Topological strings and Calabi-Yau spaces

Minxin Huang

University of Science and Technology of China

Xi'an Workshop 2014

Outline of the Talk

- Part I: background and introduction, some previous works.
  - Background and introduction
  - Toward solving compact Calabi-Yau spaces.
  - An application to counting of 5D $\mathcal{N} = 2$ black hole entropy
- Part II: Our recent works.
  - Refined topological string.
  - Nekrasov-Shatashvili limit.
  - Matrix model and β-deformation.
  - Quantum integrable system.
- Conclusion and future directions.

Part I

1.1. Background and Introduction

Reference:

"Mirror Symmetry", American Mathematical Society (August 19, 2003), by Kentaro Hori, Sheldon Katz, Albrecht Klemm, Rahul Pandharipande, Richard Thomas, Cumrun Vafa, Ravi Vakil, Eric Zaslow.

Origin of topological strings

- In the 1980's, during the first string revolution, it was realized that string theory is a consistent theory of quantum gravity.

- There are 5 perturbative formulations of consistent string theory. It is most promising to construct models of particle physics from $E_8 \times E_8$ heterotic string compactified on Calabi-Yau 3-folds (large enough gauge group, $\mathcal{N} = 1$ supersymmetry). However it turns out to be difficult to make precise quantitative physical predictions.

- String theorists continue to study Calabi-Yau manifolds. Mirror Symmetry was discovered in early 1990's. Greene et al constructed a pair of Calabi-Yau 3-folds and conjectured that the type IIA and IIB theories are dual on the pair. Candelas et al many pairs of Calabi-Yau 3-folds with the hodge numbers $h^{(1,1)}$ and $h^{(1,2)}$ exchanged, and used the duality to count spheres on Calabi-Yau 3-folds.

- Mathematicians (e.g. Yau) were intrigued by Candelas et al's calculations. Witten provided the world-sheet formulation of topological string theory. Vafa et al developed many tools for calculating topological string partition functions.

- Topological strings become an independent branch in string theory, with active research up to today.

Motivation: Why study topological strings?

- Topological strings arise in compactification of superstring theory on Calabi-Yau manifolds. Many important phenomenological quantities in 4-D effective field theory, e.g. number of chiral fermion generations, Yukawa couplings, are related to topological invariants of the compact manifold.

- Geometric engineering can relate physical (strong coupling) questions of 4-D quantum field theory to geometric questions of Calabi-Yau manifolds.

- Topological string theory is a tractable, computable sector of superstrings, and is an ideal setting to study fundamental ideas e.g. D-brane, S-duality, open/close string duality, etc.

- Mirror symmetry relates topological A-model on manifold X to topological B-model on its mirror manifold. Some very difficult mathematical problems of enumerative geometry can be easily solved by topological B-model methods. Mathematicians have studied these enumerative problems for decades or centuries.

- Relations to matrix models, quantum integrable systems, black hole physics, etc.

Some review on supersymmetry

- $\mathcal{N} = (2,2)$ supersymmetry in 2 dimension
  - Bosonic coordinates x^0, x^1, or $x^\pm := x^0 \pm x^1$.
  - Fermionic coordinates $\theta^+, \theta^-, \bar{\theta}^+, \bar{\theta}^-$. The bar is complex conjugate.
  - Superfield $F(x^0, x^1, \theta^+, \theta^-, \bar{\theta}^+, \bar{\theta}^-) = f(x^0, x^1) + \cdots$.
  - Define operators $D_\pm = \frac{\partial}{\partial \theta^\pm} - i\bar{\theta}^\pm \partial_\pm$, $\bar{D}_\pm = \frac{\partial}{\partial \bar{\theta}^\pm} - i\theta^\pm \partial_\pm$.
  - Some definitions. Chiral superfield $\bar{D}_\pm \Phi = 0$, anti-chiral superfield $D_\pm \bar{\Phi} = 0$, twisted chiral superfield $\bar{D}_+ U = D_- U = 0$, twisted anti-chiral superfield $D_+ \bar{U} = \bar{D}_- \bar{U} = 0$.

- Supersymmetric action:

  - D-term $\int d^2x d^4\theta K(F)$
  - F-term $\int d^2x d^2\theta W(\Phi) + c.c.$
  - Twisted F-term $\int d^2x d\theta^+ d\bar{\theta}^- W(U) + c.c.$

  Here W is a holomorphic function, Φ and U are chiral and twisted chiral superfields.

- Example: theory of a chiral superfield

$$\begin{aligned} S &= S_{kinetic} + S_{potential} \\ &= \int d^2x d^4\theta \bar{\Phi}\Phi + \int d^2x d^2\theta W(\Phi) + c.c. \end{aligned}$$

Non-linear Sigma Models and Landau-Ginzburg Models

- Consider a Kahler manifold M with complex coordinates $\Phi^i, \bar{\Phi}^i$, Kahler potential $K(\Phi^i, \bar{\Phi}^i)$, Kahler metric $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K(\Phi^i, \bar{\Phi}^i)$. We can construct $\mathcal{N} = (2,2)$ Lagrangian with kinetic term

$$\mathcal{L} = \int d^4\theta K(\Phi^i, \bar{\Phi}^i).$$

  The bosonic part is $\mathcal{L} = -g_{i\bar{j}}(\phi^k, \bar{\phi}^k)\partial^\mu \phi^i \partial_\mu \bar{\phi}^i + \cdots$. This is known as the non-linear sigma model on M.

- We can add a F-term $\mathcal{L}_W = \int d^2\theta W(\Phi^i) + c.c.$ This is known as the Landau-Ginzburg model.

- Vector R-symmetry $U(1)_V$ and axial R-symmetry $U(1)_A$

$$
\begin{aligned}
U(1)_V : \quad F(x^\mu, \theta^\pm, \bar{\theta}^\pm) &\rightarrow e^{i\alpha q_V} F(x^\mu, e^{-i\alpha}\theta^\pm, e^{i\alpha}\bar{\theta}^\pm), \\
U(1)_A : \quad F(x^\mu, \theta^\pm, \bar{\theta}^\pm) &\rightarrow e^{i\beta q_A} F(x^\mu, e^{\mp i\beta}\theta^\pm, e^{\pm i\beta}\bar{\theta}^\pm),
\end{aligned}
$$

  where q_V and q_A are R-charges assigned to the super field F.

- Somethings special.

  - The $U(1)_V$ R-symmetry is broken classically unless W is quasi-homogeneous, i.e. one can assign R-charges such as $W(\lambda^{q^i}\Phi^i) = \lambda^2 W(\Phi^i)$.

  - The $U(1)_A$ R-symmetry is broken by anomaly if $c_1(M) \neq 0$.

- Witten showed that non-linear sigma models and Landau-Ginzburg models can be derived from the gauged linear sigma model, leading to the Calabi-Yau/Landau-Ginzburg correspondence. Topology change in Calabi-Yau spaces is a smooth physical process when one includes stringy degrees of freedom.

Topological twist

- So far we consider a flat two-dimensional world-sheet. The supersymmetry is lost on a general curved Riemann surface. In order to preserve supersymmetry, we implement a trick known as the topological twist.

- The 2d Lorentz group is $SO(1,1)$. Consider the Euclidean version $SO(2)$. We replace the 2-d Euclidean group $SO(2) = U(1)_E$ by the diagonal group of $U(1)_E \times U(1)_R$, where $U(1)_R$ is a global R-symmetry.

- Topological twist modifies the spins of operators, and make the theory invariant under local change of world sheet metric.

- Choice of R-symmetry

$$U(1)_R = U(1)_V \rightarrow \text{A-twist} \rightarrow \text{A-model}$$
$$U(1)_R = U(1)_A \rightarrow \text{B-twist} \rightarrow \text{B-model}$$

A-model and B-model

- Physical operators are defined by the Q-cohomology, where Q is the nilpotent supercharge

  – A-model: $Q_A = \bar{Q}_+ + Q_-$

  – B-model: $Q_B = \bar{Q}_+ + \bar{Q}_-$

  Here the Q_A and Q_B become scalar under A-twist and B-twist respectively.

- For A-model, the physical operators correspond to different forms in M, and Q_A-cohomology is the de Rahm cohomology.

- For B-model, the physical operators are

$$w_{\bar{i}_1\cdots\bar{i}_p}^{j_1\cdots j_q} d\bar{z}^{\bar{i}_1}\cdots d\bar{z}^{\bar{i}_p}\frac{\partial}{\partial z^{j_1}}\cdots\frac{\partial}{\partial z^{j_q}}.$$

  The Q_B-cohomology is identified with the Dolbeault cohomology group $\bigoplus_{p,q=0}^{n} H^{(0,p)}(M, \wedge^q T_M)$.

- Physical observables are correlation functions of the physical operators.

Coupling to gravity: topological string

- Consider A-twist sigma model on M, with the worldsheet be a general Riemann surface Σ of genus g. Suppose the physical operator $\mathcal{O}_i$ correspond to differential form of (p_i, q_i) type, the fermion number counting and index theorem gives the selection rule that $\langle \mathcal{O}_1 \cdots \mathcal{O}_s \rangle = 0$ unless

$$\sum_{i=1}^{s} p_i = \sum_{i=1}^{s} q_i = \int_\beta c_1(M) + \mathrm{dim}_C(M)(1-g)$$

  where $\beta \in H_2(M, \mathbb{Z})$ is the homology class of the image of Σ.

- $c_1(M) > 0$ is not very interesting since for a given genus only a finite set of maps contribute.

- So we should consider Calabi-Yau manifold $c_1(M) = 0$. Naively the correlator vanish for $g \geq 2$. However to do string theory, we integrate over the complex structure moduli of the world-sheet. The dimension of complex structure moduli of a Riemann surface of genus $g \geq 2$ is

$$\text{Dim}_C(\mathcal{M}_g) = 3(g-1) \tag{1}$$

The counting cancels the contribution from fermionic zero mode when the target space is Calabi-Yau three-fold.

- We define the topological string amplitude for $g \geq 2$

$$F_g = \int_{\mathcal{M}_g} \langle d\Sigma \rangle, \tag{2}$$

where $d\Sigma$ is the measure on moduli space $\mathcal{M}_g$, can be written in terms of Beltrami differentials and supersymmetry currents.

- For genus zero, the three-point function is non-trivial

$$C_{i,j,k} = \langle \mathcal{O}_i \mathcal{O}_j \mathcal{O}_k \rangle, \tag{3}$$

where $\mathcal{O}_{i,j,k}$ are operators of $(1,1)$ type, corresponding to elements in $H^{(1,1)}(M,\mathbb{Z})$. The degree zero contribution is the classical intersection. We define the pre-potential

$$C_{i,j,k} = \partial_i \partial_j \partial_k F^{(0)} \tag{4}$$

- Genus one require no insertion. However one insertion is required if we integrate over the complex structure

$$\partial_i F^{(1)} = \int \mathcal{M}_{(1,1)} \langle \mathcal{O}_i d\Sigma \rangle \tag{5}$$

- We are interested in the topological string partition function

$$Z = \exp\left(\sum_{g=0}^{\infty} \lambda^{2g-2} F^{(g)}(t_i)\right)$$

where t_i are Kahler moduli in the case of A-model, and complex structure moduli in the case of B-model.

- Topological A-model counts holomorphic curves in target space X, and has a rigorous mathematical formulation known as Gromov-Witten theory. Topological B-model is a complex structure deformation theory known as Kodaira-Spencer theory,

Mathematical definition

- Consider a Kahler manifold X and $\beta \in H_{(1,1)}(X, \mathbb{Z})$. Define moduli space of stable maps of genus g with n marked points, $\bar{\mathcal{M}}_{g,n}(X, \beta)$. Yong-Bin Ruan, Gang Tian, 1995; M. Kontsevich, 1995.

- The Gromov-Witten invariant is defined

$$N_g^\beta = \int_{\bar{\mathcal{M}}_{g,0}(X,\beta)} 1 \tag{6}$$

  The genus g amplitude is $F^{(g)}(t_i) = \sum_\beta N_g^\beta e^{-\beta \cdot t}$.

- The virtual dimension

$$\mathrm{vdim}(\bar{\mathcal{M}}_{g,n}(X, \beta)) = \int_\beta c_1(X) + (\mathrm{dim}_C(X) - 3)(1 - g) + n \tag{7}$$

- The Gromov-Witten invariant is well defined is the virtual dimension vanishes. So again we see that the Calabi-Yau threefold is the most interesting.

Calabi-Yau threefolds: compact and non-compact

- Compact Calabi-Yau. Let us illustrate with a famous example: the quintic. Consider an ambient space, $\mathbb{CP}^4$ with coordinates

$$(X_1, X_2, X_3, X_4, X_5) \sim \lambda(X_1, X_2, X_3, X_4, X_5). \tag{8}$$

  A hypersurface parametrized by a degree 5 polynomial of X_i is a Calabi-Yau space, e.g. the Fermat quintic

$$X_1^5 + X_2^5 + X_3^5 + X_4^5 + X_5^5 = 0 \tag{9}$$

- Counting of complex structure parameters. There are $\binom{9}{4} = 126$ degree 5 monomials. The Calabi-Yau is the same by a $GL(5, C)$ transformation of the coordinates which has 25 parameters. So the number of complex structure parameters is $h^{(1,2)} = 126 - 25 = 101$.

- More generally, we can consider a complete intersection M, parametrized by r polynomials of degrees $d_1, \cdots, d_r$ in a weighted projective space $\mathbb{WCP}^n(w_1, w_2, \cdots, w_{n+1})$.

- The Adjunction formula: The total Chern class

$$C(T_M) = \frac{\prod_{i=1}^{n+1}(1 + w_i K)}{\prod_{k=1}^{r}(1 + d_k K)} = \sum_i c_i(M) K^i, \qquad (10)$$

where K is the Kahler class, and in the expansion we find the Chern class $c_i(M)$ of the manifold M.

