Entanglement Entropy and Black Holes

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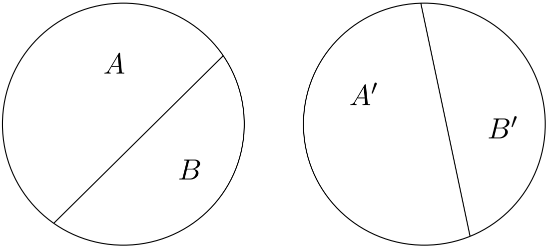
Quantum Entanglement

- Given a system in some (pure) state $|\Psi\rangle \in \mathcal{H}$, $\rho = |\Psi\rangle\langle\Psi|$
- Divided it into two sub-systems $A \cup B$, with $\mathcal{H}_A \otimes \mathcal{H}_B = \mathcal{H}$



- In general

$$|\Psi\rangle = \sum_{i,j} c_{ij} |i\rangle_A \otimes |j\rangle_B, \quad |\cdot\rangle_{A,B} \in \mathcal{H}_{A,B}$$

- $c_{ij} \cong a_i \cdot b_j \overset{\text{def}}{\Rightarrow} A, B$ not entangled; otherwise QE happens

- For non-entangled A, B

$$|\Psi\rangle = \sum_{i,j} a_i b_j |i\rangle_A \otimes |j\rangle_B = |\phi\rangle_A \otimes |\psi\rangle_B \quad \text{separable states}$$

$$|\phi\rangle_A \equiv \sum_i a_i |i\rangle_A, \quad |\psi\rangle_B \equiv \sum_j b_j |j\rangle_B$$

- Measurements performed on A do not alter the state of B

$$\mathcal{O}_A |i\rangle_A = \lambda_i |i\rangle_A, \quad |\phi\rangle_A \xrightarrow{\text{collapse}} a_1 |1\rangle_A \quad \text{with probability } |a_1|^2$$

$$|\phi\rangle_A \xrightarrow{\text{collapse}} a_2 |2\rangle_A \quad \text{with probability } |a_2|^2$$

$$\vdots$$

$$|\Psi\rangle = |\phi\rangle_A \otimes |\psi\rangle_B \xrightarrow{\text{collapse}} a_i |i\rangle_A \otimes \underbrace{|\psi\rangle_B}_{\uparrow\ \text{unchanged}} \quad \text{with probability } |a_i|^2$$

- This is not the case for entangled subsystems

- 量子纠缠态: $|\uparrow\downarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A \otimes |\downarrow\rangle_B - |\downarrow\rangle_A \otimes |\uparrow\rangle_B)$

$$c_{ij} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \det c_{ij} = \frac{1}{2} \Rightarrow c_{ij} \neq a_i b_j$$

- EPR 佯谬 (Einstein, Podolsky, and Rosen, 1935)

$(\bullet\bullet)$ $\quad t \to \infty \quad$ $\odot \longleftarrow$ "spooky action at a distance" $\longrightarrow \odot$

$|\uparrow\downarrow\rangle$ 测量到 $|\uparrow\rangle_A$ / $|\downarrow\rangle_A$ 可推断 $|\downarrow\rangle_B$ / $|\uparrow\rangle_B$

MAY 15, 1935 PHYSICAL REVIEW VOLUME 47

Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?

A. EINSTEIN, B. PODOLSKY AND N. ROSEN, *Institute for Advanced Study, Princeton, New Jersey*

(Received March 25, 1935)

In a complete theory there is an element corresponding to each element of reality. A sufficient condition for the reality of a physical quantity is the possibility of predicting it with certainty, without disturbing the system. In quantum mechanics in the case of two physical quantities described by non-commuting operators, the knowledge of one precludes the knowledge of the other. Then either (1) the description of reality given by the wave function in quantum mechanics is not complete or (2) these two quantities cannot have simultaneous reality. Consideration of the problem of making predictions concerning a system on the basis of measurements made on another system that had previously interacted with it leads to the result that if (1) is false then (2) is also false. One is thus led to conclude that the description of reality as given by a wave function is not complete.

- Bohr 的回应基于 “互补原理”

OCTOBER 15, 1935 PHYSICAL REVIEW VOLUME 48

Can Quantum-Mechanical Description of Physical Reality be Considered Complete?

N. BOHR, *Institute for Theoretical Physics, University, Copenhagen*
(Received July 13, 1935)

It is shown that a certain "criterion of physical reality" formulated in a recent article with the above title by A. Einstein, B. Podolsky and N. Rosen contains an essential ambiguity when it is applied to quantum phenomena. In this connection a viewpoint termed "complementarity" is explained from which quantum-mechanical description of physical phenomena would seem to fulfill, within its scope, all rational demands of completeness.

- Not all physicists buy Bohr's arguments

PHYSICAL REVIEW VOLUME 85, NUMBER 2 JANUARY 15, 1952

A Suggested Interpretation of the Quantum Theory in Terms of "Hidden" Variables. I

DAVID BOHM*
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey
(Received July 5, 1951)

VOLUME 49, NUMBER 2 PHYSICAL REVIEW LETTERS 12 JULY 1982

Experimental Realization of Einstein-Podolsky-Rosen-Bohm *Gedankenexperiment*: A New Violation of Bell's Inequalities

Alain Aspect, Philippe Grangier, and Gérard Roger
Institut d'Optique Théorique et Appliquée, Laboratoire associé au Centre National de la Recherche Scientifique, Université Paris-Sud, F-91406 Orsay, France
(Received 30 December 1981)

- 熵用于衡量信息的不完整

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|, \quad \sum_i p_i = 1, \quad 0 \le p_i \le 1$$

$$S = -\mathrm{Tr}(\rho \log \rho) = -\sum_i p_i \log p_i$$

纯态 $\rho = |\psi\rangle\langle\psi| \ \Rightarrow \ S = 0$

等几率混合态 $\rho = \frac{1}{N}\sum_{i=1}^{N} |\psi_i\rangle\langle\psi_i| \ \Rightarrow \ S = \log N = S_{\max}$

