Quantum integrable systems

Wen-Li Yang
Northwest University

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- Superstring and N=4 Super Yang-Mills theories: Ads/CFT correspondence (Maldacena 1998) (anomalous dimensions...)
- N=2 Super Yang-Mills theories (Nekrasov 2009) (superpotential and Yang-Yang action...)
- Low-dimensional Condensed matter physics: Cold atom systems, Quantum gas...
- Statistical physics: 2D statistical models and some stochastic processes ASEP,

- The study of magnetic chains initiated by Bethe in the early 1930s (Coordinate Bethe ansaz)
- The works of Onsanger and Baxter in 2D classical statistical mechanics in 1940s and the early 1970s (The T-Q ansatz and Yang-Baxter equation)
- The Scattering theory in many body problem with factorizable S-matrix (Yang) in the later 1960s (Yang-Baxter equation)
- The Quantum Inverse Scattering Method (QISM) developed by the school of Leningrad (Faddeev) in the end of 1970s. (Algebraic Bethe ansatz)
- Analytic Bethe ansatz, Functional Bethe ansatz, ...

QISM and its generalizations Bethe ansatz methods have been proven to be very successful in

- Exact solutions to quantum integrable systems with $U(1)$-symmetry

However, there exist a class of quantum integrable systems without $U(1)$-symmetry

- The XYZ spin chain with odd number of sites (Baxter 1970s)
- The anisotropic spin torus
- Quantum spin chains with unparallel boundary fields

The boundaries break the $U(1)$-symmetry $\Rightarrow$ the conventional Bethe ansatz fail.

There have been numerous efforts to approach the exact solutions to this class quantum integrable systems over last 20 years.

Joint works with J. -P. Cao, K.-J. Shi and Y.-P. Wang:

- *Phys. Rev. Lett.* **111** (2013), 137201 [arXiv:1305.7328]
- Nucl. Phys. **B 875** (2013), 152 [arXiv:1306.1742]
- arXiv:1307.0280
- Nucl. Phys. **B 977** (2013), 152 [arXiv:1307.2023]

Plan of the talk

- Classical integrable systems
  - Liouville Theorem.
  - Lax integrability.
- Quantum integrable systems
  - 2D statistical mechanics.
- Quantum Spin Chains with $U(1)$-symmetry
  - Periodic boundary condition and twisted boundary condition.
  - Open chain with parallel boundary fields.
- Quantum Spin Chains without $U(1)$-symmetry
  - Antiperiodic boundary condition.
  - Open chain with unparallel boundary fields.
- Summary

Let (p_i, q_i) $(i = 1, \ldots, n)$ be coordinates of the phase space of a classical mechanics system. The equation of motion is given by the Hamiltonian equations

$$\begin{cases} \dot{q}_i = \{H, q_i\} = \frac{\partial H}{\partial p_i}, \\ \dot{p}_i = \{H, p_i\} = -\frac{\partial H}{\partial q_i}. \end{cases} \tag{1}$$

If there exist n functional independent and Poisson commutative conserved charges $\{I_i | i = 1, \ldots, n\}$ (usually called action variables) , namely,

$$\{H, I_i\} = 0 = \{I_i,\ I_j\}, \quad \forall i,\ j, \tag{2}$$

$$\prod_{i=1}^{n} \wedge dI_i \neq 0, \tag{3}$$

there exists a canonical transformation $(q_i, p_i) \leftrightarrow (A_i, I_i)$, where A_i are the so-called angular variables. The equation of motion (the Hamiltonian equations), in terms of new variables, reads

$$\begin{cases} \dot{I}_i = \{H, I_i\} = 0, \\ \dot{A}_i = \{H, A_i\} = -\frac{\partial H}{\partial I_i} = \Omega_i(I_1, \ldots, I_n). \end{cases} \tag{4}$$

The solution to the equations is

$$\begin{cases} I_i(t) = I_i(0), \\ A_i = A_i(0) + \Omega_i(I_1(0), \ldots, I_n(0))t. \end{cases} \tag{5}$$

The solution to the original E.O.M (1) is obtained by the inverse of the canonical transformation

$$(q_i, p_i) \longleftarrow (A_i, I_i). \tag{6}$$

If there exists a Lax pair (M, L), where M and L are 2 $m \times m$ matrices with the matrix elements are functions of q_i and p_i such that

$$(1) \Leftrightarrow \dot{L} = [M, L], \tag{7}$$

then (M, L) is called the Lax representation of the (1). It is easy to check that the quantities $C_i = tr\{(L)^i\}$ are conserved. Moreover, if the Lax operator L satisfies the following Poisson bracket

$$\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2], \tag{8}$$

where

$$L_1 = L \otimes \mathrm{id}, \; L_2 = \mathrm{id} \otimes L, \quad r_{12} \in \mathrm{End}(V \otimes V), \quad r_{21} = P_{12} r_{12} P_{12},$$

where P_{12} is the permutation operator, then the associated conserved charges $\{C_i\}$ are Poisson commutative. In the case that C_i are functional independent each other and the number is equal to n, we conclude that

$$\text{Liouville integrability} \Leftrightarrow \text{Lax integrability}$$

Let us introduce a Lax pair (M, L) as follows

$$L = \sum_{i=1}^{n} I_i H_i + 2 I_i A_i E_i = \begin{pmatrix} \square_1 & & & \\ & \square_2 & & \\ & & \ddots & \\ & & & \square_n \end{pmatrix}, \tag{9}$$

$$M = -\sum_{i=1}^{n} \frac{\partial H}{\partial I_i} E_i = \begin{pmatrix} \square'_1 & & & \\ & \square'_2 & & \\ & & \ddots & \\ & & & \square'_n \end{pmatrix}, \tag{10}$$

where

$$\square_i = \begin{pmatrix} I_i & 2 I_i A_i \\ & -I_i \end{pmatrix}, \quad \square'_i = \begin{pmatrix} & -\frac{\partial H}{\partial I_i} \\ & \end{pmatrix}.$$

Then we have

$$\dot{L} = [M,\ L] \Leftrightarrow (25) \Leftrightarrow (1). \tag{11}$$

The Poisson bracket of the Lax operator is

$$\{L_1,\, L_2\} = 2\sum_{i=1}^{n} I_i(H_i \otimes E_i - E_i \otimes H_i) = [r_{12},\, L_1] - [r_{21},\, L_2]\,, \qquad (12)$$

$$r_{12} = -\sum_{i=1}^{n} E_i \otimes H_i. \qquad (13)$$

The Lax equation (7) is covariant by the following gauge transformation

$$(M, L) \longrightarrow (\tilde{M}, \tilde{L}), \quad \begin{cases} \tilde{L} = gLg^{-1}, \\ \tilde{M} = gMg^{-1} + \dot{g}g^{-1}. \end{cases} \qquad (14)$$

