Recent Developments in Entanglement (Rényi) entropy

陈斌

北京大学物理学院

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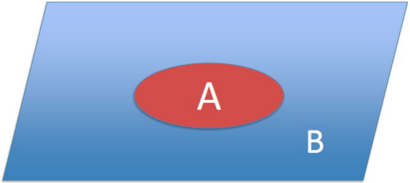
- T. Barrella, X. Dong, S. A. Hartnoll, and V. L. Martin, "Holographic entanglement beyond classical gravity," arXiv:1306.4682 [hep-th].
- Earlier works by J. Cardy et.al., D. Fursaev, M. Headrick and many others.

Outline

- Review of Entanglement entropy(EE) and Rényi entropy(RE)
- Replica trick
- Holographic Entanglement entropy
- Generalized gravitational entropy
- Rényi entropy in AdS_3/CFT_2
- Einstein equation from HEE
- Conclusion and discussions

Entanglement entropy

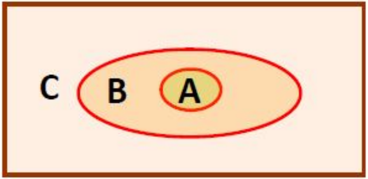


- Divide the system to be A and B such that $\mathcal{H}_{tot} = \mathcal{H}_A \otimes \mathcal{H}_B$ and $\rho_{tot} = |\Psi\rangle\langle\Psi|$
- Reduced density matrix: $\rho_A = \mathrm{tr}_B \rho_{tot}$
- von Neumann entanglement entropy: $S_A = -\mathrm{tr}\rho_A \ln \rho_A$
- It is the entropy for an observer who is only accessible to A and not to B
- Simplest case: two spin system
  1. $|\Psi\rangle = (|\uparrow\rangle_A + |\downarrow\rangle_A) \otimes (|\uparrow\rangle_B + |\downarrow\rangle_B)/2 \Rightarrow S_A = 0$
  2. Entangled state: $|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A \otimes |\downarrow\rangle_B + |\downarrow\rangle_A \otimes |\uparrow\rangle_B +) \Rightarrow S_A = \log 2$

Properties

- For a pure state $|\Psi\rangle$, $S_A = S_B$, otherwise $S_A \neq S_B$
- The thermal entropy could be obtained as a particular case of EE, just taking A as the whole system
- Subadditivity: $S_{A+B} \leq S_A + S_B$
- Strong subadditivity(SSA): Lieb-Ruskai 1973

$$S_{A+B+C} + S_B \leq S_{A+B} + S_{B+C} \tag{1.1}$$



Physical implication

- It is hard to be observed directly in Lab.
- It has been computed numerically in CM systems: spin chains, lattice models, ...
- Encodes valuable information of the system: dynamical d.o.f.
- Various applications: as quantum order parameter in CM, characterize non-equilibrium states,...
- A bridge between gravity and QFT, in particular CFT (as we will see soon)
- A new window to study AdS/CFT correspondence, especially AdS_3/CFT_2.

Rényi entropy

- More generally one can define the Rényi entanglement entropy, or in short the Rényi entropy, of A and B as

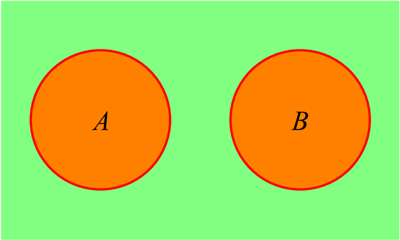
$$S_A^{(n)} = -\frac{1}{n-1} \log \mathrm{Tr}_A \rho_A^n.$$

- It is easy to see that the entanglement entropy and the Rényi entropy are related by

$$S_A = \lim_{n \to 1} S_A^{(n)}.$$

- The relation provides a practical way to compute EE (Recall that $S_A = -\mathrm{tr} \rho_A \ln \rho_A$)

Rényi mutual information



- Choose two subsystems A and B which are not necessarily each other's complement
- Define the Rényi mutual information of A and B

$$I^{(n)}_{A,B} = S^{(n)}_{A} + S^{(n)}_{B} - S^{(n)}_{A+B}.$$

- For $n = 1$, it is called mutual information, which measures the entanglement between A and B: two entangled systems are correlated because they share an amount of information that is not foreseen classically
- From subadditivity, we know $I(A, B) \geq 0$

EE in QFT

- The vacuum in QFT is highly entangled
- Consider a QFT on a $(d+1)$-dim. manifold $R \times M$, where R is time direction
- Subsystem: a d-dim. submanifold $A \in M$ at a fixed time
- In this case, the EE S_A is called the geometric entropy as it depends on the geometry of A L.Bombelli et.al. 1986, M. Srednicki 9304048

$$S_A = \gamma \frac{\text{Area}(\partial A)}{\epsilon^{d-1}} + \text{subleading terms}$$

where ∂A is the boundary of A, ϵ is the UV cutoff and γ is a constant depending on the system

- This suggests that entanglement between A and B occurs at the boundary most strongly

Remarks

- The area law is for the local QFT, could be proved rigorously for free field theories Plenio et.al. 2004,2005
  1. It holds for both ground state states and finite temperature systems
  2. It is violated for highly excited states
  3. Two exceptions: 2D CFT and QFT with Fermi surfaces
  4. Volume law in non-local QFT Shiba et.al. 2013
- The Rényi entropy could be defined similarly
- In a sense, the entanglement entropy is a generalization of "Wilson loop"
- It is really hard to compute in QFT, even for free field theory

Replica trick

- The standard way is to use replica trick J. Callan et.al. 9401072
- Here, we only focus on the 2D CFT, which provides more analytic results
- In Euclidean path-integral, the ground state wave-functional is represented by Figures from T. Takayanagi's lecture in 7th Asian winter school

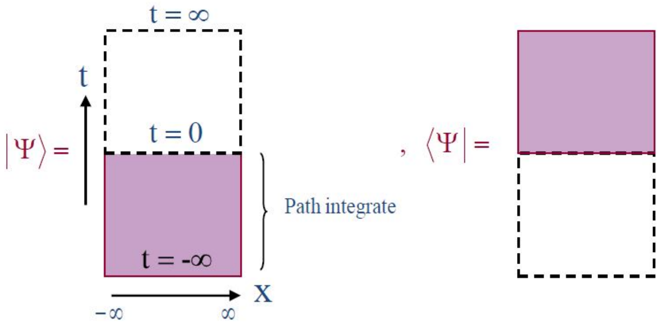
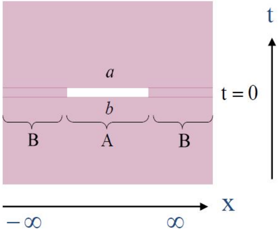


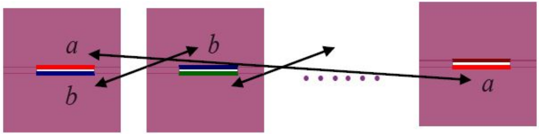
Figure: cf. T. Takayanagi

Replica trick II

$[\rho_A]_{ab} =$

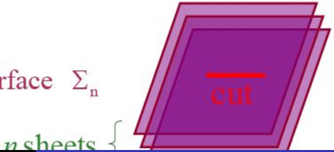


$\mathrm{Tr}(\rho_A)^n =$



= a path integral over

n-sheeted Riemann surface Σ_n



Replica trick III

- Replica trick: computation in product orbifold $(\mathrm{CFT})_n/Z_n$
- Branch points: twist operators with dimension

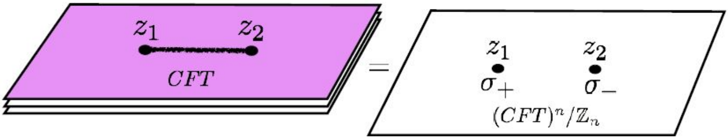
$$h = \bar{h} = \frac{c}{24}\left(n - \frac{1}{n}\right).$$

- One interval case

$$\mathrm{Tr}\rho_A^n = \langle \sigma(\ell,\ell)\tilde{\sigma}(0,0)\rangle_C = c_n \ell^{-\frac{c}{6}\left(n-\frac{1}{n}\right)},$$

from which the Rényi entropy for one interval could be read P. Calabrese and J.L. Cardy 0405152

$$S_n = \frac{c}{6}\left(1 + \frac{1}{n}\right)\log\frac{\ell}{\epsilon},$$

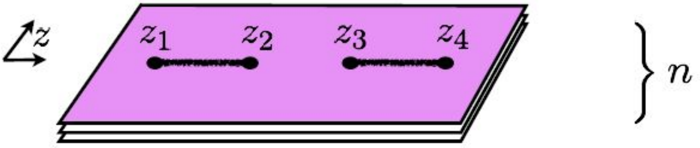


Multi-intervals

- In the case of N intervals, there are more branch cuts so that the Riemann surface is of genus $(n-1)(N-1)$, where n is the number of replica
- If we have multiple intervals $A = [z_1, z_2] \cup \cdots \cup [z_{2N-1}, z_{2N}]$,

$$\mathrm{Tr}\rho_A^n = \langle \sigma(z_{2N}, \bar{z}_{2N})\tilde{\sigma}(z_{2N-1}, \bar{z}_{2N-1}) \cdots \sigma(z_2, \bar{z}_2)\tilde{\sigma}(z_1, \bar{z}_1)\rangle_C.$$

- It is very difficult to compute (partition function on higher genus RS)
- Nevertheless, in the case that the intervals are short, we may use operator product expansion(OPE) to compute



