Towards a Mathematical Definition of 5D Wilson Loops

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Based on

(1) Shuai Guo, XW, Longting Wu, in progress

and

(2) Min-xin Huang, Kimyeong Lee, XW, 2205.02366

(3) XW, 2305.09171

(4) Min-xin Huang, Sheldon Katz, Albrecht Klemm, XW, in progress

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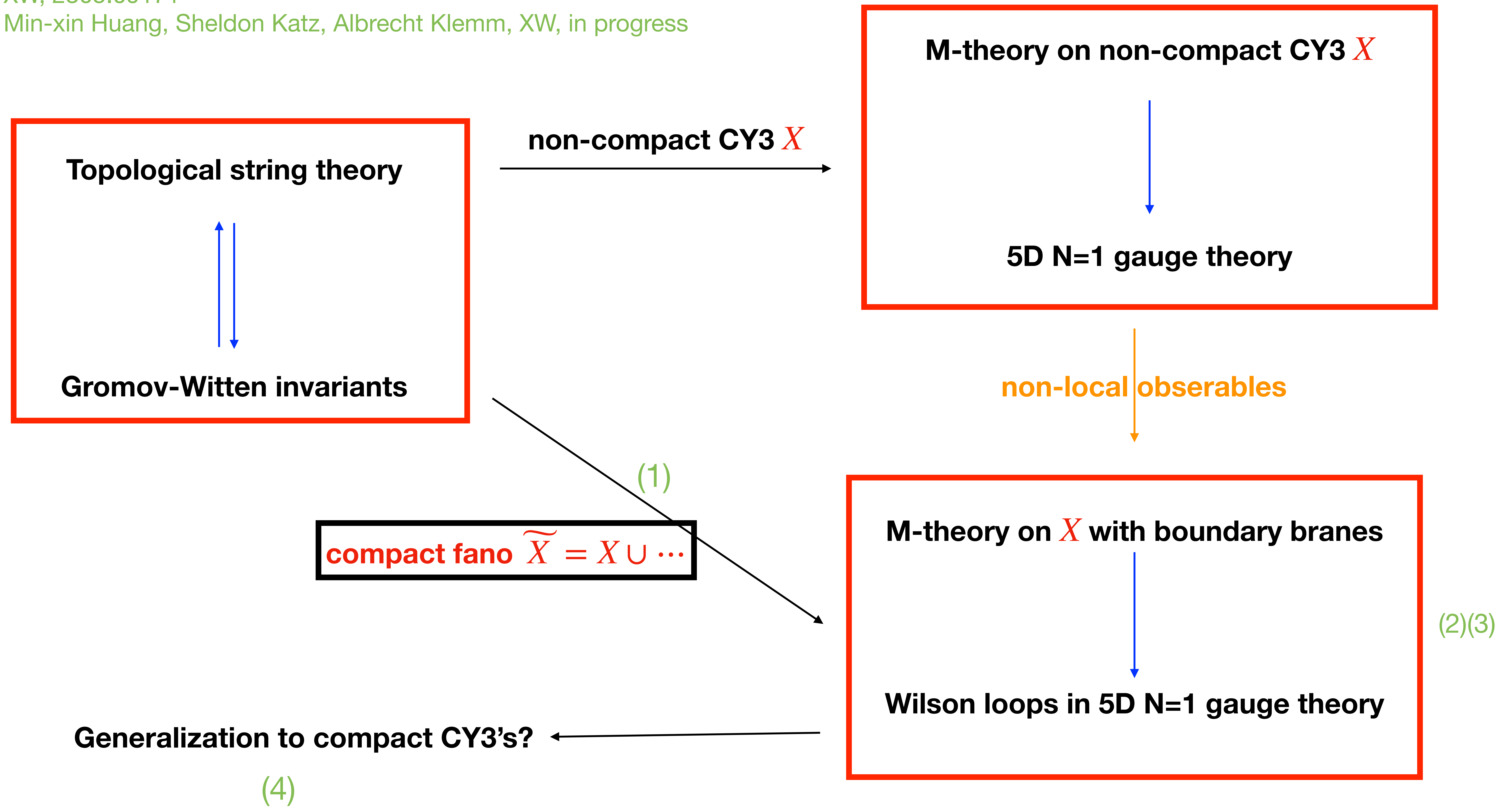
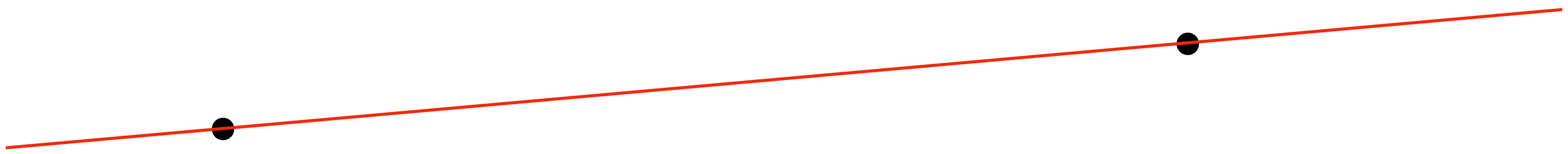