Under the gauge transformation

$$\text{Invariants}: \quad C_i = tr(L^i), \tag{15}$$

$$\text{Covariants}: \quad \left\{\tilde{L}_1,\, \tilde{L}_2\right\} = \left[\tilde{r}_{12},\, \tilde{L}_1\right] - \left[\tilde{r}_{21},\, \tilde{L}_2\right]. \tag{16}$$

$$\text{Covariants}: \quad \dot{\tilde{L}} = [\tilde{M},\, \tilde{L}], \tag{17}$$

$$\text{Covariants}: \quad \dot{\tilde{L}} = \{C_i, \tilde{L}\} = [\tilde{M}_i,\, \tilde{L}]. \tag{18}$$

Under some particular gauge

$$r_{12} = -r_{21}, \quad \text{and} \quad \frac{\partial}{\partial q_i} r_{12} = \frac{\partial}{\partial p_i} r_{12} = 0,$$
$$\{L_1, L_2\} = [r_{12}, L_1 + L_2].$$

The Jacobi-identity of the above Poisson bracket leads to the classical Yang-Baxter equation

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0. \quad (19)$$
$$\Downarrow$$
$$\{L_1, L_2\} = [r_{12}, L_1 + L_2]. \quad (20)$$
$$\Downarrow$$
$$\dot{L} = \{tr(L^i), L\} = [M_i, L].$$

There are many integrable classical mechanics systems, such as Toda chains, Calogero-Moser systems, Ruijsenaars-Schneider systems...

For a 2D dimensional classical completely solvable field theories, its equation of motion is equivalent to the zero-curvature condition of its Lax representation

$$E.O.M \Leftrightarrow [\partial_t - U(x|\lambda), \partial_x + V(x|\lambda)] = 0, \quad 0 \le x \le L, \tag{21}$$

with periodic boundary condition: $V(0|\lambda) = V(L|\lambda)$ and $U(0|\lambda) = U(L|\lambda)$. $U(x|\lambda)$ and $V(x|\lambda)$ are $n \times n$ matrix with the matrix elements as functions of local fields at point x. The Lax representation of E.O.M can also rewritten in the following form

$$\partial_t\{L(x|\lambda)\} = [U(x|\lambda),\ L(x|\lambda)], \quad L(x|\lambda) = \partial_x + V(x|\lambda). \tag{22}$$

Let us introduce the classical monodromy matrix

$$T(\lambda) \overset{def}{=} T(L,0|\lambda) = \mathcal{P}\exp\left\{-\int_0^L V(z|\lambda)dz\right\}, \tag{23}$$

$$\tau(\lambda) = tr\left\{T(\lambda)\right\}, \quad \partial_t\tau(\lambda) = 0. \tag{24}$$

The transfer matrix $\tau(\lambda)$ play a role of the generating function of the conserved charges:

$$\tau(\lambda) = \sum_{i=0} I_i\lambda^i. \tag{25}$$

$$
\begin{aligned}
\partial_t(T(L,0|\lambda)) &= \int_0^L dxT(L,x|\lambda)\partial_t\left\{-V(x|\lambda)\right\}T(x,0|\lambda) \\
&\overset{(21)}{=} \int_0^L dxT(L,x|\lambda)\partial_x\left\{U(x|\lambda)\right\}T(x,0|\lambda) \\
&\quad + \int_0^L dx\left\{T(L,x|\lambda)V(x|\lambda)\right\}U(x|\lambda)T(x,0|\lambda) \\
&\quad - \int_0^L dxT(L,x|\lambda)U(x|\lambda)\left\{V(x|\lambda)T(x,0|\lambda)\right\} \\
&= \int_0^L dx\partial_x\left\{T(L,x|\lambda)V(x|\lambda)U(x|\lambda)T(x,0|\lambda)\right\} \\
&= U(L|\lambda)T(L,0|\lambda) - T(L,0|\lambda)U(0|\lambda) = [T(\lambda), U(0|\lambda)]
\end{aligned}
$$

Therefore $\partial_t(\tau(\lambda)) = \partial_t\left\{tr(T(\lambda))\right\} = 0$

Let us use the lattice version (N sites and lattice spacing $\Delta = \frac{L}{N}$) of the classical completely solvable field theories. The coordinate of the n-th site of the lattice thus obtained is $x_n = n\Delta$ and let us introduce the L-operator

$$L(n|\lambda) = \mathcal{P}\exp\left\{-\int_{x_{n-1}}^{x_n} V(z|\lambda)dz\right\} = 1 - V(x_n|\lambda)\Delta + O(\Delta^2). \qquad (26)$$

For the most of completely solvable models the L-operator satisfies the following fundamental Poisson bracket

$$\{L_1(n|\lambda),\, L_2(m|\mu)\} = [r_{12}(\lambda-\mu),\, L_1(n|\lambda)L_2(m|\mu)]\,\delta_{n,m}. \qquad (27)$$

Then the monodromy matrix is given by the local L-operators as follow

$$T(\lambda) = L(N|\lambda)L(N-1|\lambda)\ldots L(1|\lambda), \qquad (28)$$

and it satisfies the following Poisson bracket

$$\{T_1(\lambda),\, T_2(\mu)\} = [r_{12}(\lambda-\mu),\, T_1(\lambda)T_2(\mu)] \Longrightarrow \{\tau(\lambda),\, \tau(\mu)\} = 0. \qquad (29)$$

Thus the conserved charges $\{I_i\}$ are the action variables of the model,

$$\partial_t(I_i) = 0, \quad \{I_i,\, I_j\} = 0.$$

The $r_{12}(\lambda)$ is called the classical r-matrix with spectral which satisfies the classical Yang-Baxter equation

$$[r_{12}(\lambda_{12}),\, r_{13}(\lambda_{13})] + [r_{12}(\lambda_{12}),\, r_{23}(\lambda_{23})] + [r_{13}(\lambda_{13}),\, r_{23}(\lambda_{23})] = 0. \quad (30)$$

Here we adopt the notation $\lambda_{ij} = \lambda_i - \lambda_j$. The classical Yang-Baxter equation ensures the Jacobi-identity of the fundamental Poisson bracket.
Let us take the sine-Gordon (SG) model as an example.

The action of the SG model is

$$S[\phi] = \int_0^L dx \left(\frac{1}{2}(\partial_t \phi)^2 - (\partial_x \phi)^2 - \frac{m^2}{\beta^2}(1 - \cos \beta\phi) \right),$$

with the periodic boundary condition $\phi(0,t) = \phi(L,t)$. The equation of motion is

$$(\partial_t^2 - \partial_x^2)\phi + \frac{m^2}{\beta} \sin \beta\phi = 0.$$

The Lax representation

$$V(x|\lambda) = \begin{pmatrix} i\beta\pi(x)/4 & (m/2)\sinh(-\lambda + i\beta\phi(x)/2) \\ (m/2)\sinh(\lambda + i\beta\phi(x)/2) & -i\beta\pi(x)/4 \end{pmatrix},$$

where $\pi(x) = \partial_t \phi(x)$, $\{\pi(x), \phi(y)\} = \delta(x-y)$.