- Even in 2D field theory, the computation is very difficult: partition function on a higher genus Riemann surface
- A few exceptions in 2D CFT:
  - One single interval C. Holzhey et.al. 9403108

$$S_A = \frac{c}{3} \log \frac{l}{\epsilon}$$

    where c is the central charge
  - The situations of a compactified circle or an infinite system at finite temperature
  - Rényi entropy P. Calabrese and J.L. Cardy 0405152

$$S_n = \frac{c}{6}\left(1+\frac{1}{n}\right) \log \frac{\ell}{\epsilon},$$

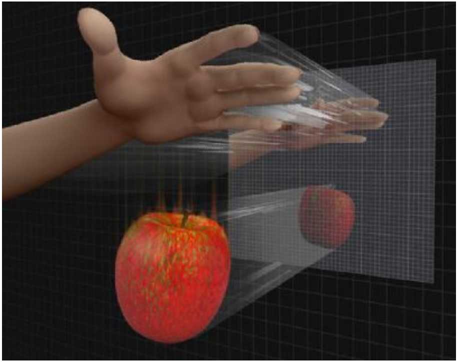
  - A free boson on a compactified circle
  - Ising model
- In higher dim., very limited knowledge
- Asking help from gravity, in the light of AdS/CFT correspondence

Holographic principle in quantum gravity

- Black hole entropy: (in Einstein gravity) Bekenstein-Hawking 1970s

$$S = \frac{k_B c^3}{4 G_N \hbar} Area(Horizon) \tag{2.1}$$

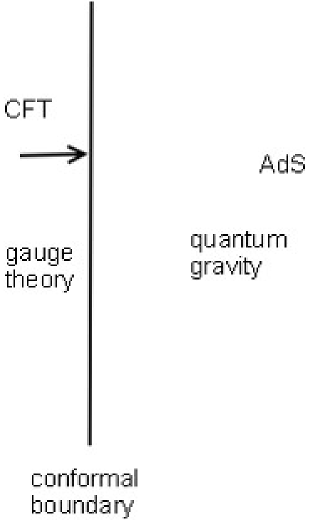
- Holographic principle: quantum gravity in any volume is naturally formulated in terms of d.o.f. on its surface, one per Planck area 't Hooft 1993, L. Susskind 1994



AdS/CFT correspondence

Quantum gravity in AdS spacetime is dual to a CFT at AdS boundary J. Maldacena 1997

- A concrete realization of holographic principle
- A new definition of quantum gravity
- Strong-weak duality: provides new way to study the strong coupling problem in QFT $\Rightarrow$ AdS/QCD, AdS/CMT etc.

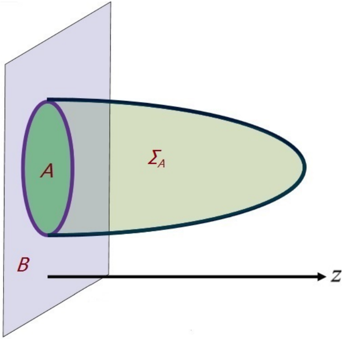


Holographic entanglement entropy Ryu and Takayanagi 2006

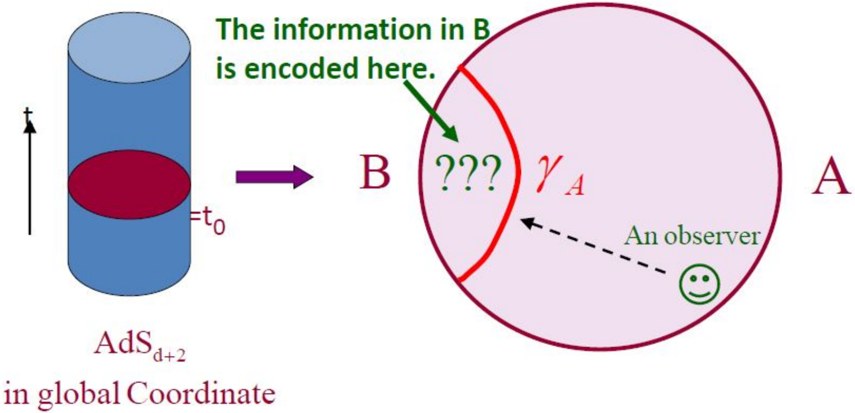
- AdS/CFT: A field theory could be holographically described by a higher-dim. gravity
- Ryu and Takayanagi(2006): Find a codimension two minimal surface Σ_A in the bulk that is homogeneous to A
- The entanglement entropy (for Einstein gravity)

$$S_A = \frac{\mathrm{Area}(\Sigma_A)}{4G_N}$$

- The area law is reminiscent of black hole entropy

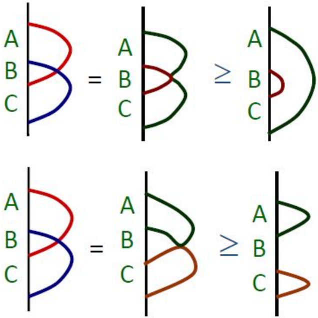


Motivation of EE



Remarks on HEE

- RT formula has passed some nontrivial tests
  1. Satisfies the area law from its definition
  2. Reproduce one interval EE in 2D CFT
  3. Conformal anomaly
  4. Obeys SSA: $S_{A+B+C} + S_B \leq S_{A+B} + S_{B+C}$



$$\Rightarrow S_{A+B} + S_{B+C} \geq S_{A+B+C} + S_B$$

$$\Rightarrow S_{A+B} + S_{B+C} \geq S_A + S_C$$

Generalization

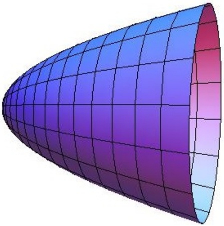
- It has been intensely studied since its proposal
  1. Covariant RT for dynamical spacetime: extremal surfaces in the Lorentzian spacetime. If there are more than one extremal surfaces, pick the one with smallest area. Hubeny et.al. 2007
  2. In the presence of black hole: the minima surfaces may wrap the horizon, in which the thermal Bekenstein-Hawking entropy contributes to the EE, so that $S_A \neq S_B$
  3. Higher curvature case Huang et.al. 2011, de Boer et.al. 2011
  4. High spin gravity de Boer et.al. 2013, Ammon et.al. 2013
- Central issue: how to prove it?

Proof of HEE

- In $2+1$ dimension, RT formula has been proven recently by T. Hartman (1303.6955) and T. Faulkner (1303.7221) independently
- Moreover, the quantum corrections to the HRE has been computed in T. Barrella et.al. (1306.4682)
- Such corrections have been confirmed by direct CFT computation BC and J-j. Zhang (1309.5453)
- In higher dimension ($d \geq 3$), it has been shown recently by A. Lewkowycz and J. Maldacena (1304.4926) from generalized gravitational entropy point of view (see also D.V. Fursaev (0606184) and H. Casini et.al. (1102.0440))
- The basic idea is related to the conical singularity method in computing the BH entropy

Black hole entropy

- The Euclidean black hole $ds^2 = r^2 d\tau^2 + dr^2 + \cdots$



- $\tau \sim \tau + 2\pi$ to make geometry regular
- Correspondingly the temperature is just $T = T_H$
- A nice way to derive the Bekenstein-Hawking entropy is to use the conical singularity method, which could be understood from the replica trick

Replica trick Fursaev-Solodukhin 9501127

- von Neumann entropy

$$S = -\mathrm{tr}\rho \ln \rho = -(\partial_n - 1) \ln \mathrm{tr}\rho^n|_{n=1}$$

- With the partition function $Z_n = \mathrm{tr}\rho^n$

$$S = -n\partial_n(\log[Z(n)] - n\log[Z(1)])_{n=1}$$

- Classically, the saddle point approximation gives $\ln Z_n = -I_n$
- In order to use the above formula to compute entropy, one has to consider the spacetime with a conical singularity, $\tau \sim \tau + 2\pi n$
- Near the singularity, the spacetime is a product $C_n \times \Sigma$, where Σ is the horizon surface. Near Σ

$$\tilde{R}_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + 2\pi(1-n)\epsilon_{\mu\nu}\epsilon_{\rho\sigma}\delta_\Sigma$$

where $n_{(i)}$ are two orthonormal vectors orthogonal to the horizon surface Σ and

$$\epsilon^{\mu\nu} = n^\mu_{(1)}n^\nu_{(2)} - n^\mu_{(2)}n^\nu_{(1)}$$

- From the action of the classical solution, one may read the entropy

Wald formula

For a general gravity action $I = -\int_{\mathcal{M}} d^{d+1}x\sqrt{g}L$, the black hole entropy could be derived by using the same trick. This leads to famous Wald formula

$$S_W = 2\pi \int_{\Sigma} d^{d-1}y\sqrt{h}\frac{\partial L}{\partial R_{\mu\nu\rho\sigma}}\epsilon_{\mu\nu}\epsilon_{\rho\sigma}$$

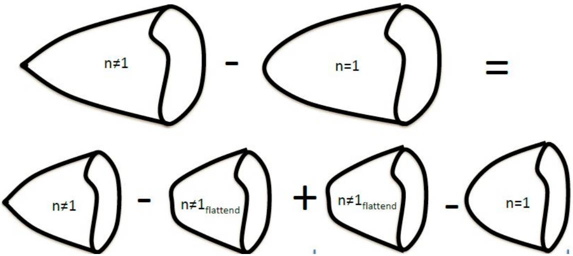
- Actually, in the case of stationary black hole, there is always a U(1) Killing symmetry along τ such that we are allowed to consider an infinitesimal deficit angle
- Only valid when the extrinsic curvatures of the embedding of Σ in $\mathcal{M}$ vanish

Generalized gravitational entropy(GGE) Lewkowycz-Maldacena 1304.4926

- Different from BH, the boundary has no U(1) symmetry
- One may still apply the replica trick to compute the entropy Fursaev 0606184
- Corresponding to the boundary manifold after n replica, there could be a bulk configuration M_n, which may be not well-defined Headrick 1006.0047
- The difference between smooth geometry corresponding to $tr_A \rho_A^n$ and the singular geometry resulted from orbifolding is of order $\mathcal{O}((n-1)^2)$, due to Einstein eq.
- Therefore $S_{GGE} = S_{EE}$
- For Rényi entropy, one has to find smooth geometry, as we will show

Generalized gravitational entropy II

- Instead of working with M_n, one may work directly with its orbifold M_n/Z_n directly
- As the boundary of M_n keeps the replica symmetry, after orbifolding, it is the same as the original boundary
- But in the bulk, the Z_n fixed points form a co-dim. 2 surface Σ_n with opening angle $2\pi/n$
- Question: how to determine the Σ_n, especially at $n \to 1$ limit?