- 引进约化密度矩阵 $\rho_A = \mathrm{Tr}_B \rho$ 和纠缠熵 $S_A = -\rho_A \mathrm{Tr} \rho_A$
- 当 A, B 不发生纠缠时，$S_A = S_B = 0$

$$|\Psi\rangle = |\phi\rangle_A \otimes |\psi\rangle_B, \ \rho = |\Psi\rangle\langle\Psi| = (|\phi_A\rangle\langle\phi_A|) \otimes (|\psi_B\rangle\langle\psi_B|)$$

$$\rho_A = (|\phi_A\rangle\langle\phi_A|) \sum_j (\langle j_B|\psi_B\rangle\langle\psi_B|j_B\rangle) = \||\psi_B\rangle\|^2 \cdot |\phi_A\rangle\langle\phi_A|$$

故以 ρ_A 描述子系统 A 给出纯态 $\Rightarrow \ S_A = 0$

- 推论：若 $S_A \neq 0$，则两个子系统之间必然有纠缠

Entanglement Entropy in QFTs

- 纠缠熵非零的例子

$$|\Psi\rangle = |\uparrow\downarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A \otimes |\downarrow\rangle_B - |\downarrow\rangle_A \otimes |\uparrow\rangle_B)$$

$$\begin{aligned}
& \rho_A = \mathrm{Tr}_B\rho = \langle\uparrow_B|\Psi\rangle\langle\Psi|\uparrow_B\rangle + \langle\downarrow_B|\Psi\rangle\langle\Psi|\downarrow_B\rangle \\
= \ & \left(-\frac{1}{\sqrt{2}}|\downarrow_A\rangle\right)\left(-\frac{1}{\sqrt{2}}\langle\downarrow_A|\right) + \left(\frac{1}{\sqrt{2}}|\uparrow_A\rangle\right)\left(\frac{1}{\sqrt{2}}\langle\uparrow_A|\right) \\
= \ & \frac{1}{2}(|\uparrow_A\rangle\langle\uparrow_A| + |\downarrow_A\rangle\langle\downarrow_A|) \qquad \text{等几率混合态} \\
\Rightarrow \ & S_A = -\mathrm{Tr}\rho_A \log\rho_A = \log 2 \neq 0
\end{aligned}$$

- 谐振子链

$$H = \frac{1}{2m}\sum_{i=1}^{N} p_i^2 + \frac{m}{2}\sum_{i,j=1}^{N} q_i K_{ij} q_j \quad \begin{cases} [q_j, p_k] = i\delta_{jk} \\ [q_j, q_k] = [p_j, p_k] = 0 \end{cases}$$

$$O^T O = 1, \quad K = O^T \mathrm{diag}(\omega_1^2, \cdots, \omega_N^2) O$$

- 经正则变换 $\tilde{q} = Oq$, $\tilde{p} = O^T p$

$$H = \frac{1}{2m}\sum_{i=1}^{N}\tilde{p}_i^2 + \frac{m}{2}\sum_{i=1}^{N}\omega_i^2\tilde{q}_i^2 \qquad \begin{cases} [\tilde{q}_j, \tilde{p}_k] = i\delta_{jk} \\ [\tilde{q}_j, \tilde{q}_k] = [\tilde{p}_j, \tilde{p}_k] = 0 \end{cases}$$

- 假定整个系统处于基态

$$\begin{aligned} \Psi_0(q_1, \cdots, q_N) &= \prod_{j=1}^{N}\left[\left(\frac{m\omega_j}{\pi}\right)^{1/4} e^{-\frac{m\omega_j\tilde{q}_j^2}{2}}\right] \\ &= \left(\frac{m}{\pi}\right)^{N/4}(\det K)^{1/8}\exp\left[-\frac{m}{2}\sum_{i,j=1}^{N} q^i (K^{1/2})_{ij} q^j\right] \end{aligned}$$

- 密度矩阵元

$$\rho(q, q') = \langle q|\Psi_0\rangle\langle\Psi_0|q'\rangle = \Psi_0(q)\Psi_0^*(q')$$

- 把 N 个谐振子拆分成两个子系统

$$A = \{q^1, \cdots, q^n\} \equiv \{q_-\}, \quad B = \{q^{n+1}, \cdots, q^N\} \equiv \{q_+\}$$

- 拆分后的角频率 $\Omega \equiv K^{1/2}$、波函数和密度矩阵

$$
\begin{aligned}
&\Omega = \begin{pmatrix} W_{--} & W_{+-}^T \\ W_{+-} & W_{++} \end{pmatrix} \\
&q^T K^{1/2} q = q_-^T W_{--} q_- \; + \; q_+^T W_{++} q_+ \; + \; 2 q_+^T W_{+-} q_- \\
&\Psi_0 \propto \exp\left[-\frac{m}{2} \left(q_-^T W_{--} q_- \; + \; q_+^T W_{++} q_+ \; + \; 2 q_+^T W_{+-} q_- \right) \right] \\
&\rho(q, q') = \Psi_0(q_+, q_-) \Psi_0(q'_+, q'_-)
\end{aligned}
$$

- 约化密度矩阵 $\rho_B = \mathrm{Tr}_B \rho$ 来自对子系统 A 的自由度求迹

$$
\begin{aligned}
\rho_B(q_+, q'_+) &= \int d^n q_- \; \Psi_0(q_+, q_-) \Psi_0(q'_+, q_-) \\
\propto & \quad \exp\left[-\frac{m}{2} \left(q_+^T W_{++} q_+ \; + \; q_+'^T W_{++} q'_+ \right) \right] \\
\times & \quad \int d^n q_- \exp\left\{ -m \left[q_-^T W_{--} q_- \; + (q_+^T + q_+'^T) W_{+-} q_- \right] \right\}
\end{aligned}
$$