The corresponding L-operator of the lattice SG model is

$$L(n|\lambda) = \begin{pmatrix} 1 - i\beta p_n/4 & (\Delta m/2)\sinh(\lambda - (i\beta\phi_n/2)) \\ -(\Delta m/2)\sinh(\lambda + (i\beta\phi_n/2)) & 1 + i\beta p_n/4 \end{pmatrix} + \ldots,$$

where

$$\phi_n = \frac{1}{\Delta}\int_{x_{n-1}}^{x_n} \phi(x)\, dx, \quad p_n = \int_{x_{n-1}}^{x_n} \pi(x)\, dx, \quad \{p_n, \phi_n\} = \delta_{nm}.$$

The corresponding classical r-matrix

$$r(\lambda) = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & \gamma\coth\lambda & -\frac{\gamma}{\sinh\lambda} & 0 \\ \hline 0 & -\frac{\gamma}{\sinh\lambda} & \gamma\coth\lambda & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

where $\gamma = \beta^2/8$.

Summary: from a solution to the classical Yang-Baxter equation, we can construct

$$L(n|\lambda) = \begin{pmatrix} \alpha_n(\lambda) & \beta_n(\lambda) \\ \gamma_n(\lambda) & \delta_n(\lambda) \end{pmatrix} \Rightarrow T(\lambda) = L(N|\lambda)\dots L(1|\lambda), \tag{31}$$

such that

$$\{L_1(n|\lambda),\ L_2(m|\mu)\} = [r_{12}(\lambda-\mu), L_1(n|\lambda)L_2(m|\mu)]\delta_{n,m}, \tag{32}$$
$$\{T_1(\lambda),\ T_2(\mu)\} = [r_{12}(\lambda-\mu), T_1(\lambda)T_2(\mu)]. \tag{33}$$

The associated transfer matrix $\tau(\lambda) = tr(T(\lambda)) = \sum_{i=0} I_i\lambda^i$ is the generating function of the action variables, i.e.,

$$\partial_t(I_i) = \{H, I_i\} = 0, \quad \{I_i,\ I_j\} = 0, \tag{34}$$

Each of the conserved charge leads to the equation of motion having a Lax representation

$$\partial_t(L(n|\lambda)) = \{I_i, L(n|\lambda)\} = U^{(i)}(n+1|\lambda)L(n|\lambda) - L(n|\lambda)U^{(i)}(n|\lambda). \tag{35}$$

The continuous version: $\partial_t(L(x|\lambda)) = [U(x|\lambda),\ L(x|\lambda)],\ L(x|\lambda) = \partial_x + V(x|\lambda)$.

Quantumization leads to

$$\hat{L}(n|\lambda)=\begin{pmatrix}\hat{\alpha}_n(\lambda) & \hat{\beta}_n(\lambda)\\ \hat{\gamma}_n(\lambda) & \hat{\delta}_n(\lambda)\end{pmatrix}\Rightarrow \hat{T}(\lambda)=\hat{L}(N|\lambda)\ldots\hat{L}(1|\lambda),$$

such that

$$R(\lambda-\mu)\hat{L}_1(n|\lambda)\hat{L}_2(n|\mu)=\hat{L}_2(n|\lambda)\hat{L}_1(n|\mu),\quad [\hat{L}_1(n|\lambda),\hat{L}_2(m|\mu)]=0,\ m\neq n$$
$$R_{12}(\lambda-\mu)\hat{T}_1(\lambda)\hat{T}_2(\mu)=\hat{T}_2(\mu)\hat{T}_1(\lambda)R_{12}(\lambda-\mu).$$

The quantum version of the transfer matrix $\hat{\tau}(\lambda)=tr(T\hat{(}\lambda))$ forms the generating function of the commutative families, i.e.,

$$[\hat{\tau}(\lambda),\,\hat{\tau}(\mu)]=0.$$

The partition function of a statistical model on a two-dimensional lattice is defined by the following:

$$Z = \sum \exp\{-\frac{E}{kT}\},$$

where E is the energy of the system and the summation is taken over all possible configurations under the particular boundary condition such as the periodic boundary condition. The model we consider here has six allowed local bulk vertex configurations

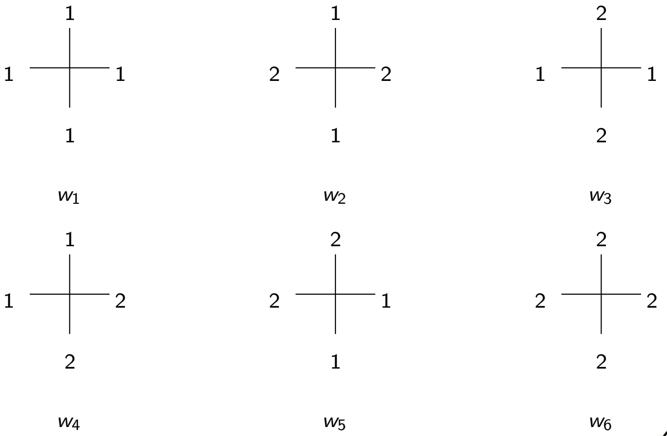


Figure 1. Vertex configurations and their associated Boltzmann weights.

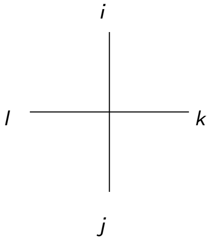
Each of the six bulk configurations is assigned a statistical weight (or Boltzmann weight) w_i (see Figs. 1). Then the partition function of the model can be rewritten as

$$Z = \sum {w_1}^{n_1}\, {w_2}^{n_2}\, {w_3}^{n_3}\, {w_4}^{n_4}\, {w_5}^{n_5}\, {w_6}^{n_6},$$

where the summation is over all possible configurations with n_i. The bulk Boltzmann weights which we consider here have Z_2-symmetry, i.e.,

$$a \equiv w_1 = w_6, \quad b \equiv w_2 = w_3, \quad c \equiv w_4 = w_5, \tag{36}$$

and the variables a, b, c satisfy a function relation, or equivalently, the local Boltzmann weights $\{w_i\}$ can be parameterized by the matrix elements of the six-vertex



$$= \quad R^{j\,l}_{i\,k}(u), \qquad i, j, k, l = 1, 2.$$

Figure 2. The Bolzmann weights and elements of the six-vertex R-matrix.

