Holography, Matrices, and Cosmology

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Getting Started

As is widely recognized, there are some intriguing relations between Einstein's gravitational theory and the ordinary Yang-Mills theory. The former looks somewhat similar to the latter, — both of which rest on local symmetries, reflecting the needs of redundent degrees of freedom in a covariant description of the dynamics. This similarity in fact constitutes the major source of many attempts to formulate gravity as a gauge theory [1]. However, turning to quantum mechanics, an essential difference between gauge fields and gravity appears: Yang-Mills theory is renormalizable in four dimensions, while quantum theory of gravity is ill-behaved at the perturbative level, even when supersymmetry is taken into account [2]. In spite of this difference, one still expects that there should exist a unified framework within which gravity and gauge fields can be described in terms of a single theory, hoping that such a theory will eventually provide the correct (and thus nonperturbative) definition of quantum gravity.

A possible unified approach to gauge fields and gravity is the Kaluza-Klein formalism [3]. The starting point here consists of the "pure" gravitational theory on some $4+n$-dimensional manifold $M_4 \times K_n$,

$$S = \frac{1}{16\pi G_{4+n}} \int_{M_4 \times K_n} d^{4+n}x \sqrt{g}\, R. \tag{1}$$

By the standard compactification procedure, one ends up with a four-dimensional effective theory whose massless sector includes Yang-Mills fields as well as gravity on M_4. The gauge symmetries are determined by isometries of the internal space K_n; the four-dimensional Newton's constant G_4 depends linearly on the one in higher dimensions, via

$$G_{4+n} = G_4 \cdot \mathrm{Vol}(K_n). \tag{2}$$

Within this framework, we treat gravitational fields (in higher dimensions) as fundamental fields, and Yang-Mills fields are regarded as induced objects — which arise only after compactification taking place. Thus, in the Kaluza-Klein approach we have, sketchily

$$\text{(Higher-dimensional) Gravity } \supset \text{ (4-dimensional) Gauge Theory.} \tag{3}$$

The above theory with extra dimensions itself does not give a useful mechanism to avoid divergences. In fact, since the dimension of the gravitational coupling constant G_{4+n} takes the form l_p^{2+n} (where l_p denotes the planck length scale on $M_4 \times K_n$), perturbative quantization of gravity in higher dimensions will lead to even worse UV-behavior. The problem may be resolved by incorporating superstrings [4][5]. Actually, combination of the Kaluza-Klein idea with strings provides the most promising way to construct a unified theory with plusible low-energy limits.

There is another (perhaps less popular) possible framework to unify gauge interactions and gravity. This is the so called "induced gravity" theory due to Adler and Zee [6]. In such a theory, Yang-Mills and other matter are considered as fundamental fields, while gravity is not — it is supposed to be "induced" from those fundamental fields. In particular, the Newton's constant G_4 should be calculated from quantum fluctuations of the gauge fields, and the gravitational interactions should be thought of as a kind of the van der Waals force, arising from the "unscreened" or "residual" interactions of the more fundamental forces, namely the strong, weak and electric-magnetic interactions. Thus, sketchily, induced gravity claims that

$$\text{Gravity } \subset \text{ Gauge Theory.} \tag{4}$$

Adler-Zee's induced gravity has several attractive features (at least qualitatively). First, it provides a simple explanation of why the observed gravitational interactions are so weak compared to the other three interactions, and why gravity is a long-range force. Second, since the fundamental degrees of freedom in induced gravity include only gauge and matter fields, the theory is renormalizable. (However, as indicated in [7], when we acturally try to compute the effective Newton's constant, the theory loses its predictibility due to a Borel non-summable divergent series. This problem could be solved by considering supersymmetric extensions of induced gravity.)

Now, if we take both the possible relations (3)–(4) seriously, then it is quite natural to expect a sort of equivalence

$$\text{Gravity } \Longleftrightarrow \text{ Gauge Theory} \tag{5}$$

between gauge theory and gravity. Actually, there is a *holographic principle* [8] supporting the existence of such a duality, which highlights some unsusual quantum behavior of gravity and, in certain simple cases, it can be realized microscopically within string/M theory [9][10].