- 求解本征方程 $\int dq'_+ \rho_B(q_+, q'_+) \varphi_n(q'_+) = p_n \varphi_n(q_+)$

- 纠缠熵 $S_B = -\sum_n p_n \log p_n$ 的表达式

$$S_B = \mathrm{Tr}\left[\log \frac{1-\Lambda/2+\sqrt{1-\Lambda}}{1-\Lambda+\sqrt{1-\Lambda}} - \frac{\Lambda \log \frac{\Lambda}{2-\Lambda+2\sqrt{1-\Lambda}}}{2(1-\Lambda+\sqrt{1-\Lambda})} \right]$$

$$\Lambda = (W_{++})^{-1/2} W_{+-} (W_{--})^{-1} W_{-+} (W_{++})^{-1/2}$$

- 如定义 $\Upsilon = \Lambda(1-\Lambda)^{-1} = -(\Omega^{-1})_{+-}\Omega_{-+}$

$$S_B = \mathrm{Tr}\left[\log \frac{\sqrt{\Upsilon}}{2} + \sqrt{1+\Upsilon} \log \left(\frac{1}{\sqrt{\Upsilon}} + \sqrt{1+\frac{1}{\Upsilon}} \right) \right]$$

- 自由标量场即连续分布的谐振子链

$$H = \frac{1}{2} \int d^d \vec{x} \left[-\left(\frac{\delta}{\delta \varphi(\vec{x})} \right)^2 + (\nabla \varphi(\vec{x}))^2 + m^2 \varphi(\vec{x})^2 \right]$$

$$\equiv -\frac{1}{2} \int d^d \vec{x} \left(\frac{\delta}{\delta \varphi(\vec{x})} \right)^2 + \frac{1}{2} \int d^d \vec{x} d^d \vec{y} \varphi(\vec{x}) K(\vec{x}, \vec{y}) \varphi(\vec{y})$$

$$K(\vec{x}, \vec{y}) = \left(-\Delta_{\mathbb{R}^d} + m^2 \right) \delta^d(\vec{x} - \vec{y})$$

- 基态波函数

$$\Psi_0[\varphi(\vec{x})] = \mathcal{N} \exp\left[-\frac{1}{2}\int d^d\vec{x} d^d\vec{y}\, \varphi(\vec{x})\Omega(\vec{x},\vec{y})\varphi(\vec{y})\right]$$

$$\begin{aligned}\int d^d\vec{z}\,\Omega(\vec{x},\vec{z})\Omega(\vec{z},\vec{y}) &= \left(-\Delta_{\mathbb{R}^d} + m^2\right)\delta^d(\vec{x}-\vec{y}) \\ &= \langle\vec{x}|\left(-\Delta_{\mathbb{R}^d} + m^2\right)|\vec{y}\rangle\end{aligned}$$

$$\hat{\Omega}^2 = -\Delta_{\mathbb{R}^d} + m^2 \;\Rightarrow\; \Omega(\vec{x},\vec{y}) = \left\langle \vec{x}\middle|\sqrt{-\Delta + m^2}\middle|\vec{y}\right\rangle$$

- 热核表示 ($\mathrm{Re}(s) > 0$)

$$\left\langle \vec{x}|e^{-\tau(-\Delta+m^2)}|\vec{y}\right\rangle = \frac{1}{(4\pi\tau)^{d/2}} e^{-m^2\tau - \frac{(\vec{x}-\vec{y})^2}{4\tau}}$$

$$\left\langle \vec{x}\middle|(-\Delta + m^2)^{-s}\middle|\vec{y}\right\rangle = \frac{1}{\Gamma(s)}\int_0^\infty d\tau\tau^{s-1}\left\langle \vec{x}\middle|e^{-\tau(-\Delta+m^2)}\middle|\vec{y}\right\rangle$$

$$= \frac{2^{1-s}}{(2\pi)^{d/2}\Gamma(s)}\left(\frac{m}{|\vec{x}-\vec{y}|}\right)^{\frac{d}{2}-s} K_{\frac{d}{2}-s}(m|\vec{x}-\vec{y}|)$$

- 解析延拓到 $s=-1/2$

$$\Omega(\vec{x},\vec{y}) = -2\left(\frac{m}{2\pi|\vec{x}-\vec{y}|}\right)^{\frac{d+1}{2}} K_{\frac{d+1}{2}}(m|\vec{x}-\vec{y}|)$$

$$\Omega^{-1}(\vec{x},\vec{y}) = \frac{1}{\pi}\left(\frac{m}{2\pi|\vec{x}-\vec{y}|}\right)^{\frac{d-1}{2}} K_{\frac{d-1}{2}}(m|\vec{x}-\vec{y}|)$$

- 将空间拆分为两个区域 $\mathbb{R}^d = V_+ \cup V_-$，$\vec{x}_\pm \in V_\pm$，整个系统分解成 V_- 中场自由度全体 A 及 V_+ 中场自由度全体 B
- 在真空密度矩阵 ρ 中对 A 取迹即得约化密度矩阵 ρ_B；计算其本征值进而得到 S_B
- 正如谐振子情形，纠缠熵可用下列“矩阵”表达

$$\Upsilon(\vec{x}_+,\vec{y}_+) = -\int [d\vec{x}_-]\Omega^{-1}(\vec{x}_+,\vec{x}_-)\Omega(\vec{x}_-,\vec{y}_+)$$

$$= \frac{m^d}{2^{d-1}\pi^{d+1}}\int [d\vec{x}_-]\frac{K_{\frac{d-1}{2}}(m|\vec{x}_+-\vec{x}_-|)K_{\frac{d+1}{2}}(m|\vec{x}_--\vec{y}_+|)}{|\vec{x}_+-\vec{x}_-|^{\frac{d-1}{2}}|\vec{x}_--\vec{y}_+|^{\frac{d+1}{2}}}$$