Let us introduce the R-matrix $R(u) \in \mathrm{End}(V \otimes V)$ according the Bolzmann weights of six-vertex model as follow

$$R(u) = \sum_{i,j,k,l=1}^{2} R_{i\,k}^{j\,l}(u) E^{ij} \otimes E^{kl} \tag{37}$$

$$= \begin{pmatrix} a(u) & & & \\ & b(u) & c(u) & \\ & c(u) & b(u) & \\ & & & a(u) \end{pmatrix}. \tag{38}$$

For any $A \in \mathrm{End}(V)$ and the R-matrix let us embed them in $\mathrm{End}(V \otimes V \otimes \cdots \otimes V)$ as

$$A \hookrightarrow A_\alpha = 1 \otimes 1 \otimes \cdots 1 \otimes A \otimes 1 \otimes \cdots$$

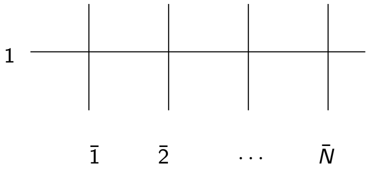
$$R(u) \hookrightarrow R_{\alpha,\beta}(u) = \sum_{i,j,k,l=1}^{2} R_{i\,k}^{j\,l}(u) E_\alpha^{ij} E_\beta^{kl}.$$

Let us introduce one-row monodormy matrix and the associated transfer matrix

$$T_1(u) = R_{1\,\bar{N}}(u)R_{1\,\bar{N-1}}(u)\ldots R_{1\,\bar{1}}(u), \quad (39)$$

$$t(u) = tr_1\{T_1(u)\}. \quad (40)$$

The monodromy matrix $T_1(u)$ can be figured as



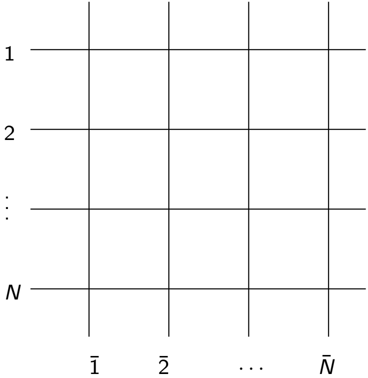


Figure 3. The six-vertex model with periodic boundary condition.

The partition function of the six vertex model with periodic boundary condition is given in terms of the transfer matrix

$$Z = tr_{\bar{1},\bar{2},\ldots,\bar{N}}(t(u))^N. \quad (41)$$

If the functions $a(u)$, $b(u)$ and $c(u)$ are given by

$$a(u) = \sinh(u+\eta), \quad b(u) = \sinh u, \quad c(u) = \sinh \eta,$$

the associated R-matrix satisfies the Yang-Baxter equation (YBE)

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v). \quad (42)$$

The YBE gives rise to the following RTT relation

$$R_{12}(u-v)\, T_1(u)\, T_2(v) = T_2(v)\, T_1(u)\, R12(u-v),$$

which leads to the commutativity of the transfer matrix:$[t(u), t(v)] = 0$.

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However, there exist a class of quantum integrable systems without $U(1)$-symmetry

- The XYZ spin chain with odd number of sites (Baxter 1970s)
- The anisotropic spin torus (1990s)
- Quantum spin chains with unparallel boundary fields (Sklyanin 1988)

The boundaries break the $U(1)$-symmetry $\Rightarrow$ the conventional Bethe ansatz fail.

There have been numerous efforts to approach the exact solutions to this class quantum integrable systems over last 20 years. A systemic method to solve this class model is still missing.

The Hamiltonian of the closed XXZ chain is

$$H = -\frac{1}{2}\sum_{k=1}^{N}\left(\sigma_k^x\,\sigma_{k+1}^x + \sigma_k^y\,\sigma_{k+1}^y + \cosh\eta\,\sigma_k^z\,\sigma_{k+1}^z\right),$$

where

$$\sigma_{N+1}^{\alpha} = \sigma_1^{\alpha}, \quad \alpha = x,\, y,\, z.$$

The system is **integrable**, i.e., there exist enough conserved charges

$$i\hbar\frac{\partial}{\partial t}h_i = [H, h_i] = 0, \qquad i = 1, \ldots .$$

and

$$[h_i, h_j] = 0.$$

It is convenient to introduce a generation function of these charges, the so-called transfer matrix

$$t(u) = \sum_{i=0} h_i u^i.$$

Then

$$[t(u),\ t(v)] = 0, \qquad H \propto \frac{\partial}{\partial u} \ln t(u)|_{u=0,\,\theta_j=0} + const,$$

or

$$H \propto h_0^{-1} h_1 + const,$$
$$h_0 = P_{N\,1} P_{N-1\,1} \ldots P_{2\,1},$$
$$h_0\, \sigma_i^{\alpha}\, h_0^{-1} = \sigma_{i+1}^{\alpha}.$$

The eigenstates and the corresponding eigenvalues can be obtained by Quantum Inverse Scattering Method (QISM). In the framework of QISM, the monodromy matrix $T(u)$

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix},$$

has played a central role. It is built from the six-vertex R-matrix of

$$T_0(u) = R_{0N}(u-\theta_N)\ldots R_{01}(u-\theta_1),$$

where the well-known six-vertex R-matrix is given by

$$R(u) = \frac{1}{\sinh\eta}\begin{pmatrix} \sinh(u+\eta) & & & \\ & \sinh u & \sin\eta & \\ & \sinh\eta & \sin u & \\ & & & \sinh(u+\eta) \end{pmatrix}.$$

The transfer matrix is $t(u) = trT(u) = A(u) + D(u)$.

In the case of N=1,

$$A(u)=\frac{1}{\sinh\eta}\begin{pmatrix}\sinh(u-\theta_1+\eta) & \\ & \sinh(u-\theta_1)\end{pmatrix},\quad B(u)=\begin{pmatrix} & \\ 1 & \end{pmatrix},$$

$$C(u)=\begin{pmatrix} & 1\\ & \end{pmatrix},\quad D(u)=\frac{1}{\sinh\eta}\begin{pmatrix}\sinh(u-\theta_1) & \\ & \sinh(u-\theta_1+\eta)\end{pmatrix}.$$

In the case of N=2

$$A(u)=A_2(u)A_1(u)+B_2(u)C_1(u),\quad B(u)=A_2(u)B_1(u)+B_2(u)D_1(u),$$
$$C(u)=C_2(u)A_1(u)+D_2(u)C_1(u),\quad D(u)=C_2(u)B_1(u)+D_2(u)D_1(u).$$

$$\vdots$$

The R-matrix satisfies the Yang-Baxter equation (YBE)

$$R_{12}(u-v)\,R_{13}(u)\,R_{23}(v) = R_{23}(v)\,R_{13}(u)\,R_{12}(u-v). \tag{43}$$

The above fundamental relation leads to the following so-called RLL relation between the monodromy matrix

$$R_{0\,0'}(u-v)\,T_0(u)\,T_{0'}(v) = T_{0'}(v)\,T_0(u)\,R_{0\,0'}(u-v). \tag{44}$$

This leads to $[t(u), t(v)] = 0$.

