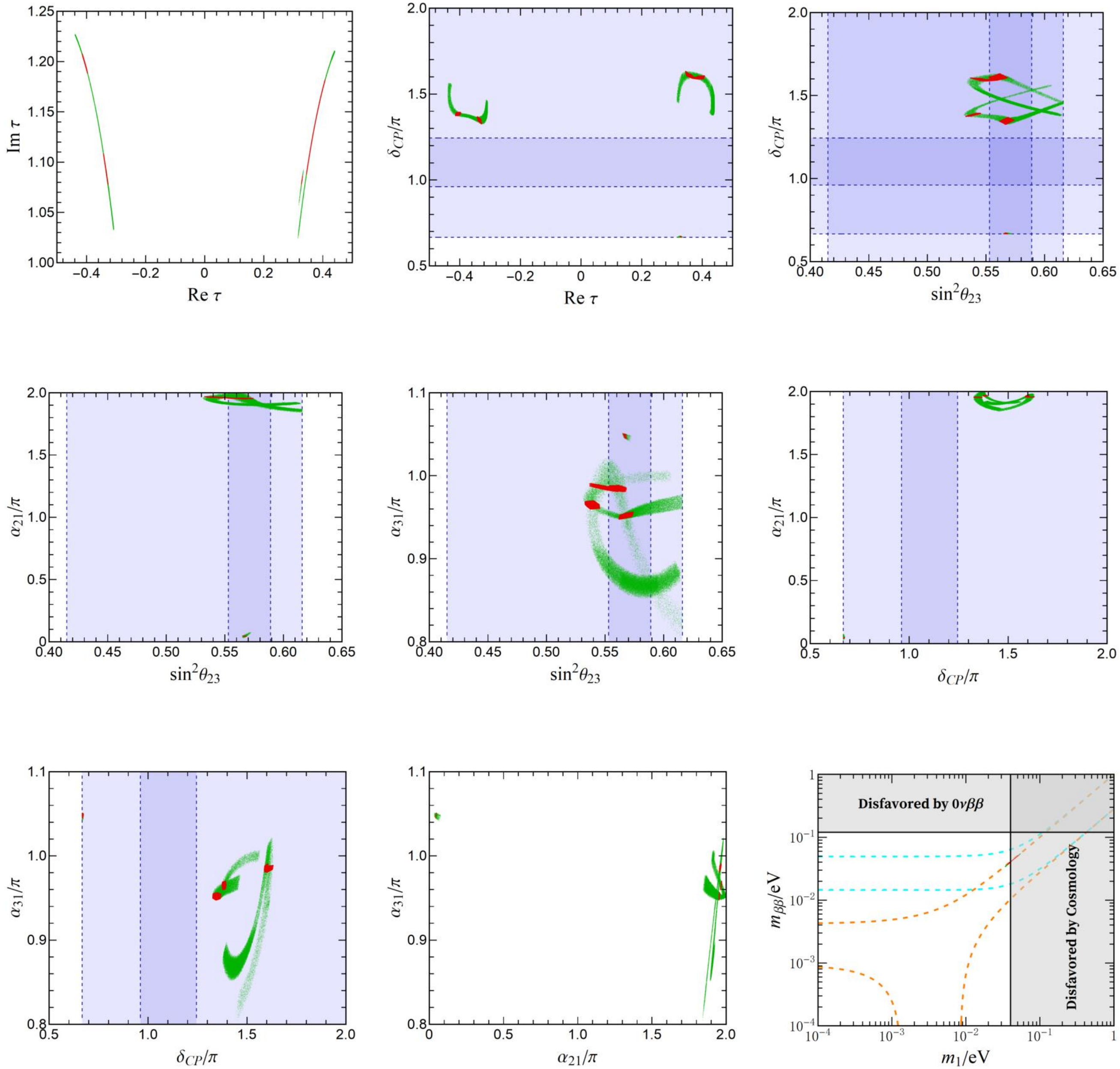
The minimum of χ^2

$$\chi^2_{\min} = 2.914$$

The predictions for various observable quantities

$$\sin^2\theta_{13} = 0.02216, \quad \sin^2\theta_{12} = 0.304, \quad \sin^2\theta_{23} = 0.568, \quad \delta_{CP} = 1.347\pi,$$
$$\alpha_{21} = 1.954\pi, \quad \alpha_{31} = 0.952\pi, \quad m_1 = 38.275\text{meV}, \quad m_2 = 39.229\text{meV},$$
$$m_3 = 63.101\text{meV}, \quad \sum_i m_i = 140.604\text{meV}, \quad m_{\beta\beta} = 38.557\text{meV}.$$



The correlations between the neutrino parameters

Model building based on Γ(2) modular symmetry with finite modular group T′

$$\mathcal{W}_\nu = \frac{g_1}{2\Lambda} Y'^{(2)}_{\mathbf{1}^0_0\, i} \left(LL\right)_{\mathbf{1}^0_0} H_u H_u + \frac{g_2}{2\Lambda} Y'^{(2)}_{\mathbf{1}^0_0\, ii} \left(LL\right)_{\mathbf{1}^0_0} H_u H_u + \frac{g_3}{2\Lambda} Y'^{(2)}_{\mathbf{1}^0_2} \left(LL\right)_{\mathbf{1}^0_1} H_u H_u$$