- 零质量极限 $m \to 0$

$$\Upsilon(\vec{x}_+, \vec{y}_+) = \begin{cases} \dfrac{\Gamma(\frac{d+1}{2})\Gamma(\frac{d-1}{2})}{2\pi^{d+1}} \displaystyle\int \frac{[d\vec{x}_-]}{|\vec{x}_+ - \vec{x}_-|^{d-1}|\vec{x}_- - \vec{y}_+|^{d+1}} \\ -\dfrac{1}{\pi^2} \displaystyle\int dx_- \frac{\log(x_+ - x_-)}{(x_- - y_+)^2}, \quad d = 1 \end{cases}$$

- 以 $(1+1)$-维理论为例：将 $x \in \mathbb{R}$ 分解成 $x_- < 0$ 和 $x_+ > 0$ 两部分，考虑本征值问题

$$\int_0^\infty dy_+ \Upsilon(x_+, y_+)\psi(y_+) = E\psi(x_+)$$

$$\Rightarrow \quad -\frac{1}{\pi^2} \int_0^\infty dy_+ \int_{-\infty}^0 dx_- \frac{\log(x_+ - x_-)}{(x_- - y_+)^2} \psi(y_+) = E\psi(x_+)$$

$$\Rightarrow \quad -\int_0^\infty dy_+ \int_0^\infty dz_+ \frac{\log(x_+ + z_+)}{\pi^2 (y_+ + z_+)^2} \psi(y_+) = E\psi(x_+)$$

- 围道积分给出 $E = E(\omega) \equiv 1/\sinh^2(\pi\omega)$

$$\psi(x) = C_+ \exp\left(i\omega \log x\right) + C_- \exp\left(-i\omega \log x\right)$$

- 紫外截断 $x=\epsilon$ 处取 Dirichlet 边界条件

$$\psi(x) = \sin(\omega \log(x/\epsilon))$$

- 通过引进红外截断 L 将谱离散化

$$\psi(L) = 0 \;\Rightarrow\; \omega = \omega(E_n) = \frac{n\pi}{\log(L/\epsilon)}$$

- In the limit $L/\epsilon \to \infty$, the eigenvalues become dense; the eigenvalue density is defined by

$$\frac{d\mu(E)}{dE} = \sum_k \delta(E - E_k) \;\Rightarrow\; \mu(E) = \sum_k \theta(E - E_k)$$

- Let $E_0 < E_1 < E_2 < \cdots$, one has

$$\mu(E_n) = \sum_{k<n} \theta(E_n - E_k) = n = \frac{\log(L/\epsilon)}{\pi}\omega(E_n)$$

$$\Rightarrow \quad \mu(E) \approx \frac{\log(L/\epsilon)}{\pi}\omega(E) \;\Rightarrow\; \mu'(E) = \frac{\omega'(E)}{\pi}\log(L/\epsilon)$$

- 对本征值E_k 求迹:

$$\mathrm{Tr} f(\Upsilon) = \sum_k f(E_k) = \int dE \frac{d\mu(E)}{d\mu} f(E)$$

$$= \frac{\log(L/\epsilon)}{\pi} \int dE \omega'(E) f(E) = \frac{\log(L/\epsilon)}{\pi} \int d\omega f(E(\omega))$$

- 现取

$$f(E) = \log \frac{\sqrt{E}}{2} + \sqrt{E+1} \log \left(\frac{1}{\sqrt{E}} + \sqrt{1 + \frac{1}{E}} \right);$$

$$\log \frac{\sqrt{E}}{2} = -\log(2 \sinh \pi\omega) = -\pi\omega - \log(1 - e^{-2\pi\omega})$$

$$\sqrt{E+1} = 1 + \frac{2}{e^{2\pi\omega} - 1}, \quad \log \left(\frac{1}{\sqrt{E}} + \sqrt{1 + \frac{1}{E}} \right) = \pi\omega$$

- 最终得到的纠缠熵

$$
\begin{aligned}
S_B &= \frac{\log(L/\epsilon)}{\pi}\int_0^\infty d\omega\left[\frac{2\pi\omega}{e^{2\pi\omega}-1}-\log(1-e^{-2\pi\omega})\right]\\
&= \frac{\log(L/\epsilon)}{\pi}\left[2\pi\cdot\frac{1}{24}-\left(-\frac{\pi}{12}\right)\right]=\frac{1}{6}\log(L/\epsilon)
\end{aligned}
$$

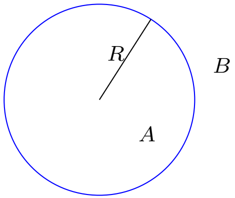
- 更一般的 CFT_2 (Calabrese and Cardy, arXiv:hep-th/0405152)

$$
S_B=\frac{c}{3}\log\left[\frac{L}{\pi\epsilon}\sin\left(\frac{\pi\ell}{L}\right)\right],\quad \ell=|B|
$$

- 可进一步考虑近视界极限下的 BTZ 黑洞，在 CFT_2 中导出 S_B (Caputa, Jejjala and Soltanpanahi, arXiv:1309.7852)
- BTZ+双区间 $A\cup B$，红外 CFT_2 的计算结果为(Bai, Xu and YHG, arXiv:1312.6374; cf. 全息计算结果 Headrick, arXiv:1006.0047)

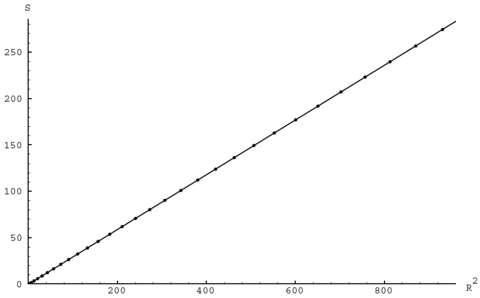
$$
I_{A:B}=\frac{c}{6}\left[\log\frac{L^2}{\ell(2L+\ell)}+\log\frac{\sinh^2 L}{\sinh(2L+\ell)\sinh\ell}\right],\quad L\gg\ell
$$