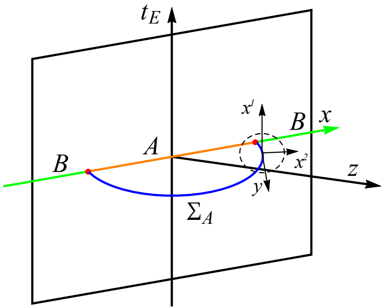


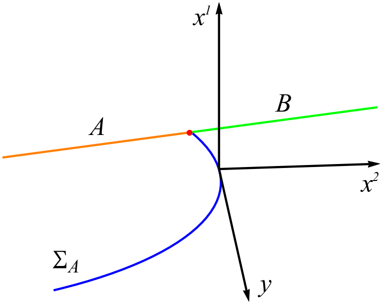
Cosmic string(brane) method I

- As $n \to 1$, the spacetime is produced by a light cosmic string, which induce a conical singularity
- At the vicinity of a hypersurface, the metric could be locally

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta + (h_{ij} + 2x^\alpha K_{(\alpha)ij}) dy^i dy^j + \cdots$$

with $g_{\alpha\beta} dx^\alpha dx^\beta = (dx^1)^2 + (dx^2)^2 = dr^2 + r^2 d\phi^2 = dz d\bar{z}$





Cosmic string(brane) method II

- There could be extrinsic curvatures for the embedding of Σ in $\mathcal{M}$,
- Replica trick $n = 1 + \epsilon$ with ϵ being infinitesimally small
- Conical singularity localized on Σ_n
- Squashed Conical geometry

$$ds^2 = e^{2\rho} g_{\alpha\beta} dx^\alpha dx^\beta + (h_{ij} + 2x^\alpha K_{(\alpha)ij}) dy^i dy^j + \cdots$$

with $\rho = -\epsilon \ln r = -\frac{\epsilon}{2} \ln(z\bar{z})$

- Expand around both $\epsilon = 0$ and $r = 0$, and focus on $\partial_\alpha \rho \sim \frac{\epsilon}{r}$ terms in zz and $\bar{z}\bar{z}$ components of Einstein equation

$$8\pi T_{zz} = 2K_{(z)} \partial_z \rho + \cdots$$
$$8\pi T_{\bar{z}\bar{z}} = 2K_{(\bar{z})} \partial_{\bar{z}} \rho + \cdots$$

- This gives the minimal area condition $K_{(\alpha)} = 0, \quad \alpha = 1, 2$

Proof of RT formula

- As in BH case, the nontrivial contribution to the EE is from the boundary of Σ_1
- The Einstein-Hilbert action of the configuration gives RT formula

$$S_{EE} = \frac{\text{Area of } \Sigma_1}{4G_N}$$

- From the variation of the RT functional, we certainly obtain the minimal surface condition

Motivation

- In Einstein gravity, GGE = HEE
- One interesting question: higher derivative gravity?
  1. Such terms appear as α'-correction in string theory
  2. For black hole, the entropy is given by the Wald functional, or equivalently Jacobson-Myers functional
  3. However, for HEE, the situation is less clear
  4. It is even more unclear for GGE
- For simplicity, let us focus on Gauss-Bonnet(GB) gravity, even though all the arguments could be applied to Lovelock gravity without trouble

Gauss-Bonnet gravity

- The action and the equation of motion are

$$I_{GB} = -\frac{1}{16\pi G}\int_{\mathcal{M}} d^{d+1}x\sqrt{g}\,[R - 2\Lambda + \lambda L_{GB}] + \cdots$$
$$L_{GB} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$$

$$R_{\mu\nu} + 2\lambda(RR_{\mu\nu} - 2R_{\mu\rho}R_{\nu}{}^{\rho} - 2R^{\rho\sigma}R_{\rho\mu\sigma\nu} + R_{\mu\rho\sigma\lambda}R_{\nu}{}^{\rho\sigma\lambda})$$
$$-\frac{1}{2}g_{\mu\nu}\left[R - 2\Lambda + \lambda(R^2 - 4R_{\rho\sigma}R^{\rho\sigma} + R_{\rho\sigma\lambda\tau}R^{\rho\sigma\lambda\tau})\right] = 8\pi G T_{\mu\nu}$$

- There are at most second derivatives of metric
- Black hole entropy from Wald formlula

$$S_{GB} = \frac{1}{4G}\int_{\Sigma} d^{d-1}y\sqrt{h}\,(1 + 2\lambda\mathfrak{R})$$

- We call it Wald functional

- Definition of the projected curvature

$$\mathfrak{R}_{\mu\nu\rho\sigma} \equiv h_{\mu}^{\lambda} h_{\nu}^{\tau} h_{\rho}^{\kappa} h_{\sigma}^{\omega} R_{\lambda\tau\kappa\omega}$$
$$\mathfrak{R}_{\mu\nu} \equiv h^{\rho\sigma} \mathfrak{R}_{\rho\mu\sigma\nu}$$
$$\mathfrak{R} \equiv h^{\mu\nu} \mathfrak{R}_{\mu\nu}$$

- The induced metric

$$h_{\mu\nu} = g_{\mu\nu} - n_{(1)\mu} n_{(1)\nu} - n_{(2)\mu} n_{(2)\nu}$$

- Black hole horizon has vanishing extrinsic curvature
- Black hole entropy is just

$$S_{GB} = \frac{1}{4G} \int_{\Sigma} d^{d-1} y \sqrt{h} (1 + 2\lambda \mathcal{R})$$

- $\mathcal{R}$ is the intrinsic curvature of Σ
- It can be got from Hamiltonian method Jacobson-Myers 9305016
- We call it Jacobson-Myers functional

HEE in higher curvature gravity

- The area of Σ is expected to be replaced by another functional
- Wald functional or Jacobson-Myers functional?
- Or another functional with differences proportional to extrinsic curvatures
- It was not clear for general higher curvature gravity, few months ago
- For GB and more general Lovelock gravity it was suggested that it was Jacobson-Myers functional, rather than Wald functional, which gives HEE

  de Bore-Kluxizi-Parnachev 1101.5781, Hung-Myers-Smolkin 1101.5813
- JM functional reproduced successfully the universal contribution to EE for CFT, relating to the trace anomaly
- Σ could be not a minimal surface, what is it?

Σ in Gauss-Bonnet gravity BC and JJ. Zhang 1305.6767

- The functional for HEE is

$$S_{GB} = \frac{1}{4G}\int_{\Sigma} d^{d-1}y\sqrt{h}(1+2\lambda\mathcal{R}) + \frac{\lambda}{G}\int_{\partial\Sigma} d^{d-2}y\sqrt{\sigma}\mathcal{K}$$

- The variation of the functional gives Σ
- Evaluate the functional at Σ gives the HEE
- The embedding of Σ in $\mathcal{M}$ is given by $x^\mu = X^\mu(y)$
- The induced metric on Σ is $h_{ij} = \partial_i X^\mu \partial_j X^\nu g_{\mu\nu}$
- The extremal condition of the Jacobson-Myers functional for Gauss-Bonnet gravity is equivalent to

$$\left[h^{ij} - 4\lambda\left(\mathcal{R}^{ij} - \frac{1}{2}\mathcal{R}h^{ij}\right)\right]K_{(\alpha)ij} = 0, \quad \alpha = 1,2$$

- Setting $\lambda = 0$ we could get the minimal surface condition

Constraint equation from replica trick BC and JJ. Zhang 1305.6767

- Assume the metric near the string behaves as before

$$ds^2 = e^{2\rho} g_{\alpha\beta} dx^\alpha dx^\beta + (h_{ij} + 2x^\alpha K_{(\alpha)ij}) dy^i dy^j + \cdots$$

with $\rho = -\epsilon \ln r = -\frac{\epsilon}{2} \ln(z\bar{z})$

- Expand around both $\epsilon = 0$ and $r = 0$, and focus on $\partial_\alpha \rho \sim \frac{\epsilon}{r}$ terms in zz and $\bar{z}\bar{z}$ components of Einstein equation
- In the end, we obtain the constraint equation

$$\left[h^{ij} - 4\lambda \left(\mathfrak{R}^{ij} - \frac{1}{2} \mathfrak{R} h^{ij} \right) \right] K_{(\alpha)ij} = 0, \quad \alpha = 1, 2$$

- In maximal symmetry cases this gives minimal surface
- Compare with equation from Jacobson-Myers functional

$$\left[h^{ij} - 4\lambda \left(\mathcal{R}^{ij} - \frac{1}{2} \mathcal{R} h^{ij} \right) \right] K_{(\alpha)ij} = 0, \quad \alpha = 1, 2$$