- For a hypersurface in $\mathbb{CP}^4$ of degree d, we have

$$C(T_M) = \frac{(1+K)^5}{1+dK} = 1 + (5-d)K + (10 - 5d + d^2)K^2 + \cdots . \quad (11)$$

So the Calabi-Yau condition $c_1 = 0$ is $d = Hori5$, i.e. quintic hypersurface.

- There is at least one Kahler parameter correspond to the volume. If there is no other fixed point in the construction, the hypersurface or complete intersection has exactly one Kahler parameter, $h^{(1,1)} = 1$. This is known as one-parameter model. The A-model is simplest in this case.

- There are 13 one-parameter models that are hypersurface or complete intersection in weighted projective space. Denote the Calabi-Yau space as $X_{d_1,d_2,\cdots,d_k}(w_1,\cdots w_l)$. The 4 hypersurfaces are

$$X_5(1^5), \quad X_6(1^4,2), \quad X_8(1^4,4), \quad X_{10}(1^3,2,5)$$

- Non-compact Calabi-Yau threefold: the simplest example $\mathbb{C}^3$, also known as local Calabi-Yau manifold.

- We take a compact complex dimension two manifold, e.g. $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, del Pezzo surfaces. These manifolds have positive curvature. We introduce anti-canonical line bundle which can cancel the curvature, so the resulting manifold is a (non-compact) Calabi-Yau threefold. Example: $\mathcal{O}(-3) \to \mathbb{P}^2$.

- Topological strings on non-compact Calabi-Yau threefolds are easier to compute than the compact cases. Many are completely solvable.

- This is due to the simplification of Calabi-Yau moduli space, a $\mathcal{N} = 2$ special Kahler geometry with the metric

$$G_{i\bar{j}} = \partial_i \bar{\partial}_{\bar{j}} K. \tag{12}$$

  For non-compact model one can choose a gauge such that the Kahler potential is constant in holomorphic limit.

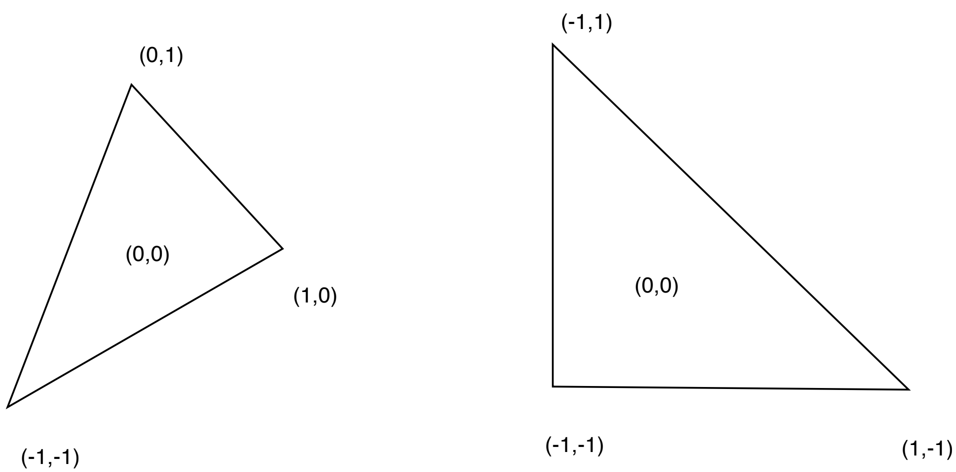
- The Calabi-Yau manifolds can be also constructed by polytope method. V. Batyrev, Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties, J. Algebraic Geom. 3 (1994),

- In this construction, the mirror symmetry is realized as a dual pair of reflexive polytopes. In this approach, the toric variety $\mathbb{P}_{\Delta}$ is defined by an n-dimensional convex integral polytope $\Delta \in \mathbb{R}_n$, containing the origin $(0, \cdots, 0)$. An integral polytope is a polytope whose vertices are integral, and is called reflexive if its dual defined by

$$\Delta^* = \{(x_1, \cdots, x_n)| \sum_{i=1}^{n} x_i y_i \geq -1 \text{ for all } (y_1, y_2, \cdots, y_n) \in \Delta\} \quad (13)$$

  is again an integral polytope.

- An example of reflexive polygon in 2-dimension. The first polygon represent the complex projective space $\mathbb{P}^2$.



- In Batyrev's construction, we use 4-d polytopes as the ambient space for compact Calabi-Yau threefolds. The hodge numbers $h^{(1,1)}$ and $h^{(1,2)}$ are exchanged for the Calabi-Yau pairs of the reflexive polytopes.

Mirror Symmetry: the quintic example

- Every Calabi-Yau threefold M has a mirror Calabi-Yau threefold W, their hodge numbers are exchanged

$$h^{(1,1)} \Leftrightarrow h^{(1,2)}$$

- Consider the sub-family of quintics defined by

$$\sum_i X_i^5 - 5\psi \prod_i X_i = 0$$

  Consider the group action $G : X_i \to \lambda^{k_i} X_i$, where λ is a fifth root of unity and $\sum_i k_i = 0(\text{mod}5)$. The group action perverse the equation, and due to scale invariance of the coordinates we have $G = (\mathbb{Z}_5)^3$. The mirror $\tilde{M}$ of the quintic manifold can be constructed by orbifold method

$$\tilde{M} = (\sum_i X_i^5 - 5\psi \prod_i X_i)/G \qquad (14)$$

- The mirror $\tilde{M}$ have hodge numbers $h^{(1,1)} = 101$, $h^{(1,2)} = 1$.

- The parameter ψ becomes the complex structure parameter of the mirror. The B-model is described by deformation theory, and the periods and mirror map are computed by the Picard-Fuchs differential equation

$$\{(\psi\partial_\psi)^4 - \psi^{-1}(\psi\partial_\psi - \frac{1}{5})(\psi\partial_\psi - \frac{2}{5})(\psi\partial_\psi - \frac{3}{5})(\psi\partial_\psi - \frac{4}{5})\}\omega = 0$$

  The equation can be solved by asymptotic series at $\psi = \infty$,

$$\vec{\Pi} = \begin{pmatrix} \int_{B_1}\Omega \\ \int_{B_2}\Omega \\ \int_{A^1}\Omega \\ \int_{A^2}\Omega \end{pmatrix} = \begin{pmatrix} F_0 \\ F_1 \\ X_0 \\ X_1 \end{pmatrix} = \omega_0 \begin{pmatrix} 2F^{(0)} - t\partial_t F^{(0)} \\ \partial_t F^{(0)} \\ 1 \\ t \end{pmatrix}$$

  The mirror map has a logarithmic behavior

$$2\pi i t(\psi) = -\log(5^5\psi) + \frac{154}{625\psi} + \frac{28713}{390625\psi^2} + \cdots$$

- We find the expansion of prepotential

$$F^{(0)} = -\frac{\kappa}{6}t^3 + P_2(t) + \sum_{m,d=1}^{\infty} \frac{n_d}{m^3} e^{-mdt} \qquad (15)$$

- The coefficient of e^{-dt} in the prepotential $F^{(0)}$ are Gromov-Witten invariants, but they are in general rational numbers. By rewriting the expansion in the form of (15), the numbers n_d turns out to be integers.

- The m^3 factor takes into account the "bubbling contributions" to higher degree invariants from low degree invariants. The numbers n_d correspond to the notion of counting the number of spheres in the Calabi-Yau manifold. It is a special case of Gopakumar-Vafa invariants at genus zero.

- Computing the number of sphere in quintic is a difficult question in algebraic geometry. In 1990, mathematicians know the low degree numbers $n_1 = 2875$, $n_2 = 609250$. Candelas et al's mirror symmetry calculations immediately gives n_d for all degrees d.

- Mathematicians try to compute n_3 but initially get the wrong number due to computer program error. They confirm Candelas et al's result after correcting the bug.

Proof of mirror conjecture (genus zero)

- Kontsevich (1995) developed the theory of Gromov-Witten invariants and the localization method to compute them. Two group of mathematicians later proved that the results are the same with those from mirror symmetry.
  A. Givental, 1996; B. Lian, K. Liu, and S.-T. Yau, Mirror principle I, II, III, 1997, 1999, 1999.

- Quotes from Yau's book "The Shape of Inner Space: String Theory and the Geometry of the Universe's Hidden Dimensions"', 2012.
  "Here again we venture into one of those areas of controversy ... We scrutinized his paper very carefully and were not alone in finding it hard to follow ... My colleagues and I also failed in our attempt to reconstruct Giventals entire argument, despite our attempts to contact him and piece together the steps we found most puzzling ... I believe the best thing to say at this point is that collectively the two papers constitute a proof of the mirror conjecture and to leave it at that "

How to compute

- Topological B-model methods: Picard-Fuchs equation for complex structure deformation, Candelas, De La Ossa, Green and Parkes, 1991); BCOV holomorphic anomaly equations compute higher genus topological string amplitudes, Bershadsky, Cecotti, Ooguri and Vafa, hep-th/9309140.

- Mathematical computation of Gromov-Witten invariant by localization techniques, Kontsevich et al.

- Large-N open/close string duality relates topological strings to Chern-Simons gauge theory in 3-manifolds, Gopakumar, Vafa; Further developed into topological vertex formalism, Aganagic, Klemm, Marino, Vafa.

- Heterotic/type II duality, applied to Calabi-Yaus with K3 fibration (Antoniadis et al).

- Counting of BPS states, i.e. Gopakumar-Vafa invariants (Katz, Klemm, Vafa).

- Matrix model techniques (Eynard, Orantin).

- Topological strings on non-compact toric Calabi-Yaus, e.g. $\mathcal{O}(-1) \to \mathbb{P}^1 \times \mathbb{P}^1$, $\mathcal{O}(-3) \to \mathbb{P}^2$, are essentially solved to all genera by topological vertex formalism.

- A long standing problem: How to solve topological strings on compact Calabi-Yau spaces? The genus zero amplitude is solved by mirror symmetry and localization method. However, At higher genus, the only available approach is the mirror symmetry and use holomorphic anomaly equation M. Bershadsky, S. Cecotti, H. Ooguri, C. Vafa (BCOV), 1993. This was done by BCOV up to genus 2.

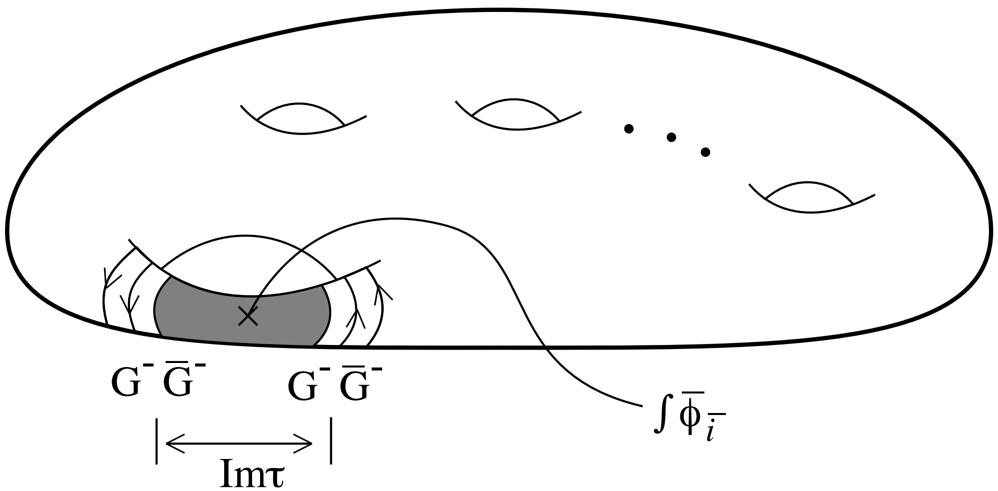
- The BCOV holomorphic anomaly equation

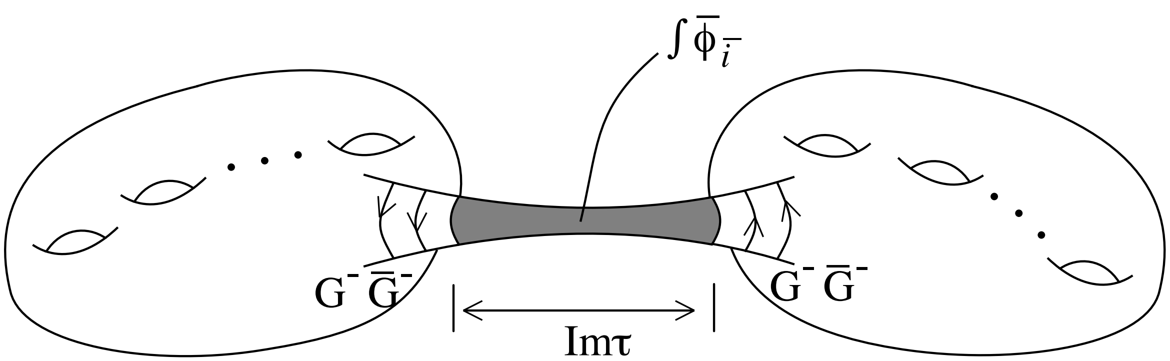
$$\bar{\partial}_{\bar{k}}\partial_m F^{(1)} = \frac{1}{2}\bar{C}_{\bar{k}}^{ij} C_{mij}^{(0)} + \left(\frac{\chi}{24} - 1\right) G_{\bar{k}m} \ ,$$

$$\bar{\partial}_{\bar{k}} F^{(g)} = \frac{1}{2}\bar{C}_{\bar{k}}^{ij} \left(D_i D_j F^{(g-1)} + \sum_{r=1}^{g-1} D_i F^{(r)} D_j F^{(g-r)} \right) \ , \quad g \geq 2$$

Holomorphic anomaly equation

- The holomorphic anomaly comes from the boundary of the moduli space of the worldsheet.





1.2. Toward solving the higher genus topological strings on compact Calabi-Yau spaces

References:

1. MH and A. Klemm, "Holomorphic Anomaly in Gauge Theories and Matrix Models," JHEP **0709**, 054 (2007) [hep-th/0605195].