- 高维时空之例：子系统 A, B 按球内、球外的场自由度划分



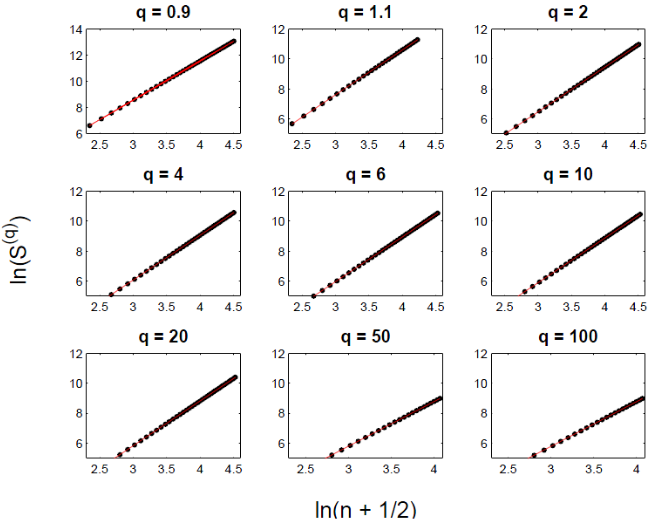
- Subsystem A: consisting of field degrees of freedom located inside a sphere $\mathcal{S}$ of radius R
- Subsystem B: {field degrees of freedom outside $\mathcal{S}$}

- Results from numerical computation: for massless scalar with $d = 3$ and $N = 60$, Srednicki finds

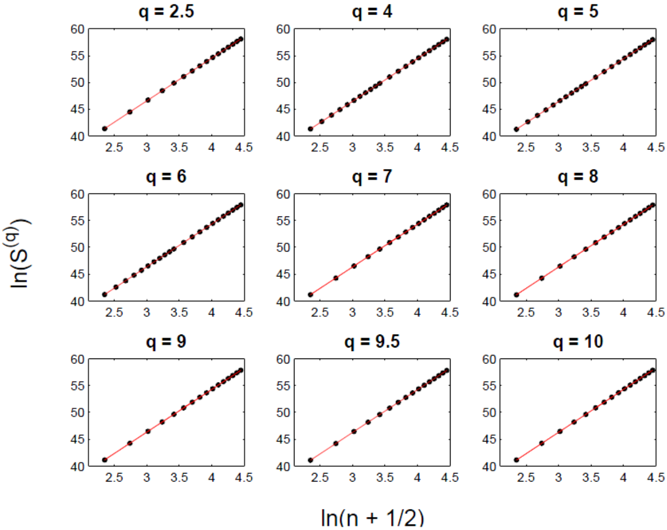


$$S \approx 0.30\epsilon^{-2}R^2$$

- Braunstein et. al. [arXiv:1110.1239] found similar numerical results, using Rényi entropy for scalars in higher dimensions
- When $d+1=5$, $S^{(q)} \sim (R/a)^3$

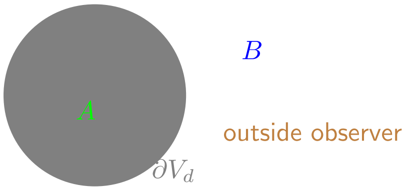


- When $d+1=10$, $S^{(q)} \sim (R/a)^8$



- Is EE related to BHE?

- Why black holes are in mixed states, and their entropy obeys the Bekenstein-Hawking area law?
  - In gravitational systems, the region inside a black hole horizon is inaccessible by outside observers; one never monitors the "black part" of degrees of freedom



  - For an outside observer the reduced density matrix $\rho_B = \mathrm{Tr}_A \rho$ may well describe a mixed state, even if the total system is in a pure state, $\rho = |\Psi\rangle\langle\Psi|$
  - The entanglement entropy S_B depends on properties shared by A and B; if S_B is geometrical in nature, $S_B \sim f(\partial V_d)$
  - Calculations in QFTs $\Rightarrow$ $S_B \propto \partial V_d$
- Callan and Wilczek (1994) suggested that entanglement entropy is the first quantum correction to a thermodynamic entropy of a black hole

- 至少对于自由场，动量空间中无纠缠
  - 依据 $|\vec{k}| < \Lambda$ 及 $|\vec{k}| > \Lambda$，将动量空间的场自由度 $\phi(\vec{k})$ 分成两组 A, B
  - 整个系统的基态波函数

$$
\begin{aligned}
\Psi_0 &\sim \exp\left[-\frac{1}{2}\int d^d\vec{k}\phi(\vec{k})\sqrt{\vec{k}^2+m^2}\phi(\vec{k})\right] \\
&\sim \left\{\bigotimes_{|\vec{k}|<\Lambda} \exp\left[-\frac{1}{2}\phi(\vec{k})\sqrt{\vec{k}^2+m^2}\phi(\vec{k})\right]\right\} \\
&\otimes \left\{\bigotimes_{|\vec{k}|>\Lambda} \exp\left[-\frac{1}{2}\phi(\vec{k})\sqrt{\vec{k}^2+m^2}\phi(\vec{k})\right]\right\} \\
&\sim \Psi_A \otimes \Psi_B
\end{aligned}
$$

  - 对 ρ 的重模式取迹，约化后仍为纯态

$$
\rho_A \sim \text{const.} \times \rho_A, \quad \rho_A = |\Psi_A\rangle\langle\Psi_A|
$$

  - 沿着重正化流不产生纠缠熵

The Replica Trick and Black Holes

- Ground states are obtainable by solving $H\Psi_0 = E_0\Psi_0$ (the Hamiltonian or real time approach); alternatively, one may invoke path-integral representation of Ψ_0
- According to Feynman-Kac formula, $\langle q_f, t_f | q_i, t_i \rangle$ is given by

$$\langle q_f | e^{-i(t_f - t_i)H} | q_i \rangle = \mathcal{N} \int_{q(t_i)=q_i}^{q(t_f)=q_f} [dq] \exp\left[i \int_{t_i}^{t_f} dt \mathcal{L}(q, \dot{q}) \right]$$

