Eddington Inspired Born-Infeld (EiBI) Gravity

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Outline

- 1. **Introduction to EiBI gravity**
- 2. **Strong gravitational lensing by charged EiBI BH**
- 3. **Cosmology in EiBI gravity**
- 4. **Braneworld in EiBI gravity**
- 5. **Conclusions**

This talk is based on:

Banados and Ferreira, PRL 105 (2010) 011101.
YX Liu, K Yang, H Guo, and Y Zhong, PRD 85 (2012) 124053.
Escamilla-Rivera, Banados, and Ferreira, PRD 85 (2012) 087302.
K Yang, XL Du, and YX Liu, PRD 88 (2013) 124037.
Lagos, Banados et al, PRD 89 (2014) 024034.
XL Du, K Yang, XH Meng, and YX Liu, arXiv:1403.0083.
QM Fu, L Zhao, K Yang, and YX Liu, preparing.
SW Wei, YX Liu, and K Yang, preparing.

1915: General Relativity (metric theory)

$$S_{\mathrm{EH}}[g] = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,[R(g) - 2\Lambda] \qquad (1)$$

- **It provides precise descriptions to a variety of phenomena in our universe for almost a century.**
- **It also suffers various troublesome theoretical problems: dark matter/energy, nonrenormalization, singularity...**

Modified Gravity

- Scalar-tensor (Brans-Dicke) gravity
- F(R) gravity, critical gravity (Hong Lü), and other general higher-order theories
- Horava-Lifschitz gravity
- Models of extra dimensions: KK, ADD, RS, DGP
- Born-Infeld Gravity
- Bimetric theories
- $\cdots$

Four formalisms of gravity theories:

- **Metric-affine formalism:** $\Gamma^{\lambda}_{\mu\nu}$ and $g_{\mu\nu}$ independent,

$$S_G = S_G[g_{\mu\nu}, \Gamma^{\lambda}_{\mu\nu}], \quad S_M = S_M[g_{\mu\nu}, \Gamma^{\lambda}_{\mu\nu}, \psi]. \tag{2}$$

The hypermomentum: $\triangle_P^{MN} \equiv -\frac{2}{\sqrt{-|g_{KL}|}} \frac{\delta S_M}{\delta \Gamma^P{}_{MN}}$.

T_{MN} does not represents the usual meaning of a energy-momentum-stress tensor, the hypermomentum also describes matter characteristics.

- **Palatini formalism:** $\Gamma^{\lambda}_{\mu\nu}$ and $g_{\mu\nu}$ independent,

$$S_G = S_G[g_{\mu\nu}, \Gamma^{\lambda}_{\mu\nu}], \quad S_M = S_M[g_{\mu\nu}, \psi]. \tag{3}$$

- **Metric formalism:** $\Gamma^{\lambda}_{\mu\nu} = \left\{ {\lambda \atop \mu\nu} \right\}$, only $g_{\mu\nu}$,

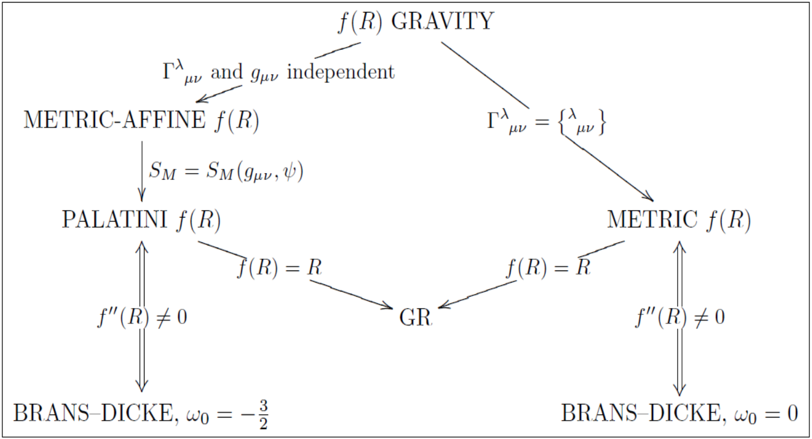
$$S_G = S_G[g_{\mu\nu}], \quad S_M = S_M[g_{\mu\nu}, \psi]. \tag{4}$$

- **Purely affine formalism:** only $\Gamma^{\lambda}_{\mu\nu}$

$$S_G = S_G[\Gamma^{\lambda}_{\mu\nu}], \quad S_M = S_M[\Gamma^{\lambda}_{\mu\nu}, \psi]. \tag{5}$$

Example: Eddington gravity.

An example: Classification of $f(R)$ theories of gravity



Taken from [T.P. Sotiriou, CQG 23(2006)1253].

1924: Eddington gravity (purely affine gravity)

[A.S. Eddington, The mathematical Theory of Relativity, Cambridge Univ. Press, 1924]

$$S_{\text{Edd}}(\Gamma) = \frac{1}{16\pi G}\frac{2}{\kappa}\int d^4x\sqrt{-|\kappa R_{\mu\nu}(\Gamma)|}, \tag{6}$$

- The independent field is a symmetric affinity $\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu}$.
- $R_{\mu\nu}$ is the symmetric part of its Ricci tensor.
- The EoMs are given by $\nabla_\lambda\left(\kappa R_{\mu\nu}(\Gamma)\right) = 0$.
- If we let $g_{\mu\nu} \equiv \kappa R_{\mu\nu}(\Gamma)$, then the EoMs become $\nabla_\lambda g_{\mu\nu} = 0$, which is equivalent to

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}(g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho}). \tag{7}$$

- Further, let $\Lambda = \frac{1}{\kappa}$, the EoMs can be written as

$$R_{\mu\nu}(g) = \Lambda g_{\mu\nu}, \tag{8}$$

  where $R_{\mu\nu}(g)$ is constructed with the metric $g_{\mu\nu}$ now.
- Eddington's theory is equivalent to GR with Λ.
- But it is incomplete because matter is not included.

Duality of $S_{\rm EH}[g]$ and $S_{\rm Edd}[\Gamma]$

- Consider the Palatini action for gravity with Λ

$$S_{\rm P}[g,\Gamma] = \frac{1}{8\pi G}\int d^4x\sqrt{-g}\left(g^{\mu\nu}R_{\mu\nu}(\Gamma) - 2\Lambda\right). \qquad (9)$$

- Eliminating the connection using its own EoM gives

$$S_{\rm EH}[g] = \frac{1}{8\pi G}\int d^4x\sqrt{-g}\left[R(g) - 2\Lambda\right]. \qquad (10)$$

- Eliminating the metric yields ($\Lambda \neq 0$) [Annals Phys. 162(1985)31]

$$S_{\rm Edd}[\Gamma] = \frac{1}{8\pi G}\frac{2}{\kappa}\int d^4x\sqrt{-|\kappa R_{\mu\nu}(\Gamma)|}. \quad (\kappa = 1/\Lambda) \qquad (11)$$

- $S_{\rm P}[g,\Gamma]$ is called the Parent action, while $S_{\rm EH}[g]$ and $S_{\rm Edd}[\Gamma]$ are its daughters.
- $S_{\rm EH}[g]$ and $S_{\rm Edd}[\Gamma]$ are said to be dual to each other, and in many respects they are equivalent.

1934: Vector Born-Infeld theory

[M.Born, Proc.R.Soc.London A 143(1934)410; M.Born and M.Infeld, Proc.R.Soc.London A 144(1934)425].

$$S_{\mathbf{VBI}} = -\frac{1}{2\lambda^2}\int d^4x\sqrt{-|g_{\mu\nu} + \lambda F_{\mu\nu}|}. \tag{12}$$

- The BI theory is a theory of nonlinear electrodynamics. It reduces to Maxwell theory for small amplitudes.
- The EoMs are of second order.
- The singularity of electric field for a point charge at the origin is removed. So does the divergence of the electron's self-energy.
- But magnetic field for a point magnetic charge at the origin and vector potential are still singular.
- The nonabelian extensions were found in string theory.

1998: Born-Infeld gravity(metric theory)

[Deser and Gibbons, CQG 15 (1998) L35]

$$S_{\mathrm{BI}}[g] = \int d^4x \sqrt{-|g_{\mu\nu} - l^2 R_{\mu\nu}(g) + X_{\mu\nu}(R)|}. \qquad (13)$$

- **A gravitational analog of the BI electrodynamics.**
- $X_{\mu\nu}$ **contains quadratic or higher terms in curvature.**
- **It must be chosen such that the action is free of ghost.**
- **For the vector BI theory, the EoMs are of second order. For the spin-2 BI gravity theory this is not automatic and requires the addition of** $X_{\mu\nu}(R)$.