$$+ \frac{g_4}{2\Lambda} \left(\left(LL\right)_{\mathbf{3}^0_1} Y'^{(2)}_{\mathbf{3}^0\, i} \right)_{\mathbf{1}^0_0} H_u H_u + \frac{g_5}{2\Lambda} \left(\left(LL\right)_{\mathbf{3}^0_1} Y'^{(2)}_{\mathbf{3}^0\, ii} \right)_{\mathbf{1}^0_0} H_u H_u + \frac{g_6}{2\Lambda} \left(\left(LL\right)_{\mathbf{3}^0_1} Y'^{(2)}_{\mathbf{3}^0\, iii} \right)_{\mathbf{1}^0_0} H_u H_u .$$

$$\mathcal{W}_e = \alpha_1 E^c_1 \left(L Y'^{(2)}_{\mathbf{3}^0\, i} \right)_{\mathbf{1}^0_0} H_d + \alpha_2 E^c_1 \left(L Y'^{(2)}_{\mathbf{3}^0\, ii} \right)_{\mathbf{1}^0_0} H_d + \alpha_3 E^c_1 \left(L Y'^{(2)}_{\mathbf{3}^0\, iii} \right)_{\mathbf{1}^0_0} H_d$$

$$+ \beta_1 \left(\left(L E^c_d \right)_{\mathbf{2}^0_1} Y'^{(1)}_{\mathbf{2}^0_2} \right)_{\mathbf{1}^0_0} H_d + \beta_2 \left(\left(L E^c_d \right)_{\mathbf{2}^0_2} Y'^{(1)}_{\mathbf{2}^0_1\, i} \right)_{\mathbf{1}^0_0} H_d + \beta_3 \left(\left(L E^c_d \right)_{\mathbf{2}^0_2} Y'^{(1)}_{\mathbf{2}^0_1\, ii} \right)_{\mathbf{1}^0_0} H_d .$$

5. Conclusions

- A new route towards finite modular groups is proposed

- Higher weight modular forms, up to weight 6

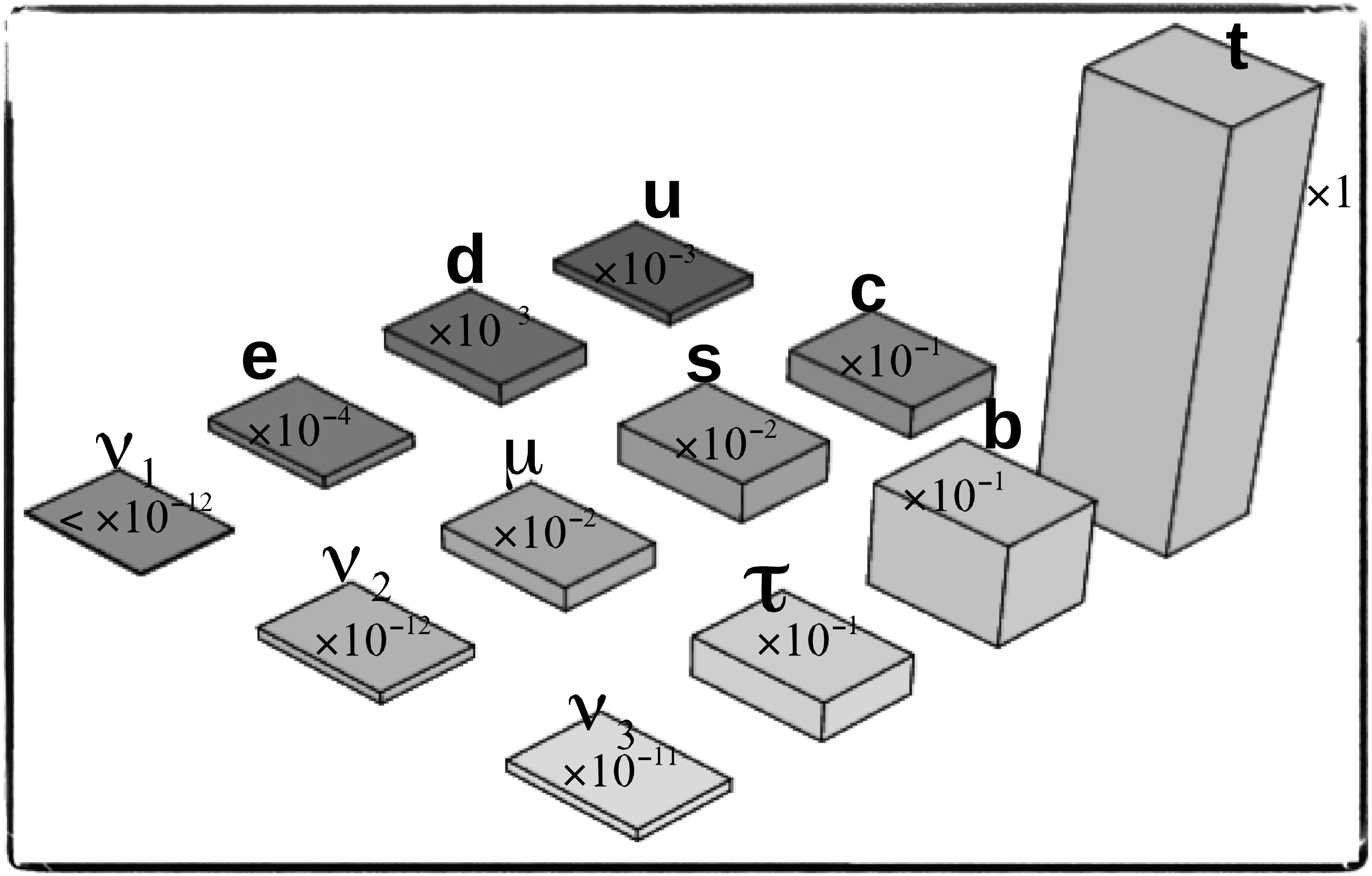
- Five types of modular Γ'_6 flavour models