Holography

A complete explanation of the holographic principle would require many extra spaces, so here we will keep our discussion very brief and incomplete (for an extensive review of the holography, see [11]). Roughly speaking, this principle suggests that gravity in higher dimensions is equivalent to a certain QFT (without gravity, *e.g.* pure Yang-Mills theory) in lower dimensions. This differs from the usual Kaluza-Klein theory since after the K-K compactification one gets, in addition to the gauge theory, a sector containing gravity in lower dimensions, which is missing in the holographic theory. When holography is concerned, the higher dimensional spacetime where gravity lives is called the “bulk space”, while the lower dimensional spacetime in which the ordinary QFT is defined sometimes referes to the “boundary space”. Holography implies the existence of a duality (5) between the bulk and the boundary theories.

One characteristic feature of holography is the UV-IR relations. Using an IR-cutoff $L_{\max}$ in the bulk amounts to introducing a UV-cutoff $E_{\max}$ on the boundary, and, in its simplest case, the relation between $L_{\max}$ and $E_{\max}$ reads [12]

$$E_{\max} \sim L_{\max}/l^2 \tag{6}$$

where l denotes some fundamental length scale of the theory, such as the Planck length l_p or the string length scale $l_s \equiv \sqrt{\alpha'}$. Thus, the holographic principle predicts not only the equivalence between the bulk and the boundary theories without cutoffs, but it also predicts the equivalence between the regulated versions of these theories. In particular, the number of degrees of freedom N_{dof} in the two (regulated) theories should be the same.

Consider, for example, a theory of gravity in some $D+1$ dimensional spacetime M_{D+1}. The spacelike part of M_{D+1} constitutes a codimension one subspace $V_D \subset M_{D+1}$. If this theory could be quantized following the ordinary procedure as in any QFT, then the total number of degrees of freedom would be counted by $N_{\text{dof}} \sim V_D/l_p^D \sim (L/l_p)^D$, where L is the (regulated) size of the gravitational system. But this is a wrong answer, since according to the Bekenstein-Hawking formula, N_{dof} cannot exceed the entropy of a black hole with the horizon area $A_{D-1} \sim L^{D-1}$:

$$N_{\rm dof} \leq S_{\rm BH} = \frac{A_{D-1}}{4G_{D+1}} \sim (L/l_p)^{D-1}. \tag{7}$$

The above formula shows that the quantum theory of gravity, if it exists, is indeed very different from any other QFT, and its degrees of freedom are apparently localized on some surface A_{D-1} of codimension 2 in the spacetime, much less than the naively expected result. If we compare (7) with the entropy of the ordinary quantum field system living in some D-dimensional spacetime $M_D \sim A_{D-1} \times \mathbb{R}$, we find that the number of degrees of freedom in the bulk theory and that in the boundary theory match perfectly, upto an order of one numeric factor. Hence, it is plusible to make the holographic assumption that quantum theory of gravity can be described by an ordinary QFT in lower dimensions. The UV-IR relation (6) may then be roughly understood as follows: On the bulk side, the more stable a black hole is — and thus the larger its gravitational radius L becomes, the more precisely a point can be localized on the horizon, since for L large we can neglect the fuzziness of the horizon due to backreaction of the Hawking radiation. For an observer in the boundary theory, however, localizing a point more precisely requires higher energy. Thus we must have a certain UV-IR relation like (6).

As another simple consequence of (7), notice that each degree of freedom can store a single bit of information, so the information density is at most given by $N_{\rm dof}/A_{D-1} \sim 1/G_{D+1}$, and hence it cannot exceed one bit per Planck area. If we keep the number of degrees of freedom fixed, then there will exist a lower bound on the size L of the gravitational system:

$$L \geq \text{const.}\, N^{\frac{1}{D-1}} \cdot l_p. \tag{8}$$

For example, in eleven dimensional supergravity we have $D + 1 = 11$, so that (8) becomes $L \geq N^{\frac{1}{9}} \cdot l_p$, where l_p is the Planck length in eleven dimensions.

Given a higher dimensional gravity system, the holographic principle itself does not tell us how to construct a concrete model of the corresponding boundary theory. Explicit construction of such a boundary theory calls for the details of bulk dynamics at the microscopic level. Since the bulk involves gravity, its microscopic description should be a closed string theory. The non-gravitational boundary theory, on the other hand, should come from the low-energy limit of open strings. So to get a microscopic understanding of the holographic principle, we have to compare closed string theory with open string theory carefully, hoping that they are somehow equivalent or at least can be used to describe some common objects.