Gromov-Witten theory

- How many straight lines pass between two points in the plane?



Gromov-Witten theory

- How many conics pass through five points in the plane?
  - Answer: 1

- How many degree d curves pass through $3d-1$ points in a plane?
  - Degree d Gromov-Witten invariants of $\mathbb{P}^2$ with $(3d-1)$ points insertion.

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Gromov-Witten theory

- The curve counting problem can be translated to the path integral of a two-dimensional (2,2) non-linear sigma model.

- The theory has two R-symmetries $U(1)_V$ and $U(1)_A$, they give A-twist and B-twist respectively, with BRST operator $Q = Q_A$ or $Q = Q_B$

- In the A-twist, a general correlation function [Witten, '91]

$$\langle \prod_a \mathcal{O}_a \rangle_n = e^{-2\pi n t} \int_{B_n} D\phi \, D\chi \, D\psi \;\; e^{-it\{Q, \int V\}} \cdot \prod_a \mathcal{O}_a .$$

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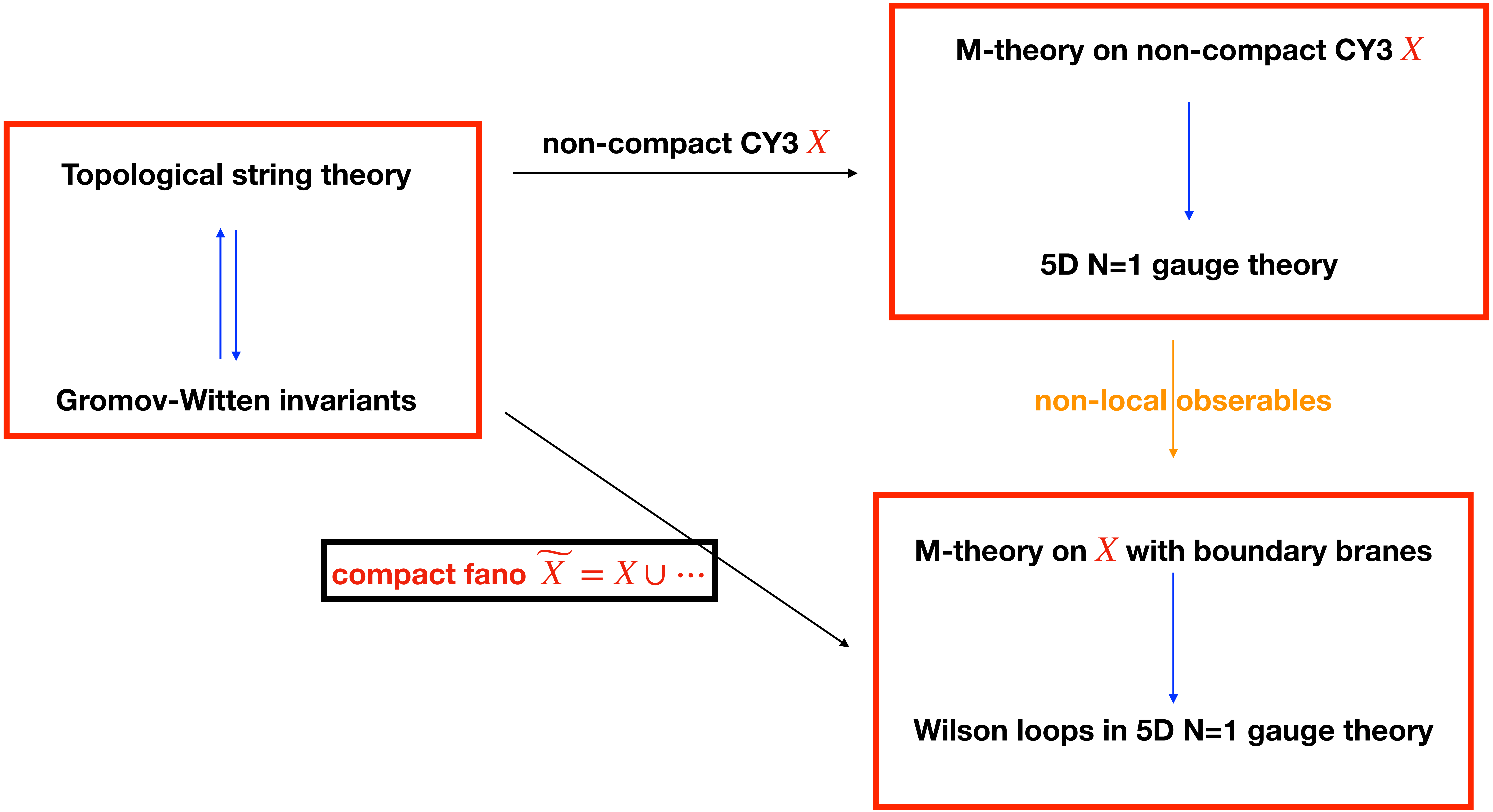
- The operator is defined in the BRST cohomology class: $\mathcal{O}_a \to \mathcal{O}_a + \{Q, S_a\}$. It has a one-to-one correspondence to the de Rham cohomology of the target space

- Reduced to holomorphic map configurations:
    - Gromov-Witten invariants.

Gromov-Witten theory

- We also have the B-twist, which provides a different understanding of the same integral. Mirror symmetry
- When the target space is not a Calabi-Yau, e.g. Fano, the axial R-symmetry is anomalous at the quantum level. There is no B-model for Fano.
- A-model is valid for any Kahler target space


Topological string theory
Gromov-Witten invariants
non-compact CY3 X
M-theory on non-compact CY3 X
5D N=1 gauge theory
non-local observables
compact fano $\widetilde{X} = X \cup \cdots$
M-theory on X with boundary branes
Wilson loops in 5D N=1 gauge theory

5D SQFTs

- Consider the compactification of M-theory on a Calabi-Yau threefold X
- If the CY3 X is non-compact, the gravity in the low-energy physics is decoupled
- we get a five-dimensional supersymmetric quantum field theory (5D SQFT) with eight supercharges
  - Hypermultiplet: Higgs branch
  - Vector multiplet: A_μ, ϕ, Coulomb branch
- **Coulomb branch**: the scalar field ϕ gets the expectation value in the Cartan subalgebra of the gauge group, which breaks the gauge group to $U(1)^r$