- A modular invariant model for N'=2 and N''=6

谢谢！

Backup

Mass ordering



$$\frac{d}{d\tau}\log\alpha_{i,j}(-1/\tau)=\frac{i\pi}{28}\left(1-\frac{1}{\tau^2}\right)+\frac{1}{2\tau}+\frac{d}{d\tau}\log\alpha^S_{i,j}(\tau),$$

$$\frac{d}{d\tau}\log\alpha_{i,j}(\tau+1)=\frac{d}{d\tau}\log\alpha^T_{i,j}(\tau),$$

The modular functions

$$Y(x_{1,-1},\ldots,x_{1,6};x_{2,-1},\ldots,x_{2,6};x_{3,-1},\ldots,x_{3,6}\,|\,\tau)\equiv\sum_{i,j}x_{i,j}\frac{d}{d\tau}\log\alpha_{i,j}(\tau),\quad \text{with}\sum_{i,j}x_{i,j}=0$$

$$S:Y(x_{1,-1},\ldots,x_{1,6};x_{2,-1},\ldots x_{2,6};x_{3,-1},\ldots x_{3,6}\,|\,\tau)\xrightarrow{S}Y(x_{1,-1},\ldots,x_{1,6};x_{2,-1},\ldots x_{2,6};x_{3,-1},\ldots x_{3,6}\,|-1/\tau)$$
$$=\tau^2Y(x_{1,0},x_{1,-1},x_{1,6},x_{2,3},x_{3,2},x_{3,5},x_{2,4},x_{1,1};x_{2,0},x_{2,-1},x_{2,6},x_{3,3},x_{1,2},x_{1,5},x_{3,4},x_{2,1};$$
$$x_{3,0},x_{3,-1},x_{3,6},x_{1,3},x_{2,2},x_{2,5},x_{1,4},x_{3,1}\,|\,\tau),$$

$$T:Y(x_{1,-1},\ldots,x_{1,6};x_{2,-1},\ldots x_{2,6};x_{3,-1},\ldots x_{3,6}\,|\,\tau)\xrightarrow{T}Y(x_{1,-1},\ldots,x_{1,6};x_{2,-1},\ldots x_{2,6};x_{3,-1},\ldots x_{3,6}\,|\,\tau+1)$$
$$=Y(x_{1,-1},x_{1,6},x_{1,0},x_{1,1},x_{1,2},x_{1,3},x_{1,4},x_{1,5};x_{2,-1},x_{2,6},x_{2,0},x_{2,1},x_{2,2},x_{2,3},x_{2,4},x_{2,5};$$
$$x_{3,-1},x_{3,6},x_{3,0},x_{3,1},x_{3,2},x_{3,3},x_{3,4},x_{3,5}\,|\,\tau),$$

Modular forms of weight 2 and level 7

$$Y_{\mathbf{7}}^{(2)}(\tau)=\frac{-i}{2\sqrt{2}\pi}\begin{pmatrix}\frac{1}{2\sqrt{2}}Y\left(7,-\mathbf{v}_0;7,-\mathbf{v}_0;7,-\mathbf{v}_0\,|\,\tau\right)\\ Y(0,\mathbf{v}_1;0,\mathbf{v}_1;0,\mathbf{v}_1\,|\tau)\\ Y(0,\mathbf{v}_2;0,\mathbf{v}_2;0,\mathbf{v}_2\,|\tau)\\ Y(0,\mathbf{v}_3;0,\mathbf{v}_3;0,\mathbf{v}_3\,|\tau)\\ Y(0,\mathbf{v}_4;0,\mathbf{v}_4;0,\mathbf{v}_4\,|\tau)\\ Y(0,\mathbf{v}_5;0,\mathbf{v}_5;0,\mathbf{v}_5\,|\tau)\\ Y(0,\mathbf{v}_6;0,\mathbf{v}_6;0,\mathbf{v}_6\,|\tau)\end{pmatrix},$$

$$Y_{\mathbf{8a}}^{(2)}(\tau)=\begin{pmatrix}\frac{1}{\sqrt{2}}Y\left(2(c_1-c_3),\mathbf{0};2(c_2-c_1),\mathbf{v}_0;2(c_3-c_2),-\mathbf{v}_0\,|\,\tau\right)\\ -\frac{1}{\sqrt{6}}Y\left(1+6c_2,-2\mathbf{v}_0;1+6c_3,\mathbf{v}_0;1+6c_1,\mathbf{v}_0\,|\,\tau\right)\\ Y\left(0,\mathbf{v}_1;0,\mathbf{0};0,-\mathbf{v}_1\,|\,\tau\right)\\ Y\left(0,\mathbf{0};0,-\mathbf{v}_2;0,\mathbf{v}_2\,|\,\tau\right)\\ Y\left(0,-\mathbf{v}_3;0,\mathbf{v}_3;0,\mathbf{0}\,|\,\tau\right)\\ Y\left(0,\mathbf{v}_4;0,-\mathbf{v}_4;0,\mathbf{0}\,|\,\tau\right)\\ Y\left(0,\mathbf{0};0,\mathbf{v}_5;0,-\mathbf{v}_5\,|\,\tau\right)\\ Y\left(0,-\mathbf{v}_6;0,\mathbf{0};0,\mathbf{v}_6\,|\,\tau\right)\end{pmatrix},$$