2010: Eddington Inspired Born-Infeld (EiBI) gravity (metric-affine or Palatini theory) [Bañados and Ferreira, PRL 105 (2010) 011101]

$$S_{\text{EiBI}}[g,\Gamma,\Phi] = \frac{1}{16\pi G}\frac{2}{\kappa}\int d^4x\left(\sqrt{-|g_{\mu\nu}+\kappa R_{\mu\nu}(\Gamma)|}-\lambda\sqrt{-g}\right)+S_M, \quad (14)$$

$$S_M = S_M[g,\Gamma,\Phi]\ (\textbf{metric-affine})\ \ \textbf{or}\ \ S_M = S_M[g,\Phi]\ (\textbf{Palatini}). \quad (15)$$

- **We mainly consider the Palatini theory.**
- **When** $\kappa R \gg g$ **and** $S_M = 0$, $S_{\text{EiBI}} \to S_{\text{Edd}}$.
- **When** $\kappa R \ll g$, **the EiBI gravity reproduces 'GR':**

$$S_{\text{EiBI}} \approx \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left(R-2\Lambda_{\text{eff}}+\frac{\kappa}{4}R^2-\frac{\kappa}{2}R^{\mu}_{\ \nu}R^{\nu}_{\ \mu}+\mathcal{O}(\kappa^2)\right)+S_M, \quad (16)$$

where $\Lambda_{\text{eff}} \equiv (\lambda-1)/\kappa$.

- **In the nonrelativistic limit, the EiBI theory gives the modified Poisson equation** $\nabla^2\Phi = -\frac{1}{2}\rho - \frac{\kappa}{4}\nabla^2\rho$.
- **It reproduces Einstein gravity precisely within the vacuum but deviates from it in the presence of source.**

2010: Eddington Inspired Born-Infeld (EiBI) gravity (metric-affine or Palatini theory) [Bañados and Ferreira, PRL 105 (2010) 011101]

$$S_{\text{EiBI}}[g,\Gamma,\Phi] = \frac{1}{16\pi G}\frac{2}{\kappa}\int d^4x\left(\sqrt{-|g_{\mu\nu}+\kappa R_{\mu\nu}(\Gamma)|}-\lambda\sqrt{-g}\right)+S_M, \quad (17)$$

$$S_M = S_M[g,\Gamma,\Phi]\ \text{(metric-affine)}\ \text{or}\ S_M = S_M[g,\Phi]\ \text{(Palatini)}. \quad (18)$$

- When $\lambda = 1$ (no cosmological constant), the theory can be formulated as a bimetric-like theory [Delsate and Steinhoff, PPL 109(2012)021101]:

$$\begin{aligned} S &= \frac{1}{2}\int d^4x\sqrt{-q}\left(R[q]+\frac{2}{\kappa}\right) \\ &+ \frac{1}{2\kappa}\int d^4x\left(\sqrt{-q}q^{\mu\nu}g_{\mu\nu}-2\sqrt{-g}\right)+S_M[g,\Phi]. \end{aligned} \quad (19)$$

- The metric representing the physical spacetime is only $g_{\mu\nu}$ because that is the one coupled to matter.

The EoMs for EiBI gravity

- By varying the action with respect to the metric gives

$$\sqrt{-|g_{\mu\nu}+\kappa R_{\mu\nu}|}\left[(g_{\mu\nu}+\kappa R_{\mu\nu})^{-1}\right]^{\alpha\beta}=\sqrt{-|g_{\mu\nu}|}\left(\lambda g^{\alpha\beta}-\kappa T^{\alpha\beta}\right), \tag{20}$$

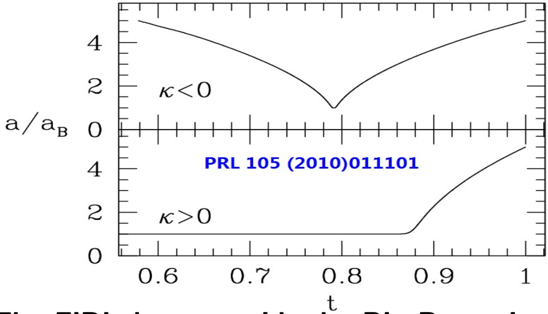
- Variation to the connection gives $\nabla_\sigma(g_{\mu\nu}+\kappa R_{\mu\nu}(\Gamma))=0$.
- By introducing an auxiliary metric $q_{\mu\nu}=g_{\mu\nu}+\kappa R_{\mu\nu}(\Gamma)$, we have $\nabla_\sigma q_{\mu\nu}=0$ and so Γ is just the Christoffel symbol of $q_{\mu\nu}$.
- Then the gravitational equations are rewritten as

$$\sqrt{-q}\, q^{\alpha\beta} = \lambda\sqrt{-g}\, g^{\alpha\beta}-\kappa\sqrt{-g}\, T^{\alpha\beta}, \tag{21}$$
$$q_{\mu\nu} = g_{\mu\nu}+\kappa R_{\mu\nu}(q). \tag{22}$$

The above equations and matter field equations form a complete set of equations of the theory.

- Here, $q^{\mu\nu}q_{\nu\lambda}=\delta^\mu_\lambda$, $g^{\mu\nu}g_{\nu\lambda}=\delta^\mu_\lambda$. $T^{\alpha\beta}=g^{\alpha\mu}g^{\beta\nu}T_{\mu\nu}$.

- **Does the EiBI gravity theory remove singularities?**
- **For a homogeneous and isotropic universe coupling to an ideal fluid, there is a maximum density ρ_B, which corresponds to a minimum length a_B in cosmology.**



- **The EiBI theory avoids the Big Bang singularity, and the universe may be entirely singularity free.**
- **This type of evolution is interesting because it eliminates a physical divergence but also generates an early stage of inflation without including any unknown type of matter.**

Motivation:

- What's the properties of a charged back hole, early universe, and braneworld in the EiBI gravtiy theory?
- Are the linear perturbations stable in early universe in EiBI gravity?
- Are the linear perturbations stable in braneworld model in EiBI gravity? Can 4D Newtonian potential be recovered on the brane?

So, we discuss

- strong gravitational lensing by the charged EiBI BH,
- the full linear perturbations of the EiBI cosmology, and
- the linear tensor perturbation of the EiBI braneworld and localization of gravity on the brane.

2. Strong gravitational lensing by charged EiBI BH

We study the charged black hole solution in EiBI gravity.

- **Consider an electromagnetic field with** $\mathcal{L}_M = -\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}$. **The energy-momentum is**

$$T_{\mu\nu} = \frac{1}{4\pi}(F_{\mu\sigma}F^{\sigma}_{\nu} - \frac{1}{4}g_{\mu\nu}F_{\sigma\rho}F^{\sigma\rho}). \tag{23}$$

We only consider the electrostatic field with $A_0 \neq 0$.

- **Assume a static spherically symmetric spacetime metric**

$$ds^2 = -\psi^2(r)f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\vartheta^2 + \sin^2\vartheta d\phi^2), \tag{24}$$

- **and the auxiliary metric** $q_{\mu\nu}$

$$ds'^2 = -G^2(r)F(r)dt^2 + \frac{dr^2}{F(r)} + H^2(r)(d\vartheta^2 + \sin^2\vartheta d\phi^2). \tag{25}$$

From the EoMs

$$q_{\mu\nu} = g_{\mu\nu} + \kappa R_{\mu\nu}, \quad (26)$$

$$\sqrt{-q}q^{\mu\nu} = \lambda\sqrt{-g}g^{\mu\nu} - 8\pi\kappa\sqrt{-g}\,T^{\mu\nu}, \quad (27)$$

and the Maxwell equations, we have

$$G(r) = \psi\left(\lambda - \frac{\kappa Q^2}{r^4}\right), \;\; H(r) = r\sqrt{\lambda + \frac{\kappa Q^2}{r^4}}, \;\; F(r) = f\left(\lambda - \frac{\kappa Q^2}{r^4}\right)^{-1}. \quad (28)$$

$$4\frac{G'}{G}\frac{H'}{H} + 2\frac{F'}{F}\frac{H'}{H} + 3\frac{G'}{G}\frac{F'}{F} + 2\frac{G''}{G} + \frac{F''}{F} = \frac{2}{\kappa F}\left(\frac{1}{\lambda - \frac{\kappa C_0^2}{r^4}} - 1\right), \quad (29)$$

$$4\frac{H''}{H} + 2\frac{F'}{F}\frac{H'}{H} + 3\frac{G'}{G}\frac{F'}{F} + 2\frac{G''}{G} + \frac{F''}{F} = \frac{2}{\kappa F}\left(\frac{1}{\lambda - \frac{\kappa C_0^2}{r^4}} - 1\right), \quad (30)$$

$$-\frac{1}{H^2F} + \frac{F'}{F}\frac{H'}{H} + \frac{G'}{G}\frac{H'}{H} + \frac{H'^2}{H^2} + \frac{H''}{H} = \frac{1}{\kappa F}\left(\frac{1}{\lambda + \frac{\kappa C_0^2}{r^4}} - 1\right). \quad (31)$$