Perturbatively, the theory of closed strings differs from open string theory in many ways. They have a different spectrum and a different low-energy limit — closed strings reduce to (super) gravity in the low-energy approximation, while the effective theory of open strings contains (super) Yang-Mills fields. In addition to these differences, the perturbative structures of the two kinds of strings look quite asymmetric: for topological reasons, one can have a perturbative theory that consists of closed strings only, but any consistent open string theory must also include a closed string sector. This asymmetry manifests even at the massless level. In fact, while pure gravity can exist without gauge or other matter fields, Yang-Mills theory will necessalily couple to gravity at a sufficiently high energy scale. Thus, perturbatively, closed string theory looks far from equivalent to open string theory.

One may ask whether the above asymmetry will disappear if nonperturbative effects are taken into account. In other words, can the nonperturbative sector of closed strings contain open string excitations? To see nonperturbative effects of closed strings, one must increase the string coupling constant g_s. This amounts to increasing the ten-dimensioanl Newton's constant

$$G_{10} = 8\pi^6 g_s^2 \alpha'^4 \tag{9}$$

in the low-energy effective theory. Evidently, when G_{10} is taken to be very large, gravitational interactions become so strong that horizons will appear. If there is some object crossing such a horizon, one cannot observe that object as a whole, but only its part outside the horizon. In this case a closed string crossing the horizon looks just like an open string whose ends are at the horizon, — there are no observable meanings to talk about the hidden part of closed strings within a horizon! Of course, in order that such a configuration may happen, one requires that at least some of the horizons should be preserved when going from the effective gravitational theory to the original closed string theory. Technically this problem can be simplified by considering BPS solutions in supergravity [13], which have the physical interpretation as certain extremal black holes. The resulting spacetime structure do not totally break down supersymmetry, so they are preserved at the stringy level, constituting D(irichlet) p-branes [14]. In other words, D branes provide a natural microscopic description of extremal black holes [15]. These are solitonic objects in string theory carrying the Ramond-Ramond charges.

This suggests that closed string theory contains a nonperturbative sector consisting of open strings which end at some Dirichlet p-branes. The presence of such D branes makes the structures of open and closed strings quite symmetric. Consequently, we have two ways to see the low-energy behavior of D branes, one from closed strings and the other from open strings. In the closed string description, the low-energy limit of D branes is considered as the charged sources of extremal black holes. This description is valid only for observers far from D branes (*i.e.* $L \gg l_s$), since at that scale open strings are too heavy ($M \sim L/l_s^2 \gg 1/l_s$) to be excited, and the resulting "bulk" theory contains only gravitational excitations. From the open string point of view, we have another effective description of D p-branes, which is based on ten-dimensional supersymmetric Yang-Mills theory dimensionally reduced to the brane worldvolume [16]. The latter (or roughly the "boundary theory") is valid only for observers very close to the branes (*i.e.* $L \ll l_s$), so that he can find the light modes ($M \sim L/l_s^2 \ll 1/l_s$) of open strings propagating as the worldvolume gauge fields.

Of course, either the bulk gravity or the boundary gauge theory so obtained only describes a simplified D brane system where some couplings to other excitations are truncated. One has no *a priori* reasons to expect these descriptions to be equivalent to each other, since in general the truncations are quite different, made at different scales. In particular these two theories may have different global symmetries and contain different numbers of degrees of freedom. However, in some cases [9][10] we can take a suitable decoupling limit, under which the bulk theory becomes equivalent to the boundary theory, so that the exact gravity/gauge theory correspondence is achieved and can be tested explicitly. One such example is the BFSS matrix model [9] which we will briefly review now.

The BFSS Theory

Consider a system of N D 0-branes. When these branes are decoupled from all the other degrees of freedom, they can have a "boundary" description as the dimensional reduction of $D = 10$ supersymmetric $U(N)$ gauge theory to 0+1 dimensions. Thus, the bosonic degrees of freedom of this theory consist of nine $N \times N$ hermitian matrices, $X^i(t)$, with their super partners, $\theta(t)$. The Lagrangian reads [9]

$$\mathcal{L} = \frac{1}{2R}\left[\sum_{i=1}^{9} \mathrm{Tr}\dot{X}^i\dot{X}^i + 2\theta^T\dot{\theta} - \frac{1}{2l_s^4}\sum_{i\neq j}\mathrm{Tr}[X^i, X^j]^2 - \frac{2}{l_s^2}\theta^T\gamma_i[\theta, X^i]\right], \quad R \equiv g_s l_s \qquad (10)$$