- They are obviously different for general non-minimal surface

Remarks

- Obviously, the constraint equations from replica trick and JM functional are different
- In the case of static geometry, the difference between two constraint equations becomes

$$-2\lambda\left[K_{(2)}K_{(2)}K_{(2)}-3K_{(2)}K_{(2)ij}K_{(2)}^{ij}+2K_{(2)ij}K_{(2)}^{jk}K_{(2)k}{}^{i}\right]$$

- This is consistent with the result from another groupA. Bhattacharyya et.al. 1305.6694
- When the cubic terms of extrinsic curvatures are much smaller than the linear term, the difference is negligible
- Our results do not contradict with the results in Myers et.al. 1101.5813
- It seems that GGE and HEE(Jacobson-Myers functional) are in conflict in Lovelock gravity

Regularized squashed cone

- Come back to the metric:

$$ds^2 = e^{2\rho} g_{\alpha\beta} dx^\alpha dx^\beta + (h_{ij} + 2x^\alpha K_{(\alpha)ij}) dy^i dy^j + \cdots$$

with $\rho = -\epsilon \ln r = -\frac{\epsilon}{2} \ln(z\bar{z})$

- It turns out that this "regularized" squashed cone metric is not regular enough
- It has curvature singularity near $r = 0$
- One has to make further regularization on the extrinsic curvature partsFursaev et.al. 1306.4000

$$ds^2 = \frac{r^2 + b^2 n^2}{r^2 + b^2} dr^2 + r^2 d\tau^2 + (h_{ij} + r^n \cos(\tau) K_{(r)ij} + r^n \sin(\tau) K_{(\tau)ij})$$

- With this regularized metric, the e.o.m. of cosmic string is exactly the same as the one read from JM functionalA. Bhattacharyya et.al. 1308.5748
- In Lovelock gravity, **GGE = HEE**!

Further development

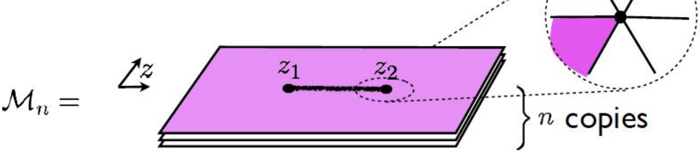
- The key point in the above treatment is the regularized squashed conic geometry near the embedded surface
- The regularization factor before the extrinsic curvature is essential
- More interestingly, with such regularization, people proposed the functional for the HEE in other higher curvature gravity Fursaev et.al. 1306.4000, Xi Dong 1310.5713
- In particular, the functional could be more conveniently written as the Wald functional plus correction terms depending on the extrinsic curvatures

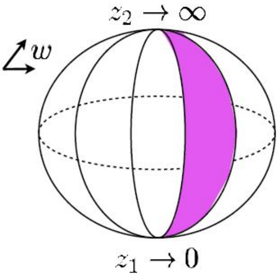
Rényi entropy in AdS_3/CFT_2

- More analytic results for Rényi entropy in AdS_3/CFT_2
- On gravity side, the configurations with replica symmetry at the boundary could be constructed explicity
- On the CFT side, 2D CFT is more tractable, and has been studied for some time
- The precise match is possible

One-interval case

Branched covering:





Conformal transformation:

$$w^n = \left(\frac{z - z_1}{z - z_2}\right)$$

$$Z(\mathcal{M}_n) \sim Z(\text{sphere}) \quad ??$$

Almost but not quite!

Rényi entropy of a single interval

- Recall that

$$ds^2 = dzd\bar{z} = e^{2\phi} dwd\bar{w} = e^{2\phi} d\hat{s}^2$$

- The partition function is

$$Z(ds^2) = e^{S_L(\phi)} Z(d\hat{s}^2)$$

where

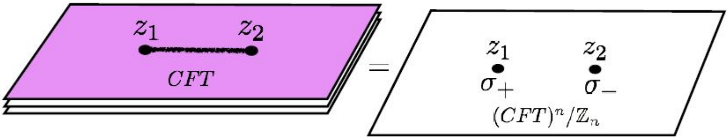
$$S_L = \frac{c}{6}\left(n - \frac{1}{n}\right)\ln(|z_1 - z_2|/\epsilon)$$

- This is nothing but the Weyl anomaly
- It leads to the Rényi entropy of a single interval

$$S_n = \frac{c}{6}\left(1 + \frac{1}{n}\right)\ln(|z_1 - z_2|/\epsilon)$$

- The similar idea has been applied to the computation of semi-classical partition functions of various gravitational configurations in AdS_3 gravity Krasnov(2000), Zograf and Takhtadzhyan(1988)

Another way I



$$Z_{\mathcal{M}_n}(ds^2) = <\sigma_+(z_1)\sigma_-(z_2)> |_{(CFT)^n/Z_n}$$

- On a complex plane, $< T(w) >_{\text{w-plane}} = 0$, therefore,

$$< T(z) >_{\text{z-plane}} = \frac{c}{12}\{w, z\}$$

where the Schwarzian is defined as

$$\{w, z\} = \frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2$$

Another way II

- As a result

$$n < T(z) >_{\text{z-plane}} = \sum_{i=1,2} \frac{h_n}{(z-z_i)^2} + \frac{\gamma_i}{(z-z_i)}$$

with

$$h_n = \frac{c}{12}(n - n^{-1}), \qquad \gamma_1 = -\gamma_2 = \frac{2h_n}{z_1 - z_2}$$

- On the other hand

$$n < T(z) >_{\text{z-plane}} = \frac{< T_{\text{orb}} \sigma_+(z_1)\sigma_-(z_2) >}{< \sigma_+(z_1)\sigma_-(z_2) >}\Big|_{(CFT)^n/Z_n}$$

Another way III

- From conformal Ward identity

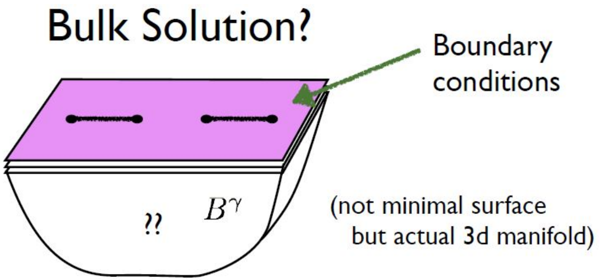
$$< T_{\rm orb}\sigma_+(z_1)\sigma_-(z_2) >= \left(\sum_i \frac{h_\sigma}{(z-z_i)^2} + \frac{\partial_i}{(z-z_i)}\right) < \sigma_+(z_1)\sigma_-(z_2) >$$

- So we can identify
  1. The scaling dimension of twist operator: $h_\sigma = h_n = \frac{c}{12}(n - n^{-1})$
  2. Accessory parameter:

$$\gamma_1 = \frac{\partial}{\partial z_1} < \sigma_+(z_1)\sigma_-(z_2) >= \frac{2h_n}{z_1 - z_2} \qquad (4.1)$$

- Integrating the accessory parameter gives us the Rényi entropy!
- This holds for the multiple intervals as well

Proof of RT formula in AdS_3: A sketch T. Faulkner 1303.7221



- Find the bulk gravity solutions B^γ such that $\partial B^\gamma = \Sigma_n$
- Key point: all solutions of AdS_3 gravity could be obtained by $B^\gamma = H_3/\Gamma_\gamma$, where Γ_γ is the subgroup of isometry $SL(2, C)$
- Consider the handlebody solutions, Γ_γ is the schottky group
- Γ_γ acts on C such that $C/\Gamma_\gamma = \Sigma_n$
- The classical bulk action reproduces RT formula (for multi-intervals)

$$\frac{\partial S_n}{\partial z_i} = -\frac{cn}{6}\gamma_i, \tag{4.2}$$

Schottky uniformization

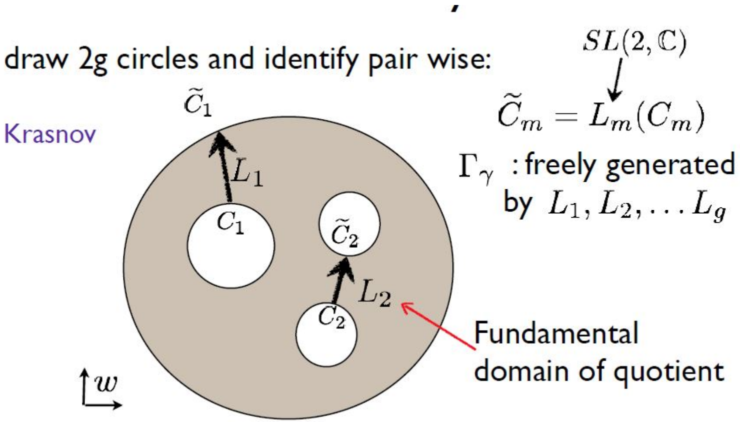


Figure: c.f. Faulkner

Bulk solutions

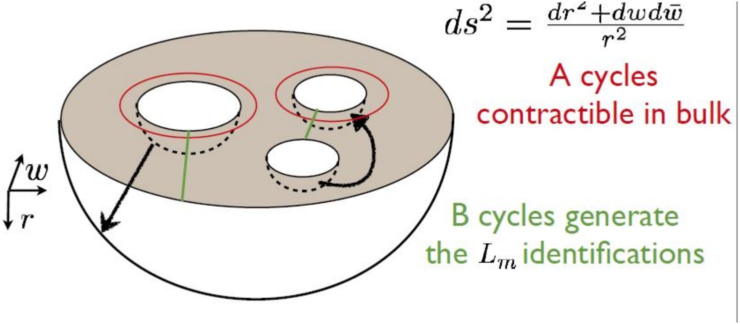


Figure: c.f. Faulkner

Remarks

- In other words, the holographic Renyi entropy(HRE) is given by the classical action of the corresponding gravitational configurations
- γ_i is fixed by the monodromy problem of an ordinary differential equation
- For the same Σ_n, there could be more than one B^γ
- In the classical gravity limit, keep only the solution of least action
- This formula is universal, even for other 3D gravity theory with a AdS_3 vacuum CB et.al. 1401.0261
- An independent proof by T. Hartman (1303.6955) used the CFT techniques
- The RT formula is the classical contribution to the HRE
- Recall that in AdS_3/CFT_2, $c = \frac{3l}{2G}$
- In the large c limit, we may discover the weak gravity result, even with quantum correction
- Why quantum correction?