- **The solution is**

$$\psi(r) = \frac{r^2}{\sqrt{r^4 + (\kappa/\lambda)Q^2}}, \qquad E(r) = \frac{Q}{\sqrt{r^4 + (\kappa/\lambda)Q^2}}, \tag{32}$$

$$f(r) = \frac{r\sqrt{\lambda r^4 + \kappa Q^2}}{\lambda r^4 - \kappa Q^2}\left[-\int \frac{(\Lambda r^4 - r^2 + Q^2)(\lambda r^4 - \kappa Q^2)}{r^4\sqrt{\lambda r^4 + \kappa Q^2}} dr - 2\sqrt{\lambda}M\right]. \tag{33}$$

- **In the limit of $r \to \infty$,**

$$\psi(r) \to 1, \; E(r) \to \frac{Q}{r^2}, \; f(r) \to 1 - \frac{2M}{r} + \frac{(1 - \kappa\Lambda(1 + \kappa\Lambda/2))}{(1 + \kappa\Lambda)}\frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2. \tag{34}$$

- **In the limit of $r \to 0$,**

$$\psi(r) \to \frac{r^2}{\sqrt{\kappa}Q}, \qquad E(r) \to \frac{1}{\sqrt{\kappa}}, \qquad f(r) \to \frac{Q^2}{3r^2}. \tag{35}$$

- **When $Q = 0$, we have $\psi(r) = 1$, $f(r) = 1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2$.**
- **When $\kappa = 0$, we get the RN-AdS/dS solution:** $\psi(r) = 1$, $E(r) = \frac{Q}{r^2}$, $f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2$.

2. Strong gravitational lensing by charged EiBI BH

- **Asymptotic flat solution ($\lambda = 1$ or $\Lambda = 0$):**

$$\psi(r) = \frac{r^2}{\sqrt{r^4 + \kappa Q^2}}, \qquad E(r) = \frac{Q}{\sqrt{r^4 + \kappa Q^2}}, \tag{36}$$

$$f(r) = \frac{r\sqrt{r^4 + \kappa Q^2}}{r^4 - \kappa Q^2}\left[\frac{(3r^2 - Q^2)\sqrt{r^4 + \kappa Q^2}}{3r^3} + \frac{1}{3}\sqrt{\frac{Q^3}{\sqrt{\kappa}\pi}}\Gamma^2(1/4) + \frac{4}{3}\sqrt{\frac{iQ^3}{\sqrt{\kappa}}}F\left(i\text{arcsinh}(\sqrt{\frac{i}{\sqrt{\kappa}Q}}r), -1\right) - 2M\right]. \tag{37}$$

where $F(x, m)$ is the elliptic integral of the first kind.

- **In the limit of $r \to \infty$,**

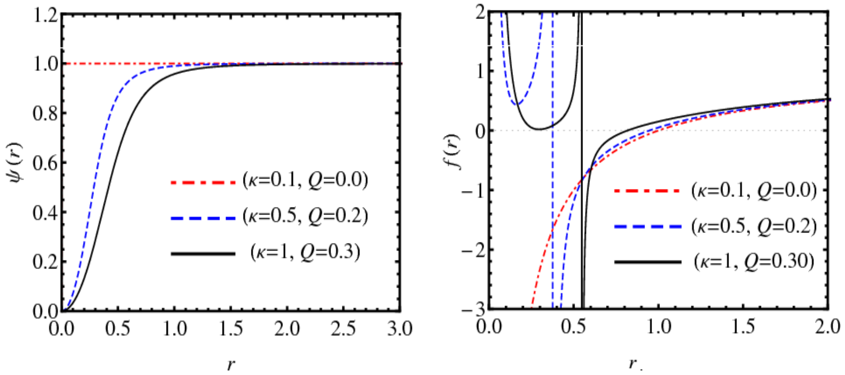
$$\psi(r) \to 1, \qquad E(r) \to \frac{Q}{r^2}, \qquad f(r) \to 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{2\kappa Q^2}{r^4} + \mathcal{O}(r^{-5}). \tag{38}$$

- **In the limit of $r \to 0$,**

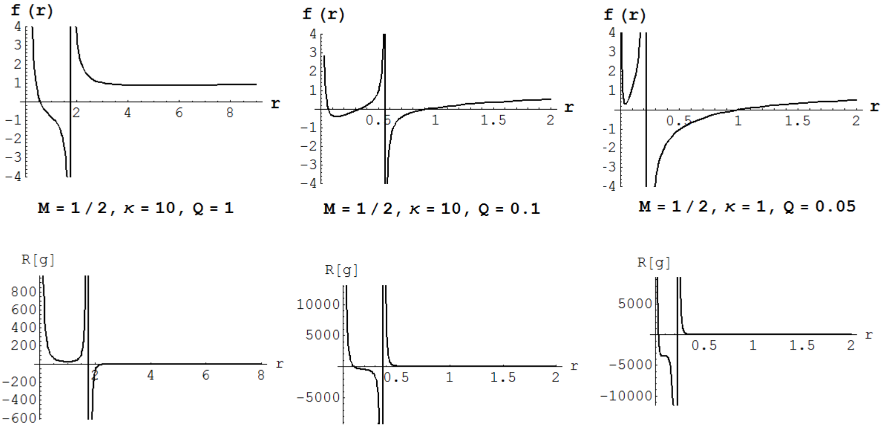
$$\psi(r) \to \frac{r^2}{\sqrt{\kappa}Q}, \qquad E(r) \to \frac{1}{\sqrt{\kappa}}, \qquad f(r) \to \frac{Q^2}{3r^2}. \tag{39}$$

- **In the limit of $r \to \sqrt{\sqrt{\kappa}Q}$,**

$$\psi(r) \to \frac{1}{\sqrt{2}}, \qquad E(r) \to \frac{1}{\sqrt{2\kappa}}, \qquad f(r) \to \infty. \tag{40}$$

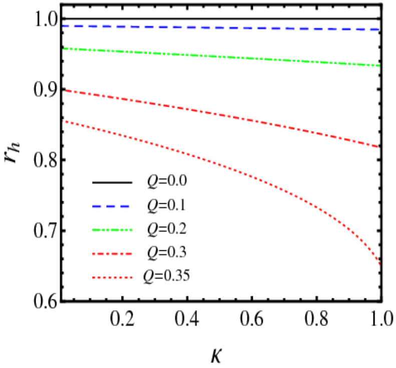


The metric functions $\psi(r)$ and $f(r)$.



- $R[g] = g^{\mu\nu} R[g]_{\mu\nu} \propto (r^4 - \kappa Q^2)^{-3}$.
- $g^{\mu\nu} R[q]_{\mu\nu} = g^{\mu\nu}(q_{\mu\nu} - g_{\mu\nu})/\kappa = 8/\kappa$.
- $R[q] = q^{\mu\nu} R[q]_{\mu\nu} = 8\frac{(r^4+\kappa Q^2/\sqrt{2})(r^4-\kappa Q^2/\sqrt{2})}{(r^4+\kappa Q^2)(r^4-\kappa Q^2)}$.

The outer horizon r_h as a function of κ.



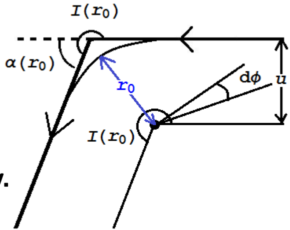
- **The outer horizon r_h decreases with κ and Q.**
- **For RN black hole in GR, the outer horizon r_+ decreases with Q ($r_\pm = M \pm \sqrt{M^2 - Q^2}$).**

Next, we consider strong gravitational lensing by the charged EiBI BH.

- Rewrite the spacetime metric as

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + C(r)(d\theta^2 + \sin^2\theta d\phi^2), \quad (41)$$
$$A(r) = \psi^2(r)f(r), \quad B(r) = f(r)^{-1}, \quad C(r) = r^2. \quad (42)$$

Consider that a photon incomes from infinity with the impact parameter u, reaches a minimum distance r_0, then returns to infinity.