which has the fermionic conservative charges $Q^\alpha = \frac{1}{R}\mathrm{Tr}(\gamma_i{}^{\beta\alpha}\theta_\beta\dot{X}^i - \frac{i}{2l_s^2}\gamma_{ij}{}^{\beta\alpha}\theta_\beta[X^i, X^j])$ and $q^\alpha = \frac{1}{R}\mathrm{Tr}\theta^\alpha$. The canonical relations $\{\theta^\alpha, \theta^\beta\} = R\delta^{\alpha\beta}$ implies that upto gauge transformations these charges form the SUSY algebra

$$\{Q^\alpha, Q^\beta\} = H\delta^{\alpha\beta},$$
$$\{q^\alpha, q^\beta\} = \frac{N}{R}\delta^{\alpha\beta},$$

$$\{Q^\alpha, q^\beta\} = \frac{\mathrm{Tr}\dot{X}^i}{R}{\gamma_i}^{\alpha\beta} \tag{11}$$

here H is the Hamiltonian derived from (10). The bosonic light modes correspond to diagonal elements x_a^i ($1 \le a \le N$) of the matrix X^i, which can be thought of as position variables of the ath D 0-brane in 9D spacial directions $i = 1, \cdots, 9$. By integrating out heavy modes (*i.e.*, off-diagonal elements of X^i), the effective interaction between x_a and x_b takes the form [17]

$$V(r, v) \sim \frac{l_s^9}{R^3}\frac{v^4}{r^7}, \quad r = |x_a - x_b|, \quad v = |\dot{x_a} - \dot{x_b}|. \tag{12}$$

Hence, effectively, the boundary theory is a supersymmetric extension of the N-body particles living in 9+1 dimensional spacetime, with the velocity-dependent pair interactions (12).

Let us turn to the "bulk" description, where D 0-branes are contained in the nonperturbative sector of closed strings (specifically, IIA strings). Such solitonic objects will become light and hence can be seen in the low-energy effective theory provided the string coupling is sufficiently strong. When $g_s \to \infty$, the IIA string theory goes to $D = 11$ M-theory, in which a new dimension x^{11} of size $R = g_s l_s$ gets uncompactified, and the BPS state of N D 0-branes becomes a Kaluza-Klein excitation with the longitudinal momentum $p_{11} = N/R$. For generic p_{11} these K-K modes will couple to some other degrees of freedom in the bulk in a rather complicated way, so it is not immediately possible to identify the low-energy limit of M-theory (namely $D = 11$ supergravity) with the boundary matrix theory considered in the foregoing paragraph, — the matrix theory is supposed to describe D 0-branes decoupled from all other degrees of freedom. Indeed, the global symmetry of M theory in flat background should contain $D = 11$ super Lorentz group, which is definitely different from the global symmetry (11) in matrix theory.

The key observation made by Banks, Fischler, Shenker and Susskind [9] is that taking the infinite momentum frame (IMF) $p_{11} \to \infty$ in M-theory will force these K-K modes (or D 0-branes) to decouple from all other degrees of freedom. The decoupling limit obviously requires us to send N to infinity. Thus, as in [9], one expects that M-theory in IMF is equivalent to the large N matrix model (10). Note that $p_{11} \to \infty$ is a kinda non-relativistic limit, under which the super Lorentz symmetry in M-theory reduces to the super Galilean symmetry, whose algebra coincides exactly with (11) after the identification

$$p_{11} \longleftrightarrow \frac{N}{R}, \quad p_i \longleftrightarrow \frac{\mathrm{Tr}\dot{X}^i}{R}, \qquad E \equiv \frac{p_i^2 + M^2}{2p_{11}} \longleftrightarrow H \tag{13}$$

where p_i is the transverse momentum and E the light-cone energy of a fundamental particle in M-theory. Such particles should describe supergravitons in the low-energy limit. Since

$\dot{X}^i/R$ is also the canonical momentum conjugate to X^i in matrix theory, (13) leads to the correspondence between a BPS bound state of N D 0-branes and a supergraviton in 11D with the longitudinal monentum $p_{11} \sim N/R$. Moreover, the two-body potential (12) produces the correct long-range interaction between two supergravitons in flat background [9], which gives a satisfactory way to calculate the 11D Newton's constant $G_{11} \sim l_p^9$ using matrix quantum mechanics, and the result turns out to be consistent with the IIA strings/M-theory duality:

$$l_p = g_s^{1/3} l_s. \tag{14}$$

As reviewed by various authors [18], M(atrix) theory in flat spacetime has in fact passed over many other consistent checks.