$$Y_{\mathbf{8b}}^{(2)}(\tau)=\begin{pmatrix}\frac{1}{\sqrt{2}}Y\left(2(c_3-c_2),-\mathbf{v}_0;2(c_1-c_3),\mathbf{0};2(c_2-c_1),\mathbf{v}_0\,|\,\tau\right)\\ -\frac{1}{\sqrt{6}}Y\left(1+6c_1,\mathbf{v}_0;1+6c_2,-2\mathbf{v}_0;1+6c_3,\mathbf{v}_0\,|\,\tau\right)\\ Y\left(0,-\mathbf{v}_1;0,\mathbf{v}_1;0,\mathbf{0}\,|\,\tau\right)\\ Y\left(0,\mathbf{v}_2;0,\mathbf{0};0,-\mathbf{v}_2\,|\,\tau\right)\\ Y\left(0,\mathbf{0};0,-\mathbf{v}_3;0,\mathbf{v}_3\,|\,\tau\right)\\ Y\left(0,\mathbf{0};0,\mathbf{v}_4;0,-\mathbf{v}_4\,|\,\tau\right)\\ Y\left(0,-\mathbf{v}_5;0,\mathbf{0};0,\mathbf{v}_5\,|\,\tau\right)\\ Y\left(0,\mathbf{v}_6;0,-\mathbf{v}_6;0,\mathbf{0}\,|\,\tau\right)\end{pmatrix},$$

$$\mathbf{v}_k\equiv(1,\rho^{6k},\rho^{5k},\rho^{4k},\rho^{3k},\rho^{2k},\rho^{k})$$

$$\mathbf{0}\equiv(0,0,0,0,0,0,0)$$

	$\mathbf{1}_k^r$	$\mathbf{2}_k$	$\mathbf{2}_k^r$	$\mathbf{3}^r$	$\mathbf{4}_k$	$\mathbf{6}$
S	$(-1)^r$	$\frac{1}{2}\begin{pmatrix}-1 & \sqrt{3}\\ \sqrt{3} & 1\end{pmatrix}$	$(-1)^r\boldsymbol{a_2}$	$(-1)^r\boldsymbol{a_3}$	$\frac{1}{2}\begin{pmatrix}-\boldsymbol{a_2} & \sqrt{3}\boldsymbol{a_2}\\ \sqrt{3}\boldsymbol{a_2} & \boldsymbol{a_2}\end{pmatrix}$	$\frac{1}{2}\begin{pmatrix}-\boldsymbol{a_3} & \sqrt{3}\boldsymbol{a_3}\\ \sqrt{3}\boldsymbol{a_3} & \boldsymbol{a_3}\end{pmatrix}$
T	$(-1)^r\omega^k$	$\omega^k\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$	$(-1)^r\omega^{k+1}\boldsymbol{b_2}$	$(-1)^r\boldsymbol{b_3}$	$\omega^{k+1}\begin{pmatrix}\boldsymbol{b_2} & 2\\ 2 & -\boldsymbol{b_2}\end{pmatrix}$	$\begin{pmatrix}\boldsymbol{b_3} & 3\\ 3 & -\boldsymbol{b_3}\end{pmatrix}$

Table 5. The representation matrices of the generators S and T for the twenty-one irreducible representations of Γ'_6 in the chosen basis, where $\omega = e^{2\pi i/3}$, $r = 0, 1$, $k = 0, 1, 2$ and matrices $\boldsymbol{a_2}$, $\boldsymbol{b_2}$, $\boldsymbol{a_3}$ and $\boldsymbol{b_3}$ are shown in eq. (A.5).

$$\mathbf{1}_i^r \otimes \mathbf{1}_j^s = \mathbf{1}_m^t, \quad \mathbf{1}_i^r \otimes \mathbf{2}_j = \mathbf{2}_m, \quad \mathbf{1}_i^r \otimes \mathbf{2}_j^s = \mathbf{2}_m^t, \quad \mathbf{1}_i^r \otimes \mathbf{3}^s = \mathbf{3}^t,$$

$$\mathbf{1}_i^r \otimes \mathbf{4}_j = \mathbf{4}_m, \quad \mathbf{1}_i^r \otimes \mathbf{6} = \mathbf{6}, \quad \mathbf{2}_i \otimes \mathbf{2}_j = \mathbf{1}_m^0 \oplus \mathbf{1}_m^1 \oplus \mathbf{2}_m, \quad \mathbf{2}_i \otimes \mathbf{2}_j^r = \mathbf{4}_m,$$

$$\mathbf{2}_i \otimes \mathbf{3}^r = \mathbf{6}, \quad \mathbf{2}_i \otimes \mathbf{4}_j = \mathbf{2}_m^0 \oplus \mathbf{2}_m^1 \oplus \mathbf{4}_m, \quad \mathbf{2}_i \otimes \mathbf{6} = \mathbf{3}^0 \oplus \mathbf{3}^1 \oplus \mathbf{6},$$

$$\mathbf{2}_i^r \otimes \mathbf{2}_j^s = \mathbf{1}_m^t \oplus \mathbf{3}^t, \quad \mathbf{2}_i^r \otimes \mathbf{3}^s = \mathbf{2}_0^t \oplus \mathbf{2}_1^t \oplus \mathbf{2}_2^t, \quad \mathbf{2}_i^r \otimes \mathbf{4}_j = \mathbf{2}_m \oplus \mathbf{6},$$