2. MH, A. Klemm and S. Quackenbush, "Topological string theory on compact Calabi-Yau: Modularity and boundary conditions," Lect. Notes Phys. **757**, 45 (2009) [hep-th/0612125].

Quote: "... heroic efforts" (to solve topological strings), C. Vafa, string 2012 review talk, Munich.

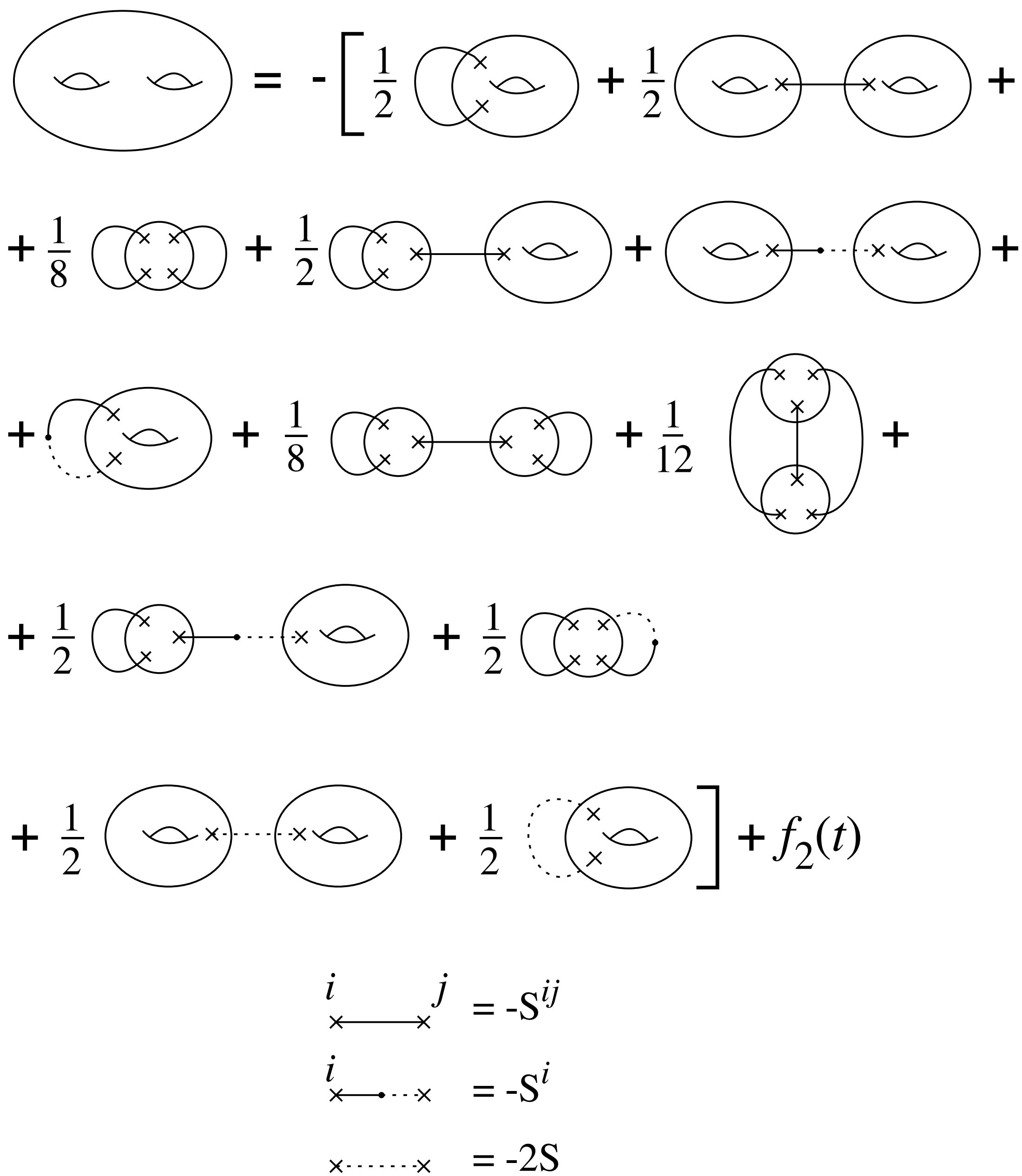
- It is difficult to push the BCOV methods to higher genus. Two major difficulties are the the followings.

  1. Holomorphic ambiguity problem. The holomorphic anomaly equation only determine $F^{(g)}$ recursively in terms of lower genus results up to a holomorphic ambiguity, a meromorphic function in the moduli space with a finite number of unknown constants. One need find alternative ways to fix these unknown constants.

  2. Computational complexity in BCOV method: the number of diagrams grows exponentially with genus. A normal laptop can handle the computation only up to about genus 6, even for the simplest one parameter models such as the quintic.

- The calculation was pushed up to genus 3 for the quintic, using further information from the counting of BPS states known as Gopakumar-Vafa invariants. Katz, Klemm, Vafa, hep-th/9910181.

- An example of BCOV diagrams, at genus 2.



- We made some important progress Huang, Klemm, Quackenbush, hep-th/0612125.

  1. We solve the holomorphic anomaly equation directly without the BCOV Feynman diagrams, by using the idea of formulating topological strings as polynomials Yamaguchi, Yau, hep-th/0406078. The computational complexity of the method grows only polynomially in genus.

  2. We discover boundary conditions at the conifold point of the moduli space, i.e. the "gap" condition c.f. Huang, Klemm, hep-th/0605195, which fix the holomorphic ambiguity to a large extend.

- We are able to solve a class of one-parameter Calabi-Yau models to very high genus, e.g. in principle to genus 51 for the quintic.

Topological strings as polynomials

Yamaguchi and Yau, hep-th/0406078

- Define the following generators

$$A_p := \frac{(\psi\partial_\psi)^p G_{\psi\bar{\psi}}}{G_{\psi\bar{\psi}}}, \quad B_p := \frac{(\psi\partial_\psi)^p e^{-K}}{e^{-K}}, \quad (p = 1, 2, 3, \cdots)$$

$$C := C_{\psi\psi\psi}\psi^3, \quad X := \frac{1}{1-\psi}$$

These generators satisfy the derivative relations

$$\psi\partial_\psi A_p = A_{p+1} - AA_p, \quad \psi\partial_\psi B_p = B_{p+1} - BB_p, \quad \psi\partial_\psi X = X(X-1)$$

- The independent generators are (A_1, B_1, B_2, B_3, X). One can use the Picard-Fuchs equation and special geometry relation to show B_4 and A_2 are polynomials of (A_1, B_1, B_2, B_3, X).

- Define the topological string amplitudes in "Yukawa coupling frame"

$$P_g := C^{g-1}F^{(g)}, \qquad P_g^{(n)} = C^{g-1}\psi^n C^{(g)}_{\psi^n}$$

- We have the initial data and recursion relation in n

$$\begin{aligned} P^{(3)}_{g=0} &= 1 \\ P^{(1)}_{g=1} &= -\frac{31}{3}B + \frac{1}{12}(X-1) - \frac{1}{2}A + \frac{5}{3} \\ P_g^{(n+1)} &= \psi\partial_\psi P_g^{(n)} - [n(A+1) + (2-2g)(B - \frac{1}{2}X)]P_g^{(n)} \end{aligned}$$

- Define a change of variable

$$(A_1, B_1, B_2, B_3, X) \to (u, v_1, v_2, v_3, X)$$

by the followings

$$\begin{aligned} &B = u, \quad A = v_1 - 1 - 2u, \quad B_2 = v_2 + uv_1, \\ &B_3 = v_3 - uv_2 + uv_1X - \frac{2}{5}uX \end{aligned}$$

- The anti-holomorphic derivative of the generators can be related to each other. Only $\partial_{\bar{\psi}} A_1$ and $\partial_{\bar{\psi}} B_1$ are independent. The BCOV holomorphic anomaly equations are

$$\begin{aligned} \frac{\partial P_g}{\partial u} &= 0 \\ (\frac{\partial}{\partial v_1} - u\frac{\partial}{\partial v_2} - u(u+X)\frac{\partial}{\partial v_3})P_g &= -\frac{1}{2}(P^{(2)}_{g-1} + \sum_{r=1}^{g-1} P^{(1)}_r P^{(1)}_{g-r}) \end{aligned}$$

- The Main Proposition: Each P_g, $(g \geq 2)$ is a degree $3g-3$ inhomogeneous polynomial of v_1, v_2, v_3, X, where one assigns the degree $1, 2, 3, 1$ for v_1, v_2, v_3, X, respectively. Yamaguchi and Yau.

- The number of terms n_g in P_g grows polynomially with genus g.

$$n_g \preceq (3g-3)^4$$

- The generators (A_i, B_i, X) are modular functions of the monodromy group of the quintic, a subgroup of $Sp(4, Z)$.

- We use the holomorphic anomaly equation to compute the P_g recursively, up to a holomorphic ambiguity

$$f^{(g)} = \sum_{i=0}^{3g-3} c_i X^i$$

The degree is fixed by the maximal degree of the poles at the conifold point.

- There are $3g - 2$ unknown constants at each genus g.

Boundary conditions

- There are three singular points in the complex structure moduli space: $\psi = 0$, $\psi = 1$, $\psi = \infty$.

- We can expand the topological strings around these singular points. In the holomorphic limit, the Kahler potential and metric go like

$$e^{-K} \sim \omega_0, \qquad G_{\psi\bar{\psi}} \sim \partial_\psi t,$$

  So in the holomorphic limit, the generators A_p and B_p are

$$A_p = \frac{(\psi\partial_\psi)^p(\partial_\psi t)}{\partial_\psi t}, \qquad B_p = \frac{(\psi\partial_\psi)^p \omega_0}{\omega_0},$$

- The period ω_0 and mirror map t can be solved asymptotically at each singular point of the moduli space by the Picard-Fuchs equation.

- Boundary condition at the orbifold point $\psi = 0$. The Picard-Fuchs equation has 4 power series solutions that go like $\omega_0 \sim \psi^{\frac{1}{5}}$, $\omega_1 \sim \psi^{\frac{2}{5}}$, $\omega_2 \sim \psi^{\frac{3}{5}}$, $\omega_3 \sim \psi^{\frac{4}{5}}$.

- The topological string amplitudes are

$$F^{(g)}_{\text{orbifold}} = \lim_{\bar{\psi}\to 0} \omega_0^{2(g-1)} \left(\frac{1-\psi}{\psi}\right)^{g-1} P_g \sim \frac{P_g}{\psi^{\frac{3}{5}(g-1)}}$$

We expect $F^{(g)}_{\text{orbifold}}$ to be regular at the orbifold point, based on earlier works (e.g. Katz, Klemm , Vafa).

- P_g is a power series of ψ, starting from a constant. This imposes

$$\lceil \frac{3}{5}(g-1) \rceil$$

number of conditions on the holomorphic ambiguity in P_g.

- Boundary condition at the conifold point $\psi = 1$. Picard-Fuchs equation around $z = \psi - 1$ have four solutions that go like

$$\vec{\Pi} = \begin{pmatrix} \omega_0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} 1 + \mathcal{O}(z) \\ z + \mathcal{O}(z^2) \\ z^2 + \mathcal{O}(z^3) \\ \omega_1 \log(z) + \mathcal{O}(z^4) \end{pmatrix}$$

- We define a dual mirror map $t_D = \frac{\omega_1}{\omega_0}$. We find the topological strings around the conifold point has a "gap" structure in the t_D coordinate

$$\begin{aligned} F^{(g)}_{\text{conifold}} &= \lim_{\bar{z} \to 0} \omega_0^{2(g-1)} \left(\frac{1-\psi}{\psi}\right)^{g-1} P_g \\ &= \frac{(-1)^{g-1} B_{2g}}{2g(2g-2) t_D^{2g-2}} + \mathcal{O}(t_D^0), \end{aligned}$$

This fixes $2g - 2$ coefficients in the holomorphic ambiguity.

- An arbitrary change of the basis $\omega_0 \to \omega_0 + b_1\omega_1 + b_2\omega_2$ does not affect this gap like structure.

- The leading coefficients of the conifold expansion were actually pointed out long time ago, Ghoshal, Vafa, hep-th/9506122. The gap condition is first observed recently in the context of $SU(2)$ Seiberg-Witten theory, Huang, Klemm, hep-th/0605195.

- Near the conifold point of the moduli space, a D3-brane wrapping a vanishing 3-cycle appears as a charged, BPS, extremal, and nearly massless black hole in space-time, Strominger, hep-th/9504090.

- A physical explanation of the gap condition: Integrating out the massless black hole state in a graviphoton background...

- Gopakumar-Vafa-Schwinger Computation: In $\mathcal{N} = 2$ supergravity, we integrate out a charged BPS hypermultiplet of $e = m = \frac{t}{\lambda}$, and Lorentz Group $SO(4) = SU(2)_L \times SU(2)_R$ representation

$$[(\frac{1}{2}, 0) + 2(0,0)] \bigotimes (j_L, j_R)$$

in a graviphoton background where the self-dual part of the graviphoton field strength is $F_+ = \lambda$.

- The Gopakumar-Vafa-Schwinger Computation generates the following term in the effective action

$$S = \int d^4x F(t,\lambda) R_+^2,$$

$$\text{where} \quad F(t,\lambda) = \int_\epsilon^\infty \frac{ds}{s} \frac{\text{Tr}\,(-1)^F \exp(-st)\exp(-2s\lambda\sigma_L)}{(2\sin(\frac{s\lambda}{2}))^2}$$

- In type IIB compactification near the conifold, there is only one light particle: the massless black hole.

- The topological string near the conifold should be, (up to regular terms of the period t),

$$F(\lambda,t)=\int_{\epsilon}^{\infty}\frac{ds}{s}\;\frac{\exp(-st)}{(2\sin(\frac{s\lambda}{2}))^2}=\sum(\frac{\lambda}{t})^{2g-2}\frac{(-1)^{g-1}B_{2g}}{2g(2g-2)}+\mathcal{O}(t^0)$$

This is precisely the gap condition.