In terms of the matrix elements of the monodromy matrix, the RLL relation read

$$B(u)B(v) = B(v)B(u), \tag{45}$$

$$A(u)B(v) = \frac{\sinh(u-v-\eta)}{\sinh(u-v)} B(v)A(u) + \frac{\eta}{\sinh(u-v)} B(u)A(v), \tag{46}$$

$$D(u)B(v) = \frac{\sinh(u-v+\eta)}{\sinh(u-v)} B(v)D(u) - \frac{\eta}{\sinh(u-v)} B(u)D(v), \tag{47}$$

$$\vdots$$

There exists a quasi-vacuum state (or reference state) $|\Omega\rangle$ such that

$$|\Omega\rangle = |\uparrow, \ldots, \uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \ldots \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{48}$$

$$A(u)|\Omega\rangle = a(u)|\Omega\rangle = |\Omega\rangle, \quad D(u)|\Omega\rangle = d(u)|\Omega\rangle = \prod_{i=1}^{N} \frac{\sinh(u-\theta_i)}{\sinh(u-\theta_i+\eta)}|\Omega\rangle, \tag{49}$$

$$C(u)|\Omega\rangle = 0, \quad B(u)|\Omega\rangle \neq 0. \tag{50}$$

Let us introduce the Bethe state

$$|\lambda_1, \ldots, \lambda_M\rangle = B(\lambda_1) \ldots B(\lambda_M) |\Omega\rangle. \tag{51}$$

The action of the transfer matrix reads

$$\begin{aligned} t(u)|\lambda_1, \ldots, \lambda_M\rangle &= \prod_{i=1}^{M} \frac{\sinh(u - \lambda_i - \eta)}{\sinh(u - \lambda_i)} a(u)|\lambda_1, \ldots, \lambda_M\rangle \\ &+ \prod_{i=1}^{M} \frac{\sinh(u - \lambda_i + \eta)}{\sinh(u - \lambda_i)} d(u)|\lambda_1, \ldots, \lambda_M\rangle \\ &+ \text{unwanted terms.} \end{aligned}$$

If the parameters $\{\lambda_i\}$ needs satisfy Bethe ansatz equations,

$$\prod_{k\neq j}\frac{\sinh(\lambda_j-\lambda_k+\eta)}{\sinh(\lambda_j-\lambda_k-\eta)}=\prod_{j=1}^{N}\frac{\sinh(\lambda_j-\theta_k+\eta)}{\sinh(\lambda_j-\theta_k)},\qquad j=1,\ldots,M. \tag{52}$$

Then the Bethe states become the common eigenstates of $t(u)$ with eigenvalue $\Lambda(u)$

$$t(u)|\lambda_1,\ldots,\lambda_M\rangle=\Lambda(u)|\lambda_1,\ldots,\lambda_M\rangle,$$

where $\Lambda(u)=\Lambda(u;\lambda_1,\ldots,\lambda_M)$ is given by

$$\Lambda(u) = a(u)\prod_{i=1}^{M}\frac{\sinh(u-\lambda_i-\eta)}{\sinh(u-\lambda_i)}+d(u)\prod_{i=1}^{M}\frac{\sinh(u-\lambda_i+\eta)}{\sinh(u-\lambda_i)}, \tag{53}$$

$$= a(u)\frac{Q(u-\eta)}{Q(u)}+d(u)\frac{Q(u+\eta)}{Q(\eta)}, \tag{54}$$

where

$$Q(u)=\prod_{i=1}^{M}\sinh(u-\lambda_i). \tag{55}$$

For the higher spin chains (e.g. spin-s), one can introduce the Bethe state

$$|\lambda_1, \ldots, \lambda_M\rangle^{(s)} = B^{(s)}(\lambda_1) \ldots B^{(s)}(\lambda_M) \, |\Omega^{(s)}\rangle. \tag{56}$$

If the parameters satisfy the associated Bethe ansatz equations, the above state becomes the eigenstate with an eigenvalue $\Lambda(u)$

$$\Lambda(u) = a^{(s)}(u) \frac{Q(u-\eta)}{Q(u)} + d^{(s)}(u) \frac{Q(u+\eta)}{Q(\eta)}. \tag{57}$$

The method has succeeded in solving the spectrum of the quantum integrable closed spin chains arising from the R-matrix of other quantum affine (super)algebras. However, the exact solutions to an integrable model should mean that

- **To diagonalize the Hamiltonian**
- **To compute correlation functions of local operators**

Following the works of Kitanine et al (Nucl. Phys. B 554 (1999), 647) and Gohmann et al (J. Phys. A 33 (2000), 1199), all local operators can be expressed in terms of the entries of the monodromy matrix $T(u)$, i.e. $A(u)$, $B(u)$, $C(u)$, $D(u)$.

Eample

$$\sigma_j^+ = \prod_{k=1}^{j-1}(A(\theta_k) + D(\theta_k))\ C(\theta_j)\ \prod_{k=j+1}^{N}(A(\theta_k) + D(\theta_k)).$$

Thus the problem of computations of correlation functions reduces to calculations of scalar products

$$\langle\Omega|\prod_{k=1}^{M} C(u_k)\,|\lambda_1,\ldots,\lambda_M\rangle = \langle\Omega|\prod_{k=1}^{M} C(u_k)\ \prod_{k=1}^{M} B(\lambda_k)\,|\Omega\rangle$$

The scalar products were first expressed in terms of some determinant by Korepin (Commun. Math. Phys. 86 (1982), 391):

$$\langle\Omega|\prod_{k=1}^{M}C(u_k)\ \prod_{k=1}^{M}B(\lambda_k)\,|\Omega\rangle = s(\{u_k\};\{\lambda_k\})\times\det N(\{u_k\};\{\lambda_k\}).$$

where $N(\{u_k\};\{\lambda_k\})$ is a $M\times M$ matrix whose entries are function of u_k and λ_k.

Maillet et al (Am. Math. Soc. Transl. 201 (2000), 137; Nucl. Phys. B 567 (2000), 554; The proceedings of ICM of 2006) developed a method by using the Drinfeld twists (or Factorizing F-matrices) to rederive the above results.

Generalizing the work of Maillet et al, we have obtained the determinant representations of scalar products of Bethe states for the quantum closed spin chain associated with $gl(2|1)$, namely, the supersymmetric t-J model (see JHEP 12 (2004), 038; Commun. Math. Phys. 264 (2006), 87; Commun. Math. Phys. 268 (2006), 505). Our results can be generalized to closed spin chain associated with A type (super) algebra.