- The ground state wave function $\Psi_0(q_f, t_f)$ is determined by its initial value $\psi_0(q_i)$ at $t = t_i$

$$\begin{aligned} \Psi_0(q_f, t_f) &= \int dq_i \langle q_f | e^{-i(t_f - t_i)H} | q_i \rangle \psi_0(q_i) \\ &= \mathcal{N} \int dq_i \psi_0(q_i) \int_{q(t_i)=q_i}^{q(t_f)=q_f} [dq] \exp\left[i \int_{t_i}^{t_f} dt \mathcal{L}(q, \dot{q}) \right] \end{aligned}$$

- One may suppose interactions are adiabatically turn off when Euclidean time $it_i \to \pm\infty$

- In such a limit, one can solve the free Schrödinger equation to find the initial values of energy states

$$-\frac{1}{2m}\left(\frac{\partial}{\partial q_i}\right)^2 \psi_k(q_i) = E_k \psi_k(q_i) \;\Rightarrow\; \begin{cases} E_k = \dfrac{k^2}{2m} \\ \psi_k \sim e^{iq_i \cdot k} \end{cases}$$

- Ground state: $E_k = 0,\ \psi_0(q_i) = \mathrm{const.}$
- So $\Psi_0(q_f, t_f)$ allows a path integral representation (Euclidean formulism)

$$\begin{aligned}\Psi_0(q_f, t_f) &= \mathcal{N}' \int dq_i \int_{q(-\infty)=q_i}^{q(t_f)=q_f} [dq] \exp\left(-\int_{-\infty}^{t_f} dt\, \mathcal{L}_E\right) \\ &= \mathcal{N}' \int_{q(t_f)=q_f} [dq] \exp\left(-\int_{-\infty}^{t_f} dt\, \mathcal{L}_E\right)\end{aligned}$$

- For time-independent H, one may take $t_f = 0$

$$\Psi_0(q) \propto \int [dq] \exp\left(-\int dt\, \mathcal{L}_E\right)$$

summing over paths $q(t)$ with $t \leq 0$ and $q(0) = q$

- The density matrix $\rho(q, q') = \Psi_0^*(q)\Psi_0(q')$ is a product of two integrals of this type, one may let the Euclidean time run from 0 to $\pm\infty$, respectively, in the two integrals
- Divide whole system into two sets, $q = \{q_-^i, q_+^\alpha\}$, with inside dof q_-^i $(i \in A)$ and outside dof q_+^α $(\alpha \in B)$; the reduced density matrix is given by tracing out q_-^i

$$\rho_B(q_+, q'_+) = \int [dq_-] \Psi_0^*(q_+, q_-) \Psi_0(q'_+, q_-) \tag{1}$$

- Think of (1) as a functional integral over paths

$$q(t) = \begin{cases} (q_-^i(t), q_+^\alpha(t)), & 0 \le t < \infty \\ (q_-'^i(t), q_+'^\alpha(t)), & -\infty < t \le 0 \end{cases}$$

with b.c. $\begin{cases} q^\alpha(0^+) = q_+^\alpha \\ q'^\alpha(0^-) = q_+'^\alpha \end{cases}$ $\quad q^i(0^+) = q'^i(0^-)$ continious at $t = 0$

$$\rho_B(q_+, q'_+) = \mathcal{N} \int [dq(t)] \exp\left(-\int_{-\infty}^{\infty} dt\, \mathcal{L}_E\right)$$

- Replacing $q^i \to \phi(\vec{x})$, the reduced density matrix $\rho_B(\phi_+^1, \phi_+^2)$ in QFT is the path integral over fields defined on a singular spacetime constructed from $(A \cup B) \times (\mathbb{R}^+ \cup \mathbb{R}^-)$, by gluing $(A, 0^-)$ and $(A, 0^+)$, with b.c.

$$\phi(\vec{x}, 0^-)|_{\vec{x}\in B} = \phi_+^1, \quad \phi(\vec{x}, 0^+)|_{\vec{x}\in B} = \phi_+^2$$

- The trace $\mathrm{Tr}\rho_B^n$ is the functional integral on the n-sheeted manifold, which has a negative deficit angle $\delta = 2\pi(1-n)$ along the surface ∂A
- According to the "replica trick", entanglement entropy is computed by

$$\begin{aligned} S_B &= -\mathrm{Tr}(\rho_B \log \rho_B) = \left(-\frac{d}{dn} + 1\right) \log \mathrm{Tr}\rho_B^n \Big|_{n=1} \\ &= \left(2\pi \frac{d}{d\delta} + 1\right) \log Z_\delta \Big|_{\delta=0} \end{aligned}$$

- The general expression for thermodynamic entropy reads

$$S = -\left(\beta\frac{\partial}{\partial\beta} - 1\right) Z(\beta), \quad Z(\beta) = \mathrm{Tr} e^{-\beta H}$$

- Compare to black hole entropy: For general β, there is a $2D$ conical curvature singularity at the horizon of deficit angle $\delta = 2\pi(\beta_H - \beta)/\beta_H$; this singularity disappears at $\beta = \beta_H$ or, equivalently, at $\delta = 0$

$$S_{BH} = \left(2\pi\frac{d}{d\delta} + 1\right) \log Z_\delta \Big|_{\delta=0}$$

- So the Bekenstein-Hawking entropy and the entanglement entropy in QFTs measure similar responses of the Euclidean path integral to the introduction of a conical singularity
- Spacetime geometries are different in the two approaches: EE of QFT is computed in flat space, while BHE is calculated in curved space