$$\mathbf{2}_i^r \otimes \mathbf{6} = \mathbf{4}_0 \oplus \mathbf{4}_1 \oplus \mathbf{4}_2, \quad \mathbf{3}^r \otimes \mathbf{3}^s = \mathbf{1}_0^t \oplus \mathbf{1}_1^t \oplus \mathbf{1}_2^t \oplus \mathbf{3}_1^t \oplus \mathbf{3}_2^t,$$

$$\mathbf{3}^r \otimes \mathbf{4}_i = \mathbf{4}_0 \oplus \mathbf{4}_1 \oplus \mathbf{4}_2, \quad \mathbf{3}^r \otimes \mathbf{6} = \mathbf{2}_0 \oplus \mathbf{2}_1 \oplus \mathbf{2}_2 \oplus \mathbf{6}_1 \oplus \mathbf{6}_2,$$

$$\mathbf{4}_i \otimes \mathbf{4}_j = \mathbf{1}_m^0 \oplus \mathbf{1}_m^1 \oplus \mathbf{2}_m \oplus \mathbf{3}^0 \oplus \mathbf{3}^1 \oplus \mathbf{6},$$

$$\mathbf{4}_i \otimes \mathbf{6} = \mathbf{2}_0^0 \oplus \mathbf{2}_1^0 \oplus \mathbf{2}_2^0 \oplus \mathbf{2}_0^1 \oplus \mathbf{2}_1^1 \oplus \mathbf{2}_2^1 \oplus \mathbf{4}_0 \oplus \mathbf{4}_1 \oplus \mathbf{4}_2\,,$$

$$\mathbf{6} \otimes \mathbf{6} = \mathbf{1}_0^0 \oplus \mathbf{1}_1^0 \oplus \mathbf{1}_2^0 \oplus \mathbf{1}_0^1 \oplus \mathbf{1}_1^1 \oplus \mathbf{1}_2^1 \oplus \mathbf{2}_0 \oplus \mathbf{2}_1 \oplus \mathbf{2}_2 \oplus \mathbf{3}_S^0 \oplus \mathbf{3}_A^0 \oplus \mathbf{3}_S^1 \oplus \mathbf{3}_A^1 \oplus \mathbf{6}_S \oplus \mathbf{6}_A\,,$$

: Modular symmetry

$$\underbrace{\overline{\Gamma} \,/\, \overline{\Gamma}(N)}_{\Gamma_N}$$

Bottom-up approach

We will choose N & scan τ

For top-down, see e.g.:

Kobayashi et al., 1804.06644
Kobayashi, Tamba, 1811.11384
de Anda et al., 1812.05620
Baur et al., 1901.03251
Kariyazono et al., 1904.07546

$$\overline{\Gamma}(N) \equiv \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \; \gamma \in \overline{\Gamma} \;\wedge\; (\gamma = \mathbb{1}) \bmod N \right\}$$

$\mathcal{N}$=1 SUSY modular invariant theories

- known since late 1980s

S. Ferrara, D. Lust, A. D. Shapere and S. Theisen, Phys. Lett. B **225** (1989) 363.

S. Ferrara, .D. Lust and S. Theisen, Phys. Lett. B **233** (1989) 147.

focus on Yukawa interactions and $\mathcal{N}$=1 global SUSY

$$\mathcal{S} = \int d^4x d^2\theta d^2\bar{\theta}\ K(\Phi, \bar{\Phi}) + \int d^4x d^2\theta\ w(\Phi) + h.c.$$

$$\Phi = (\tau, \varphi)$$

Kahler potential, kinetic terms

superpotential, holomorphic function of Φ Yukawa interactions

S invariant if

$$\begin{cases} w(\Phi) \to w(\Phi) \\ K(\Phi, \bar{\Phi}) \to K(\Phi, \bar{\Phi}) + f(\Phi) + f(\overline{\Phi}) \end{cases}$$

- invariance of the Kahler potential easy to achieve. For example:

$$K(\Phi, \bar{\Phi}) = -h \log(-i\tau + i\bar{\tau}) + \sum_I (-i\tau + i\bar{\tau})^{-k_I} |\varphi^{(I)}|^2$$

minimal K

- extension to $\mathcal{N}$=1 SUGRA straightforward: ask invariance of $G=K+\log|w|^2$

Few facts about (level-N) Modular Forms

- transformation property under the modular group

$$f_i(\gamma\tau) = (c\tau + d)^k \rho(\gamma)_{ij} f_j(\tau)$$

unitary representation of the finite modular group $\Gamma_N \equiv \overline{\Gamma}/\overline{\Gamma}(N)$

- q-expansion

$$f(\tau + N) = f(\tau) \Rightarrow f(\tau) = \sum_{i=0}^{\infty} a_n q_N^n \qquad q_N = e^{\frac{i2\pi\tau}{N}}$$

- $k < 0 \Rightarrow f(\tau) = 0$

  $k = 0 \Rightarrow f(\tau) =$ constant

  $k > 0$ (even integer) $\Rightarrow f(\tau) \in \mathcal{M}_k(\Gamma(N))$ finite-dimensional linear space

- ring of modular forms generated by few elements

$$\mathcal{M}(\Gamma(N)) = \bigoplus_{k=0}^{\infty} \mathcal{M}_{2k}(\Gamma(N))$$

an explicit example in a moment

Kinetic Term

Kinetic term of the modulus τ $$\frac{\left|\partial_\mu \tau\right|^2}{\langle -i\tau + i\bar{\tau}\rangle^2}$$