The Hamiltonian of the XXZ chain with twisted boundary condition is

$$H = -\frac{1}{2}\sum_{k=1}^{N}\left(\sigma_k^x\,\sigma_{k+1}^x + \sigma_k^y\,\sigma_{k+1}^y + \cosh\eta\,\sigma_k^z\,\sigma_{k+1}^z\right),$$

where

$$\sigma_{N+i}^{\alpha} = e^{i\phi\sigma_i^z}\,\sigma_i^{\alpha}\,e^{-i\phi\sigma_i^z}, \quad \alpha = x,\, y,\, z.$$

The phase factor ϕ can be arbitrary complex number. The system is **integrable**, i.e., the corresponding transfer matrix can be constructed as

$$t(u) = tr(e^{i\phi\sigma^z}\,T(u)) = e^{i\phi}A(u) + e^{-i\phi}D(u).$$

The transfer matrix can diagonalized by algebraic Bethe ansatz similar as that of periodic case. The Bethe state is the same as (51), namely,

$$|\lambda_1, \ldots, \lambda_M\rangle = B(\lambda_1)\ldots B(\lambda_M)\,|\Omega\rangle.$$

If the parameters $\{\lambda_i\}$ satisfies Bethe ansatz equations,

$$\prod_{k\neq j}\frac{\sinh(\lambda_j-\lambda_k+\eta)}{\sinh(\lambda_j-\lambda_k-\eta)}=e^{2i\phi}\prod_{j=1}^{N}\frac{\sinh(\lambda_j-\theta_k+\eta)}{\sinh(\lambda_j-\theta_k)},\qquad j=1,\ldots,M. \tag{58}$$

Then the Bethe states become the common eigenstates of $t(u)$ with eigenvalue $\Lambda(u)$

$$\begin{aligned}\Lambda(u) &= e^{i\phi}a(u)\prod_{i=1}^{M}\frac{\sinh(u-\lambda_i-\eta)}{\sinh(u-\lambda_i)}+e^{-i\phi}d(u)\prod_{i=1}^{M}\frac{\sinh(u-\lambda_i+\eta)}{\sinh(u-\lambda_i)},\\ &= e^{i\phi}a(u)\frac{Q(u-\eta)}{Q(u)}+e^{-i\phi}d(u)\frac{Q(u+\eta)}{Q(\eta)}.\end{aligned} \tag{59}$$

The determinant representations of scalar products can given by similar form as those of periodic case.

The Hamiltonian of the XXZ chain with antiperiodic boundary condition is

$$H = -\frac{1}{2}\sum_{k=1}^{N}\left(\sigma_k^x\,\sigma_{k+1}^x + \sigma_k^y\,\sigma_{k+1}^y + \cosh\eta\,\sigma_k^z\,\sigma_{k+1}^z\right),$$

where

$$\sigma_{N+i}^{\alpha} = \sigma_i^x\,\sigma_i^{\alpha}\,\sigma_i^x, \quad \alpha = x,\, y,\, z.$$

The system is **integrable**, i.e., the corresponding transfer matrix can be constructed as

$$t(u) = tr(\sigma^x T(u)) = B(u) + C(u).$$

The model is a typical integrable without $U(1)$ symmetry. Most of conventional Bethe ansatz method fails to give the solution because of the lack of a proper vacuum (or reference) state.

Recently, we give a solution to the spectrum problem of the corresponding transfer matrix in

- J. Cao et al, *Phys. Rev. Lett.* **111** (2013), 137201 [arXiv:1305.7328].

Let $|\Psi\rangle$ be an eigenstate of the transfer matrix with an eigenvalue

$$t(u)|\Psi\rangle = \Lambda(u)|\Psi\rangle.$$

Due to the fact that $|\Psi\rangle$ does not depend on u, we can derive the following properties which can determine $\Lambda(u)$ completely

$$\Lambda(u), \text{ as a function of } u, \text{ is a trigonometric polynomial of degree } N-1, \quad (60)$$

$$\Lambda(u+i\pi) = (-1)^{N-1}\Lambda(u), \quad (61)$$

$$\Lambda(\theta_j)\,\Lambda(\theta_j-\eta) = -f_1(\theta_j)f_2(\theta_j-\eta), \quad j=1,\ldots,N. \quad (62)$$

The functions $f_1(u)$, $f_2(u)$ are

$$f_1(u) = \prod_{i=1}^{N} \sinh(u-\theta_i+\eta), \quad f_2(u) = f_1(u-\eta). \quad (63)$$

The solution to the above equations is given by

$$\Lambda(u) = e^u f_1(u)\frac{Q_1(u-\eta)}{Q_2(u)} - e^{-u-\eta} f_2(u)\frac{Q_2(u+\eta)}{Q_1(u)} - b(u)\frac{f_1(u)f_2(u)}{Q_1(u)Q_2(u)}, \quad (64)$$

$$Q_1(u) = \prod_{j=1}^{M}\sinh(u-\mu_j), \quad Q_2(u) = \prod_{j=1}^{M}\sinh(u-\nu_j),$$

$$b^{(e)}(u) = e^{u+\phi_1} - e^{-u-\eta+\phi_2}, \quad b^{(o)}(u) = e^{2u+\phi_1} + e^{-2u-2\eta+\phi_2}.$$

$$\phi_1 = \sum_{j=1}^{N}\theta_j - M\eta - 2\sum_{j=1}^{M}\mu_j, \quad \phi_2 = -\sum_{j=1}^{N}\theta_j + M\eta + 2\sum_{j=1}^{M}\nu_j,$$

The parameters $\{\mu_j\}$ and $\{\nu_j\}$ satisfy the associated Bethe ansatz equations

$$f_2(\nu_j) = \frac{e^{\nu_j}}{b(\nu_j)} Q_1(\nu_j - \eta)Q_1(\nu_j), \quad j = 1, \ldots, M, \quad (65)$$

$$f_1(\mu_j) = -\frac{e^{-\mu_j-\eta}}{b(\mu_j)} Q_2(\mu_j + \eta)Q_2(\mu_j), \quad j = 1, \ldots, M. \quad (66)$$

Open XXZ chain Hamiltonian

$$
\begin{aligned}
H &= -\frac{1}{2}\sum_{k=1}^{N-1}\left(\sigma_k^x\sigma_{k+1}^x + \sigma_k^y\sigma_{k+1}^y + \cosh\eta\,\sigma_k^z\sigma_{k+1}^z\right)\\
&\quad + f_1^x\sigma_1^x + f_1^y\sigma_1^y + f_1^z\sigma_1^z\\
&\quad + f_N^x\sigma_N^x + f_N^y\sigma_N^y + f_N^z\sigma_N^z
\end{aligned}
$$

The model is **integrable**. If the components of boundary fields are parameterized by

$$
F_1 = (f_1^x, f_1^y, f_1^z) = \frac{\sinh\eta}{\sinh\alpha_-\cosh\beta_-}(\coth\alpha_-\sinh\beta_-, \cosh\theta_-, i\sinh\theta_-)
$$

$$
F_N = (f_N^x, f_N^y, f_N^z) = \frac{\sinh\eta}{\sinh\alpha_+\cosh\beta_+}(-\coth\alpha_+\sinh\beta_+, \cosh\theta_+, i\sinh\theta_+).
$$

The corresponding transfer matrix $t(u)$ can be constructed by the six-vertex R-matrix and the associated K-matrices, i.e.,

$$
t(u) = tr(\mathbb{T}(u)) = tr\left(K^+(u)T(u)K^-(u)T^{-1}(-u)\right),
$$

where the K-matrices $K^{\pm}(u)$ are the most general solutions of the reflection equation and its dual.