Matrices in de Sitter Space

Several candidates for the holographic description of de Sitter gravity were proposed (see [19] for a review); here I will only recall a model [20] that mimics M(atrix) theory in flat background. In the flat space case bulk symmetries are described by the (super) Poincaré group; mathematically, taking IMF amounts to performing the non-relativistic Inönü-Wigner contraction of this kind of symmetries, which gives rise to the (super) Galilean algebra (11). As long as the non-relativistic symmetries are found out, we can construct the matrix quantum mechanics in a strightforward way. Now in de Sitter space one could do similar things.

So let us consider the non-relativistic limit of the de Sitter group. This will result in the Newton-Hooke symmetries [20][21] whose Lie algebra generators contain d momentum operators P_i, $d(d-1)/2$ angular momentum operators J^{ij}, d boosts K^i, as well as a Hamiltonian H. The algebra so constructed can be viewed as a curved space generalization of the Galilean algebra. Explicitly, spacetime transformations generated by P_i, J^{ij}, K^i, H take the following form:

$$\begin{aligned} x^i &\rightarrow x'^i = \mathcal{R}^i{}_j \cdot x^j + v^i R \sinh\frac{t}{R} + a^i \cosh\frac{t}{R} \\ t &\rightarrow t' = t + b \end{aligned} \tag{15}$$

where R denotes the "size" of de Sitter space, $\mathcal{R} = (\mathcal{R}^i{}_j) \in SO(d)$ is a space rotation generated by the angular momentum operators, v^i is a "velocity" corresponding to the boosts, and a^i, b are spacetime translations generated by the momentum operators and the Hamiltonian, respectively. It is clear that (15) reduces to the usual Galilean transformations in the flat space limit $R \to \infty$.

We can now write down a matrix model with (15) as its symmetry group. The Lagrangian looks like:

$$\mathcal{L} = \mathrm{Tr}\left\{\frac{m}{2}\,(\dot{X}^i)^2 + \frac{m}{2R^2}\,(X^i)^2 + \frac{m}{4g^2R^4}\,[X^i, X^j]^2 + \cdots\right\}. \tag{16}$$

In the above expression, m is the mass of some "matrix particle" described by the non-commuting spatial coordinates X^i, $g \sim G\Lambda^{(d-1)/2}$ is the dimensionless coupling constant, and "$\cdots$" denotes other terms, possibly including those proportional to a central extension of the Newton-Hooke algebra or even comming from fermionic contributions. Our convention is that the cosmological constant term in the Hilbert-Einstein action takes the form $I = \frac{\Lambda}{16\pi G}\int d^{d+1}x\sqrt{-g}$, so that $\Lambda = d(d-1)/2R^2$ has mass dimension 2.

This model is indeed manifestly invariant under the Newton-Hooke transformations

$$X^i \to X^i + \left(v^i R \sinh\frac{t}{R} + a^i \cosh\frac{t}{R}\right)\mathbf{1}_{N\times N}. \tag{17}$$

Note that (17) acts only on the center of mass of the system, $X^i_{\mathrm{c.m.}}$, and keeps the part X^i_{rel} of relative motion intact, where $X^i_{\mathrm{c.m.}}$ and X^i_{rel} are defined by

$$X^i_{\mathrm{c.m.}} = \frac{\mathrm{Tr}X^i}{N}\,\mathbf{1}_{N\times N}, \quad X^i_{\mathrm{rel}} = X^i - X^i_{\mathrm{c.m.}}. \tag{18}$$

Since $X^i_{\mathrm{c.m.}}$ is a multiple of the identity and X^i_{rel} is a traceless matrix, we have $\mathrm{Tr}(X^i_{\mathrm{c.m.}}X^i_{\mathrm{rel}}) = [X^i_{\mathrm{c.m.}}, (\cdots)] = 0$, and therefore (16) is decomposed into two decoupled pieces. The commutator terms appear only in the relative motion part of the Lagrangian.

After rescaling $X^i \to gR^2X^i$ (so that the redefined matrix variables X^i have mass dimension 1), the Lagrangian (16) gives rise to the following classical equations of motion

$$\frac{d^2X^i}{dt^2} + [X^j, [X^j, X^i]] = \frac{2\Lambda}{d(d-1)}X^i. \tag{19}$$

Presumably, such equations of motion should describe an empty universe with a positive cosmological constant. Some interesting cosmological consequences of (19) are explored in [22][23].

Incorporation of Matter

Now we are led to the following question [24]: how can one incorporate matter sources into the above matrix model? In order to give a complete answer, one has to solve a rather difficult problem, namely formulating M(atrix) theory in nontrivial background spacetime geometries.