Modular transformation $$\tau' = \frac{a\tau + b}{c\tau + d}, ad - bc = 1$$

- numerator

$$\partial_\mu \tau' = \frac{(a\partial_\mu \tau)(c\tau + d) - (a\tau + b)(c\partial_\mu \tau)}{(c\tau + d)^2} = \frac{(ad - bc)\partial_\mu \tau}{(c\tau + d)^2} = \frac{\partial_\mu \tau}{(c\tau + d)^2}$$

- denominator

$$\tau' - \bar{\tau}' = \frac{(a\tau + b)(c\bar{\tau} + d) - (a\bar{\tau} + b)(c\tau + d)}{|c\tau + d|^2} = \frac{(ad - bc)(\tau - \bar{\tau})}{|c\tau + d|^2} = \frac{\tau - \bar{\tau}}{|c\tau + d|^2}$$

$$\Rightarrow \quad \frac{\left|\partial_\mu \tau'\right|^2}{\langle -i\tau' + i\bar{\tau}'\rangle^2} = \frac{\left|\partial_\mu \tau\right|^2}{\langle -i\tau + i\bar{\tau}\rangle^2}$$ Modular invariant

Guidelines for model building

Using minimality as a guiding principle...

- **RGEs** need to be considered, depend on $\tan\beta$
- **SUSY-breaking** corrections can be made negligible via separation of scales (power counting argument)

Feruglio and Criado, 1807.01125

CP & T Violation

Under CPT invariance, CP- and T-violating asymmetries are identical:

$$P\left(\nu_\alpha \to \nu_\beta\right) - P\left(\overline{\nu}_\alpha \to \overline{\nu}_\beta\right) = P\left(\nu_\alpha \to \nu_\beta\right) - P\left(\nu_\beta \to \nu_\alpha\right)$$
$$= 16\mathcal{J} \sum_\gamma \epsilon_{\alpha\beta\gamma} \sin\frac{\Delta m^2_{21} L}{4E} \sin\frac{\Delta m^2_{31} L}{4E} \sin\frac{\Delta m^2_{32} L}{4E}$$

Comments:

★ **CP / T violation cannot show up in the disappearance neutrino oscillation experiments ($\alpha = \beta$);**

★ **CP / T violation is a small three-family flavor effect;**

★ **CP / T violation in normal lepton-number-conserving neutrino oscillations depends only upon the Dirac phase of *V*; hence such oscillation experiments cannot tell us whether neutrinos are Dirac or Majorana particles.**

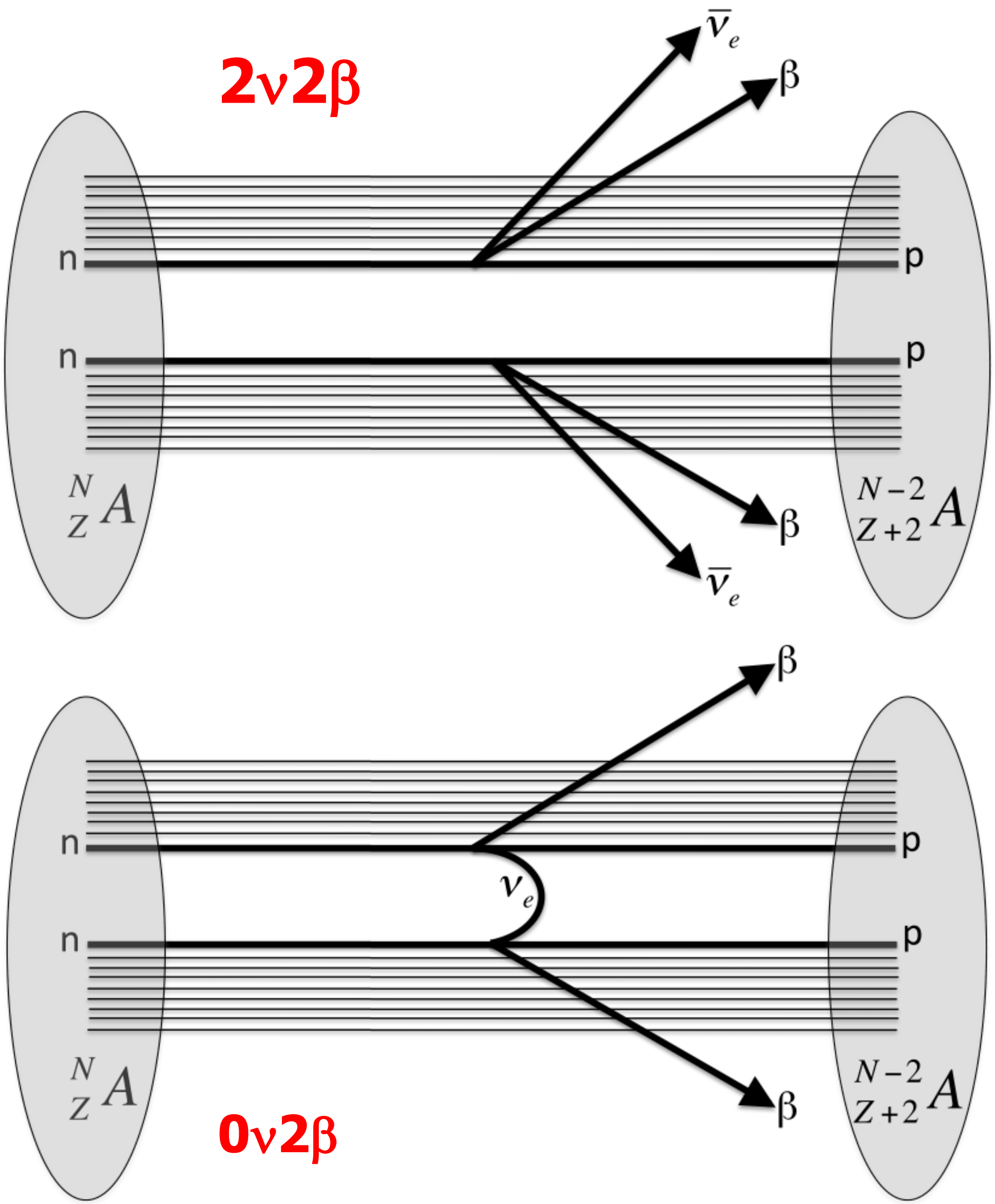
$$J = \sin\theta_{12}\cos\theta_{12}\sin\theta_{23}\cos\theta_{23}\sin\theta_{13}\cos^2\theta_{13}\sin\delta \le 1/6\sqrt{3} \approx 9.6\%$$