The K-matrices are given by

$$K^-(u) = \begin{pmatrix} K_{11}^-(u) & K_{12}^-(u) \\ K_{21}^-(u) & K_{22}^-(u) \end{pmatrix},$$
$$K_{11}^-(u) = 2\left(\sinh(\alpha_-)\cosh(\beta_-)\cosh(u) + \cosh(\alpha_-)\sinh(\beta_-)\sinh(u)\right),$$
$$K_{22}^-(u) = 2\left(\sinh(\alpha_-)\cosh(\beta_-)\cosh(u) - \cosh(\alpha_-)\sinh(\beta_-)\sinh(u)\right),$$
$$K_{12}^-(u) = e^{\theta_-}\sinh(2u), \quad K_{21}^-(u) = e^{-\theta_-}\sinh(2u), \tag{67}$$

and

$$K^+(u) = K^-(-u-\eta)\big|_{(\alpha_-,\beta_-,\theta_-)\to(-\alpha_+,-\beta_+,\theta_+)}. \tag{68}$$

The Hamiltonian can be expressed in terms of the transfer matrix

$$H = \sinh\eta \frac{\partial \ln t(u)}{\partial u}|_{u=0,\theta_j=0} - N\cosh\eta - \tanh\eta\sinh\eta.$$

For the very special case of $F_1 = (f_1^x, f_1^y, f_1^z) = (0, 0, f_1^z)$ and $F_N = (0, 0, f_N^z)$, namely,

$$H = -\frac{1}{2}\sum_{k=1}^{N-1}\left(\sigma_k^x\sigma_{k+1}^x + \sigma_k^y\sigma_{k+1}^y + \cosh\eta\sigma_k^z\sigma_{k+1}^z\right) + f_1^z\sigma_1^z + f_N^z\sigma_N^z$$

the model was solved by Sklyanin (J. Phys. A 21 (1988), 2375). The boundary QISM has failed to solve the spectral problem of the general case for many years. However, it can be solved by a generalized boundary QISM developed (Fan et al Nucl. Phys. B 478 (1996), 723, Cao et al Nucl. Phys. B 663 (2003), 487) for some case. In these cases, a local vacuum state does exist and the corresponding Bethe sates have similar structure as that of closed but with a different quasi-particle creation operator $\mathcal{B}(u)$ and reference state $\tilde{\Omega}$. The corresponding Bethe states are

$$|\lambda_1, \ldots, \lambda_M\rangle = \mathcal{B}(\lambda_1)\ldots\mathcal{B}(\lambda_M)\,|\tilde{\Omega}\rangle,$$

where the parameters $\{\lambda_i\}$ needs satisfy the associate Bethe ansatz equations, which can be written in the following form (see **Yang** et al (Nucl. Phys. B 698 (2004), 503-516; JHEP 04 (2007), 044))

$$\frac{\partial}{\partial\lambda_k}\mathcal{F}(u; \{\lambda_j\})|_{u=\lambda_k} = 0, \qquad k = 1, \ldots, M.$$

The associated scalar product can also be expressed in terms of some determinant (see **Yang** et al (JHEP 01 (2011), 006))

Eample

$$\langle\Omega|\prod_{k=1}^{M}\mathcal{C}(u_k)\ \prod_{k=1}^{M}\mathcal{B}(\lambda_k)\,|\Omega\rangle = s(\{u_k\};\{\lambda_k\})\times \det\mathcal{N}(\{u_k\};\{\lambda_k\}),$$

where the $M\times M$ matrix $\mathcal{N}(\{u_k\};\{\lambda_k\})$ can be written as

$$\mathcal{N}(\{u_k\};\{\lambda_k\})_{k,\alpha} = \frac{\partial^2}{\partial\lambda_k\,\partial u_\alpha}\mathcal{F}(\{u_\alpha\};\{\lambda_j\}),\qquad \alpha,k=1,\ldots,M.$$

For the generic $F_1 = (f_1^x, f_1^y, f_1^z)$, $F_N = (f_N^x, f_N^y, f_N^z)$ and generic anisotropic parameter Δ , the model has not been solved since Sklyanin's work in 1988 until the recent works of Cao et al:

- J. Cao et al, *Nucl. Phys.* **B 875** (2013), 152-165 [arXiv:1306.1742];
- J. Cao et al, *Nucl. Phys.* **B 877** (2013), 152-175 [arXiv:1307.2023].

In the solutions, we can give the eigenvalues in terms of some parameters which satisfy associated Bethe ansatz equations, without the explicit expressions of the corresponding eigenstates (such as Bethe states). However, in principle we can reconstruct the corresponding eigenstates with some recursion relations.

Besides the quantum Yang-Baxter equation, the R-matrix satisfies

$$\text{Initial condition}: \; R_{12}(0) = P_{12}, \tag{69}$$

$$\text{Unitarity relation}: \; R_{12}(u)R_{21}(-u) = -\frac{\sinh(u+\eta)\sinh(u-\eta)}{\sinh\eta\sinh\eta} \times \mathrm{id}, \tag{70}$$

$$\text{Crossing relation}: \; R_{12}(u) = V_1 R_{12}^{t_2}(-u-\eta)V_1, \quad V = -i\sigma^y, \tag{71}$$

$$\text{PT-symmetry}: \; R_{12}(u) = R_{21}(u) = R_{12}^{t_1\,t_2}(u), \tag{72}$$

$$Z_2\text{-symmetry}: \quad \sigma_1^i\sigma_2^i R_{1,2}(u) = R_{1,2}(u)\sigma_1^i\sigma_2^i, \quad \text{for } i = x, y, z, \tag{73}$$

$$\text{Antisymmetry}: \; R_{12}(-\eta) = -(1-P) = -2P^{(-)}. \tag{74}$$

In addition to reflection equations, the K-matric satisfies

$$K^-(0) = \frac{1}{2}tr(K^-(0)) \times \mathrm{id}, \quad K^-(\frac{i\pi}{2}) = \frac{1}{2}tr(K^-(\frac{i\pi}{2})) \times \sigma^z.$$

These properties and the quasi-periodic properties of R-matrix and K-matrices imply

$$
\begin{aligned}
t(-u-\eta) &= t(u), \quad t(u+i\pi) = t(u), \\
t(0) &= -2^3 \sinh\alpha_- \cosh\beta_- \sinh\alpha_+ \cosh\beta_+ \cosh\eta \\
&\quad \times \prod_{l=1}^{N} \frac{\sinh(\eta-\theta_l)\sinh(\eta+\theta_l)}{\sinh\eta \sinh\eta} \times \mathrm{id}, \\
t(\frac{i\pi}{2}) &= -2^3 \cosh\alpha_- \sinh\beta_- \cosh\alpha_+ \sinh\beta_+ \cosh\eta \\
&\quad \times \prod_{l=1}^{N} \frac{\sinh(\frac{i\pi}{2}+\theta_l+\eta)\sinh(\frac{i\pi}{2}+\theta_l-\eta)}{\sinh\eta \sinh\eta} \times \mathrm{id}, \\
\lim_{u\to\pm\infty} t(u) &= -\frac{\cosh(\theta_- - \theta_+)e^{\pm[(2N+4)u+(N+2)\eta]}}{2^{2N+1}\sinh^{2N}\eta} \times \mathrm{id} + \ldots,
\end{aligned}
$$

and the very operator identity

$$
t(\theta_j)t(\theta_j-\eta) = -\frac{\sinh^2\eta\, \Delta_q^{(o)}(\theta_j)}{\sinh(2\theta_j+\eta)\sinh(2\theta_j-\eta)}, \quad \Delta_q^{(o)}(u) = \delta(u) \times \mathrm{id}.
$$