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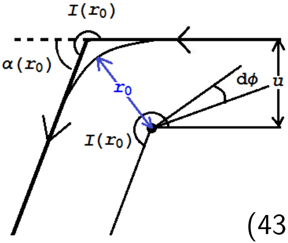


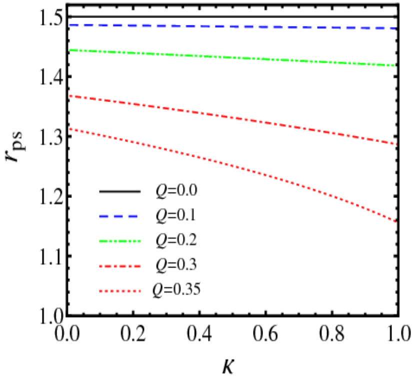
- **The deflection angle is**

$$\alpha(r_0) = I(r_0) - \pi, \tag{43}$$

$$I(r_0) = \int d\phi = 2\int_{r_0}^{\infty} \frac{\dot{\phi}}{\dot{r}} dr = 2\int_{r_0}^{\infty} \frac{\sqrt{B/C}dr}{\sqrt{CA_0/(C_0A) - 1}}. \tag{44}$$

- **When r_0 approaches some certain points, the photon can complete one loop or more than one loop before reaching the observer.**
- **When r_0 approaches the radius r_{ps} of the photon sphere, the photon will surround the black hole all the time if there is no perturbation.**

The equation determining the radius of the photon sphere r_{ps} is $r\,A'(r) = 2A(r)$. $r_{\text{ps}} = 3/2$ for $Q = 0$.



The radius of photon sphere r_{ps} as a function of κ.

Consider the lensing that the photon passes very close the photon sphere. The deflection angle can be expanded with a logarithmic term [PRD 66(2002)103001],

$$\alpha(\theta) = -a_1 \log\Big(\frac{\theta D_{\mathbf{OL}}}{u_{\mathbf{ps}}} - 1\Big) + a_2. \tag{45}$$

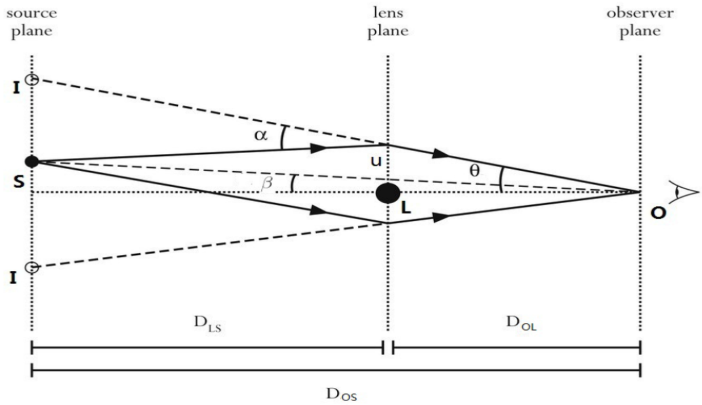
The strong deflection limit coefficients a_1, a_2 depend on $r_{\mathbf{ps}}$,

$$a_1 = \frac{R(0, r_{\mathbf{ps}})}{2\sqrt{\chi_2(r_{\mathbf{ps}})}}, \quad R(z, r_{\mathbf{ps}}) = \frac{2r_{\mathbf{ps}}\sqrt{ABC_{\mathbf{ps}}}}{C(1-z^2)}, \tag{46}$$

$$a_2 = -\pi + I_{\mathbf{R}}(r_{\mathbf{ps}}) + a_1 \log \frac{r_{\mathbf{ps}}^2(2A(r_{\mathbf{ps}}) - r_{\mathbf{ps}}^2 A''(r_{\mathbf{ps}}))}{u_{\mathbf{ps}} r_{\mathbf{ps}} A^{3/2}(r_{\mathbf{ps}})}. \tag{47}$$

(48)

Lens geometry



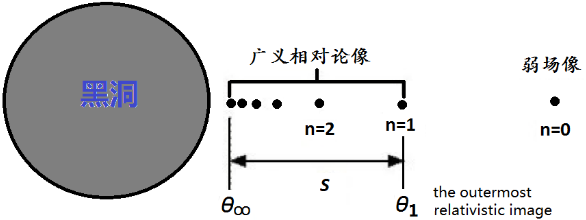
The lens equation is [Phys. Rev. **D 62**, 084003 (2000)]

$$\tan\beta = \tan\theta - \frac{D_{LS}}{D_{OS}}\left[\tan(\alpha-\theta)+\tan\theta\right]. \quad (49)$$

Relation between strong field limit coefficients and observable quantities [PRD 66(2002)103001]

$$\theta_\infty = \frac{u_{ps}}{D_{OL}}, \quad s = \theta_1 - \theta_\infty = \theta_\infty e^{\frac{a_2 - 2\pi}{a_1}}, \quad \tilde{r} = e^{2\pi/a_1}. \qquad (50)$$

where $\tilde{r}$ is the ratio between the flux of the first image and the sum of the others.



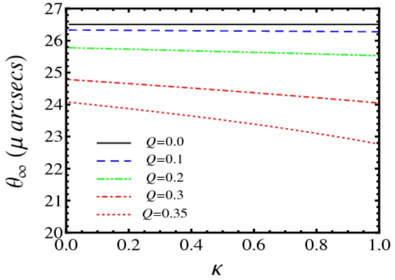
An example

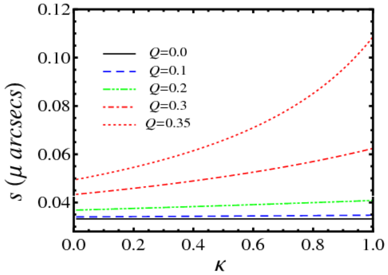
- **The lens is supposed to be the supermassive black hole located at the center of our Milk Way.**
- **It is described by the EiBI black hole metric (24).**
- **The mass of the black hole is estimated to be $M = 4.4 \times 10^6 M_\odot$ with $M_\odot$ the mass of the sun.**
- **The distance from us is around $D_{\mathbf{OL}} = 8.5$ kpc.**

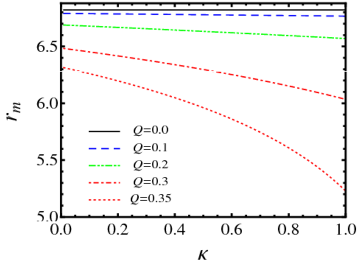
κ	Q	θ_∞	s	r_m	u_{ps}/R_s	a_1	a_2
Sch-BH	0.0	26.510	0.0332	6.8219	2.598	1.000	-0.4002
RN-BH	0.1	26.311	0.0340	6.7909	2.581	1.005	-0.3993
	0.2	25.779	0.0368	6.9899	2.526	1.020	-0.3972
	0.3	24.788	0.0433	6.4858	2.429	1.052	-0.3965
	0.35	24.084	0.0493	6.3190	2.360	1.080	-0.4001
0.1	0.1	26.326	0.0341	6.7886	2.580	1.004	-0.3994
	0.2	25.751	0.0372	6.6789	2.524	1.021	-0.3974
	0.3	24.722	0.0445	6.4522	2.423	1.057	-0.3982
	0.35	23.980	0.0518	6.2609	2.350	1.090	-0.4068
0.5	0.1	26.304	0.0344	6.7791	2.578	1.006	-0.3995
	0.2	25.658	0.0387	6.6327	2.515	1.029	-0.3989
	0.3	24.445	0.0507	6.2975	2.396	1.083	-0.4102
	0.35	23.519	0.0658	5.9611	2.305	1.144	-0.4445
1.0	0.1	26.277	0.0347	6.7669	2.575	1.008	-0.3997
	0.2	25.531	0.0408	6.5698	2.502	1.038	-0.4017
	0.3	24.049	0.0624	6.0350	2.357	1.130	-0.4473
	0.35	22.764	0.1087	5.2286	2.231	1.305	-0.6902

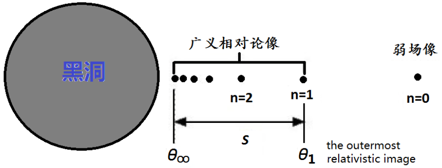
Numerical estimation for the observables and the strong deflection limit coefficients. $r_m = 2.5 \log \tilde{r}$.