2-flavor Approximation

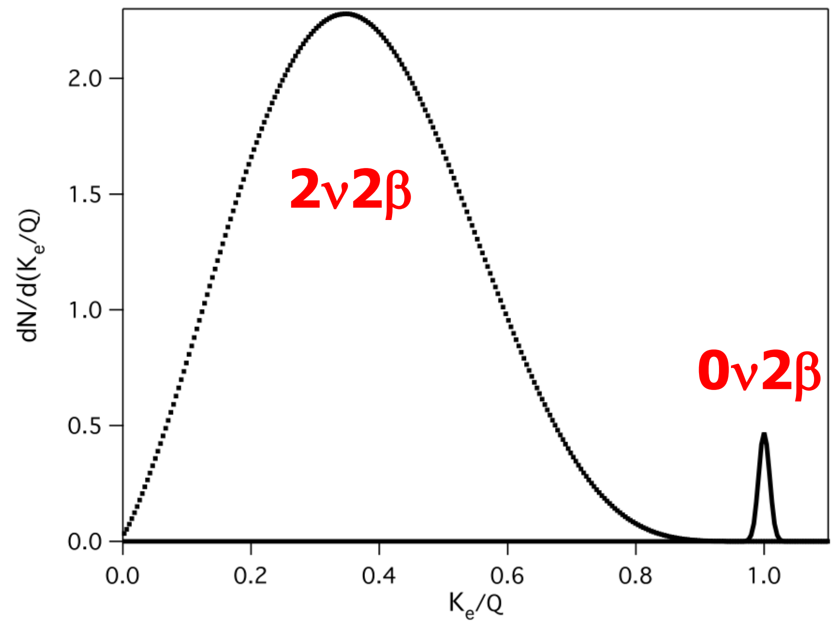
Solar, reactor, atmospheric and accelerator ν oscillation experiments:

Experiment	Survival probability	Oscillation factor
Solar $\nu_e \to \nu_e$	$1 - \sin^2 2\theta_{12} \sin^2\left(1.27\frac{\Delta m^2_{21} L}{E}\right)$	$\sin^2 2\theta_{12} = 4\lvert V_{e1}\rvert^2 \lvert V_{e2}\rvert^2$
KamLAND $\overline{\nu}_e \to \overline{\nu}_e$	$1 - \sin^2 2\theta_{12} \sin^2\left(1.27\frac{\Delta m^2_{21} L}{E}\right)$	$\sin^2 2\theta_{12} = 4\lvert V_{e1}\rvert^2 \lvert V_{e2}\rvert^2$
Atmospheric $\nu_\mu \to \nu_\mu$	$1 - \sin^2 2\theta_{23} \sin^2\left(1.27\frac{\Delta m^2_{32} L}{E}\right)$	$\sin^2 2\theta_{23} = 4\lvert V_{\mu 3}\rvert^2 \left(1 - \lvert V_{\mu 3}\rvert^2\right)$
K2K $\nu_\mu \to \nu_\mu$	$1 - \sin^2 2\theta_{23} \sin^2\left(1.27\frac{\Delta m^2_{32} L}{E}\right)$	$\sin^2 2\theta_{23} = 4\lvert V_{\mu 3}\rvert^2 \left(1 - \lvert V_{\mu 3}\rvert^2\right)$
MINOS $\nu_\mu \to \nu_\mu$	$1 - \sin^2 2\theta_{23} \sin^2\left(1.27\frac{\Delta m^2_{32} L}{E}\right)$	$\sin^2 2\theta_{23} = 4\lvert V_{\mu 3}\rvert^2 \left(1 - \lvert V_{\mu 3}\rvert^2\right)$
CHOOZ $\overline{\nu}_e \to \overline{\nu}_e$	$1 - \sin^2 2\theta_{13} \sin^2\left(1.27\frac{\Delta m^2_{32} L}{E}\right)$	$\sin^2 2\theta_{13} = 4\lvert V_{e3}\rvert^2 \left(1 - \lvert V_{e3}\rvert^2\right)$