- According to Callan and Wilczek (1994),
  - “If the Bekenstein-Hawking and geometrical quantum entropies are to be just different orders of approximation to the same thing, the conical geometries must be exactly the same.”
  - “We will achieve this by taking the large mass limit, in which the Schwarzschild metric goes into Rindler space. In this limit, curvatures go to zero, the area of the horizon goes to infinity (so that the quantity of interest becomes entropy per unit horizon area), and the two conical geometries match precisely”
  - “Our theorem, then, is that the (appropriately defined) geometric entropy of a free field in flat space is just the quantum correction to the Bekenstein-Hawking entropy of Rindler space. By introducing curvature in the space on which the free field lives, we could compute the quantum correction to the entropy of a finite mass black hole.”
- However, “free fields” in curved spacetime are not actually free; we will consider a coupled system
  - N. Bai, X.-B. Xu and YHG, “One-Loop Divergences and Conical Singularity Corrections of Quantized Maxwell-Dilaton Theory”

Einstein-Maxwell-Dilaton Theory

- EMD in 4-dim is obtainable from 5-dim gravitational theory by dimensional reduction, thus its field content being

$$\begin{pmatrix} g_{\mu\nu} & \kappa e^{2\sigma} A_\mu \\ \kappa e^{2\sigma} A_\mu & e^{2\sigma} \end{pmatrix}$$

4d metric	$g_{\mu\nu}$
$U(1)$ gauge potential	A_μ
dilaton	σ

- Euclidean action

$$I = \int d^4x \sqrt{g} \left[R - \frac{1}{2}(\nabla\sigma)^2 - \frac{1}{4} e^{\gamma\sigma} F_{\mu\nu} F^{\mu\nu} - 2\Lambda e^{-\delta\sigma} \right]$$

coupling constants: $\gamma,\ \delta,\ \Lambda$

- Fixing $g_{\mu\nu}$ as a classical background, consider quantum fluctuations of "matter fields" around their classical configurations

$$\sigma \to \sigma + \varphi, \quad A_\mu \to A_\mu + a_\mu$$

- 1-loop corrections to $I[g_{\mu\nu},\Phi]$:

$$\Phi = \Phi_0 + \phi, \quad \Phi_0 \text{ obeys the classical EOM}$$

$$\begin{aligned} I[\Phi] &= I[\Phi_0] + \int d^4x \, \frac{\delta I[\Phi]}{\delta \Phi(x)}\bigg|_{\Phi_0} \phi(x) \\ &\quad + \frac{1}{2}\int d^4x d^4x' \, \frac{\delta^2 I[\Phi]}{\delta\Phi(x)\delta\Phi(x')}\bigg|_{\Phi_0} \phi(x)\phi(x') + \cdots \\ &\approx \; I[\Phi_0] + \int d^4x \phi(x) \hat{\mathcal{D}}[g_{\mu\nu},\Phi_0]_x \phi(x) \end{aligned}$$

$$e^{-I_{\text{eff}}[\Phi_0]} = \int [d\phi] e^{-I[\Phi_0+\phi]} \approx e^{-I[\Phi_0]} \det[\hat{\mathcal{D}}]^{-1/2}$$

$$I_{\text{eff}}[\Phi] = I[\Phi] + \frac{1}{2}\log\det[\hat{\mathcal{D}}] = I[\Phi] + \frac{1}{2}\mathrm{Tr}\log\hat{\mathcal{D}}$$

- Heat kernel method: introducing the Schwinger proper time s

$$\left(\frac{\partial}{\partial s} + \hat{\mathcal{D}}_{x'}\right) K(x,x';s) = 0, \quad K(x,x';s) \sim \langle x | e^{-s\hat{\mathcal{D}}} | x' \rangle$$

- Schwinger proper time representation

$$\sum_j e^{-s\lambda_j} = \mathrm{Tr} e^{-s\hat{\mathcal{D}}} = \int d^4x \sqrt{g} K(x, x; s)$$

$$\sum_j \log \lambda_j \sim -\sum_j \int_0^\infty \frac{ds}{s} e^{-s\lambda_j}$$

$$\Delta I = -\frac{1}{2} \int_0^\infty \frac{ds}{s} \int d^4x \sqrt{g} K(x, x; s)$$

$$\Delta \mathcal{L} = -\frac{1}{2} \int_0^\infty \frac{ds}{s} K(x, x; s)$$

- If $\hat{\mathcal{D}} = -\Delta_{\mathbb{R}^n} + m^2$

$$K(x, x'; s) = \frac{1}{(4\pi s)^{n/2}} \exp\left(-\frac{(x - x')^2}{4s} - m^2 s \right)$$

- In general, for small proper time s (with $b_0 = 1$, $b_j \Rightarrow b_{j+1}$)

$$K(x, x'; s) \sim \frac{1}{(4\pi s)^{n/2}} e^{-\frac{(x-x')^2}{4s} - m^2 s} \sum_{j=0}^\infty b_j(x, x') s^j$$

- UV divergence comes from contributions at $s = \epsilon^2 \approx 0$; terms with $j > n/2$ are UV regular, so in 4-dim the divergent terms are indexed by $j = 0, 1, 2$

$$(\Delta\mathcal{L})_{\rm div} = -\frac{1}{32\pi^2}\left(\frac{1}{2}a_0\epsilon^{-4} + a_1\epsilon^{-2} + 2a_2\log\frac{\Lambda_{\rm IR}}{\epsilon}\right)$$