This problem is closely related to certain subtle issues such as background independence and covarianting M(atrix) dynamics, which we know little at present. Our approach is therefore very preliminary.

One natural guess from holography is that matrix equations such as (19) could be considered as a microscopic version of Einstein's gravitational theory. Thus, if a matter component is present, it should couple to (19) via its energy-momentum tensors $T_{\mu\nu}$. Lorentz invariance of matter sources requires that the quantity really entering into the equations should be the trace $T^{\mu}{}_{\mu}$. Since we are interested in cosmology here, we can assume that the matter is described by a pefect fluid, so that $T^{\mu}{}_{\mu} = \rho + d \cdot p$ depends only on time t. Let us restrict ourselves to the case of three spatial dimensions ($d = 3$), though extension to other dimensions should be straightforword. The equations of motion (19) are then modified by a new term, proportional to $(\rho + 3p)$:

$$\frac{d^2 X^i}{dt^2} + [X^j, [X^j, X^i]] = \left(\frac{\Lambda}{3} - \frac{4\pi G}{3}(\rho + 3p) \right) X^i. \tag{20}$$

Note that as a matrix model, (20) itself is not in a Lorentz covariant form.

We now consider possible solutions to the above equations of motion, paying particular attention to those relevant to cosmology. Thus, according to the cosmological principle, we may take the homothetic ansatz [22][23][25] $X^i(t) = a(t)M^i$, where $a(t)$ is a scale factor characterizing the spatial size of the universe and M^i denote the co-moving matrix coordinates, which are independent of time. With this ansatz (20) becomes

$$\left(\frac{\ddot{a}(t)}{a^3(t)} + \frac{4\pi G}{3a^2(t)}(\rho + 3p) - \frac{\Lambda}{3a^2(t)} \right) M^i = -[M^j, [M^j, M^i]], \tag{21}$$

so we get a couple of conditions

$$\mu M^i + [M^j, [M^j, M^i]] = 0 \tag{22}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} + \mu a^2 \tag{23}$$

with μ being a numeric constant. Note that (23) looks very similar to the second Friedmann equation with matter sources ρ and p, plus the standard cosmological constant term $\Lambda/3$ as well as a somewhat exotic "matrix term" μa^2.

The first Friedmann equation can be derived from (23) as a first integral. To see this, let us multiply both sides of (23) by $a\dot{a}$ and rewrite the equation as:

$$\frac{1}{2}\frac{d}{dt}\dot{a}^2 = -\frac{2\pi G}{3}(\rho + 3p)\frac{d}{dt}a^2 + \frac{\Lambda}{6}\frac{d}{dt}a^2 + \frac{\mu}{4}\frac{d}{dt}a^4 \tag{24}$$

Since $\rho + 3p$ depends only on t, we can think of it as a function of a. Thus, introducing an auxiliary function $F \equiv F(a^2)$ such that

$$\frac{dF(a^2)}{da^2} = -(\rho + 3p) \tag{25}$$

the first term in the right hand side of (24) then becomes a total derivative $\frac{2\pi G}{3}\frac{dF(a^2)}{dt}$. It follows that (23) has the first integral

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{4\pi G}{3}\frac{F(a^2)}{a^2} + \frac{\Lambda}{3} + \frac{\mu a^2}{2} \tag{26}$$

where k is an integration constant. Comparing this with the standard Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} + \frac{\mu a^2}{2} \tag{27}$$

and using the definition (25), we find

$$\rho = \frac{F(a^2)}{2a^2}, \quad p = -\frac{1}{3}\left(\frac{dF(a^2)}{da^2} + \frac{F(a^2)}{2a^2}\right). \tag{28}$$

This is the condition that our matrix model will reproduce the Friedmann cosmology with matter sources.

A First Look at Matrix Cosmology

Given an equation of state of the matter source, the condition (28) will impose some constraints on the admissible form of the auxiliary function $F(a^2)$. Suppose, for example, that the universe described by our matrix model has only one matter component, obeying the simple equation of state:

$$p = w\rho, \quad w \equiv const. \tag{29}$$

Then (28) gives rise to the following differential equation for $F(a^2)$:

$$F'(a^2) + \frac{1+3w}{2a^2}F(a^2) = 0 \tag{30}$$

The general solutions of this differential equation can be easily found

$$F(a^2) = c \cdot a^{-(1+3w)}, \quad c = const. \tag{31}$$

and, accordingly, we have

$$\rho = \frac{c}{2} \cdot a^{-3(1+w)}, \quad p = \frac{wc}{2} \cdot a^{-3(1+w)}. \tag{32}$$