Let $|\Psi\rangle$ be a common eigenstate of the transfer matrix with an eigenvalue $\Lambda(u)$, then

$$\Lambda(-u-\eta) = \Lambda(u), \quad \Lambda(u+i\pi)=\Lambda(u), \tag{75}$$

$$\Lambda(0) = -2^3 \sinh\alpha_- \cosh\beta_- \sinh\alpha_+ \cosh\beta_+ \cosh\eta \tag{76}$$

$$\times \prod_{l=1}^{N} \frac{\sinh(\eta-\theta_l)\,\sinh(\eta+\theta_l)}{\sinh\eta\,\sinh\eta}, \tag{77}$$

$$\Lambda(\frac{i\pi}{2}) = -2^3 \cosh\alpha_- \sinh\beta_- \cosh\alpha_+ \sinh\beta_+ \cosh\eta \tag{78}$$

$$\times \prod_{l=1}^{N} \frac{\sinh(\frac{i\pi}{2}+\theta_l+\eta)\sinh(\frac{i\pi}{2}+\theta_l-\eta)}{\sinh\eta\,\sinh\eta}, \tag{79}$$

$$\lim_{u\to\pm\infty}\Lambda(u) = -\frac{\cosh(\theta_- - \theta_+)e^{\pm[(2N+4)u+(N+2)\eta]}}{2^{2N+1}\sinh^{2N}\eta} + \ldots, \tag{80}$$

$$\Lambda(\theta_j)\Lambda(\theta_j-\eta) = -\frac{\sinh^2\eta\,\delta(\theta_j)}{\sinh(2\theta_j+\eta)\sinh(2\theta_j-\eta)}. \tag{81}$$

The function $\delta(u)$ is given by

$$\begin{aligned}\delta(u) &= -2^4\frac{\sinh(2u-2\eta)\sinh(2u+2\eta)}{\sinh\eta\,\sinh\eta}\sinh(u+\alpha_-)\sinh(u-\alpha_-)\cosh(u+\beta_-)\\ &\quad\times\cosh(u-\beta_-)\sinh(u+\alpha_+)\sinh(u-\alpha_+)\cosh(u+\beta_+)\cosh(u-\beta_+)\\ &\quad\times\prod_{l=1}^{N}\frac{\sinh(u+\theta_l+\eta)\sinh(u-\theta_l+\eta)\sinh(u+\theta_l-\eta)\sinh(u-\theta_l-\eta)}{\sinh(\eta)\sinh(\eta)\sinh(\eta)\sinh(\eta)}.\end{aligned}$$

Moreover, it follows that $\Lambda(u)$, as an entire function of u, is a trigonometric polynomial of degree $2N+4$. Hence (75)-(81) completely determine the function $\Lambda(u)$. For this purpose, let us introduce the following functions:

$$\begin{aligned}A(u) &= \prod_{l=1}^{N}\frac{\sinh(u-\theta_l+\eta)\,\sinh(u+\theta_l+\eta)}{\sinh\eta\,\sinh\eta},\\ a(u) &= -2^2\frac{\sinh(2u+2\eta)}{\sinh(2u+\eta)}\sinh(u-\alpha_-)\cosh(u-\beta_-)\\ &\quad\times\sinh(u-\alpha_+)\cosh(u-\beta_+)A(u),\\ d(u) &= a(-u-\eta).\end{aligned}$$

$$\Lambda(u) = a(u)\frac{Q_1(u-\eta)}{Q_2(u)} + d(u)\frac{Q_2(u+\eta)}{Q_1(u)} + \frac{2c\sinh(2u)\sinh(2u+2\eta)}{Q_1(u)Q_2(u)}A(u)A(-u-\eta), \quad (82)$$

where the functions $Q_1(u)$ and $Q_2(u)$ are some trigonometric polynomials parameterized by N Bethe roots $\{\mu_j|j=1,\ldots,N\}$ as follows,

$$Q_1(u) = \prod_{j=1}^{N}\frac{\sinh(u-\mu_j)}{\sinh(\eta)}, \quad Q_2(u)=Q_1(-u-\eta). \quad (83)$$

the parameters $\bar{c}$ is determined by the boundary parameters and μ_j

$$c=\cosh((N+1)\eta+\alpha_-+\beta_-+\alpha_++\beta_++2\sum_{j=1}^{N}\mu_j)-\cosh(\theta_--\theta_+). \quad (84)$$

The N parameters $\{\mu_j\}$ satisfy the associated Bethe ansatz equations

$$\frac{2c\sinh(2\mu_j)\sinh(2\mu_j+2\eta)\,A(\mu_j)A(-\mu_j-\eta)}{d(\mu_j)Q_2(\mu_j)Q_2(\mu_j+\eta)}=-1,\ j=1,\ldots,N, \quad (85)$$

and with the following selection rule for the roots of the above equations

$$\mu_j\neq\mu_l \quad \text{and} \quad \mu_j\neq-\mu_l-\eta. \quad (86)$$

$$\Lambda(u) = a(u)\frac{Q_1(u-\eta)}{Q_2(u)} + d(u)\frac{Q_2(u+\eta)}{Q_1(u)} + \frac{2^3 c \sinh(2u)\sinh(2u+2\eta)}{Q_1(u)Q_2(u)}\frac{\sinh u \sinh(u+\eta)}{\sinh\eta\sinh\eta}A(u)A(-u-\eta), \quad (87)$$

$$Q_1(u) = \prod_{j=1}^{N+1}\frac{\sinh(u-\mu_j)}{\sinh(\eta)}, \quad Q_2(u) = Q_1(-u-\eta). \quad (88)$$

$$c = \cosh((N+3)\eta + \alpha_- + \beta_- + \alpha_+ + \beta_+ + 2\sum_{j=1}^{N+1}\mu_j) - \cosh(\theta_- - \theta_+). \quad (89)$$

The $N+1$ parameters $\{\mu_j\}$ satisfy the associated Bethe ansatz equations

$$\frac{2^3 c \sinh(2\mu_j)\sinh(2\mu_j+2\eta)\,A(\mu_j)A(-\mu_j-\eta)}{d(\mu_j)Q_2(\mu_j)Q_2(\mu_j+\eta)} = -\frac{\sinh\eta\sinh\eta}{\sinh\mu_j\sinh(\mu_j+\eta)}, \quad j = 1, \ldots, N+1, \quad (90)$$

with the very selection rule (86) for the roots of the above equations.

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- complete solvable field theories

Thank for your attentions