2. Strong gravitational lensing by charged EiBI BH



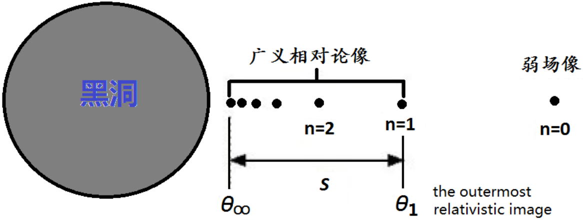






Conclusion

- We found a charged EiBI black hole solution.
- With the increase of κ and Q, θ_∞ and r_m decrease, while s increases.



3. Cosmology in EiBI gravity

3.1 The background metrics

- **The background space-time metric is**

$$
\begin{aligned}
ds^2 &= g_{\mu\nu} dx^\mu dx^\nu && (51)\\
&= -dt^2 + a^2(t)\delta_{ij} dx^i dx^j. && (52)
\end{aligned}
$$

- **The background auxiliary metric is**

$$
\begin{aligned}
ds'^2 &= q_{\mu\nu} dx^\mu dx^\nu && (53)\\
&= -X^2(t)dt^2 + Y^2(t)a^2(t)\delta_{ij} dx^i dx^j. && (54)
\end{aligned}
$$

3.2 The background field equations

- **The 1st field equation reads**

$$\sqrt{-|q_{\mu\nu}|}q^{\mu\nu} = \lambda\sqrt{-|g_{\mu\nu}|}g^{\mu\nu} - \kappa\sqrt{-|g_{\mu\nu}|}\,T^{\mu\nu}, \qquad (55)$$

where $T^{\mu\nu} = Pg^{\mu\nu} + (P+\rho)u^{\mu}u^{\nu}$**, i.e.,**

$$\frac{Y^3}{X} = \lambda + \kappa\rho, \qquad XY = \lambda - \kappa P. \qquad (56)$$

- **The 2nd field equation** $q_{\mu\nu} = g_{\mu\nu} + \kappa R_{\mu\nu}(q)$**:**

$$X^2 = 1 + 3\kappa\Big[\frac{\ddot{a}}{a} + \frac{\ddot{Y}}{Y} - \frac{\dot{a}}{a}\frac{\dot{X}}{X} + 2\frac{\dot{a}}{a}\frac{\dot{Y}}{Y} - \frac{\dot{X}}{X}\frac{\dot{Y}}{Y}\Big], \qquad (57)$$

$$Y^2 = 1 + \kappa\frac{Y^2}{X^2}\Big(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} - \frac{\dot{a}}{a}\frac{\dot{X}}{X} + 6\frac{\dot{a}}{a}\frac{\dot{Y}}{Y} - \frac{\dot{X}}{X}\frac{\dot{Y}}{Y} + \frac{\ddot{Y}}{Y} + 2\frac{\dot{Y}^2}{Y^2}\Big).$$

3.3 The perturbed metrics

- **The perturbed space-time metric is**

$$\begin{aligned} d\tilde{s}^2 &= \tilde{g}_{\mu\nu}dx^\mu dx^\nu = (g_{\mu\nu} + H_{\mu\nu})dx^\mu dx^\nu \\ &= (-1 + h_{00}(x))dt^2 + a^2(t)(\delta_{ij} + h_{ij}(x))dx^i dx^j \\ &\quad +2h_{0i}(x)dtdx^i. \end{aligned} \tag{58}$$

- **The perturbed auxiliary metric is**

$$\begin{aligned} d\tilde{s}'^2 &= \tilde{q}_{\mu\nu}dx^\mu dx^\nu = (q_{\mu\nu} + \Pi_{\mu\nu})dx^\mu dx^\nu \\ &= X^2(t)(-1 + \gamma_{00}(x))dt^2 + a^2(t)Y^2(t)(\delta_{ij} + \gamma_{ij}(x))dx^i dx^j \\ &\quad +2Y^2(t)\gamma_{0i}(x)dtdx^i. \end{aligned} \tag{59}$$

3.4 The perturbation of the energy-momentum tensor

- **The perturbation of** $T_{\mu\nu} = Pg^{\mu\nu} + (P+\rho)u^{\mu}u^{\nu}$

$$\delta T^{00} = \delta\rho + \rho h_{00}, \quad (60)$$

$$\delta T^{i0} = -a^{-2}\rho h_{0i} + a^{-2}(P+\rho)\delta u_i, \quad (61)$$

$$\delta T^{ij} = a^{-2}\delta P\delta_{ij} - a^{-2}P h_{ij}. \quad (62)$$

3.5 Decomposition of the perturbed metric and δu_i

- **The scalar-vector-tensor decomposition of $h_{\mu\nu}$**

$$h_{00} = -E, \qquad h_{i0} = \partial_i F + G_i, \tag{63}$$
$$h_{ij} = A\delta_{ij} + \partial_i\partial_j B + \partial_j C_i + \partial_i C_j + D_{ij}, \tag{64}$$

where $\partial^i C_i = \partial^i G_i = 0$, $\partial^i D_{ij} = 0$, and $D_i{}^i = 0$.

- **The scalar-vector decomposition of δu_i**

$$\delta u_i = \partial_i \delta u + \delta U_i, \quad (\partial^i \delta U_i = 0). \tag{65}$$

3.6 The first-order perturbations of the field equations

[Yang, Du, and Liu, PRD 88(2013)124037]