This form agrees completely with the form of matter sources in the standard cosmology (e.g. $w = 1/3 \leftrightarrow$ radiation, $w = 0 \leftrightarrow$ "dust", $w < 0 \leftrightarrow$ dark energy, $w < -1 \leftrightarrow$ phantom etc.). Of course, in general the energy density ρ will contain different components, each with a different equation of state $p_I = w_I \rho_I$. This can be easily realized in our model if the auxiliary function takes the general form:

$$F(a^2) = \sum_I c_I a^{-(1+3w_I)}, \quad c_I = const. \tag{33}$$

Consequently, in the regime when the μa^2 term in (27) is negligible, our model will share with the convetional appraoches essentially all the same features, at least at the classical level. In order that (27) are consistent with astrophysical observations, the constant μ should not be too large. This raises a new problem similar to the (old) cosmological constant problem. Pehaps these two apparently different problems originate from one source.

From Eq.(27) we see that the μ-term is not important in the early universe, when the scale factor $a(t)$ is small. It will have considerable effects in later time universe, however. There may have two different cases that should be considered separately: (i) $\mu > 0$, and (ii) $\mu < 0$. In the first case, the μ-component can be equivalently described by a phantom energy source, with $w_\mu = -5/3$. The appearance of such a source seems to be compatible with the most classical tests of cosmology based on current data [26]. The second case ($\mu < 0$) seems physically problematic since it describes a certain "anti-phantom" energy with $\rho_\mu < 0$ and $p_\mu = -5\rho_\mu/3 > 0$. When such a source dominates the universe at later time, the scale factor will have a maximal value determined by $a_{max} \sim (2k/|\mu|)^{1/4}$ (corresponding to a closed universe $k \sim 1$), and there will have no real function solutions to the Friedmann equation when a exceeds this maximal value.

As indicated in [23], the case (ii) corresponds to the most symmetric solutions to the static matrix equations (22). In fact, let J^i be a basis of $SU(2)$ generators (in the N-dimensional irreducible representation), satisfying the standard commutation relations $[J^i, J^j] = i\epsilon^{ijk} J_k$. One find that $M^i = \sqrt{-\frac{\mu}{2}} J^i$ solve (22), and the heimitian condition for M^i requires $\mu < 0$. This solution defines a fuzzy sphere of radius

$$\sum_i (M^i)^2 = -\frac{\mu}{8}\,(N^2 - 1) \sim \frac{k}{4a_{max}^4}\,(N^2 - 1) \tag{34}$$

As we mentioned ealier, our theoretical understanding of the matrix model is rather incomplete, and we expect that problems with $\mu < 0$ may be resolved within a more accurate formulation. Some unusual but possible couplings that we omitted in (20) may have positive phantom energies that could cancel out the anti-phantom contribution from the μ-term.

Consider, for instance, a rank four "RR background field" $F^{(4)}_{0ijk} \sim -f\epsilon_{ijk}$ coupled to our matrix system, similar to that studied by Myers [27]. This would add an additional term $\frac{i}{3}F^{(4)}_{0ijk}X^iX^jX^k$ to the Lagrangian, so that the equations of motion (20) receive a correction:

$$\frac{d^2X^i}{dt^2} + [X^j,[X^j,X^i]] + if\epsilon_{ijk}[X^j,X^k] = \left(\frac{\Lambda}{3} - \frac{4\pi G}{3}(\rho+3p)\right)X^i. \tag{35}$$

Solving this equation by the ansatz $X^i = \sqrt{-\frac{\mu}{2}}\,a(t)\,J^j$, one finds the following Friedmann-like equation

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} + \frac{\mu a^2}{2} + \frac{\sqrt{-2\mu}\,f}{3}\,a \tag{36}$$

The new term, $\frac{\sqrt{-2\mu}\,f}{3}\,a$, corresponds to a phantom source ($w_f = -4/3$) with positive energy density (though it cannot be accuturely used to cancel out the μ-component contribution).

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References

[1] See, *e.g.*, R. Utiyama, "Invariant theoretical interpretation of interaction", Phys. Rev. **101** (1956) 1597.

[2] S. Deser, "Nonrenormalizability of (last hope) $D = 11$ supergravity, with a terse survey of divergences in quantum gravities", `hep-th/9905017`.

[3] See M. J. Duff, B. E. W. Nilsson, and C. N. Pope, "Kaluza-Klein supergravity", Phys. Rep. **130** (1986) 1, and references therein.