- Boundary conditions at infinity $\psi = \infty$. The constant map contribution of manifold M, Faber, Pandharipande, math.ag/9810173,

$$\lim_{t\to\infty} F^{(g)}_{\text{A-model}} = \frac{(-1)^{g-1} B_{2g} B_{2g-2}}{4g(2g-2)(2g-2)!}\chi(M)$$

- The world sheet instanton corrections

$$F^{(g)}_{\text{instanton}} = \sum_{\beta\in H_2(M,\mathbb{Z})} r^{(g)}_{\beta} \exp(2\pi i t\beta)$$

where $r^{(g)}_{\beta}$ are rational numbers, known as the Gromov-Witten invariants of holomorphic maps.

- Re-organize the world sheet instanton contributions

$$\sum_{g=0}^{\infty} \lambda^{2g-2} F^{(g)}_{\text{instanton}} = \sum_{g=0}^{\infty}\sum_{\beta}\sum_{m=1}^{\infty} n^{(g)}_{\beta} (\frac{e^{2\pi i t\beta m}}{m})(2\sin\frac{m\lambda}{2})^{2g-2}$$

- The Gopakumar-Vafa invariants $n_\beta^{(g)}$ are integers counting BPS D0-D2 brane bound states.

- The quintic example: one kahler modulus, $\beta = d$ is the degree of the holomorphic map. The GV invariants

g	d=1	d=2	d=3	d=4	d=5
0	2875	609250	317206375	242467530000	229305888887625
1	0	0	609250	3721431625	12129909700200
2	0	0	0	534750	75478987900
3	0	0	0	8625	-15663750
4	0	0	0	0	49250
5	0	0	0	0	1100
6	0	0	0	0	10
7	0	0	0	0	0

- Boundary condition: at each genus, the Gopakumar-Vafa invariants vanish $n_d^{(g)} = 0$ for low degree d holomorphic maps.

Summary of Boundary Conditions at genus g

- Holomorphic ambiguity: $3g-2$ unknown constants.

- The expansion around orbifold point $\psi = 0$ provides $\lceil \frac{3}{5}(g-1) \rceil$ boundary conditions.
  The expansion around conifold point $\psi = 1$ provides $2g-2$ boundary conditions.
  The large complex structure modulus/large volume limit $\psi = \infty$ provides $a_g + 1$ boundary conditions, where a_g is the number of low degree vanishing GV invariants at genus g, sensitive to specific models.

- Count the number of unknown constants

$$3g-2-(\lceil \frac{3}{5}(g-1) \rceil + 2g-2+1+a_g) = [\frac{2}{5}(g-1)] - a_g$$

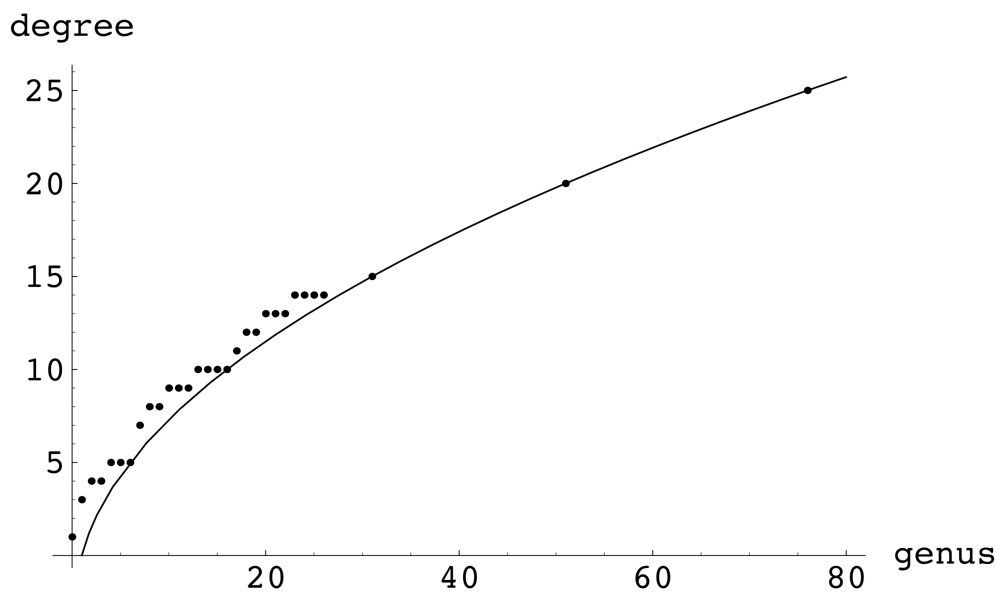
- We have enough/redundant data to compute topological strings if

$$a_g \geq [\frac{2}{5}(g-1)]$$

- This is true for low genus, (up to $g \sim 51$ for the quintic) . However, asymptotically

$$a_g \sim \sqrt{g}, \quad \text{when} \quad g \to \infty$$

So far our calculation is limited only by the power of our computational facilities.



- The analysis can be straightforwardly generalized to one-parameter Calabi-Yau models, realized as hypersurfaces or complete intersections in weight projective spaces.

$$X_5(1^5) \quad X_6(1^4,2), \quad X_8(1^4,4), \quad X_{10}(1^3,2,5), \quad X_{3,3}(1^6),$$
$$X_{4,2}(1^6), \quad X_{3,2,2}(1^7), \quad X_{2,2,2,2}(1^8) \quad X_{4,3}(1^5,2), \quad X_{4,4}(1^4,2^2),$$
$$X_{6,2}(1^5,3), \quad X_{6,4}(1^3,2^2,3), \quad X_{6,6}(1^2,2^2,3^2).$$

- We solve all these 13 models to very high genus. The singular behaviors around the conifold point is universal.

- On the other hand, we discover a rich variety of singularity structures around the orbifold point. The 13 models fall into 4 classes.

Four cases

(1). No massless charged state. The F^g are regular at the orbifold point $\psi = 0$, imposing boundary conditions. This includes models $X_5(1^5)$, $X_6(1^4,2)$, $X_8(1^4,4)$, $X_{10}(1^3,2,5)$,$X_{3,3}(1^6)$, $X_{2,2,2,2}(1^8)$, $X_{4,4}(1^4,2^2)$,$X_{6,6}(1^2,2^2,3^2)$.

(2). One massless charged state. The F^g exhibit the "gap structure" similar to the conifold point, imposing boundary conditions. This includes models $X_{4,2}(1^6)$, $X_{6,2}(1^5,3)$.

(3). Two massless charged states. The interactions between massless states destroy the "gap structure", no boundary conditions at the orbifold point. This includes models $X_{3,2,2}(1^7)$.

(4). Multiple massless charged states. The F^g are singular with no obvious structures at the orbifold point. However the scaling of masses of these light states imposes some boundary conditions. This includes the model $X_{4,3}(1^5,2)$, $X_{6,4}(1^3,2^2,3)$.

Castelnuovo's theory

- We make many predictions for the Gopakumar-Vafa invariants. The counting of BPS states of a degree d can be calculated from the cohomology of the moduli space $\mathcal{M}$ of the D0-D2 brane bound states. This algebraic geometric counting is known as Castelnuovo's theory. (Katz, Klemm, Vafa, 1999)

- The "top genus" numbers are the easiest to calculate.

$$n_d^g = (-1)^{\dim(\mathcal{M})} \chi(\mathcal{M})$$

- Examples from the quintic:

  1. Genus $g = 6$, degree $d = 5$: $n_5^6 = 10$, $n_5^g = 0$ $(g \geq 7)$.

  2. Genus $g = 16$, degree $d = 10$: $n_{10}^{16} = -50$, $n_{10}^g = 0$ $(g \geq 17)$.

- Some basics: The moduli space of $\mathbb{P}^k$ moving in $\mathbb{P}^n$ is the Grassmannian $\mathbb{G}(k,n)$. The complex dimension and Euler number are

$$\dim(\mathbb{P}^k) = k, \quad \chi(\mathbb{P}^k) = k+1,$$
$$\dim(\mathbb{G}(k,n)) = (k+1)(n-k), \quad \chi(\mathbb{G}(k,n)) = \binom{n+1}{k+1}$$

- Consider a complete intersection of degree $(1,1,5)$ in $\mathbb{P}^4$. This is a curve of genus 6, degree 5. The moduli space of curves in the quintic is Grassmannian $\mathbb{G}(2,4)$, so we recover the BPS number

$$n_5^6 = (-1)^{3\cdot 2}\binom{5}{3} = 10$$

- Similarly, consider a complete intersection of degree $(1,2,5)$ in $\mathbb{P}^4$. This is a curve of genus 16, degree 10. The moduli space of curves in the quintic is $\mathbb{P}^4 \times \mathbb{P}^9$, so we recover the BPS number

$$n_{10}^{16} = (-1)^{4+9} 5 \cdot 10 = -50$$

1.3. Applications for black hole physics

References:

1. MH, A. Klemm, M. Marino and A. Tavanfar, "Black Holes and Large Order Quantum Geometry," Phys. Rev. D **79**, 066001 (2009) [arXiv:0704.2440 [hep-th]].

Quote: "... congratulations for settling the long outstanding conjecture on the entropy of the spinning BH ....", C. Vafa to A. Klemm, private communication.

- Compactify M-theory on a compact Calabi-Yau 3-fold. The 5-D supergravity has a BPS black hole solution (BMPV black hole) with graviphoton charge Q, angular momentum J of the $SU(2)_L \subset SO(4)$. The classical entropy of the black hole is one quarter of the horizon area

$$S = 2\pi\sqrt{Q^3 - J^2}$$

- There are R^2 correction to the black hole entropy, computable by Wald's formula,

$$\Delta S = 2\pi \int_{\text{Horizon}} \frac{\partial(\Delta\mathcal{L})}{\partial R_{\mu\nu\rho\sigma}} \epsilon_{\mu\nu}\epsilon_{\rho\sigma} \sim Q^{\frac{1}{2}}$$

- An open problem: How to count the black hole microstates? Much more difficult than the Strominger-Vafa black hole.

- Katz, Klemm, Vafa (KKV), 1999: The black hole microstates are counted by topological strings. For a black hole with 2-brane charge d and $SU(2)_L$ angular momentum $J = m$, the number of states are

$$N_d^m = \sum_r n_d^r \binom{2r+2}{r+1+m}$$

The graviphoton charge are related by the supergravity attractor equation $Q = (\frac{2}{9})^{\frac{1}{3}} \frac{d}{\sqrt{\kappa}}$, where κ is the intersection number.

- This is a very natural proposal since the Gopakumar-Vafa invariant n_d^r is a supersymmetric index that remains constant in the moduli space.

- Difficulty: For non-compact Calabi-Yaus, the KKV formula can not be reliably applied to count 5D black hole microstates, since this is not really a compactification to 5D supergravity. There were not much computations of the Gopakumar-Vafa invariants for compact Calabi-Yau available (before our paper).

- We use our new results and the KKV formula the count micro-states. Consider e.g. angular momentum $m = 0$,

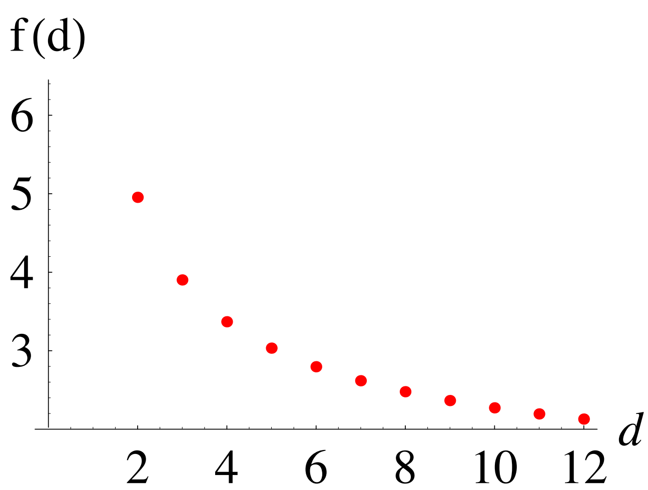
$$S = \log(N_d^0) = \frac{4\pi}{3\sqrt{2\kappa}} d^{\frac{3}{2}} + \mathcal{O}(d^{\frac{1}{2}})$$

Topological string data provide the values

$$f(d) = \frac{\log(N_d^0)}{d^{\frac{3}{2}}} = \frac{4\pi}{3\sqrt{2\kappa}} + \frac{b_1}{d} + \frac{b_2}{d^2} + \cdots$$

for d up to a finite degree.

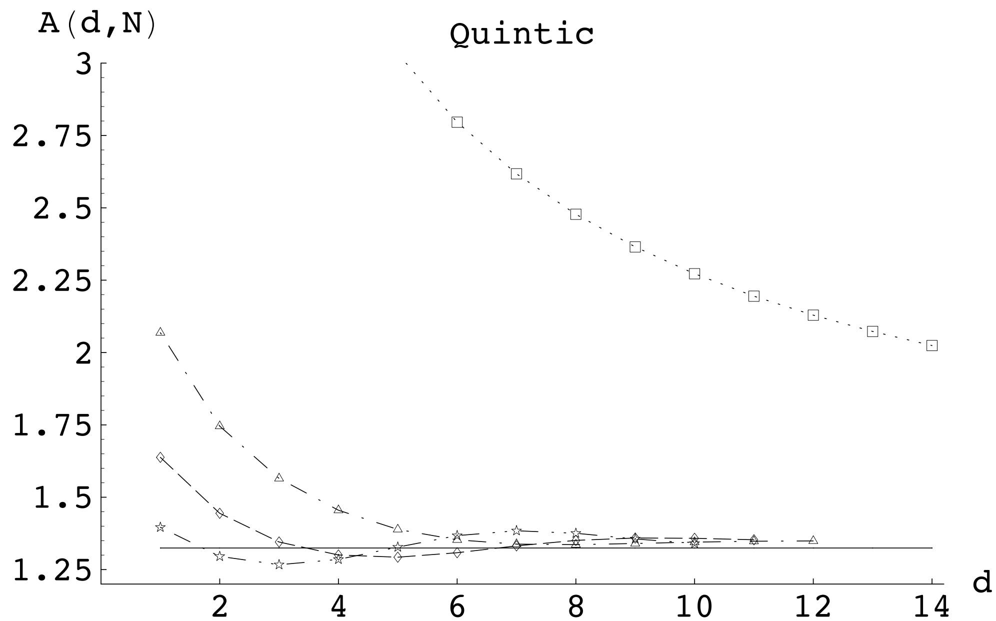
- The quintic example



- How to extrapolate? Use the Richardson extrapolation method. To cancel the sub-leading corrections up to order $1/d^N$, one defines

$$A(d,N) = \sum_{k=0}^{N} \frac{f(d+k)(d+k)^N(-1)^{k+N}}{k!(N-k)!}, \tag{16}$$

For example $A(d,1) = (d+1)f(d+1) - df(d)$.