- **7 scalar modes** $A, B, E, F, \delta\rho, \delta P, \delta u$

$$
\begin{aligned}
&\frac{1}{2}\frac{X^2}{Y^2}a^{-2}\nabla^2 E+3\left(\frac{\ddot{a}}{a}+\frac{\ddot{Y}}{Y}-\frac{\dot{a}}{a}\frac{\dot{X}}{X}+2\frac{\dot{a}}{a}\frac{\dot{Y}}{Y}-\frac{\dot{X}}{X}\frac{\dot{Y}}{Y}\right)E+\frac{3}{2}\left(\frac{\dot{a}}{a}+\frac{\dot{Y}}{Y}\right)\dot{E}\\
&-\frac{1}{2}(3\ddot{A}+\nabla^2\ddot{B})-\left(\frac{\dot{a}}{a}-\frac{1}{2}\frac{\dot{X}}{X}+\frac{\dot{Y}}{Y}\right)(3\dot{A}+\nabla^2\dot{B})+a^{-2}\nabla^2\dot{F}\\
&+a^{-2}\left(2\frac{\dot{Y}}{Y}-\frac{\dot{X}}{X}\right)\nabla^2 F-\frac{\kappa}{4}a^{-2}\frac{X^2}{Y^2}\frac{\nabla^2\delta\rho}{\lambda+\kappa\rho}-\frac{3\kappa}{4}a^{-2}\frac{X^2}{Y^2}\frac{\nabla^2\delta P}{\lambda-\kappa P}\\
&-\frac{3\kappa}{4}\partial_0\partial_0\frac{\delta\rho}{\lambda+\kappa\rho}-\frac{3\kappa}{4}\left(3\frac{\dot{a}}{a}+3\frac{\dot{Y}}{Y}-\frac{\dot{X}}{X}\right)\partial_0\frac{\delta\rho}{\lambda+\kappa\rho}+\frac{3\kappa}{4}\partial_0\partial_0\frac{\delta P}{\lambda-\kappa P}\\
&-\frac{1}{2}\left[1+3\kappa\left(\frac{\ddot{a}}{a}+\frac{\ddot{Y}}{Y}-\frac{\dot{a}}{a}\frac{\dot{X}}{X}+2\frac{\dot{a}}{a}\frac{\dot{Y}}{Y}-\frac{\dot{X}}{X}\frac{\dot{Y}}{Y}\right)\right]\left(\frac{\delta\rho}{\lambda+\kappa\rho}+\frac{3\delta P}{\lambda-\kappa P}\right)\\
&-\frac{3\kappa}{4}\left(\frac{\dot{a}}{a}+\frac{\dot{Y}}{Y}+\frac{\dot{X}}{X}\right)\partial_0\frac{\delta P}{\lambda-\kappa P}-\kappa a^{-2}\partial_0\left[\frac{P+\rho}{\lambda+\kappa\rho}\nabla^2\delta u\right]\\
&-\kappa a^{-2}\left(2\frac{\dot{Y}}{Y}-\frac{\dot{X}}{X}\right)\frac{P+\rho}{\lambda+\kappa\rho}\nabla^2\delta u=0. \qquad (66)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{a^2Y^2}{X^2}\left(\frac{\ddot{a}}{a}+\frac{\ddot{Y}}{Y}+2\frac{\dot{a}^2}{a^2}-\frac{\dot{a}}{a}\frac{\dot{X}}{X}+6\frac{\dot{a}}{a}\frac{\dot{Y}}{Y}+2\frac{\dot{Y}^2}{Y^2}-\frac{\dot{X}}{X}\frac{\dot{Y}}{Y}\right)E+\frac{1}{2}\frac{a^2Y^2}{X^2}\ddot{A}\\
&-\frac{1}{2}\frac{a^2Y^2}{X^2}\left(\frac{\dot{a}}{a}+\frac{\dot{Y}}{Y}\right)\dot{E}-\frac{1}{2}\nabla^2A-\frac{Y^2}{X^2}\left(\frac{\dot{a}}{a}+\frac{\dot{Y}}{Y}\right)\nabla^2F\\
&+\frac{1}{2}\frac{a^2Y^2}{X^2}\left(3\frac{\dot{a}}{a}+3\frac{\dot{Y}}{Y}-\frac{\dot{X}}{X}\right)\dot{A}+\frac{1}{2}\frac{a^2Y^2}{X^2}\left(\frac{\dot{a}}{a}+\frac{\dot{Y}}{Y}\right)(3\dot{A}+\nabla^2\dot{B})\\
&+\frac{\kappa}{4}\frac{a^2Y^2}{X^2}\partial_0\partial_0\frac{\delta\rho}{\lambda+\kappa\rho}-\frac{\kappa}{4}\frac{a^2Y^2}{X^2}\partial_0\partial_0\frac{\delta P}{\lambda-\kappa P}\\
&-\frac{\kappa}{4}\left(\frac{\nabla^2\delta\rho}{\lambda+\kappa\rho}-\frac{\nabla^2\delta P}{\lambda-\kappa P}\right)-\frac{1}{2}a^2\left(\frac{\delta\rho}{\lambda+\kappa\rho}-\frac{\delta P}{\lambda-\kappa P}\right)\\
&+\frac{\kappa}{2}\frac{a^2Y^2}{X^2}\left(\frac{\ddot{a}}{a}+\frac{\ddot{Y}}{Y}+2\frac{\dot{a}^2}{a^2}-\frac{\dot{a}}{a}\frac{\dot{X}}{X}+6\frac{\dot{a}}{a}\frac{\dot{Y}}{Y}+2\frac{\dot{Y}^2}{Y^2}-\frac{\dot{X}}{X}\frac{\dot{Y}}{Y}\right)\\
&\times\left(\frac{\delta\rho}{\lambda+\kappa\rho}+\frac{3\delta P}{\lambda-\kappa P}\right)+\frac{\kappa}{4}\frac{a^2Y^2}{X^2}\left(7\frac{\dot{a}}{a}+7\frac{\dot{Y}}{Y}-\frac{\dot{X}}{X}\right)\partial_0\frac{\delta\rho}{\lambda+\kappa\rho}\\
&-\frac{\kappa}{4}\frac{a^2Y^2}{X^2}\left(3\frac{\dot{a}}{a}+3\frac{\dot{Y}}{Y}-\frac{\dot{X}}{X}\right)\partial_0\frac{\delta P}{\lambda-\kappa P}+\kappa\frac{Y^2}{X^2}\left(\frac{\dot{a}}{a}+\frac{\dot{Y}}{Y}\right)\frac{P+\rho}{\lambda+\kappa\rho}\nabla^2\delta u\\
&=0.
\end{aligned}
\tag{67}
$$

$$(\frac{\dot{a}}{a}+\frac{\dot{Y}}{Y})E-\dot{A}-\frac{\kappa}{2}\partial_0\frac{\delta\rho}{\lambda+\kappa\rho}+\frac{\kappa}{2}\partial_0\frac{\delta P}{\lambda-\kappa P}+\frac{P+\rho}{\lambda+\kappa\rho}\delta u$$
$$-\frac{\kappa}{2}(\frac{\dot{a}}{a}+\frac{\dot{Y}}{Y})(\frac{\delta\rho}{\lambda+\kappa\rho}+\frac{3\delta P}{\lambda-\kappa P})=0. \tag{68}$$

$$-\frac{1}{2}E-\frac{1}{2}A+\frac{1}{2}\frac{a^2Y^2}{X^2}\ddot{B}+\frac{1}{2}\frac{a^2Y^2}{X^2}(3\frac{\dot{a}}{a}+3\frac{\dot{Y}}{Y}-\frac{\dot{X}}{X})\dot{B}-\frac{Y^2}{X^2}\dot{F}$$
$$-\frac{Y^2}{X^2}(\frac{\dot{a}}{a}-\frac{\dot{X}}{X}+3\frac{\dot{Y}}{Y})F+\kappa\frac{\delta P}{\lambda-\kappa P}+\kappa\frac{Y^2}{X^2}\partial_0(\frac{P+\rho}{\lambda+\kappa\rho}\delta u) \tag{69}$$
$$+\kappa\frac{Y^2}{X^2}(\frac{\dot{a}}{a}-\frac{\dot{X}}{X}+3\frac{\dot{Y}}{Y})\frac{P+\rho}{\lambda+\kappa\rho}\delta u=0.$$

$$\delta P=w\delta\rho \quad \text{(the state equation)}, \tag{70}$$

$$\delta\dot{\rho}+3\frac{\dot{a}}{a}(\delta\rho+\delta P)+\frac{1}{2}(P+\rho)(3\dot{A}+\nabla^2\dot{B})-a^{-2}(P+\rho)\nabla^2(F-\delta u)=0. \tag{71}$$
$$\delta P+\frac{1}{2}(P+\rho)E+(P+\rho)\delta\dot{u}+(\dot{P}+\dot{\rho})\delta u+3\frac{\dot{a}}{a}(P+\rho)\delta u=0. \tag{72}$$

The last two come from the perturbation of the conservation equation.

- **3 transverse vector modes** $C_i, G_i, \delta U_i$

$$\nabla^2 \dot{C}_i - a^{-2}\nabla^2 G_i + \kappa a^{-2}\frac{P+\rho}{\lambda+\kappa\rho}\nabla^2\delta U_i + 2\frac{P+\rho}{\lambda+\kappa\rho}\delta U_i = 0, \tag{73}$$

$$(P+\rho)\delta\dot{U}_i + (\dot{P}+\dot{\rho})\delta U_i + 3\frac{\dot{a}}{a}(P+\rho)\delta U_i = 0, \tag{74}$$

$$\begin{aligned} &\frac{a^2Y^2}{X^2}\ddot{C}_j + \frac{a^2Y^2}{X^2}(3\frac{\dot{a}}{a} + 3\frac{\dot{Y}}{Y} - \frac{\dot{X}}{X})\dot{C}_j - \frac{Y^2}{X^2}\dot{G}_j \\ &-\frac{Y^2}{X^2}(\frac{\dot{a}}{a} - \frac{\dot{X}}{X} + 3\frac{\dot{Y}}{Y})G_j + \kappa\frac{Y^2}{X^2}\partial_0(\frac{P+\rho}{\lambda+\kappa\rho}\delta U_j) \\ &+\kappa\frac{Y^2}{X^2}(\frac{\dot{a}}{a} - \frac{\dot{X}}{X} + 3\frac{\dot{Y}}{Y})\frac{P+\rho}{\lambda+\kappa\rho}\delta U_j = 0. \end{aligned} \tag{75}$$

- **1 transverse-traceless tensor mode** D_{ij}

$$-\nabla^2 D_{ij} + \frac{a^2Y^2}{X^2}\ddot{D}_{ij} + \frac{a^2Y^2}{X^2}(3\frac{\dot{a}}{a} + 3\frac{\dot{Y}}{Y} - \frac{\dot{X}}{X})\dot{D}_{ij} = 0. \tag{76}$$

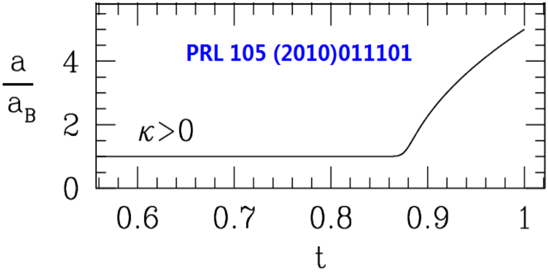
- **The perturbation equations involve**
  7 scalar modes $A, B, E, F, \delta\rho, \delta P, \delta u$,
  3 transverse vector modes $C_i, G_i, \delta U_i$, **and**
  1 transverse-traceless tensor mode D_{ij}.
- **For scalar modes, we work in the Newtonian gauge** ($B = F = 0$).
- **For vector modes, we fix the gauge freedom to eliminate** C_i.
- **All the remaining perturbed modes can be solved.**