[4] M. Green, J. Schwarz and E. Witten, "Superstring theory" (I)(II), Cambridge University Press, 1987.

[5] J. Polchinski, "String theory" (I)(II), Cambridge University Press, 1998.

[6] S. L. Adler, "Einstein gravity as a symmetry breaking effect in quantum field theory", Rev. Mod. Phys. **54** (1982) 729; A. Zee, "Calculating Newton's gravitational constant in infrared stable Yang-Mills theories", Phys. Rev. Lett. **48** (1982) 295.

[7] F. David, "A comment on induced gravity", Phys. Lett. **B138** (1984) 383; F. David and A. Strominger, "On the calculability of Newton's constant and the renormalizability of scale invariant quantum gravity", Phys. Lett. **B143** (1984) 125.

[8] G. 't Hooft, "Dimensional reduction in quantum gravity", gr-qc/9310026; L. Susskind, "The world as a hologram", J. Math. Phys. **36** (1995) 6377.

[9] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, "M theory as a matrix model: a conjecture", Phys. Rev. **D55** (1997) 112, hep-th/9610043.

[10] J. Maldacena, "The large N limit of superconformal field theories and supergravity", Adv. Theor. Math. Phys. **2** (1998) 231, hep-th/9711200; S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "Gauge theory correlators from non-critical string theory", Phys. Lett. **B428** (1998) 105, hep-th/9802109; E. Witten, "Anti-de Sitter space and holography", Adv. Theor. Math. Phys. **2** (1998) 253, hep-th/9802150.

[11] D. Bigatti and L. Susskind, "TASI lectures on the holographic principle", hep-th/0002044.

[12] L. Susskind and E. Witten, "The holographic bound in anti-de Sitter space", hep-th/9805114; A. W. Peet and J. Polchinski, "UV/IR relations an AdS Dynamics", hep-th/9809022.

[13] For a review of such solutions, see *e.g.* K. S. Stelle, "BPS branes in supergravity", hep-th/9803116.

[14] J. Polchinski, "TASI lectures on D-branes", hep-th/9611050.

[15] A. Strominger and C. Vafa, "Microscopic origin of the Bekenstein-Hawkin entropy", hep-th/9601029.

[16] E. Witten, "Bound states of strings and p-branes", hep-th/9510135.

[17] U. H. Danielsson, G. Ferretti and B. Sundborg, "D-particle dynamics and bound states", hep-th/9603081; D. Kabat and P. Pouliot, "A comment on zero-brane quantum mechanics", hep-th/9603127; M. R. Douglas, D. Kabat, P. Pouliot and S. Shenker, "D-branes and short distance in string theory", hep-th/9608024.

[18] R. Dijkgraaf, E. Verline and H. Verlinde, "Notes on matrix and micro strings", hep-th/9709107; A. Bilal, "M(atrix) theory: a pedagogical introduction", hep-th/9710136; T. Banks, "Matrix theory", hep-th/9710231; D. Bigatii and L. Susskind, "Review of Matrix theory", hep-th/9712072; W. Taylor IV, "The M(atrix) model of M-theory", hep-th/0002016.

[19] M. Spradlin, A. Strominger and A. Volovich, "Les Houches Lectures on de Sitter Space", hep-th/0110007.

[20] Y. h. Gao, "Symmetries, Matrices, and de Sitter Gravity", hep-th/0107067.

[21] G. W. Gibbons and C. E. Patricot, "Newton-Hooke Spacetime, Hpp Waves and the Cosmological Constant", Class. Quant. Grav. **20** (2003) 5225, hep-th/0201101.

[22] E. Alvarez and P. Meesen, "Newtonian M(atrix) Cosmology", Phys. Lett. **B 426** (1998) 282, hep-th/9712136.

[23] D. Z. Freedman, G .W. Gibbons and M. Schnabl, "Matrix Cosmology", hep-th/0411119.

[24] H. z. Chen, Y. h. Gao, W. s. Xu and D. f. Zeng, Work in progress.

[25] J. Hoppe, “Some Classical Solutions of Matrix Model Equations”, *Prepared for 4th Annual European TMR Conference on Integrability Nonperturbative Effects and Symmetry in Quantum Field Theory, Paris, France, 7-13 Sep 2000*.

[26] See e.g. R. R. Caldwell, “A Phantom Menace?”, astro-ph/9908168.

[27] R. C. Myers, “Dielectric branes”, JHEP 9912 (1999) 022, `hep-th/9910053`.