- For all 13 models, the KKV formula for counting micro-states confirms the macroscopic black hole prediction of leading coefficient with impressively small error of 1∼ 3 % .

Calabi-Yau	d_{max}	$A(d_{max}-3,3)$	$b_0 = \frac{4\pi}{3\sqrt{2\kappa}}$	error
$X_5(1^5)$	14	1.35306	1.32461	2.15 %
$X_6(1^4,2)$	10	1.75559	1.71007	2.66 %
$X_8(1^4,4)$	7	2.11454	2.0944	0.96 %
$X_{10}(1^3,2,5)$	5	2.99211	2.96192	1.02 %
$X_{3,3}(1^6)$	17	1.00204	0.987307	1.49 %
$X_{4,2}(1^6)$	15	1.07031	1.0472	2.21 %
$X_{3,2,2}(1^7)$	10	0.821169	0.855033	-3.96 %
$X_{2,2,2,2}(1^8)$	13	0.722466	0.74048	-2.43 %
$X_{4,3}(1^5,2)$	11	1.21626	1.2092	0.58 %
$X_{6,2}(1^5,3)$	11	1.52785	1.48096	3.17 %
$X_{4,4}(1^4,2^2)$	7	1.42401	1.48096	-3.85 %
$X_{6,4}(1^3,2^2,3)$	5	2.06899	2.0944	-1.21 %
$X_{6,6}(1^2,2^2,3^2)$	4	2.95082	2.96192	-0.37 %

- The subleading correction have been studied by supergravity analysis in the literature. We confirm the macroscopic results numerically with about 5$\sim$ 15 % error for the 13 one-parameter models.

Calabi-Yau	d_{max}	$A_1(d_{max}-3,3)$	$b_1=\frac{\pi c_2}{4\sqrt{2\kappa}}$	error
$X_5(1^5)$	14	11.2668	12.4182	-9.27 %
$X_6(1^4,2)$	10	11.9237	13.4668	-11.5 %
$X_8(1^4,4)$	7	14.0537	17.2788	-18.7 %
$X_{10}(1^3,2,5)$	5	15.2509	18.8823	-19.2 %
$X_{3,3}(1^6)$	17	9.29062	9.99649	-7.06 %
$X_{4,2}(1^6)$	15	10.0226	10.9956	-8.85 %
$X_{3,2,2}(1^7)$	10	8.45163	9.61912	-12.1 %
$X_{2,2,2,2}(1^8)$	13	7.84595	8.88577	-11.7 %
$X_{4,3}(1^5,2)$	11	9.5981	10.8828	-11.8 %
$X_{6,2}(1^5,3)$	11	12.5614	14.4394	-13.0 %
$X_{4,4}(1^4,2^2)$	7	9.70091	11.1072	-12.7 %
$X_{6,4}(1^3,2^2,3)$	5	11.1008	12.5664	-11.7 %
$X_{6,6}(1^1,2^2,3^3)$	4	11.1378	12.2179	-8.84 %

Part II

2.1. Refined topological strings

References:

1. MH and A. Klemm, "Direct integration for general Ω backgrounds," Adv. Theor. Math. Phys. **16**, no. 3, 805 (2012) [arXiv:1009.1126 [hep-th]].

2. MH, A. -K. Kashani-Poor, and A. Klemm, "The Omega deformed B-model for rigid N=2 theories," Annales Henri Poincare **14**, 425 (2013) [arXiv:1109.5728 [hep-th]].

3. MH, A. Klemm and M. Poretschkin, "Refined stable pair invariants for E-, M- and [p,q]-strings," arXiv:1308.0619 [hep-th].

Refinement: Origin and Motivation

- The effective action of 4d $\mathcal{N} = 2$ supersymmetric gauge theories is determined by a holomorphic quantity known as the prepotential $F^{(0)}$. Seiberg and Witten (1994) solved the low energy effective action using the holomorphicity and monodromy around singular points of the moduli space.

- The main parts of the low energy effective action of asymptotically free gauge theories come from instanton contributions. Nekrasov's partition function provides the formulae from direct computations of instanton contributions. It can be mathematically proven that the formalism gives the same prepotential found by Seiberg-Witten method (Nekrasov, Okounkov, Nakajima, Yoshioka).

- The Nekrasov function also contains the gravitational couplings, the coefficients of $R^2F^{(2g-2)}$ terms in the effective action. Here R and F are the curvature and graviphoton fields.

- Inspired by Nekrasov's partition function, one can study the topological string theory on Calabi-Yau manifolds with two expansion parameters, known as refined topological string theory.

- Two ways to compute refined topological string amplitudes.

  1. A-mode method: refined topological vertex (A. Iqbal, C. Kozcaz, C. Vafa), applicable to certain local toric Calabi-Yau manifolds that geometrically engineer the gauge theory.

  2. B-model method: Generalized holomorphic anomaly equation, boundary conditions (MH, A. Klemm; Krefl, Walcher).

The Nekrasov Function

- The Nekrasov function is an integral over the moduli space of instantons in the 4d gauge theory. The moduli space is naively non-compact, and the integral is naively divergent and need to be regularized. Nekrasov regularizes the integral using the so-called Ω deformations, where the two deformation parameters are ϵ_1 and ϵ_2. Formally

$$Z_{instanton}(a, \epsilon_1, \epsilon_2) = \sum_{n=0}^{\infty} \int_{\mathcal{M}_n} 1 \tag{17}$$

where $\mathcal{M}_n$ is the moduli space of n instantons in Ω background.

- In the Ω backgrounds, the integral over instanton moduli space localizes to a finite number of points. It can be written as a sum over some 2d Young tableaux. The number of the boxes of the Young tableaux is the number of instantons.

- The formula ($SU(2)$ case)

$$
\begin{aligned}
& Z_{\text{instanton}}(a,\epsilon_1,\epsilon_2) \\
= & \sum_{Y_1,Y_2}\prod_{(i,j)\in Y_1}\frac{\prod_{k=1}^{N_f}(a+\epsilon_1(i-1)+\epsilon_2(j-1)+m_k)}{E(0,Y_1,Y_1,i,j)(\epsilon-E(0,Y_1,Y_1,i,j))E(2a,Y_1,Y_2,i,j)(\epsilon-E(2a,Y_1,Y_2,i,j))} \\
& \cdot\prod_{(i,j)\in Y_2}\frac{\prod_{k=1}^{N_f}(-a+\epsilon_1(i-1)+\epsilon_2(j-1)+m_k)}{E(0,Y_2,Y_2,i,j)(\epsilon-E(0,Y_2,Y_2,i,j))E(-2a,Y_2,Y_1,i,j)(\epsilon-E(-2a,Y_2,Y_1,i,j))},
\end{aligned}
$$

where

$$E(a,Y_1,Y_2,i,j)\equiv a+\epsilon_1(Y_{1,j}^T-i+1)-\epsilon_2(Y_{2,i}-j)$$

The Y^T is the transpose of the Young tableau. The ϵ_1 and ϵ_2 are the deformation parameters in Ω backgrounds, and $\epsilon=\epsilon_1+\epsilon_2$. The m_k's are the mass parameters of the massive flavors. For the $SU(2)$ case we have one period a which is the flat coordinate at large modulus limit.

- Including the perturbative contributions, the total contribution is

$$Z = Z_{\text{pert}} Z_{\text{instanton}}$$

- The gravitational couplings can be computed by expansion of the Nekrasov's partition function $Z(a, \epsilon_1, \epsilon_2)$ around small ϵ_1, ϵ_2 parameters of the general Ω background

$$\log Z(a, \epsilon_1, \epsilon_2) = \sum_{i,j=0}^{\infty} (\epsilon_1 + \epsilon_2)^i (\epsilon_1 \epsilon_2)^{j-1} F^{(\frac{i}{2}, j)}(a)$$

- The leading term $F^{(0,0)}(a)$ is the well-known prepotential. It is determined by holomorphicity and monodromy, and can be conveniently computed by a differential equation known as the Picard-Fuchs equation.

- Here I use the pure $SU(2)$ gauge theory as an example. We studied the case of $\epsilon_1 + \epsilon_2 = 0$ in 2006, and generalized the analysis to general case in a later paper. It turns out the odd power terms $F^{(\frac{i}{2},j)}(a)$ where i an odd integer vanish.

- For the genus one case, the $F^{(0,1)}$ follows the conventional genus one BCOV (Bershadsky-Cecotti-Ooguri-Vafa, 1993) holomorphic anomaly equation, while $F^{(1,0)}$ has no holomorphic anomaly. A further boundary condition at the singular points of moduli space determine the formulae (up to an irrelevant constant)

$$
\begin{aligned}
F^{(0,1)} &= -\frac{1}{2}\log(\frac{da}{du}) - \frac{1}{12}\log(\Delta), \\
F^{(1,0)} &= \frac{1}{24}\log(\Delta)
\end{aligned}
$$

  where u is the complex modulus parameter of $SU(2)$ Seiberg-Witten gauge theory, and is the expectation value of the adjoint scalar field $u = \frac{1}{2}\mathrm{Tr}\,(\phi^2)$. The Δ is the discriminant of the Seiberg-Witten curve and the singular points of the moduli space are the loci $\Delta(u) = 0$.

- It is convenient to introduce a modular parameter τ (related to the modular parameter of the Seiberg-Witten curve). The relations between the parameter $q = e^{2\pi i\tau}$, u, a are the followings (around $u = \infty$)

$$
\begin{aligned}
a &= \frac{E_2(\tau) + \theta_3^4(\tau) + \theta_4^4(\tau)}{3\theta_2^2(\tau)} \sim q^{-\frac{1}{4}} \\
u &= \frac{\theta_3^4(\tau) + \theta_4^4(\tau)}{\theta_2^4(\tau)} \sim q^{-\frac{1}{2}}
\end{aligned}
$$

- The u parameter is defined over the complex plane, and is singular at $u = \infty$, $u = \pm 1$, where some charged particles become massless. The parameters τ, a are only defined locally in patches of the u plane. The parameters a, τ near the monopole point $u = 1$ are obtained by a S-duality transformation $\tau \to -\frac{1}{\tau}$

$$
\begin{aligned}
a_D &= \frac{2}{3\theta_4^2(\tau)}(E_2(\tau) - \theta_3^4(\tau) - \theta_2^4(\tau)) \sim q^{\frac{1}{2}} \\
u &= \frac{\theta_3^4(\tau) + \theta_2^4(\tau)}{\theta_4^4(\tau)} \sim 1
\end{aligned}
$$

- The genus one formulae are

$$F^{(0,1)} = -\log(\eta(\tau))$$
$$F^{(1,0)} = -\frac{1}{6}\log(\frac{\theta_2^2}{\theta_3\theta_4}) = \frac{1}{24}\log(u^2-1)$$

- It turns out that the topological amplitudes $F^{(g_1,g_2)}$ satisfy a generalized holomorphic anomaly equation

$$\bar{\partial}_{\bar{i}} F^{(g_1,g_2)} = \frac{1}{2}\bar{C}_{\bar{i}}^{jk}\Big(D_j D_k F^{(g_1,g_2-1)} + (\sum_{r_1=1}^{g_1}\sum_{r_2=1}^{g_2})' D_j F^{(r_1,r_2)} D_k F^{(g_1-r_1,g_2-r_2)}\Big)$$

where $g_1+g_2 \geq 2$, the prime denotes that the sum over r_1, r_2 does not include $(r_1,r_2)=0$, and $(r_1,r_2)=(g_1,g_2)$, and the first term on the right hand side is understood to be zero if $g_2=0$. This equation reduces to the ordinary BCOV holomorphic anomaly equation when $g_1=0$.

- In order to be a modular form, the second Eisenstein series $E_2(\tau)$ should be shifted by anti-holomorphic piece

$$E_2(\tau) \to E_2(\tau) - \frac{3}{\pi\tau_2}$$

- The shifted E_2 is the only anti-holomorphic piece. $F^{(g_1,g_2)}$ is a polynomial of the shifted E_2 whose coefficients are rational functions of u. The holomorphic anomaly equations become

$$\begin{aligned} & 48\frac{\partial F^{(g_1,g_2)}(E_2,u)}{\partial E_2} \\ = \; & \frac{d^2}{da^2}F^{(g_1,g_2-1)} + (\sum_{r_1=0}^{g_1}\sum_{r_2=0}^{g_2})'(\frac{dF^{(r_1,r_2)}}{da})(\frac{dF^{(g_1-r_1,g_2-r_2)}}{da}) \end{aligned}$$

- The holomorphic anomaly equation determines the $F^{(g_1,g_2)}$ up to a rational function of u, called the holomorphic ambiguity. To further determine now the holomorphic ambiguity for the pure $SU(2)$ theory, we expand the topological amplitudes around the monopole point $u=1$. We find the gap condition around this point completely fixes the holomorphic ambiguity.