For $\kappa > 0$, the approximate background solution near the maximum density ($t \to -\infty$) is given by

[Escamilla-Rivera, Banados, and Ferreira, PRD 85(2012)087302],

[Scargill, Banados, and Ferreira, PRD **86** (2012) 103533]

$$\begin{aligned} a(t) &= a_B[1 + e^{b(t-t_0)}], \\ X(t) &= 2\, e^{\frac{3}{4}b(t-t_0)}, \\ Y(t) &= 2\, e^{\frac{1}{4}b(t-t_0)}, \quad (b = (8/3\kappa)^{\frac{1}{2}}). \end{aligned} \tag{77}$$



A. Scalar perturbations ($t \to -\infty$, in the Eddington region)

[Yang, Du, and Liu, PRD 88(2013)124037]

$$A \simeq c_1 + c_2 k^2 t + c_3 e^{\frac{7}{4}b(t-t_0)}, \quad (78)$$

$$E \simeq (c_4 + c_5 k^2 t) e^{b(t-t_0)}, \quad (79)$$

$$\delta\rho \simeq (c_6 + c_7 k^2 t) e^{b(t-t_0)}, \quad (80)$$

$$\delta u \simeq c_8 + c_9 e^{\frac{7}{4}b(t-t_0)}. \quad (81)$$

- **They are stable for $k = 0$ modes (infinite wavelength limit),**
- **but unstable for $k \neq 0$ modes.**

B. **Transverse vector modes** [Yang, Du, and Liu, PRD 88(2013)124037]

$$G_i \simeq c_{10}, \quad \delta U_i \simeq c_{11}. \tag{82}$$

- They are stable in the Eddington region.

C. **Transverse-traceless tensor mode**

$$D_{ij} \simeq c_{12} t + c_{13}. \tag{83}$$

- **It causes an instability as claimed in [PRD 85(2012)087302].**

For $\kappa < 0$, the approximate background solution near the maximum density ($t \to 0$) is given by

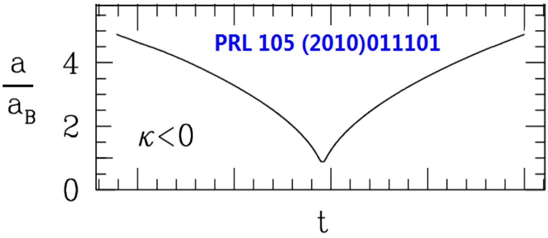
[Escamilla-Rivera, Banados, and Ferreira, PRD 85(2012)087302],

[Scargill, Banados, and Ferreira, PRD **86** (2012) 103533]

$$a = a_B\left(1 - \frac{2}{3\kappa}|t|^2\right), \quad (84)$$

$$X = (2/\sqrt{3})\,(-\kappa/2)^{1/4}\,|t|^{-\frac{1}{2}}, \quad (85)$$

$$Y = (2/\sqrt{3})\,(-2/\kappa)^{1/4}\,|t|^{\frac{1}{2}}. \quad (86)$$



The perturbations (at $t \to 0$) are approximately given by

[Yang, Du, and Liu, PRD 88(2013)124037]

$$A \simeq C_1|t|^{\frac{3}{2}} + C_2|t|^{\varepsilon}, \tag{87a}$$

$$E \simeq C_3|t|^{-\frac{1}{2}} + C_4|t|^{-2+\varepsilon}, \tag{87b}$$

$$\delta\rho \simeq C_5|t|^{\frac{3}{2}} + C_6|t|^{\varepsilon}, \tag{87c}$$

$$\delta u \simeq C_7|t|^{\frac{1}{2}} + C_8|t|^{-1+\varepsilon}. \tag{87d}$$

$$\delta U_i \simeq C_9, \quad G_i \simeq C_{10}|t|^{-2}. \tag{87e}$$

$$D_{ij} \simeq |t|^{-\frac{1}{2}}\left(C_{11}|t|^{\frac{\sqrt{1+24\varepsilon}}{2}} + C_{12}|t|^{-\frac{\sqrt{1+24\varepsilon}}{2}}\right), \tag{87f}$$

where C_i are functions of wave vector and $\varepsilon = -\frac{\kappa k^2}{12a_B^2} > 0$.

The scalar, vector, and tensor modes will all cause instabilities in the Eddington region for $\kappa < 0$.

4. Braneworld in EiBI gravity

4.1 The model

- **We consider a braneworld model in 5D EiBI theory. The brane is generated by a scalar field with the lagrangian**

$$L_M[g,\phi] = \sqrt{-|g_{PQ}|}\left[-\frac{1}{2}g^{MN}\partial_M\phi\partial_N\phi - V(\phi)\right]. \quad (88)$$

- **The ansatz for the spacetime metric is**

$$ds^2 = g_{MN}dx^M dx^N = a^2(y)\eta_{\mu\nu}dx^\mu dx^\nu + dy^2, \quad (89)$$

where $a^2(y)$ is the warp factor and y denotes the physical coordinate of extra dimension.

- **The auxiliary metric can be assumed as**

$$ds'^2 = q_{MN}dx^M dx^N = u(y)\eta_{\mu\nu}dx^\mu dx^\nu + v(y)dy^2, \quad (90)$$

- **The EoMs are**

$$u = a^2 + b\frac{uu'v' - 2v(u'^2 + uu'')}{4uv^2},$$
$$v = 1 + b\frac{uu'v' + v(u'^2 - 2uu'')}{u^2 v}. \qquad (91)$$
$$4\frac{a'}{a}\phi' + \phi'' = \frac{\partial V}{\partial \phi},$$

where $u = \Xi_+^{\frac{1}{3}}\,\Xi_-^{\frac{1}{3}} a^2$, $v = \Xi_+^{\frac{4}{3}}\,\Xi_-^{-\frac{2}{3}}$, **and** $\Xi_\pm = \lambda + b\kappa\big(V \pm \frac{1}{2}\phi'^2\big)$.

- **There are three functions and three equations. But not all of these equations are independent.**
- **In order to solve the above equations, we need to give the explicit form of** $V(\phi)$**, or give another relation between** $a(y)$ **and** $\phi(y)$**.**

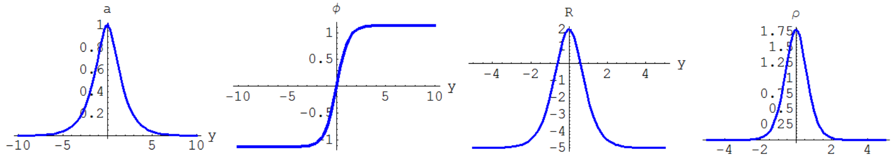
4.2 The brane solutions

- **Model A:** $\phi'(y) = Ka^{2n}(y)$

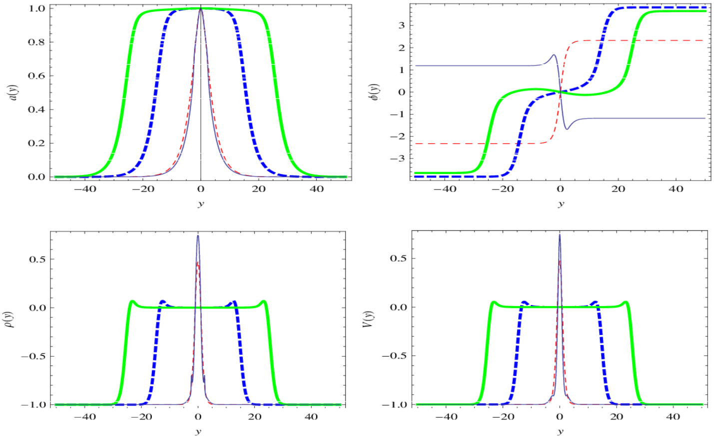
$$a(y) = \mathbf{sech}^{\frac{3}{4n}}(ky), \qquad (k = 2n/\sqrt{3b(4n+3)}) \tag{92}$$

$$\phi(y) = \frac{2K}{k}\Big(i\mathbf{E}(iky/2, 2) + \mathbf{sech}^{\frac{1}{2}}(ky)\,\mathbf{sinh}(ky)\Big), \tag{93}$$

$$V(y) = \frac{(n+2)(1+4n/3)^{3/2}}{2(n+1)^2 b\kappa}\mathbf{sech}^3(ky) - \frac{\lambda}{b\kappa}. \tag{94}$$



Model B: $\phi'(y) = K_1\, a^2(y)\big(1 - K_2\, a^2(y)\big)$



4.2 Stability of tensor fluctuations

- **The TT tensor fluctuations of the metrics:**

$$
\begin{aligned}
d\hat{s}^2 &= a^2(y)[\eta_{\mu\nu} + h_{\mu\nu}(x,y)]dx^\mu dx^\nu + dy^2, \quad (95)\\
d\hat{s}'^2 &= u(y)[\eta_{\mu\nu} + \gamma_{\mu\nu}(x,y)]dx^\mu dx^\nu + v(y)dy^2, \quad (96)
\end{aligned}
$$

where $h_{\mu\nu}$ are TT: $\partial^\mu h_{\mu\nu} = 0$, $\eta^{\mu\nu} h_{\mu\nu} = 0$.