If this is the case, …



$$^N_Z A \rightarrow ^{N-2}_{Z+2} A + 2\beta^- + 2\bar{\nu}_e$$

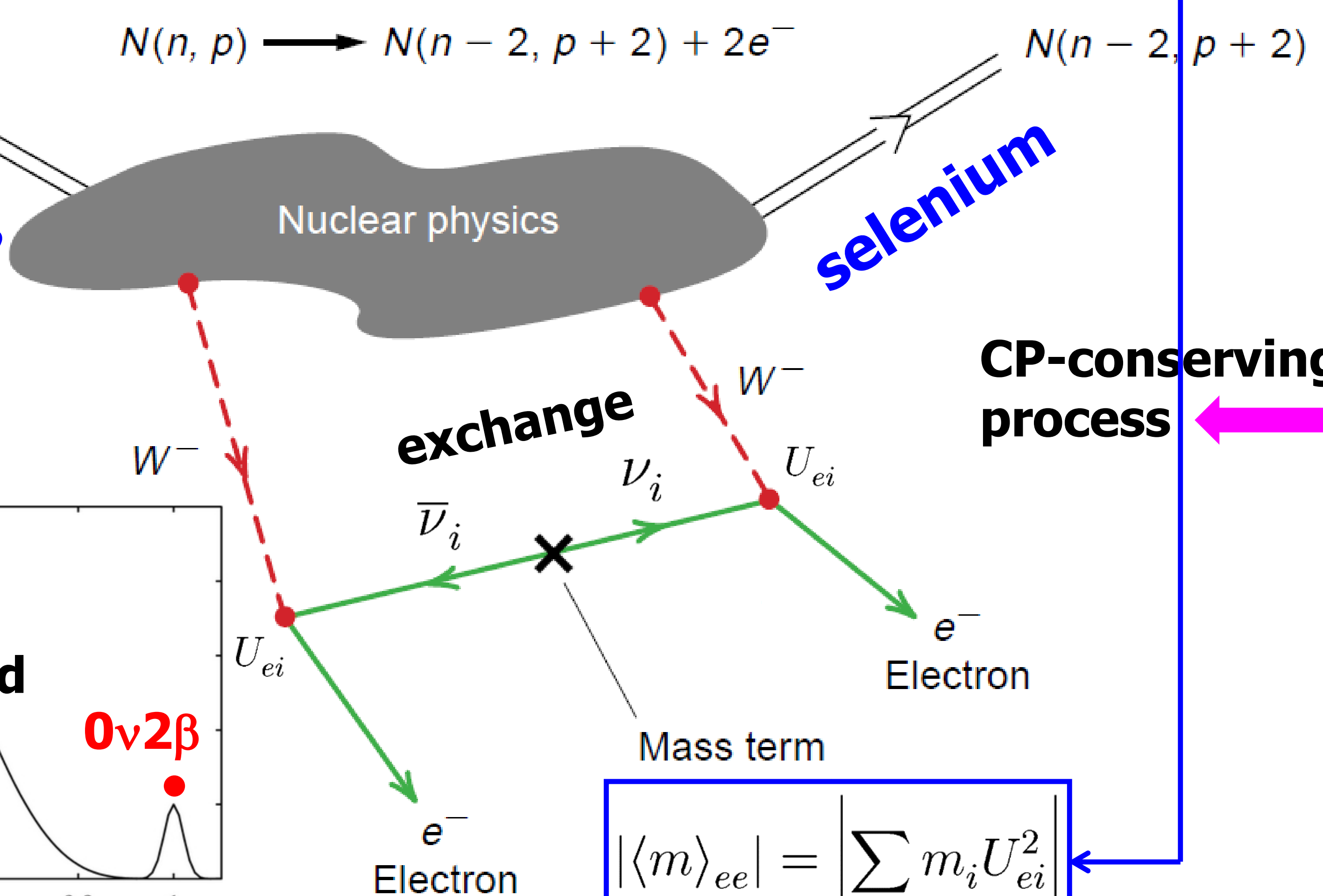


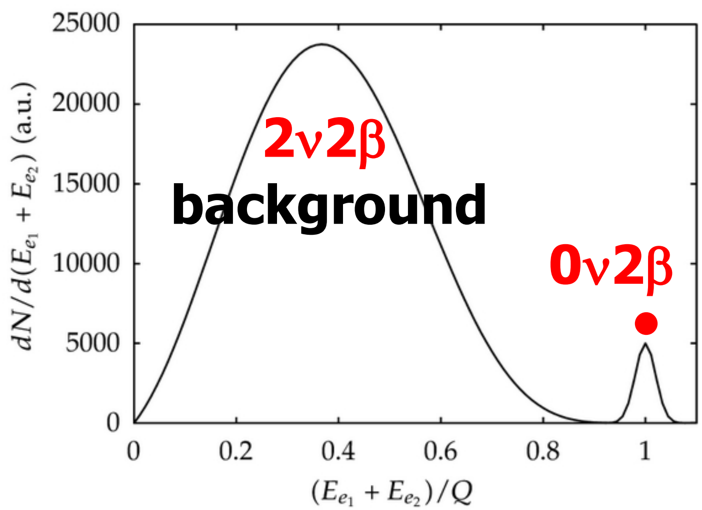
$$^N_Z A \rightarrow ^{N-2}_{Z+2} A + 2\beta^-$$

1939: 0ν2β decays

A 0ν2β decay can happen if massive ν's have the Majorana nature (Wendell Furry 1939)

$$T_{1/2}^{0\nu} = \left(G^{0\nu}\right)^{-1} \left|M^{0\nu}\right|^{-2} \left|\langle m \rangle_{ee}\right|^{-2}$$





$$\left|\langle m \rangle_{ee}\right| = \left|\sum_i m_i U_{ei}^2\right|$$