- The gap conditions at genus two are

$$
\begin{aligned}
F_D^{(0,2)} &= -\frac{1}{240a_D^2} + \mathcal{O}(a_D^0) \\
F_D^{(1,1)} &= \frac{7}{1440a_D^2} + \mathcal{O}(a_D^0) \\
F_D^{(2,0)} &= -\frac{7}{5760a_D^2} + \mathcal{O}(a_D^0)
\end{aligned}
$$

- The genus two formulae are

$$
\begin{aligned}
F^{(0,2)} &= \frac{200X^3 - 360uX^2 + (60u^2 + 180)X - 19u^3 - 45u}{12960(u^2-1)^2} \\
F^{(1,1)} &= \frac{20uX^2 - (40u^2 + 60)X + 3u^3 + 45u}{2160(u^2-1)^2} \\
F^{(2,0)} &= \frac{10u^2X + u^3 - 75u}{4320(u^2-1)^2}
\end{aligned}
$$

where $X = E_2(\tau)/\theta_2(\tau)^4$.

Our results

- We provide exact formulae summing up all instanton contributions at a given genus. We extend the calculations to include $SU(2)$ theory with $N_f = 1, 2, 3, 4$ fundamental or an adjoint hypermultiplet(s) with generic mass parameters. Our results agree with the Nekrasov partition functions.

- We find the boundary gap conditions are not applicable for the case of $N_f = 4$ or adjoint massless matters. We need to solve the theory first with mass deformation and then take the massless limit.

Calabi-Yau case

- How to define the (A-model) refined topological string? It is best to do it in the target space perspective. Consider the compactification of M-theory on a Calabi-Yau 3-fold M. The BPS particles in the remaining 5-dimension are M2 branes wrapping 2-cycles $\beta \in H_2(M,\mathbb{Z})$ of the Calabi-Yau, and are in the representation $[(1/2,0)+2(0,0)]\otimes(j_L,j_R)$ of 5d little group $SO(4)=SU(2)_L\times SU(2)_R$. Denote the number of such particles as $n^{\beta}_{j_L,j_R}$, a non-negative integer. The mass and charge of the particle is the size of the 2-cycle (denoted as a Kahler modulus parameter t).

- The number $n^{\beta}_{j_L,j_R}$ depends on the geometry of the Calabi-Yau. It can jump cross line of marginal stability when we move in the complex structure moduli space, a phenomenon known as the wall crossing. The following index is invariant.

$$n^{\beta}_{j_L}=\sum_{j_R}(-1)^{2j_R}(2j_R+1)n^{\beta}_{j_L,j_R}$$

- Further compactly the 5d theory on a circle S^1 to 4d, and also turn on the gravi-photon field $G = \epsilon_1 dx^1 \wedge dx^2 + \epsilon_2 dx^3 \wedge dx^4$ in 4d. The BPS charged particle can have momentum on the compact circle, which is the $D0$ brane charge in the language of M theory/IIA duality. The mass and charge of the particle is

$$m = t + 2\pi i n$$

where $n \in \mathbb{Z}$ is the D0 brane charge, or momentum on S^1.

- Integrating out charged particles in a gravi-photon background $G = \epsilon_1 dx^1 \wedge dx^2 + \epsilon_2 dx^3 \wedge dx^4$ generates $\int d^4x R_+^2 F$ terms in the effective action (c.f. Gopakumar-Vafa and Schwinger), where

$$F = -\int_{\epsilon}^{\infty} \frac{ds}{s} \frac{\mathrm{Tr}_{\mathcal{R}}(-1)^{\sigma_L+\sigma_R} e^{-sm} e^{-2is(\sigma_L\epsilon_L+\sigma_R\epsilon_R)}}{4\left(\sin^2\left(\frac{s\epsilon_L}{2}\right) - \sin^2\left(\frac{s\epsilon_R}{2}\right)\right)}$$

where $\epsilon_{R/L} = \epsilon_\pm = \frac{1}{2}(\epsilon_1 \pm \epsilon_2)$.

- We need to sum over all representation (j_L, j_R) of $SO(4)$, 2-cycles β with multiplicity $n^\beta_{j_L,j_R}$. This is the refined topological string amplitude. It can be computed using the refined topological vertex formalism. (Iqbal-Kozcaz-Vafa, hep-th/0701156; Awata-Kanno)

- Performing the sums, and also the integral over s, we find

$$F = \sum_{\substack{j_L,j_R=0 \\ m=1}}^{\infty} \sum_{\beta \in H_2(M,\mathbb{Z})} \frac{n^\beta_{j_L,j_R}}{m} \frac{(-1)^{2j_L+2j_R} \left(\sum_{n=-j_L}^{j_L} y_L^{mn}\right) \left(\sum_{n=-j_R}^{j_R} y_R^{mn}\right) e^{m\cdot(\beta,t)}}{4\left(\sin^2\left(\frac{m\epsilon_L}{2}\right) - \sin^2\left(\frac{m\epsilon_R}{2}\right)\right)}$$

with $y_{L/R} = e^{i\epsilon_{L/R}}$.

- Sometimes it is convenient from computational perspective to define the BPS numbers in a different basis

$$\sum n^\beta_{j_L,j_R}[J_L \otimes J_R] = n^\beta_{g_L,g_R} I_L^{g_L} \otimes I_R^{g_R}$$

where $I_{L/R} = [(1/2) + 2(0)]_{L/R}$. The numbers $n^\beta_{g_L,g_R}$ are integers if $n^\beta_{j_L,j_R}$ are, and vice versa. ($n^\beta_{g_L,g_R}$ could be negative).

The gap condition

- Consider the mirror picture in type IIB theory. At the conifold point, a D3-brane wrapping a vanishing 3-cycle becomes massless. Integrating out this charged particles generates the following singular terms in the effective action

$$F(\epsilon_1,\epsilon_2,a_D) = -\int_0^\infty \frac{ds}{s}\frac{\exp(-sa_D)}{4\sin(s\epsilon_1/2)\sin(s\epsilon_2/2)} + \mathcal{O}(a_D^0)$$

  Here we only consider singular terms as the mass of the particle $a_D \to 0$.

- We can expand the integrand in small ϵ_1, ϵ_2 and perform the integral

$$\begin{aligned} F(\epsilon_1,\epsilon_2,a_D) \;\; = \;\; & \big[-\frac{1}{12}+\frac{1}{24}(\epsilon_1+\epsilon_2)^2(\epsilon_1\epsilon_2)^{-1}\big]\log(a_D) \\ & +\big[-\frac{1}{240}(\epsilon_1\epsilon_2)+\frac{7}{1440}(\epsilon_1+\epsilon_2)^2-\frac{7}{5760}(\epsilon_1+\epsilon_2)^4(\epsilon_1\epsilon_2)^{-1}\big]\frac{1}{a_D^2} \\ & +\mathcal{O}(\frac{1}{a_D^4})+\mathcal{O}(a_D^0) \end{aligned}$$

  We see the gap structure near conifold point.

- Since the BPS number $n^{\beta}_{j_L,j_R}$ can change in the complex structure moduli space. The (A-model) refined topological string amplitudes may also jump in the complex structure moduli space, in addition to the dependence on Kahler moduli space. We consider some cases where the problem is more tractable, namely the local Calabi-Yau space where the complex structure moduli are frozen.

- Using the generalized holomorphic anomaly equation and the gap conditions, we solve the refined topological string on some well-known toric geometries: resolved conifold, local P^2 model, local $P^1 \times P^1$ model. We find agreements with the calculations from the refined topological string vertex formalism of Iqbal-Kozcaz-Vafa .

- We can also compute the refined BPS invariants for other local non-toric Calabi-Yau manifolds, such as the del Pezzo, half K3 Calabi-Yau manifolds. MH, A. Klemm and M. Poretschkin, arXiv:1308.0619.

Mathematical definition of refined invariants

- Mathematical definition of $N^{\beta}_{j_L,j_R}$ from stable pairs (J. Choi, S. Katz and A. Klemm, arXiv:1210.4403).

- The moduli space of stable pairs on a Calabi-Yau threefolds M is denoted $P_n(M,\beta)$. It has a perfect and symmetric obstruction theory, and Its virtual dimension is zero, so one can integrate it to a number.

$$\#^{vir}(P_n(M,\beta)) = \int_{[P_n(M,\beta)]^{\mathrm{vir}}} 1 \ .$$

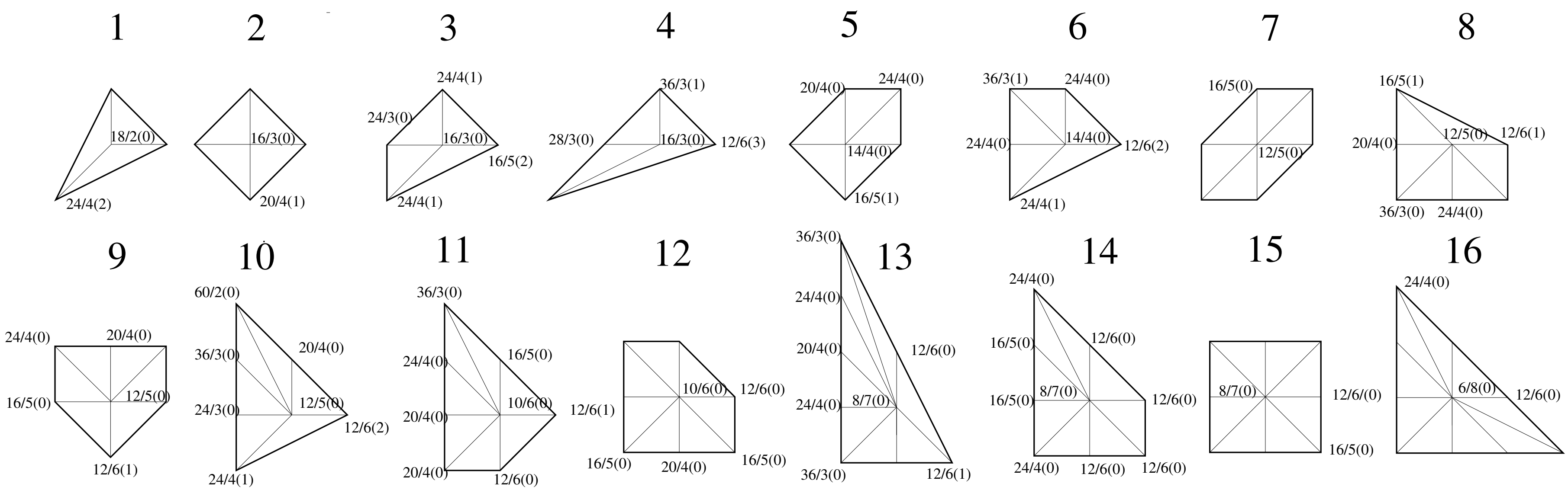
- The construction can be refined by an extension of the classical Bialynicki-Birula decomposition to the virtual case.

- Del Pezzo Calabi-Yau spaces: the total space of the fibration of the anti-canonical line bundle $\mathcal{O}(-K_B) \to B$, over a Fano variety B and their mirror manifolds.

- Del Pezzo surfaces are two-dimensional smooth Fano manifolds and enjoy a finite classification. The list is $\mathbb{P}^2$ and blow-ups of $\mathbb{P}^2$ in up to $n = 8$ points, called $\mathcal{B}_n$, as well as $\mathbb{P}^1 \times \mathbb{P}^1$.

- The d-dimensional toric Fano varieties are most easily classified by d-dimensional reflexive polyhedra. Toric almost del Pezzo surfaces are given by reflexive polyhedra in two dimensions.

- These are the 16 reflexive polyhedra Δ in two dimensions, which build 11 dual pairs (Δ, Δ^*). Polyhedron k is dual to polyhedron $17-k$ for $k=1,\ldots,6$. The polyhedra $7,\ldots,10$ are self-dual. In particular the polyhedra 1,2,3,5,6 correspond to toric del Pezzo surfaces.



- The local mirror geometries are encoded by elliptic curves. We use the well known Nagell's algorithm to transform the curves to Weierstrass form, and derived the Picard-Fuchs differential equations for the periods.

The massless cases

- The complex geometry of the mirror manifolds are described by the Picard-Fuchs differential equations

$$(\theta_z^2 + c_0 z \prod_{i=1}^{2} (\theta_z + 1 - a_i))\theta_z \int_{\gamma_i} \Omega = 0,$$

  where z is the complex structure modulus in the mirror manifold and $\theta_z = z\partial_z$. a_1, a_2 and c_0 are classical constants of the the Calabi-Yau manifolds.

- The vectors $\vec{a} = (a_1, a_2)$ satisfy $a_1 + a_2 = 1$ and are given as follows for various one-parameter families of Calabi-Yau manifolds we consider

$$\mathbb{P}^2: \ \vec{a} = (\frac{1}{3}, \frac{2}{3}), \qquad \mathbb{P}^1 \times \mathbb{P}^1: \ \vec{a} = (\frac{1}{2}, \frac{1}{2}), \qquad D_5: \ \vec{a} = (\frac{1}{2}, \frac{1}{2}),$$
$$E_6: \ \vec{a} = (\frac{1}{3}, \frac{2}{3}), \qquad E_7: \ \vec{a} = (\frac{1}{4}, \frac{3}{4}), \qquad E_8: \ \vec{a} = (\frac{1}{6}, \frac{5}{6}).$$

Higher genus amplitudes

- We discuss next the genus one amplitudes $F^{(1,0)}$ and $F^{(0,1)}$. The $F^{(1,0)}$ amplitude is holomorphic while the amplitude $F^{(0,1)}$ has a holomorphic anomaly which is determined by the genus one holomorphic anomaly equation. Both amplitudes have logarithmic cuts for the discriminant $\Delta(z) = 1 + c_0 z$ and z as

$$\begin{aligned} F^{(1,0)} &= \frac{\log(\Delta(z)) - c^{(1,0)} \log(z)}{24}, \\ F^{(0,1)} &= -\frac{1}{2}\log(\partial_z t(z)) - \frac{1}{12}(\log(\Delta(z)) + c^{(0,1)} \log(z)), \end{aligned}$$

  where we use the constants $c^{(1,0)}$ and $c^{(0,1)}$ to denote the coefficients for $\log(z)$ terms in the refined amplitudes.