- **The perturbation equations are**

$$
\frac{u}{v} h''_{\mu\nu} + \left(\frac{2u'}{v} - \frac{uv'}{2v^2}\right) h'_{\mu\nu} + \Box^{(4)} h_{\mu\nu} = 0, \quad (97)
$$

- **By making a coordinate transformation $dy = \sqrt{\frac{u(z)}{v(z)}}dz$, Eq. (97) can be rewritten as**

$$
\partial_{z,z} h_{\mu\nu} + \frac{3\partial_z u(z)}{2u(z)} \partial_z h_{\mu\nu} + \Box^{(4)} h_{\mu\nu} = 0. \quad (98)
$$

- **By making the KK decomposition $h_{\mu\nu} = \varepsilon_{\mu\nu}(x)f(z)h(z)$, where $f(z) = \exp(-\int \frac{3u'}{4u})dz$, we can get the following two equations**

$$\Box^{(4)}\varepsilon_{\mu\nu}(x) = m^2\varepsilon_{\mu\nu}(x), \tag{99}$$

$$\Big[-\partial_z^2 + U(z)\Big]h(z) = m^2 h(z). \tag{100}$$

- **The effective potential $U(z)$ is given by**

$$U(z) = \frac{2f'(z)^2}{f(z)^2} - \frac{f''(z)}{f(z)}. \tag{101}$$

- **Eq. (100) can be written as the supersymmetric form**

$$L^{\dagger} L\, h(z) = m^2\, h(z), \tag{102}$$

where $L = (-d/dz + f'/f)\,, \quad L^{\dagger} = (d/dz + f'/f)$**.**

- **As the operator $L^{\dagger}L$ is hermitian and positive definite, this ensures that $m^2 \geq 0$.**
- **Thus, the tensor perturbations are stable (no tachyonic KK modes with $m^2 < 0$).**
- **The zero mode $h^{(0)}_{\mu\nu} = \varepsilon^{(0)}_{\mu\nu}(x)$ is normalizable and is the 4D massless graviton localized on the brane.**
- **The 4D Einstein gravity is recovered on the brane at low energy.**

4.2 The brane solutions

Model A: $\phi'(y) = K a^{2n}(y)$

- **The function $f(z)$ is turned out to be**

$$f(z) = a(z)^{-\frac{4n+3}{2}}, \tag{103}$$

and the potential $U(z)$ is

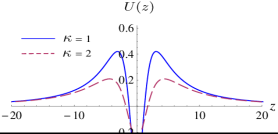
$$U(z) = \frac{(4n+3)\partial_{z,z}a}{2a} + \frac{(4n+3)(4n+1)(\partial_z a)^2}{4a^2}. \tag{104}$$

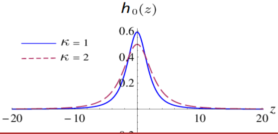
- **The Hamiltonian can be factorized as**

$$H = \left(\frac{d}{dz} + \frac{(4n+3)\partial_z a}{2a}\right)\left(-\frac{d}{dz} + \frac{(4n+3)\partial_z a}{2a}\right). \tag{105}$$

- **The zero mode is**

$$h_0(z) = N_0 a^{\frac{(4n+3)}{2}}(z). \tag{106}$$





4.2 The brane solutions

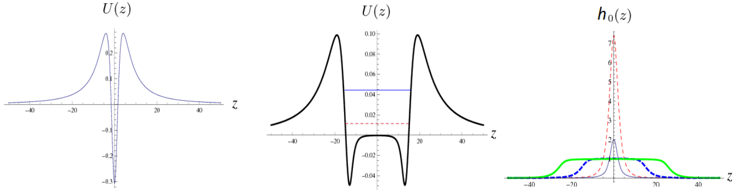
Model B: $\phi'(y) = K_1\, a^2(y)\big(1 - K_2\, a^2(y)\big)$

- **The function $f(z)$ and the potential $U(z)$ are**

$$
\begin{aligned}
f(z) &= a^{-\frac{7}{2}}(6 - K_2 a^2 + 3K_2^2 a^4)^{-\frac{1}{4}}(12 - 20K_2 a^2 + 9K_2^2 a^4)^{-\frac{1}{4}}, && (107)\\
U(z) &= \frac{3}{4} a^{-2}(6 - 8K_2 a^2 + 3K_2^2 a^4)^{-2}(12 - 20K_2 a^2 + 9K_2^2 a^4)^{-2}\Big[(60480 - 445824K_2 a^2 && (108)\\
&+ 1406496K_2^2 a^4 - 2501856K_2^3 a^6 + 2760996K_2^4 a^8 - 1944256K_2^5 a^1 0 + 856140K_2^6 a^1 2\\
&- 216216K_2^7 a^1 4 + 24057K_2^8 a^1 6)(\partial_z a)^2 + 2a(12096 - 77760K_2 a^2 + 220416K_2^2 a^4 - 359856K_2^3 a^6\\
&+ 370236K_2^4 a^8 - 245936K_2^5 a^{10} + 103080K_2^6 a^{12} - 24948K_2^7 a^{14} + 2673K_2^8 a^{16})\partial_z^2 a\Big].
\end{aligned}
$$

- **The zero mode is**

$$h_0(z) = C_0 a^{\frac{7}{2}}(6 - K_2 a^2 + 3K_2^2 a^4)^{\frac{1}{4}}(12 - 20K_2 a^2 + 9K_2^2 a^4)^{\frac{1}{4}} \quad (109)$$



For both model A and model B

- **The tensor perturbations are stable.**
- **The zero mode $h^{(0)}_{\mu\nu} = \varepsilon^{(0)}_{\mu\nu}(x)$ is normalizable and is the 4D massless graviton localized on the brane.**
- **The 4D Einstein gravity is recovered on the brane at low energy.**

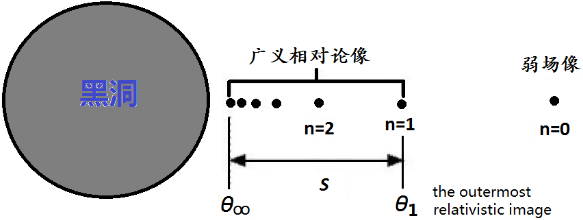
5. Conclusions

We have discussed three issues in EiBI gravity theory:

- **strong gravitational lensing by charged BH,**
- **full linear perturbations in cosmology in early universe,**
- **tensor perturbations in braneworld.**

Black hole and strong gravitational lensing

- We found a charged EiBI black hole solution.
- With the increase of κ and Q, θ_∞ and r_m decrease, while s increases.



Full linear perturbations in EiBI cosmology in early universe

- **Linearly stability of the perturbations**

	Tensor	Vector	Scalar
$\kappa < 0$	$\times$	$\times$	$\times$
$\kappa > 0$	$\times$	$\surd$	$\surd(k=0)$, $\times(k \neq 0)$

- **For these unstable (divergent) linear perturbations, the condition $|h_{\mu\nu}| \ll 1$ is not satisfied, so nonlinear perturbations should be considered.**
- **Large scale structure formation [1403.0083]:**
  The linear growth of scalar perturbations deviates from that in general relativity for modes with large k, but the deviation is largely suppressed with the expansion of the universe.

Tensor perturbations in EiBI braneworld

- **We considered braneworld models in 5D EiBI theory. The brane is generated by a scalar field.**
- **Some regular brane solutions were found.**
- **The TT tensor perturbation is always stable.**
- **The zero mode can be localized on the brane, and it is the 4D massless graviton on the brane.**
- **So, the 4D Einstein gravity is recovered on the brane at low energy.**

Thank you!