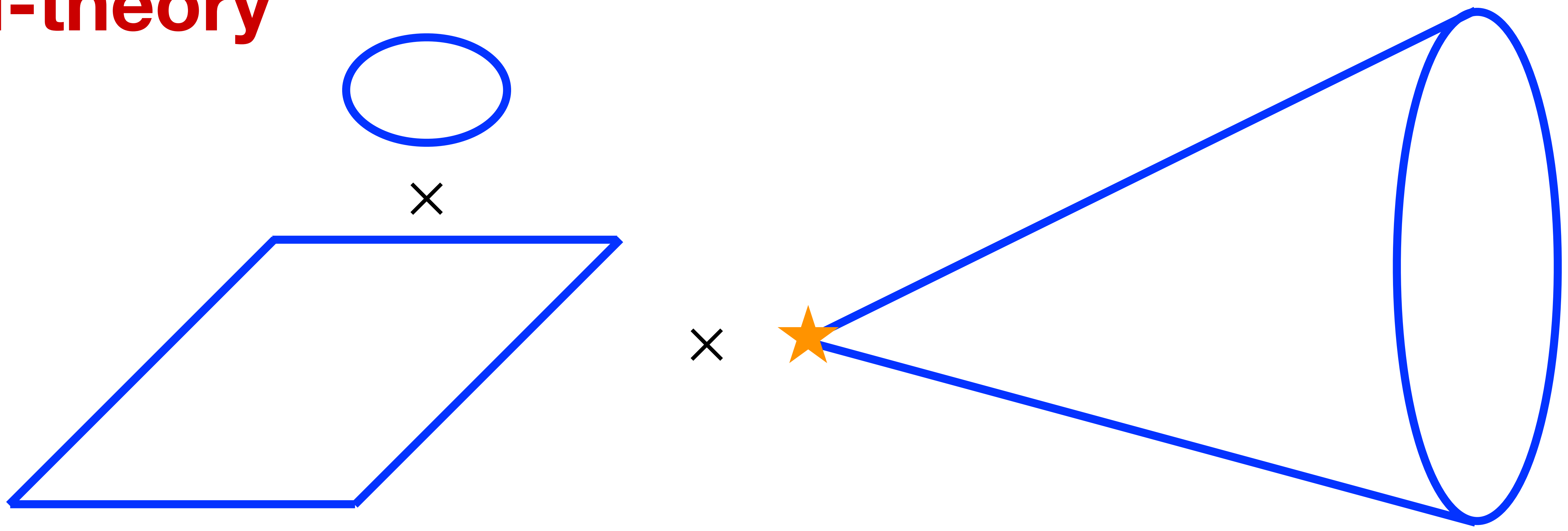
5D SQFTs

- **Coulomb branch**: the scalar field ϕ gets the expectation value in the Cartan subalgebra of the gauge group, which breaks the gauge group to $U(1)^r$.

- The scalar expectation values $\phi_i, i = 1, \cdots, r$ parametrize the moduli space on the Coulomb branch.

- The BPS particles carry non-trivial spins (j_L, j_R) under the 4D rotation group $SO(4) \simeq SU(2)_L \times SU(2)_R$, with the multiplicity labeled by $N^{\beta_i}_{j_L, j_R}$

5D SQFT from M-theory



E.g. resolved $\mathbb{C}^3/\mathbb{Z}_3$

electric BPS particles — M2-branes on holomorphic two-cycles

Coulomb moduli + mass deformations — Volume of compact + non-compact curves $\sim h^{1,1}$

Rank of the gauge group r — Betti number b_4 : # compact divisors

BPS partition function — Topological string partition function

Half-BPS Wilson loop operators

- Consider the **half-BPS** Wilson loop operator along the "time circle", with the gauge field in the representation $\mathbf{r}$

$$W_{\mathbf{r}} = \mathrm{Tr}_{\mathbf{r}} \exp\left(i \oint_{S^1} dt (A_0(t) - \phi(t)) \right)$$

- On the Coulomb branch, $G \to U(1)^r$, the notion of the representation is replaced by the electric charges $\mathbf{r} = [q_1, \cdots, q_r]$ under the abalien groups

- The expectation value

$$\langle W_{\mathbf{r}=[\mathbf{q}_1,\cdots,\mathbf{q}_r]} \rangle = e^{q_1 \phi_1 + \cdots + q_r \phi_r} \times (1 + \cdots)$$

$$\langle W^{SU(2)}_{\mathbf{2}=[-\mathbf{1}]} \rangle = e^{-\phi_1} + e^{\phi_1} + \mathrm{Inst}\,.$$