- We compute the $g \geq 2$ amplitudes by refined holomorphic anomaly and boundary conditions. Due to the orbifold singularity, there are not sufficient boundary conditions for the E_n models at high genus.

An example of E_5 del Pezzo

$2j_L \backslash 2j_R$	0
0	16

d=1

$2j_L \backslash 2j_R$	0	1
0		10

d=2

$2j_L \backslash 2j_R$	0	1	2
0			16

d=3

$2j_L \backslash 2j_R$	0	1	2	3	4
0		1		45	
1					1

d=4

$2j_L \backslash 2j_R$	0	1	2	3	4	5
0			16		144	
1						16

$d = 5$

$2j_L \backslash 2j_R$	0	1	2	3	4	5	6	7
0		10		130		456		
1					10		130	
2								10

$d = 6$

some salient features

- For degree d which is a positive integer as an element in $H_2(M,\mathbb{Z})$, there is a non-vanishing positive integer $n^d_{j_L,j_R} = \tilde{n}^d_{2j_L,2j_R}$ at the top genus $(2j_L, 2j_R) = (g_L^{top}, g_R^{top})$. All higher genus invariants vanish so the non-vanishing GV invariants form a rectangular matrix.

- In the basis of integers $\tilde{n}^d_{g_L,g_R}$, the GV invariants do not generically vanish if the genus pair lies in the rectangular matrix. However in the j-spin basis $n^d_{j_L,j_R}$, there is furthermore a large number of vanishing GV invariants $n^d_{j_L,j_R}$ inside the rectangular matrix. The genus pairs of these non-vanishing integers follow a chess board pattern.

- These patterns can help to fix the holomorphic ambiguities at low genus. The redundancies provide highly non-trivial checks of our calculations.

Half K3 model

- The topological string amplitudes on the half K3 Calabi-Yau threefold are equivalent to the partition function of the six-dimensional non-critical E-string compactified on a circle. The winding and momentum numbers of the E-string on the compactified circle correspond to the wrapping numbers n_b and d on the base and fiber in the homology classes $n_b[p] + d[f]$ in the half K3 surface.

- The geometry of the half K3 surface can be constructed by blowing up nine points on $\mathbb{P}^2$. The second homology classes in $H_2(\mathcal{B}_9, \mathbb{Z})$ consist of the class of the line $[l]$ on $\mathbb{P}^2$, and the nine exceptional classes $[e_i]$ $(i = 1, 2, , \cdots , 9)$ on the blown-up points. The non-zero intersections of the classes are $[l] \cdot [l] = 1$, $[e_i] \cdot [e_i] = -1$. In this construction, the base class p and fiber class f we mentioned above are the linear combinations

$$p = [e_9], \qquad f = 3[l] - \sum_{i=1}^{9} [e_i]$$

- The Göttsche formula is the generating function for the Betti numbers of the Hilbert scheme of d points on a complex surface S. For wrapping number $n_b = 1$ we can compute by an infinite product formula, the refined Göttsche formula.

$$\frac{1}{G^S(q,y_L,y_R)} = \prod_{n=1}^{\infty}(1-y_Ly_Rq^n)(1-y_Ly_R^{-1}q^n)(1-y_L^{-1}y_Rq^n)$$
$$(1-y_L^{-1}y_R^{-1}q^n)(1-q^n)^{b_2(S)-2}$$

- For higher wrapping number $n_b \geq 2$, we propose a refinement of the modular anomaly equation of S. Hosono, M. H. Saito and A. Takahashi.

- We can turn on 8 generic mass parameters. The topological string amplitudes are construct by Jacobi modular forms of E_8 lattice.

- By properly scaling the parameters and set some to zero, we can flow to the del Pezzo models and recover the previous results.

2.2. Nekrasov-Shatashvili limit

References:

1. MH, “On Gauge Theory and Topological String in Nekrasov-Shatashvili Limit,” JHEP **1206**, 152 (2012) [arXiv:1205.3652 [hep-th]].

2. MH, A. Klemm, J. Reuter and M. Schiereck, arXiv:1401.4723 [hep-th].

- Nekrasov-Shatashvili limit: we take one of $\epsilon_{1,2}$ to vanish, say $\epsilon_2 = 0$, and consider the expansion around $\epsilon \equiv \epsilon_1$. Nekrasov-Shatashvili conjectured that in this limit, the vacua of $\mathcal{N} = 2$ gauge theory correspond to certain quantum integrable systems.

- In this limit we can define the deformed prepotential $\mathcal{F}$ as

$$\mathcal{F}(a_i, \epsilon) = \sum_{n=0}^{\infty} \epsilon^{2n} F^{(n,0)}(a_i) \tag{18}$$

- We can prove some of our higher genus formulae for $F^{(n,0)}$ in the limit. We can derive the holomorphic anomaly equation in this limit. This also works for the non-compact Calabi-Yau manifolds, e.g. the local $\mathbb{P}^2$ model, local $\mathbb{P}^1 \times \mathbb{P}^1$ model, and the local del Pezzo Calabi-Yau models.

The Saddle Point Equation

- The deformed prepotential $\mathcal{F}$ can be computed from Nekrasov function using saddle point method (Poghossian et al).

$$qM(x-\epsilon)w(x)w(x-\epsilon) - w(x)P(x) + 1 = 0$$

where $q = \Lambda^{4-N_f}$ and $y^2 = P(x)^2 - 4qM(x)$ is the Seiberg-Witten curve. For the pure $SU(2)$ case, $P(x) = x^2 - u$ and $M(x) = 1$.

- Poghossian et al use the saddle point method to solve the deformed prepotential $\mathcal{F}(\tilde{a}, \epsilon, q)$ perturbatively in q parameter and the solution is exact in ϵ parameter. On the other hand, in order to make connection with our higher genus formulae, we need to instead solve the deformed prepotential exactly in q parameter and but perturbatively in ϵ parameter.

- The $w(x)$ is the spectral function. It can be solved perturbatively

$$
\begin{aligned}
w(x) &= \sum_{n=0}^{\infty} w_n(x)\epsilon^n, \\
w_0(x) &= \frac{P(x) - \sqrt{P(x)^2 - 4q}}{2q}, \\
w_1(x) &= \frac{x(P(x) - \sqrt{P(x)^2 - 4q})^2}{2q(P(x)^2 - 4q)}, \\
\ldots
\end{aligned}
$$

- The deformed period

$$
\begin{aligned}
\tilde{a} &= -\sum_{n=0}^{\infty} \mathrm{Res}_{x=\sqrt{u}} x\partial_x \log w(x) \\
&= a_0 + a_2\epsilon^2 + a_4\epsilon^4 + \mathcal{O}(\epsilon^6)
\end{aligned}
$$

- Here $a \equiv a_0$ is the conventional period, satisfy Picard-Fuchs differential equation

$$4(4q - u^2)\partial_u^2 a = a$$

- There are exact formulae for higher order terms.

$$\begin{aligned} a_2 &= \frac{1}{24}(\partial_u a + 2u\partial_u^2 a) \\ a_4 &= \frac{(60qu - u^3)\partial_u a + 2(300q^2 + 153qu^2 - u^4)\partial_u^2 a}{2880(u^2 - 4q)^2} \end{aligned}$$

- We can also define the deformed dual period from the integrand, with a different contour

$$\tilde{a}_D = a_{D0} + a_{D2}\epsilon^2 + a_{D4}\epsilon^4 + \mathcal{O}(\epsilon^6)$$

They satisfy the same equations as the deformed period.

Two ways to compute the deformed prepotential

- First way: use the deformed Matone relation

$$
\begin{aligned}
0 &= q\frac{d\mathcal{F}_{inst}(\tilde{a},\epsilon,q)}{dq} - \tilde{a}^2 + \\
&= \frac{1}{2}F^{(0,0)}(a) - \frac{1}{4}a\frac{\partial F^{(0,0)}(a)}{\partial a} + u \\
&\quad + \frac{\epsilon^2}{4}[a_2(a_D + 2\pi i\tau a) - a\frac{\partial F^{(1,0)}(a)}{\partial a} + \frac{1}{6}] \\
&\quad + \frac{\epsilon^4}{2}[-F^{(2,0)}(a) - \frac{a}{2}\frac{\partial F^{(2,0)}(a)}{\partial a} - \frac{a_2}{2}\partial_a(a\frac{\partial F^{(1,0)}(a)}{\partial a}) \\
&\quad + \frac{a_4}{2}(a_D + 2\pi i\tau a) + \frac{a_2^2}{4}\partial_a(a_D + 2\pi i\tau a)] + \mathcal{O}(\epsilon^6)
\end{aligned}
$$

- Each order provides a differential equation for $F^{(n,0)}$. We can prove our formulas by showing they satisfy the equations.

- Second way: use the deformed period (A. Mironov and A. Morozov).

$$\begin{aligned} 0 &= \frac{\partial \mathcal{F}(\tilde{a})}{\partial \tilde{a}} - \tilde{a}_D \\ &= \frac{\partial F^{(0,0)}(a)}{\partial a} - a_D + \epsilon^2(\partial_a F^{(1,0)}(a) - 2\pi i \tau a_2 - a_{D2}) \\ &\quad + \epsilon^4[\partial_a F^{(2,0)}(a) + a_2 \partial_a^2 F^{(1,0)}(a) - 2\pi i \tau a_4 \\ &\quad - \pi i (\partial_a \tau)(a_2)^2 - a_{D4}] + \mathcal{O}(\epsilon^6) \end{aligned}$$

- We can also show our formulas satisfy the low order equations. Disadvantage: the equation is not yet derived from Nekrasov's function.

Deriving holomorphic anomaly equation in NS limit

- We explicitly check our higher genus formulae satisfy the equations from the saddle point method up to some low genus. It would be nice to directly show the saddle point method is consistent with the holomorphic anomaly equation and gap boundary conditions, and therefore prove the equivalence to all genera.

- Under certain simple assumptions, the holomorphic anomaly equation in the Nekrasov-Shatashvili limit can be derived from the equation $\frac{\partial \mathcal{F}(\tilde{a})}{\partial \tilde{a}} = \tilde{a}_D$ for deformed dual period.

$$\partial_{E_2} F^{(n,0)} = \frac{1}{24} \sum_{l=1}^{n-1} \partial_a F^{(l,0)} \partial_a F^{(n-l,0)}$$

- Basic idea: understand how ∂_{E_2} and ∂_a commute with each others. We find

$$\partial_{E_2}\partial_a F_k = \partial_a\partial_{E_2}F_k - \frac{k\pi i}{6}(\partial_a\tau)F_k,$$

where F_k is a tensor with k lower indices.

- We expand equation for deformed dual period, and derive the holomorphic anomaly equation in NS limit by induction.

$$\begin{aligned} 0 &= \frac{\partial\mathcal{F}(\tilde{a},\epsilon)}{\partial\tilde{a}} - \tilde{a}_D \\ &= \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\partial_a^{k+1}F^{(n,0)}(a)\frac{(\tilde{a}-a)^k}{k!}\epsilon^{2n} \\ &\quad + \sum_{k=0}^{\infty}\partial_a^{k+1}(-2\pi i\tau)\frac{(\tilde{a}-a)^{k+2}}{(k+2)!} - [2\pi i\tau(\tilde{a}-a)+(\tilde{a}_D-a_D)] \end{aligned}$$

2.3. Modular Anomaly from Holomorphic Anomaly in Mass Deformed N=2 Superconformal Field Theories

References:

1. M. -x. Huang, Phys. Rev. D **87**, 105010 (2013) [arXiv:1302.6095 [hep-th]]

Modular anomaly and holomorphic anomaly

- we consider two well-known superconformal field theories, namely the $SU(2)$ $\mathcal{N}=2$ gauge theories with an adjoint hypermultiplet and with $N_f=4$ fundamental hypermultiplets.

- In the first theory the supersymmetry is enhanced to $\mathcal{N}=4$ and the gauge coupling is corrected by neither perturbative nor instanton contributions. For the second theory the gauge coupling is renormalized by instanton effects, as seen from the Nekrasov partition function.

- We turn on mass parameters in the theories, which break the conformal symmetry and keep the $\mathcal{N}=2$ supersymmetry. The first theory with mass deformation is also known as the $\mathcal{N}=2^*$ theory. In both theories the gauge coupling is renormalized by mass deformation.

- Two types of anomaly equations, namely the modular anomaly and holomorphic anomaly, have been discovered in the literature. We provide a clean solution to the long standing puzzle about their precise relation, and obtain some universal formulae.

- Modular anomaly: We can expand the instanton partition functions of the two theories around the large modulus point in the Coulomb branch, i.e. where the v.e.v. of the scalar in the vector multiplet is large. As power series of the flat coordinate a, the coefficients consist of Eisenstein series and Jacobi theta functions as shown by Minahan et al, Billo et al.

- Physically, the quasi-modularity comes from the $SL(2,\mathbb{Z})$ duality which acts on the gauge coupling constant. Here the quasi-modular forms are weighted homogenous polynomials of the Eisenstein series E_2, E_4, E_6. The E_2 series transforms with a shift under S-duality so it is not exactly modular. The modular anomaly equations relate the partial derivative of instanton partition function with respect to E_2 to lower order terms.

- Comparing with Holomorphic anomaly from topological string theory:

- It was strongly believed that these two approaches are related. However there are apparent differences between them, and no clear derivation from one to the other is available in the literature. In the modular anomaly equation, the partition functions are expanded around large Coulomb modulus point and the argument of the quasi-modular forms is the bare coupling, while the holomorphic anomaly approach gives exact amplitudes at any points of moduli space and the argument of quasi-modular forms is the renormalized gauge coupling.

- Furthermore, the modular anomaly appears already at genus zero while the holomorphic anomaly appears only at higher genus.