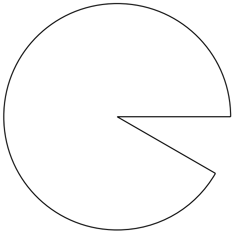
- Applying to EMD theory
  - Due to gauge invariance, the propagator $\hat{\mathcal{D}}^{-1}$ is ill-defined; one has to fix gauge and add ghost degrees of freedom

$$\mathcal{L}_{\rm gf} = -\frac{1}{2}e^{\gamma\sigma}(\nabla_\mu a^\mu)^2, \quad \mathcal{L}_{\rm ghost} = c^*\Box c$$

$$\Delta\mathcal{L} = \frac{\sqrt{g}}{16\pi^2(d-4)}\Bigg\{\Big[-\frac{\gamma^4}{4} - \frac{\gamma^2}{8} - \frac{7}{120}\Big]R^2 + \Big[\frac{3\gamma^4}{8} - \frac{\gamma^2}{24} + \frac{2}{15}\Big]R_{\mu\nu}^2$$
$$+\Big[\frac{5\gamma^4}{8} + \delta\gamma^3 + \frac{11\gamma^2}{12} - \frac{\delta^2}{6}\Big]\Lambda e^{-\delta\sigma}R + \Big[-\frac{5\gamma^4}{2} - 4\delta\gamma^3 + (\delta^2-2)\gamma^2 + \delta^4\Big]$$
$$\times\Lambda^2 e^{-2\delta\sigma} + \Big[-\frac{3\delta\gamma^3}{32} + \frac{\gamma^4}{2}\Big]\Lambda e^{(\delta-\gamma)\sigma}F^2 + \Big[-\frac{17\gamma^4}{192} - \frac{\gamma^2}{32}\Big]e^{2\gamma\sigma}F^2$$
$$+\Big[-\frac{5\gamma^4}{32} + \frac{\gamma^2}{96}\Big]e^{\gamma\delta}RF^2 + \Big[\frac{7\gamma^4}{384} - \frac{\gamma^2}{128}\Big]e^{2\gamma\sigma}(F^2)^2 - \frac{\gamma^2}{24}e^{\gamma\delta}R_{\mu\nu\rho\lambda}F^{\mu\nu}F^{\rho\lambda}\Bigg\}$$

- Conical singularity
  - 2-dim cone C_α

$$dr^2 + r^2 d\varphi^2, \quad \varphi \sim \varphi + 2\pi\alpha, \quad \alpha \neq 1 \;\Rightarrow\; \text{singular at } r = 0$$



  - The identification $\varphi \leftrightarrow \varphi + \alpha$ not only topological, but also geometrical; this requires the existence of a 1-parameter isometric group Γ, $\varphi \to \varphi + \omega$, generated by a Killing vector
  - The tip at $r = 0$ is the fixed point of this isometric group
- n-dim manifold M_α with conical singularity
  - $\exists\Gamma$, its fixed points form an $(n-2)$-dim surface Σ
  - $r \approx 0 \Rightarrow M_\alpha \approx C_\alpha \times \Sigma$, and the metric is the product metric

- Writing $x^\mu = (u, \varphi)$; Sommerfeld integral representation:

$$
\begin{aligned}
& K_{M_\alpha}(u, u', \varphi - \varphi'; s) \\
\sim \quad & \frac{i}{2\alpha} \int_C d\omega \cot(\pi \alpha^{-1} \omega) K_M(u, u', \varphi - \varphi' + \omega; s)
\end{aligned}
$$

- Proper time expansion coefficients $a_n = a_n^{\mathrm{reg}} + a_{\alpha,n}$

$$
\begin{aligned}
a_{\alpha,0} &= 0 \\
a_{\alpha,1} &= \frac{\pi}{3} \frac{(1-\alpha)(1+\alpha)}{\alpha} \int_\Sigma 1 \\
a_{\alpha,2} &= \frac{\pi}{3} \frac{(1-\alpha)(1+\alpha)}{\alpha} \int_\Sigma \Big[\Big(\gamma^2 - \frac{1}{6} \Big) R \\
& \quad + (4\Lambda\gamma(\delta - \gamma) - 2\Lambda\delta^2) e^{-\delta\sigma} - \frac{1}{2} \gamma^2 e^{\gamma\sigma} F^2 \Big] \\
& \quad \frac{\pi}{180} \frac{(1-\alpha)(1+\alpha)(1+\alpha^2)}{\alpha^3} \\
& \quad \times \int_\Sigma \Big(R_{\mu\nu} n_i^\mu n_j^\nu - 2 R_{\mu\nu\rho\lambda} n_i^\mu n_j^\nu n_i^\rho n_j^\lambda \Big)
\end{aligned}
$$

- Taking Σ_α to be a black hole solution in EMD theory, with $\alpha = \beta/\beta_H$ (Chan, Horne and Mann, gr-qc/9502042)

$$ds^2 = \beta^2 g(r) d\varphi^2 + \frac{dr^2}{g(r)}$$

- 纠缠熵

$$\begin{aligned} S_{con} &= \frac{A_\Sigma}{48\pi\epsilon^2} + \left(\frac{A_\Sigma r_h^{-2-\frac{2}{1+\gamma^2}}}{60\pi\left(\gamma^4-1\right)} \left(M r_h + \left(Q^2 - 2M r_h + 2r_h^2\right)\gamma^2 + \left(-Q^2 + r_h M + r_h^2\right)\gamma^4\right) \right) \\ & \times \log\frac{\tilde{\Lambda}}{\epsilon} + \frac{A_\Sigma}{24\pi\left(-1+\gamma^2\right)} r_h^{-3-\frac{1}{1+\gamma^2}} \left(-2r_h^3 + \left(\gamma^2 - \gamma^4\right)\left(Q^2 + 4r_h^3\right)\right) \log\frac{\tilde{\Lambda}}{\epsilon} \\ &+ \frac{A_\Sigma r_h^{-2-\frac{2}{1+\gamma^2}}}{72\pi\left(-1+\gamma^4\right)} \left(-\gamma^2 + 6\gamma^4\right)\left(2M r_h - Q^2 + Q^2\gamma^2 + r_h\left(r_h - 2M\gamma^2 + 2r_h\gamma^2\right)\right) \log\frac{\tilde{\Lambda}}{\epsilon} \\ &+ \frac{1}{18}\log\frac{\tilde{\Lambda}}{\epsilon} \end{aligned} \tag{4.70}$$