One-form symmetry action: $\phi_1 \mapsto \phi_1 + i\pi$

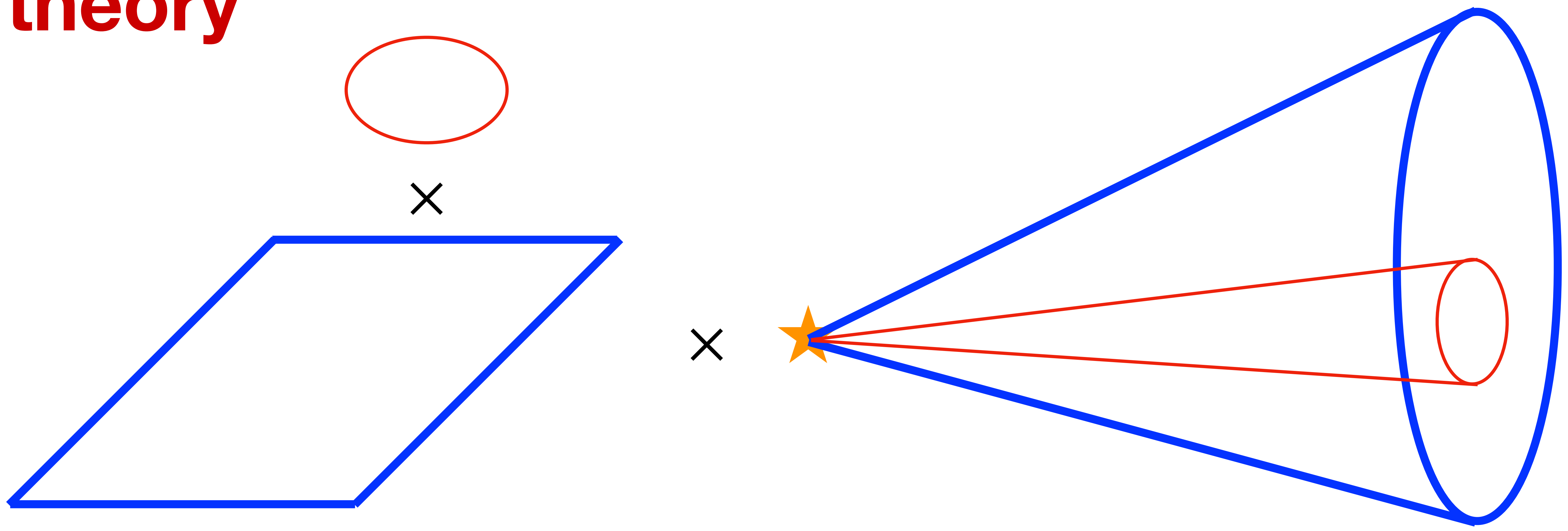
[Tian, XW, to appear]

Half-BPS Wilson loop operators

- The Wilson loop operator we consider can be realized by a **heavy, stationery electric particle** located at the origin of the space $\mathbb{R}^4$
- The worldline of that particle becomes the Wilson line along the time circle.

We now try to understand such particle(s) in the geometric descriptions

Realization in M theory



Wilson loop	Geometry X
heavy, stationery electric particle	a non-compact curve $\mathsf{C} = \mathbb{P}^1$ in CY3 extended to infinity Heavy
BPS particles	M2-branes wrapping around $C + \mathsf{C}$, $C \in H_2(X; \mathbb{Z})$
Representation	Charges of C $\quad q_i = D_i \cdot \mathsf{C}$ (D_i:) Compact divisor

Realization in M theory

- We want to consider the BPS partition function in the presence of W operators.
    - Generation function for the **Wilson loop BPS invariants** $\widetilde{N}^{C}_{j_L,j_R}$
- Without Wilson loop: BPS partition function has the (refined) Goparkumar-Vafa expansion [Gopakumar, Vafa, '98]

$$\mathcal{F}_{\text{BPS}} = \log Z_{\text{BPS}} = \sum_{\beta \in H_2(X,\mathbb{Z})} \sum_{j_L,j_R} (-1)^{2j_L+2j_R} N^{\beta}_{j_L,j_R} \frac{\chi_{j_L}(k\epsilon_-)\chi_{j_R}(k\epsilon_+)}{k\left(q_1^{1/2}-q_1^{-1/2}\right)\left(q_2^{1/2}-q_2^{-1/2}\right)} e^{-k\beta\cdot t}$$

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Dynamic contribution!