- A similar issue also appears in the studies of topological strings on a class of elliptically fibered Calabi-Yau manifolds Alim et al:2012, Klemm et al:2012. In a related paper (work in progress) we will resolve this long standing issue. The idea is similar but the details are more complicated for compact Calabi-Yau models.

2.4. Dijkgraaf-Vafa conjecture and β-deformed matrix models

References:

1. M. -x. Huang, JHEP **1307**, 173 (2013) [arXiv:1305.1103 [hep-th]].

Introduction

- One of Dijkgraaf-Vafa's conjectures (2002) states the equivalence of the following two theories:

1. The Hermitian matrix model

$$Z = e^F = \frac{1}{\mathrm{Vol}(U(N))} \int [D\Phi] e^{-\frac{\mathrm{tr}\, W(\Phi)}{g_s}}, \tag{19}$$

where Φ is a $N \times N$ Hermitian matrix and the matrix potential $W(\Phi)$ is a degree $n+1$ polynomial.

2. The B-model topological string theory on the local Calabi-Yau 3-fold, described by a curve in $\mathbb{C}^4$ coordinate (u, v, x, y)

$$uv = y^2 + W'(x)^2 + f(x), \tag{20}$$

where $f(x)$ is a degree $n-1$ polynomial whose coefficients parametrize the complex structure moduli of the Calabi-Yau geometry.

- The matrix model and the topological string free energy F are conjectured to be equal.

- We will take large N limit in the matrix model. There is a natural large N expansion of the matrix model free energy organized by the genus of the Feynman diagram. At each genus, there is a further perturbative expansion in terms of the t'Hooft parameter

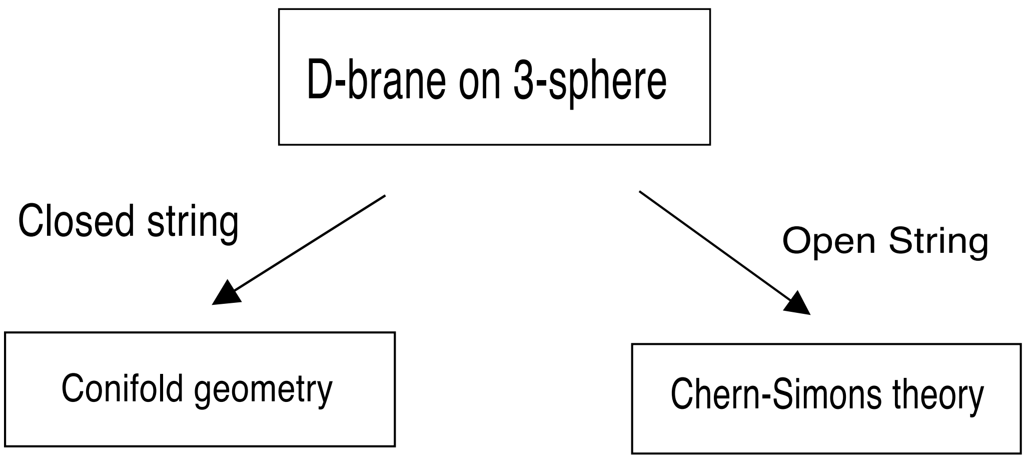
$$t_i = g_s N_i, \tag{21}$$

  where the N eigenvalues of the matrix are distributed continuously around the critical points of the matrix potential $N = \sum_i N_i$.

- The genus expansion of the matrix model is identified with the genus expansion of the topological string. The t'Hooft parameters (21) in the matrix model is identified with the periods, or flat coordinates of the Calabi-Yau geometry.

Origin of the conjecture

- The Gopakumar-Vafa duality relates the large N Chern- Simons theory on S^3 with A-model topological strings on the resolved conifold. This is an example of open-closed string duality.



- The Dijkgraaf-Vafa conjecture is morally the mirror of the Gopakumar-Vafa duality. The open string theory is a holomorphic Chern-Simon theory deformed by the potential $W(\Phi)$, which turns out to reduce to a matrix model.

Some previous works on DV conjecture

- For the genus zero case, the leading planar free energy is known as the prepotential $\mathcal{F}^{(0)}$. Dijkgraaf and Vafa showed that the both sides of the duality satisfy the period equation

$$t_i = \oint_{A_i} \lambda dx, \qquad \frac{\partial \mathcal{F}^{(0)}}{\partial t_i} = \oint_{B_i} \lambda dx, \tag{22}$$

  where $\lambda dx = \sqrt{W'(x)^2 + f(x)}dx$ is a differential one-form, reduced from the unique holomorphic 3-form of the Calabi-Yau geometry.

- The genus one case is checked by Klemm, Marino and Theisen, hep-th/0211216; Dijkgraaf, Sinkovics and Temurhan, hep-th/0211241.

- In my previous work HM and A. Klemm, hep-th/0605195, we check the conjecture at genus two. Later, Klemm, Marino and Rauch, arXiv:1002.38 pushed the calculations to much higher genus using the gap condition near the conifold divisor.

DV conjecture and β-deformation

- Recently there have been some interests in refined topological string theory, which originate from the Ω deformation in supersymmetric gauge theory. It is expected that the Dijkgraaf-Vafa conjecture can be generalized to the refined case.

- Here the refinement corresponds to the β-deformation of the matrix models.

- The topological expansion of ordinary matrix model free energy has been constructed from the spectral curves [Akemann; Eynard, Orantin, math-ph/0702045], known as the topological recursion method. It can be generalized to the β-deformed case [Chekhov, Eynard].

- However, the topological recursion method is still too difficult for many practical calculations, even more for the β-deformed case. In this paper we provide some higher genus formulae from the refined topological string method.

β-deformed matrix models

- The matrix integral can be written in terms of eigenvalues

$$Z = e^F = \frac{1}{N!(2\pi)^N} \int (\prod_i d\lambda_i) \Delta^2(\lambda) e^{-\sum_i \frac{W(\lambda_i)}{g_s}}, \tag{23}$$

where λ_i $(i = 1, 2 \cdots, N)$ are the eigenvalues of the Hermitian matrix Φ, and $\Delta(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j)$ is the well known Vandermonde determinant.

- The β-deformation replaces the matrix integrand by its β power, so the partition function becomes

$$Z(\beta) = e^{F(\beta)} = \frac{1}{N!(2\pi)^N} \int (\prod_i d\lambda_i) \Delta^{2\beta}(\lambda) e^{-\frac{\beta}{g_s} \sum_i W(\lambda_i)}. \tag{24}$$

- We find the exact formulas for the genus two amplitudes $\mathcal{F}^{(2,0)}$, $\mathcal{F}^{(0,2)}$ and $\mathcal{F}^{(1,1)}$ using the topological B-model methods of holomorphic anomaly equation and boundary conditions. These formulas are very difficult to obtain from traditional matrix model method. We check the expansions near the point $(z_1, z_2) = (0, 0)$ agree with the higher order terms from perturbative matrix model calculations.

- It would be interesting to prove our formulas from the topological recursion method of Eynard et al.

2.5. Relation with quantum integrable models

References:

1. Y. Hatsuda, M. Marino, S. Moriyama and K. Okuyama, "Non-perturbative effects and the refined topological string," arXiv:1306.173 [hep-th].

2. J. Kallen and M. Marino, "Instanton effects and quantum spectral curves," arXiv:1308.6485 [hep-th].

A spectral problem

- Consider the following integral kernel

$$\rho(x_1,x_2) = \frac{1}{\hbar}\frac{1}{\left(2\cosh\frac{x_1}{2}\right)^{1/2}}\frac{1}{\left(2\cosh\frac{x_2}{2}\right)^{1/2}}\frac{1}{2\cosh\left(\frac{\pi(x_1-x_2)}{\hbar}\right)}. \tag{25}$$

The spectral problem is

$$\int_{-\infty}^{\infty}\rho(x_1,x_2)\phi(x_2)dx_2 = \lambda\phi(x_1). \tag{26}$$

- In operator formalism, we can write

$$\widehat{\rho} = e^{-\frac{1}{2}U(\widehat{x})}e^{-T(\widehat{p})}e^{-\frac{1}{2}U(\widehat{x})}. \tag{27}$$

where $\widehat{x},\widehat{p}$ are canonical position and momentum operators $[\widehat{x},\widehat{p}] = i\hbar$, and

$$U(x) = \log\left(2\cosh\frac{x}{2}\right), \qquad T(p) = \log\left(2\cosh\frac{p}{2}\right). \tag{28}$$

- The expectation value is then $\langle x|\widehat{\rho}|x'\rangle = \rho(x,x')$, using the integral

$$\int_{-\infty}^{\infty} \frac{e^{ixp}dp}{2\cosh(\frac{p}{2})} = \frac{\pi}{\cosh(\pi x)} \tag{29}$$

- The operator $\widehat{\rho}$ has some properties.

1. it is Hermitian: $\rho(x_1,x_2) = \rho(x_2,x_1)$.

2. It is non-negative. Denote $|\psi\rangle = e^{-\frac{1}{2}U(\widehat{x})}|\phi\rangle$

$$\langle\phi|\widehat{\rho}|\phi\rangle = \langle\psi|e^{-T(\widehat{p})}|\psi\rangle = \int_{-\infty}^{\infty} \frac{dp}{2\cosh(\frac{p}{2})} \geq 0 \tag{30}$$

3. An upper bound for eigenvalues. Suppose $\phi(x)$ is normalized wave function,

$$\begin{aligned} \int_{-\infty}^{\infty} \phi(x_1)\rho(x_1,x_2)\phi(x_2)dx_1dx_2 &\leq \int_{-\infty}^{\infty} \rho(x_1,x_2)|\phi(x_1)|^2 dx_1dx_2 \\ &\leq \int_{-\infty}^{\infty} \frac{1}{\hbar}\frac{dx_2}{2\cosh\left(\frac{\pi x_2}{\hbar}\right)} = \frac{1}{2} \end{aligned}$$

- So we can write the eigenvalues of $\widehat{\rho}$ as $\lambda = e^{-E}$, where the energy levels

$$\log(2) \leq E_0 \leq E_1 \leq E_2 \leq \cdots \tag{31}$$

- Define $\widehat{\rho} = e^{-\widehat{H}}$. The classical Hamiltonian is

$$H = T(p) + U(x) = \log(4 \cosh(\frac{x}{2}) \cosh(\frac{p}{2})) \tag{32}$$

The classical ground state energy is obtained at $x = p = 0$

$$E_0 = \log(4) \tag{33}$$

- The Hamiltonian (32) turns out to be the a specialization of the curve describing the mirror of local $\mathbb{P}^1 \times \mathbb{P}^1$ Calabi-Yau manifold. It can be also regarded as a relativistic deformation of the spectral curve of the periodic Toda chain.

- It is not know how to solve the spectral problem analytically. One has to use numerical or approximate method.

- Nekrasov-Shatashvili conjecture: The spectral problem is equivalent to the Bohr-Sommerfeld quantization condition

$$\oint p(x)dx = 2\pi\hbar(n + \frac{1}{2}), \tag{34}$$

where the period integral is the deformed B-period in the Nekrasov-Shatashvili limit of the local Calabi-Yau manifold.

- The computation of the periods is familiar in mirror symmetry, by the use of Picard-Fuchs differential equation. Setting the complex structure parameters $z_1 = z_2 = e^{-2E}$ in the local $\mathbb{P}^1 \times \mathbb{P}^1$ Calabi-Yau manifold, we find the undeformed period

$$8\Pi_B\left(e^{-2E}\right) - \frac{4\pi^2}{3} = 8E^2 - \frac{4\pi^2}{3} - 8E\sum_{\ell\geq 1}\widehat{a}_n^{(0)}e^{-2\ell E} + 2\sum_{\ell\geq 1}\widehat{b}_n^{(0)}e^{-2\ell E}.$$

- Here the coefficients are written in Gamma and Digamma functions $\Gamma(x), \psi(x)$ as

$$\hat{a}_\ell^{(0)} = \frac{1}{\ell}\left(\frac{\Gamma\left(\ell+\frac{1}{2}\right)}{\Gamma(\frac{1}{2})\ell!}\right)^2 16^\ell ,$$

$$\hat{b}_\ell^{(0)} = \frac{4}{\ell}\left(\frac{\Gamma\left(\ell+\frac{1}{2}\right)}{\Gamma(\frac{1}{2})\ell!}\right)^2 16^\ell \left[\psi\left(\ell+\frac{1}{2}\right) - \psi(\ell+1) + 2\log 2 - \frac{1}{2\ell}\right].$$

- The Bohr-Sommerfeld quantization condition with undeformed period correspond to the classical energy spectrum in $\hbar \to 0$. We can check the classical ground state energy $E_0 = \log(4)$ indeed satisfy the equation

$$8E_0^2 - \frac{4\pi^2}{3} - 8E_0 \sum_{\ell\geq 1} \hat{a}_n^{(0)} e^{-2\ell E_0} + 2\sum_{\ell \geq 1} \hat{b}_n^{(0)} e^{-2\ell E_0} = 0 \qquad (35)$$

- To compute the quantum energy spectrum, one needs to use the deformed period Nekrasov-Shatashvili limit, explained in previously. The Planck constant $\hbar$ is identified with the ϵ parameter in the Nekrasov-Shatashvili limit.

- However, it turns out that this is not enough. The deformed period has singularities when $\frac{\hbar}{2\pi}$ is an integer. In J. Kallen and M. Marino, arXiv:1308.6485 [hep-th], Some non-perturbative corrections proportional to $e^{-\frac{2\pi n E}{\hbar}}$ are proposed using the ordinary topological string amplitudes. They checked that the non-perturbative corrections cancel the singularities, and match the energy spectrum from numerical calculations.

- It would be interesting to generalize to the idea to other Calabi-Yau models.

Summary and outlook

- There are many questions to study. Find more methods to compute (refined) topological string theory. Prove that the many methods are equivalent.

- Solve a compact Calabi-Yau model completely (to any genus).

- How to define non-perturbative topological strings?

Thank You