Quantum correction

- For large separation, the mutual information is vanishing

- The mutual information satisfies M. Wolf et.al. 0704.3906

$$I(A,B) \geq \frac{|<\mathcal{O}_A \cdot \mathcal{O}_B> - <\mathcal{O}_A><\mathcal{O}_B>|^2}{2|\mathcal{O}_A|^2|\mathcal{O}_B|^2}$$

- $I(A,B)$ is only vanishing to the leading order in G_N
- It should be nonzero, with quantum corrections T. Faulkner et.al. 1307.2892
- With the bulk solution in AdS_3, the 1-loop quantum correction of graviton to the Rényi entropy has been computed T. Barrella et.al. 1306.4682

Classical part of HRE in $f(R)$ BC et.al. 1401.0261

- Consider a general 3D gravity theory with a AdS_3 vacuum,

$$I = \frac{1}{16\pi G}\int d^3x\sqrt{g}\mathcal{L}(g_{\mu\nu},\nabla_\mu,R_{\mu\nu}) + I_{bndy}, \tag{4.3}$$

- Without the gravitational CS term,

$$c = \frac{3\mathcal{L}_m l^3}{8G}, \tag{4.4}$$

where $\mathcal{L}_m$ is the value of the Lagrangian density at the AdS vacuum.

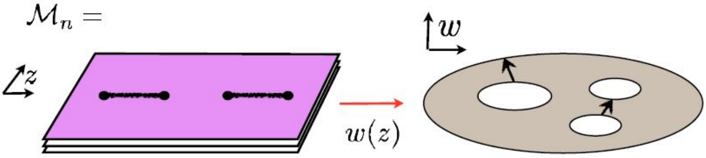
- Classical HRE

$$\frac{\partial S_n}{\partial z_i} = -\frac{cn}{6}\gamma_i, \tag{4.5}$$

- In the case with CS term,

$$\frac{\partial S_n}{\partial z_i} = -\frac{n(c_L + c_R)}{12}\gamma_i. \tag{4.6}$$

Two-interval case



- Find the coordinate $w(z)$, which is single-valued on $\mathcal{M}_n$
- It is determined by the differential equation

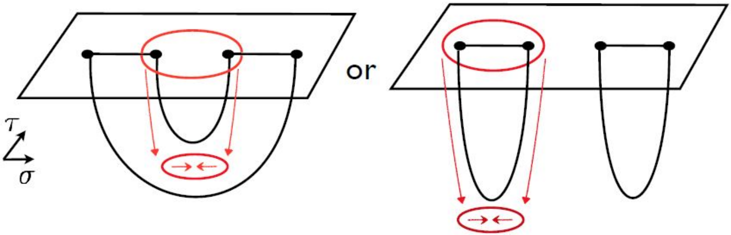
$$\psi''(z) + \sum_{i=1}^{4} \left(\frac{h_n}{(z - z_i)^2} + \frac{\gamma_i}{z - z_i} \right) \psi(z) = 0 \qquad (4.7)$$

- There are two independent solutions ψ_1 and ψ_2, and

$$w(z) = \frac{\psi_1(z)}{\psi_2(z)} \qquad (4.8)$$

Monodromy

- The solutions have monodromies:
  $\psi_1 \to a\psi_1 + b\psi_2, \psi_2 \to c\psi_1 + d\psi_2,$
  $\Rightarrow w \to L(w) = \frac{aw+b}{cw+d}, ad - bc = 1$
- The accessory parameters are determined by requiring trivial monodromy at infinity and on either the A-cycle or B-cycle



Result

For the case of two intervals with small cross ratio x, one can get the classical part of the holographic Rényi mutual information to order x^8 Faulkner 1303.7221, Barrella et.al. 1306.4682

$$
\begin{aligned}
I_n^{cl} = & \frac{(c+\bar{c})(n-1)(n+1)^2x^2}{288n^3} + \frac{(c+\bar{c})(n-1)(n+1)^2x^3}{288n^3} \\
& + \frac{(c+\bar{c})(n-1)(n+1)^2\left(1309n^4-2n^2-11\right)x^4}{414720n^7} \\
& + \frac{(c+\bar{c})(n-1)(n+1)^2\left(589n^4-2n^2-11\right)x^5}{207360n^7} \\
& + \frac{(c+\bar{c})(n-1)(n+1)^2\left(805139n^8-4244n^6-23397n^4-86n^2+188\right)x^6}{313528320n^{11}} \\
& \text{the terms proportional to } x^7 \text{ and } x^8 + O(x^9).
\end{aligned}
\tag{4.9}
$$

The classical HRE has nothing to do with the asymptotic conditions. It takes a universal form, depending only on the central charge.

1-loop correction to HRE

- The gravitational configurations for HRE are generated by the Schottky group
- Consider the fluctuations around these configurations
  1. Pure AdS_3 gravity, only massless gravitons
  2. Higher spin gravity, higher spin fluctuations
  3. For chiral gravity, only massless right-moving graviton
  4. For log gravity, massless gravitons and log mode
  5. For NMG, massless gravitons and/or log modes
- The partition function, which is just S_2, in these cases have been studied before
- For higher S_n, the strategy is the same.

1-loop correction to HRE

- As all the configurations are locally AdS_3, we may use the heat kernel method to compute the contribution
- 1-loop partition function Giombi et.al. 0804.1773, Yin 0710.2129

$$Z^{1-loop} = \prod_{\gamma\in\mathcal{P}} \prod_{s} \prod_{m=s}^{\infty} \frac{1}{|1-q_\gamma^m|}. \tag{4.10}$$

  Here the product over s is with respect to the spins of massless fields and $\mathcal{P}$ is a set of representatives of primitive conjugacy classes of the Schottky group Γ. q_γ is defined by writing the two eigenvalues of $\gamma \in \Gamma$ as $q_\gamma^{\pm 1/2}$ with $|q_\gamma| < 1$.
- The contributions of the fields with different spins could be separated.

Strategy

- Find the Schottky group Γ corresponding to $\mathcal{M}_n$
- Generate $\mathcal{P} = \{$non-repeated words up to conjugation$\}$, e.g.

$$\mathcal{P} = \{L_1, L_2, L_1^{-1}, L_2^{-2}, L_1L_2 \sim L_2L_1, ...\}$$

- Compute eigenvalues of these words and sum over their contributions
- Two-interval case $(-1,-y),(y,1)$ with small cross ratio $x = 4y/(y+1)^2$
  1. Find γ_i by imposing trivial monodromy
  2. Solve the equation for $\psi(z)$ in $|z| << 1$ and $|z| >> y$
  3. Match the solutions and construct L_i
  4. Only finitely many words contribute to each order in y
- For two intervals with small cross ratio,
  1. Metric fluctuations, up to x^8 Barrella et.al. 1306.4682
  2. Spin 3 and/or 4 fluctuations, up to x^8 BC et.al. 1312.5510
  3. Metric log mode, up to x^6 BC et.al. 1401.0261
- One interval in the torus case, both low and high temperature Barrella et.al. 1306.4682

Questions

- Q1: Is the holographic computation of quantum correction of Renyi entropy correct?
- In the large c limit, such quantum correction should correspond to the subleading terms independent of c
- We showed that for two disjoint intervals with small cross ratio x, the CFT result matches exactly with 1-loop HRE
- Q2: how about the situation with matter coupling?
- We discussed the case with higher spin fields
- The results are remarkably good ...
- Q3: how about other 3D gravity theories, CTMG or CNMG?
- Quite interesting, all in good agreement

HS/CFT correspondence

- The higher spin theory in AdS_3 is relatively easy
- It could be defined in terms of Chern-Simons theory with gauge group SL(n,R), describing the interacting fields with spin from 2 to n;
- With generalized Brown-Henneaux b.c., spin n gravity in AdS_3 has W_n asym. symmetry algebra, with the same central charge $c_L = c_R = 3l/2G$
- In our work, we considered
  1. the spin 3 HS gravity, which is dual to a CFT with W(2,3) symmetry
  2. the spin 4 HS gravity, which is dual to a CFT with W(2,3,4) symmetry
  3. the spin $\tilde{4}$ HS gravity, which is dual to a CFT with W(2,4) symmetry

Dictionary

- massless graviton $\leftrightarrow$ stress tensor
- massless spin 3 field $\leftrightarrow$ W_3 field with conformal weight $(3, 0)$ (holomorphic sector)
- massless spin 4 field $\leftrightarrow$ W_4 field with conformal weight $(4, 0)$ (holomorphic sector)

Bulk computation

- We focus on the AdS_3 vacuum, which corresponds to the vacuum of dual CFT
- The gravitational configurations corresponding to the higher genus RS due to the replica trick are the same as the ones in pure gravity
- Therefore the classical HRE is invariant
- But we must consider the other fluctuations in computing quantum correction
- This could be done using the heat kernel method
- We computed the 1-loop correction to HRE to order x^8
- The difficult part is on the CFT side

Correlators in 2D CFT

- In a 2D CFT, all the operators could be written in terms of quasiprimary fields and their derivatives
- We write the quasiprimary operators as ϕ_i with conformal weights h_i and $\bar{h}_i$
- The correlation functions of two and three quasiprimary operators on complex plane C are

$$\langle \phi_i(z_i, \bar{z}_i)\phi_j(z_j, \bar{z}_j)\rangle_C = \frac{\alpha_i \delta_{ij}}{z_{ij}^{2h_i} \bar{z}_{ij}^{2\bar{h}_i}},$$

$$\langle \phi_i(z_i, \bar{z}_i)\phi_j(z_j, \bar{z}_j)\phi_k(z_k, \bar{z}_k\rangle_C$$

$$= \frac{C_{ijk}}{z_{ij}^{h_i+h_j-h_k} z_{jk}^{h_j+h_k-h_i} z_{ik}^{h_i+h_k-h_j} \bar{z}_{ij}^{\bar{h}_i+\bar{h}_j-\bar{h}_k} \bar{z}_{jk}^{\bar{h}_j+\bar{h}_k-\bar{h}_i} \bar{z}_{ik}^{\bar{h}_i+\bar{h}_k-\bar{h}_j}},$$

with $z_{ij} \equiv z_i - z_j$ and $\bar{z}_{ij} \equiv \bar{z}_i - \bar{z}_j$.