Spectra of Harmonic oscillators

$$\frac{1}{q_{1,2}^{-1/2}-q_{1,2}^{1/2}} = \sum_n \exp\left[\left(n+\frac{1}{2}\right)\epsilon_{1,2}\right]$$

Realization in M theory

- We want to consider the BPS partition function in the presence of W operators.

  - Generation function for the **Wilson loop BPS invariants** $\widetilde{N}^{C}_{j_L,j_R}$

- With Wilson loop, $SU(2)_L \times SU(2)_R$ symmetry is not broken: modified (refined) Goparkumar-Vafa expansion with a similar spin structure

$$\mathcal{F}_{\mathrm{BPS},\{\mathsf{C}\}} \sim \sum_{\beta\in H_2(X,\mathbb{Z})} \sum_{j_L,j_R} (-1)^{2j_L+2j_R} \widetilde{N}^{\beta}_{j_L,j_R} \frac{\chi_{j_L}(\epsilon_-)\chi_{j_R}(\epsilon_+)}{\left(q_1^{1/2}-q_1^{-1/2}\right)\left(q_2^{1/2}-q_2^{-1/2}\right)} e^{-\beta\cdot t - t_{\mathsf{C}}}$$

$\mathbb{R}^4 \times S^1$

╳ **Dynamic contribution!**

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$\rightarrow \mathcal{I}$

- Here $t_{\mathsf{C}} = +\infty$. However, we can first treat it with finite mass.
- The inserted curve C must have minimal charge $\mathbf{r} = [q_1, \cdots, q_r], \quad |q_i| \leq 1$, to avoid self-interactions. We call curves with that property *primitive curves*
- Higher representations/charges are generated with distinct primitive curves $\{\mathsf{C}_i\}$

$$\mathcal{F}^{\mathrm{Wilson}}_{\mathrm{BPS}} = \mathcal{F}_{\mathrm{BPS},\{\}} + \mathcal{F}_{\mathrm{BPS},\{\mathsf{C}_1\}} M_1 + \mathcal{F}_{\mathrm{BPS},\{\mathsf{C}_2\}} M_2 + \mathcal{F}_{\mathrm{BPS},\{\mathsf{C}_1,\mathsf{C}_2\}} M_1 M_2$$

Realization in M theory

- In the large mass limit

$$Z_{\rm BPS}^{\rm Wilson} = \exp(\mathcal{F}_{\rm BPS}^{\rm Wilson}) = \exp(\mathcal{F}_{\rm BPS,\{\}}) \times \left[1 + \mathcal{F}_{\rm BPS,\{c_1\}} M_1 + \mathcal{F}_{\rm BPS,\{c_2\}} M_2 + \left(\mathcal{F}_{\rm BPS,\{c_1,c_2\}} + \mathcal{F}_{\rm BPS,\{c_1\}}\mathcal{F}_{\rm BPS,\{c_2\}}\right) M_1 M_2\right]$$

$$\langle W_{\mathbf{r}_1} \rangle \qquad \langle W_{\mathbf{r}_2} \rangle \qquad \langle W_{\mathbf{r}_1 \otimes \mathbf{r}_2} \rangle$$

- BPS sector [Kim, Kim, Kim, '21][Huang, Lee, XW, '22]

$$\mathcal{F}_{\rm BPS,\{c_1,\cdots,c_n\}} = \mathcal{I}^{n-1} \cdot \sum_{\beta \in H_2(X,\mathbb{Z})} \sum_{j_L, j_R} (-1)^{2j_L + 2j_R} \widetilde{N}^{\beta}_{j_L, j_R} \chi_{j_L}(\epsilon_-) \chi_{j_R}(\epsilon_+) e^{-\beta \cdot t}$$