OPE in 2D CFT

The OPE of two quasiprimary operators could be generally written as

$$\phi_i(z,\bar{z})\phi_j(0,0) = \sum_k C_{ij}^k \sum_{m,r\geq 0} \frac{a_{ijk}^m}{m!}\frac{\bar{a}_{ijk}^r}{r!}\frac{1}{z^{h_i+h_j-h_k-m}\bar{z}^{\bar{h}_i+\bar{h}_j-\bar{h}_k-r}}\partial^m\bar{\partial}^r\phi_k(0,0),$$

where the summation k is over all quasiprimary operators and

$$a_{ijk}^m \equiv \frac{C_{h_k+h_i-h_j+m-1}^m}{C_{2h_k+m-1}^m}, \quad \bar{a}_{ijk}^r \equiv \frac{C_{\bar{h}_k+\bar{h}_i-\bar{h}_j+r-1}^r}{C_{2\bar{h}_k+r-1}^r}, \quad C_{ij}^k \equiv \frac{C_{ijk}}{\alpha_k}$$

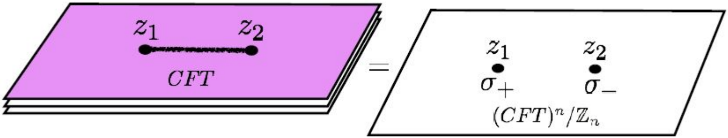
with the binomial coefficient being $C_x^y = \frac{\Gamma(x+1)}{\Gamma(y+1)\Gamma(x-y+1)}$.

CFT_n

- The replica trick requires us to study a orbifold CFT: $(\mathrm{CFT})_n/Z_n$
- The CFT_n has central charge nc with c being the central charge of CFT_1, and the stress tensors are

$$\sum_{j=0}^{n-1} T(z_j), \quad \sum_{j=0}^{n-1} \bar{T}(\bar{z}_j)$$

where $T(z_j)$, $\bar{T}(\bar{z}_j)$ are the stress tensors of the j-th copy of the original CFT and z_j is the coordinate of the j-th copy of the Riemann surface $\mathcal{R}_{n,N}$.



Quasiprimaries in CFT_n

We denote the linear independent quasiprimary operators of CFT_n as $\Phi_K(z,\bar{z})$ with conformal wights h_K and $\bar{h}_K$. The product of quasiprimary operators in each copy forms a quasiprimary operator of CFT_n,

$$\Phi_K(z,\bar{z}) = \prod_{j=0}^{n-1} \phi_{k_j}(z_j,\bar{z}_j),$$

and in this case there are

$$K = \{k_j\}, \quad \alpha_K = \prod_{j=0}^{n-1} \alpha_{k_j}, \quad h_K = \sum_{j=0}^{n-1} h_{k_j}, \quad \bar{h}_K = \sum_{j=0}^{n-1} \bar{h}_{k_j}.$$

Note that not all of the quasiprimary operators of CFT_n could be written in the above form.

General prescription M. Headrick 1006.0047, P. Calabrese et.al. 1011.5482, BC and J-j Zhang 1309.5453

When the intervals are short, we have the OPE of the twist operators

$$\sigma(z,\bar{z})\tilde{\sigma}(0,0) = c_n \sum_K d_K \sum_{m,r\geq 0} \frac{a_K^m}{m!}\frac{\bar{a}_K^r}{r!}\frac{1}{z^{2h-h_K-m}\bar{z}^{2\bar{h}-\bar{h}_K-r}}\partial^m\bar{\partial}^r\Phi_K(0,0),$$

with the summation K being over all the independent quasiprimary operators of CFT_n. Here

$$a_K^m \equiv \frac{C^m_{h_K+m-1}}{C^m_{2h_K+m-1}}, \quad \bar{a}_K^r \equiv \frac{C^r_{\bar{h}_K+r-1}}{C^r_{2\bar{h}_K+r-1}}.$$

- For a quasiprimary operator Φ_K, the OPE coefficient is

$$C_K = c_n \ell^{-\frac{c}{6}\left(n-\frac{1}{n}\right)} d_K,$$

- The OPE coefficient of its derivatives $\partial^m\bar{\partial}^r\Phi_K$ is

$$C_K^{(m,r)} = c_n \ell^{-\frac{c}{6}\left(n-\frac{1}{n}\right)+m+r} d_K \frac{a_K^m}{m!}\frac{\bar{a}_K^r}{r!}.$$

Vacuum Verma module

- For a concrete CFT model, the summation should be over all the conformal blocks
- For pure AdS_3 gravity, it is enough to consider the vacuum Verma module
- From AdS/CFT, the graviton fluctuation corresponds to the stress tensor in CFT which is in the vacuum module
- Moreover, from the study of quantum gravity in AdS_3, it has been known that the pure gravity partition function could be reproduced from the vacuum module Maloney and Witten, 0712.0155
- For HS AdS_3 gravity, it is necessary to include the quasi-primary operators from W fields

How to compute the OPE coefficients

- For usual OPE, they depend on the three point functions
- For the OPE of twist operators, we may just focus on the one interval case, in the small interval limit P. Calabrese et.al. 1011.5482
- When there is one interval $A = [0, \ell]$, we consider the expectation value of one quasiprimary operator $\Phi_K(z, \bar{z})$ on $\mathcal{R}_{n,1}$, and then we have

$$\frac{Z_n(A)}{Z^n}\langle\Phi_K(z,\bar{z})\rangle_{\mathcal{R}_{n,1}} = \langle\Phi_K(z,\bar{z})\sigma(\ell,\ell)\tilde{\sigma}(0,0)\rangle_C.$$

- Using the OPE of twist operators and the orthogonality of quasiprimary operators of CFT_n we have

$$d_K = \frac{1}{\alpha_K \ell^{h_K+\bar{h}_K}} \lim_{z\to\infty} z^{2h_K}\bar{z}^{2\bar{h}_K}\langle\Phi_K(z,\bar{z})\rangle_{\mathcal{R}_{n,1}},$$

with α_K being a normalization coefficient.

- The key ingredients in the OPE of twist operators is the coefficients α_K and d_K.

Holomorphic quasiprimary operators in CFT_1

Explicitly the holomorphic quasiprimary operators of first few levels are listed as follows.

- At level 0, it is the identity operator 1
- At level 2, there is one quasiprimary operator the stress tensor T.
- At level 4, it is $\mathcal{O} = (TT) - \frac{3}{10}\partial^2 T$.
- At level 6, they are $\mathcal{Q} = (\partial T \partial T) - \frac{2}{9}\partial^2(TT) + \frac{1}{42}\partial^4 T$ and $\mathcal{R} = \mathcal{P} + \frac{9(14c+43)}{2(70c+29)}\mathcal{Q}$, with $\mathcal{P} = (T(TT)) - \frac{1}{4}\partial^2(TT) + \frac{1}{56}\partial^4 T$.
- At level 8, more complicated construction

We use the notation $(AB)(z)$ representing the normal ordering of two operators $A(z)$ and $B(z)$. Note that at level 6, $\mathcal{P}(z)$ and $\mathcal{Q}(z)$ are not orthogonal. After using the Gram-Schmidt orthogonalization process, we get the orthogonalized operators $\mathcal{Q}(z)$ and $\mathcal{R}(z)$.

Normalization factor α_k

Firstly one define the state $|k\rangle \equiv \phi_k(0,0)|0\rangle$, with $|0\rangle$ being the vacuum state of the CFT on C, and then

$$\alpha_k = \langle k|k\rangle.$$

For example, for the operator $\mathcal{O}(z)$ we have

$$|\mathcal{O}\rangle = \left(L_{-2}L_{-2} - \frac{3}{5}L_{-4}\right)|0\rangle,$$

and then

$$\alpha_{\mathcal{O}} = \frac{c(5c+22)}{10}.$$

Similarly, for other quasiprimary operators, their normalization factors are respectively

$$\alpha_1 = 1, \quad \alpha_T = \frac{c}{2}, \quad \alpha_{\mathcal{Q}} = \frac{4c(70c+29)}{63},$$
$$\alpha_{\mathcal{R}} = \frac{3c(2c-1)(5c+22)(7c+68)}{4(70c+29)}.$$

Quasiprimaries in CFT_1

There are also the antiholomorphic quasiprimary operators $\bar{T}$, $\bar{\mathcal{O}}$, $\bar{\mathcal{Q}}$ and $\bar{\mathcal{R}}$, as well as the quasiprimary operators with mixing holomorphic and antiholomorphic parts. Explicitly, at each level $L_0 + \bar{L}_0$, we have

- At level 0, it is 1.
- At level 2, they are T and $\bar{T}$.
- At level 4, they are $\mathcal{O}$, $\bar{\mathcal{O}}$ and $T\bar{T}$.
- At level 6, they are $\mathcal{Q}$, $\mathcal{R}$, $\bar{\mathcal{Q}}$, $\bar{\mathcal{R}}$, $T\bar{\mathcal{O}}$ and $\bar{T}\mathcal{O}$.

Note that here the quasiprimary operators are just trivial multiplications of the holomorphic and antiholomorphic parts, because that the OPE of T and $\bar{T}$ has no singular terms.

Quasiprimaries in CFT_n

The quasiprimary operators are listed as below.

L_0	quasiprimary operators	degeneracies	#
0	1	1	1
2	$T(z_j)$	n	n
4	$T(z_{j_1})T(z_{j_2})$ with $j_1 < j_2$	$\frac{n(n-1)}{2}$	$\frac{n(n+1)}{2}$
	$\mathcal{O}(z_j)$	n	
5	$\mathcal{S}_{j_1 j_2}(z)$ with $j_1 < j_2$	$\frac{n(n-1)}{2}$	$\frac{n(n-1)}{2}$
6	$T(z_{j_1})T(z_{j_2})T(z_{j_3})$ with $j_1 < j_2 < j_3$	$\frac{n(n-1)(n-2)}{6}$	$\frac{n(n+1)(n+5)}{6}$
	$T(z_{j_1})\mathcal{O}(z_{j_2})$ with $j_1 \neq j_2$	$n(n-1)$	
	$\mathcal{U}_{j_1 j_2}(z)$ with $j_1 < j_2$	$\frac{n(n-1)}{2}$	
	$\mathcal{Q}(z_j)$	n	
	$\mathcal{R}(z_j)$	n	
...	...	...	...

Note that the j's listed above vary as $0 \leq j \leq n-1$, and also the operators

$$\mathcal{S}_{j_1j_2}(z) = T(z_{j_1})i\partial T(z_{j_2}) - i\partial T(z_{j_1})T(z_{j_2}),$$
$$\mathcal{U}_{j_1j_2}(z) = \frac{5}{9}\partial T(z_{j_1})\partial T(z_{j_2}) - \frac{2}{9}\partial^2 T(z_{j_1})T(z_{j_2}) - \frac{2}{9}T(z_{j_1})\partial^2 T(z_{j_2})$$

can not be factorized into the operators at different copies.
The coefficients α_K for these operators could be calculated easily

$$\alpha_{TT} = \frac{c^2}{4}, \quad \alpha_{\mathcal{S}} = 2c^2, \quad \alpha_{TTT} = \frac{c^3}{8},$$
$$\alpha_{T\mathcal{O}} = \frac{c^2(5c+22)}{20}, \quad \alpha_{\mathcal{U}} = \frac{20c^2}{9}.$$

The coefficient d_K

To compute d_K we consider the multivalued transformation

$$z \to f(z) = \left(\frac{z-\ell}{z}\right)^{1/n},$$

which maps the Riemann surface $\mathcal{R}_{n,1}$ to the complex plane C. With some efforts, we find d_K's for various operators listed above,

$$d_1 = 1, \quad d_T = \frac{n^2-1}{12n^2}, \quad d_{TT}^{j_1j_2} = \frac{1}{8n^4c}\frac{1}{s_{j_1j_2}^4} + \frac{(n^2-1)^2}{144n^2},$$

$$d_{\mathcal{O}} = \frac{(n^2-1)^2}{288n^4}, \quad d_{\mathcal{S}}^{j_1j_2} = \frac{1}{16n^5c}\frac{c_{j_1j_2}}{s_{j_1j_2}^5},$$

$$d_{TTT}^{j_1j_2j_3} = -\frac{1}{8n^6c^2}\frac{1}{s_{j_1j_2}^2 s_{j_2j_3}^2 s_{j_1j_3}^2} + \frac{n^2-1}{96n^6c}\left(\frac{1}{s_{j_1j_2}^4} + \frac{1}{s_{j_2j_3}^4} + \frac{1}{s_{j_1j_3}^4}\right) + \frac{(n^2-1)^3}{1728n^6},$$

$$d_{T\mathcal{O}}^{j_1j_2} = \frac{n^2-1}{96n^6c}\frac{1}{s_{j_1j_2}^4} + \frac{(n^2-1)^3}{3456n^6}, \quad d_{\mathcal{Q}} = -\frac{(n^2-1)^2\left(2(35c+61)n^2-93\right)}{5760n^6(70c+29)},$$

Here $s_{j_1j_2} \equiv \sin\frac{\pi(j_1-j_2)}{n}$ and $c_{j_1j_2} \equiv \cos\frac{\pi(j_1-j_2)}{n}$.

Application I: one short interval on cylinder

- We choose the coordinate of the cylinder be z and the subsystem A to be an interval $A = [0, \ell]$ with $\ell \ll L$.
- The Rényi entanglement entropy of A is known exactly P. Calabrese and J. Cardy 0405152

$$S_n = \frac{c}{6}\left(1+\frac{1}{n}\right)\log\left(\frac{L}{\pi\epsilon}\sin\frac{\pi\ell}{L}\right). \tag{4.11}$$

- From OPE of twist operators

$$\mathrm{Tr}\rho_A^n = \langle\sigma(\ell,\ell)\tilde{\sigma}(0,0)\rangle_L = c_n \ell^{-\frac{c}{6}\left(n-\frac{1}{n}\right)}\sum_K d_K \ell^{h_K+\bar{h}_K}\langle\Phi_K(0,0)\rangle_L,$$

- Due to the translational invariance, the expectation value of one operator on the cylinder $\langle\Phi_K(z,\bar{z})\rangle_L$ must be independent of the coordinates, and so the derivative terms vanish uniformly.

Finite size correction

- The holo. and anti-holo. sectors are decoupled, the computation could be simplified more

$$\mathrm{Tr}\rho_A^n = c_n \ell^{-\frac{c}{6}\left(n-\frac{1}{n}\right)} \left(\sum_K d_K \ell^{h_K} \langle \Phi_K(0) \rangle_L \right)^2 ,$$

with K being the summation over all the linear independent holomorphic quasiprimary operators.

- In the end, we could find the Rényi entanglement entropy

$$\begin{aligned} S_n &= -\frac{1}{n-1} \log \mathrm{Tr}\rho_A^n \\ &= \frac{c}{6}\left(1+\frac{1}{n}\right)\left(\log\frac{\ell}{\epsilon} - \frac{\pi^2\ell^2}{6L^2} - \frac{\pi^4\ell^4}{180L^4} - \frac{\pi^6\ell^6}{2835L^6} + O\left(\frac{\ell}{L}\right)^8\right), \end{aligned}$$

which matches (4.11) to the order of $O(\ell^6)$.

Application II: Two intervals with small cross ratio

We choose $A = [0, y] \cup [1, 1 + y]$ with y being small, and thus the cross ratio is $x = y^2$

$$
\begin{aligned}
\mathrm{Tr}\rho_A^n &= \langle \sigma(1+y, 1+y)\tilde{\sigma}(1,1)\sigma(y,y)\tilde{\sigma}(0,0)\rangle_C \\
&= c_n^2 y^{-\frac{c}{3}\left(n-\frac{1}{n}\right)} \left(\sum_K \alpha_K d_K^2 y^{2h_K} \right. \\
&\qquad \left. \sum_{m,p\geq 0} (-)^m \frac{(m+p)!}{m!p!} a_K^m a_K^p C_{2h_K+m+p-1}^{m+p} y^{m+p} \right)^2
\end{aligned}
$$

- With the coefficients d_K obtained before, the computation is straightforward but tedious

Rényi mutual information

- The Rényi mutual information is

$$\begin{aligned} I_n &= \frac{c}{3}(1+\frac{1}{n})\log\frac{y}{\epsilon} + \frac{1}{n-1}\log \mathrm{Tr}\rho_A^n, \\ &= I_n^{tree} + I_n^{1-loop} + I_n^{2-loop} + \cdots . \end{aligned}$$

- Here we have classified the contributions according to the order of the inverse of central charge $\frac{1}{c}$, which in the large c limit corresponds to tree, 1-loop, and 2-loop contributions in the gravity side
  1. $I_n^{tree} \sim \mathcal{O}(c)$ terms
  2. $I_n^{1-loop} \sim \mathcal{O}(c^0)$ terms
  3. $I_n^{2-loop} \sim \mathcal{O}(1/c)$ terms
- After some highly nontrivial summations...

Useful formulae I

Define

$$f_m(n) \equiv \sum_{j=1}^{n-1} \frac{1}{\left(\sin\frac{\pi j}{n}\right)^{2m}},$$

we need

$$f_1(n) = \frac{n^2-1}{3}, \quad f_2(n) = \frac{(n^2-1)\left(n^2+11\right)}{45},$$

$$f_3(n) = \frac{(n^2-1)\left(2n^4+23n^2+191\right)}{945},$$

$$f_4(n) = \frac{(n^2-1)\left(n^2+11\right)\left(3n^4+10n^2+227\right)}{14175},$$

$$f_5(n) = \frac{(n^2-1)\left(2n^8+35n^6+321n^4+2125n^2+14797\right)}{93555},$$

$$\sum_{0\le j_1<j_2<j_3\le n-1} \frac{1}{s^2_{j_1 j_2} s^2_{j_2 j_3} s^2_{j_1 j_3}} = \frac{n\left(n^2-1\right)\left(n^2-4\right)\left(n^2+47\right)}{2835},$$

Useful formulae II

$$\sum_{0\le j_1<j_2<j_3\le n-1}\frac{1}{s^4_{j_1j_2}s^4_{j_2j_3}s^4_{j_1j_3}}=\frac{n\left(n^2-1\right)\left(n^2-4\right)(19n^8+875n^6+22317n^4+505625n^2+5691964)}{273648375}$$

$$\sum_{0\le j_1<j_2<j_3\le n-1}\left(\frac{1}{s^4_{j_1j_2}}+\frac{1}{s^4_{j_2j_3}}+\frac{1}{s^4_{j_1j_3}}\right)=\frac{n(n^2-1)(n-2)\left(n^2+11\right)}{90},$$

$$\sum_{0\le j_1<j_2<j_3\le n-1}\left(\frac{1}{s^4_{j_1j_2}}+\frac{1}{s^4_{j_2j_3}}+\frac{1}{s^4_{j_1j_3}}\right)^2=\frac{n(n^2-1)(n-2)\left(n^2+11\right)(3n^4+8n^3+26n^2+152n+531)}{28350}$$

Mutual information: classical part

The tree part, or the classical part, being proportional to the central charge c, originates only from the vacuum module

$$
\begin{aligned}
I_n^{tree} = &\frac{c(n-1)(n+1)^2x^2}{144n^3} + \frac{c(n-1)(n+1)^2x^3}{144n^3} \\
&+ \frac{c(n-1)(n+1)^2\left(1309n^4-2n^2-11\right)x^4}{207360n^7} \\
&+ \frac{c(n-1)(n+1)^2\left(589n^4-2n^2-11\right)x^5}{103680n^7} \\
&+ \frac{c(n-1)(n+1)^2(805139n^8-4244n^6-23397n^4-86n^2+188)x^6}{156764160n^{11}} \\
&+ (\text{the terms proportional to } x^7 \text{ and } x^8) + \mathcal{O}\left(x^9\right)
\end{aligned}
$$

It matches the result in M. Headrick 1006.0047, T. Hartman 1303.6955, T. Faulkner 1303.7221 up to order x^8.

Mutual information: 1-loop correction from graviton

The quantum 1-1oop part from the stress tensor, being proportional to c^0, is

$$I_n^{(2)1-loop} = \frac{(n+1)\left(n^2+11\right)\left(3n^4+10n^2+227\right)x^4}{3628800n^7} + \frac{(n+1)\left(109n^8+1495n^6+11307n^4+81905n^2-8416\right)x^5}{59875200n^9} + \frac{(n+1)\left(1444050n^{10}+19112974n^8+140565305n^6+1000527837n^4-167731255n^2-14142911\right)x^6}{523069747200n^{11}} + (\text{the terms proportional to } x^7 \text{ and } x^8) + \mathcal{O}\left(x^9\right).$$

It matches exactly the result in M. Headrick 1006.0047, T. Barrella 1306.4682 up to order x^8.

Mutual information: 1-loop correction in W_3

The quantum 1-1oop part in CFT with W_3 symmetry, being proportional to c^0, is

$$I_n^{(2,3)1-loop} = \cdots$$
$$+\frac{(n+1)x^6(3610816n^{10}+47796776n^8+351567243n^6+2502467423n^4-412426559n^2+10856301)}{1307674368000n^{11}}$$
$$+(\text{the terms proportional to } x^7 \text{ and } x^8) + \mathcal{O}(x^9),$$

- the "$\cdots$" being the x^4, x^5 parts of $I_n^{(2)1-loop}$
- The extra contribution starts to appear from order x^6, as the conformal weight of W_3 field is three
- It exactly matches the 1-loop correction to HRE to order x^8

Mutual information: 2-loop correction

Remarkably there is also the quantum 2-loop contribution, being proportional to $1/c$,

$$I_n^{2-loop} = \frac{(n+1)\left(n^2-4\right)(19n^8+875n^6+22317n^4+505625n^2+5691964)x^6}{70053984000n^{11}c} \\ +(\text{the terms proportional to } x^7 \text{ and } x^8) + \mathcal{O}\left(x^9\right),$$

This is novel, expected to be confirmed by 2-loop computation in gravity

- When $n = 2$, the two-loop correction is vanishing, as S_2 is 1-loop exact
- When $n > 2$, there should be nonvanishing 2-loop correction Xi Yin, 0710.2129
- The extra contribution from W_3 field appears at order x^8
- Actually there is nonvanishing quantum 3-loop contribution, being proportional to $1/c^2$, for $S_n, n > 3$.

CTMG

- No local physical d.o.f. in 3D Einstein gravity;
- To have local gravitational degree of freedom, one may add higher-derivative terms;
- A simple choice is to add a gravitational Chern-Simons term, which is parity breaking and topological: S.Deser et.al. 1982

$$I_{CS} = \frac{1}{2\mu}\int d^3x\sqrt{-g}\varepsilon^{\lambda\mu\nu}\Gamma^{\rho}_{\lambda\sigma}\left(\partial_\mu\Gamma^{\sigma}_{\rho\nu} + \frac{2}{3}\Gamma^{\sigma}_{\mu\tau}\Gamma^{\tau}_{\nu\rho}\right)$$

- The AdS_3 vacuum

$$ds^2 = l^2\Big(-(dx^+)^2 - (dx^-)^2 - 2\cosh(2\rho)dx^+dx^- + d\rho^2\Big).$$

- The linear fluctuations around the AdS_3 vacuum obey a third order differential equation
- If $\mu l \neq 1$, there are two massless boundary gravitons h^L, h^R and a local massive graviton h^M.
- However, 3D TMG in AdS_3 is not well-defined for generic value μl, either because of the instability or negative energy for black hole;
- At the critical point $\mu l = \pm 1$, 3D TMG in AdS_3 could be well-defined

Chiral gravity

- Both local mode and left-moving graviton are just pure gauge at the critical point;
- The only physical d.o.f. is the right-moving boundary graviton;
- Conjecture: chiral gravity is holographically dual to a 2D chiral CFT by imposing self-consistent Brown-Henneaux B.C.;W.Li, W.Song and A. Strominger 1998

$$c_L = 0, \ c_R = \frac{3l}{G}$$

- The Brown-Henneaux B.C.

$$\begin{pmatrix} h_{++} = O(1) & h_{+-} = O(1) & h_{+\rho} = O(e^{-2\rho}) \\ h_{-+} = h_{+-} & h_{--} = O(1) & h_{-\rho} = O(e^{-2\rho}) \\ h_{\rho+} = h_{+\rho} & h_{\rho-} = h_{-\rho} & h_{\rho\rho} = O(e^{-2\rho}) \end{pmatrix}.$$

Log gravity

- There is actually a logarithmic mode at the critical point Grumiller et.al. 2008

$$h^{log} = \lim_{\mu l \to 1} \frac{h^M - h^L}{\mu l - 1}.$$

- Such mode has been excluded by Brown-Henneaux b.c.;
- However there exists another set of consistent boundary conditions, which include the log. mode

$$\begin{pmatrix} h_{++} = O(1) & h_{+-} = O(1) & h_{+\rho} = O(e^{-2\rho}) \\ h_{-+} = h_{+-} & h_{--} = O(\rho) & h_{-\rho} = O(\rho e^{-2\rho}) \\ h_{\rho+} = h_{+\rho} & h_{\rho-} = h_{-\rho} & h_{\rho\rho} = O(e^{-2\rho}) \end{pmatrix},$$

- It has been conjectured that under this set of B.C., the CTMG at the critical point is dual to a logarithmic CFT. Grumiller et.al. 2008, A. Moloney et.al. 2009
- The quantum gravity is defined with respect to the asymptotic boundary conditions

New massive gravity at critical point

$$S=\frac{1}{16\pi G}\int dx^3\sqrt{-g}[\sigma R+\frac{1}{m^2}(R_{\mu\nu}R^{\mu\nu}-\frac{3}{8}R^2)-2\lambda m^2],$$

- Various vacua, here we focus on the AdS_3
- In general, two massless graviton h^L, h^R and two massive ones $h^{m\pm}$
- At the critical point

$$2m^2l^2=-\sigma,$$

the massive modes $h^{m\pm}$ coincide with the massless modes h^L and h^R, and there appear the left- and right-moving logarithmic modes h_L^{log} and h_R^{log} Y. Liu and Y.-w. Sun 2009, D. Grumiller et.al. 2009

1. Brown-Henneaux B.C.:

$$c_L=c_R=\frac{3l}{2G_N}(\sigma+\frac{1}{2l^2m^2}).$$

2. At the critical point, Log. B.C. to include the left- and/or right-moving log mode
3. At the critical point, the central charges are vanishing

HRE in TMG and NMG

- The classical contribution is given as discussed before

$$\frac{\partial S_n}{\partial z_i} = -\frac{n(c_L + c_R)}{12}\gamma_i.$$

- The 1-loop correction is up to the fluctuations
- For example, in TMG the 1-loop thermal partition function is

$$Z^{1-loop}_{TMG} = \prod_{r=2}^{\infty} \frac{1}{|1-q^r|^2} \prod_{m=2}^{\infty} \prod_{\bar{m}=0}^{\infty} \frac{1}{1-q^m \bar{q}^{\bar{m}}}.$$

- Similarly the 1-loop correction to HRE is

$$\log Z^{1-loop}_{TMG} = -\sum_{\gamma\in P}\sum_{r=2}^{\infty} \log\left(|1-q^r_\gamma|\right) - \frac{1}{2}\sum_{\gamma\in P}\sum_{m=2}^{\infty}\sum_{\bar{m}=0}^{\infty} \log\left(1-q^m_\gamma \bar{q}^{\bar{m}}_\gamma\right).$$

- For NMG, even the central charge is vanishing such that the classical HRE is zero, the 1-loop correction is not vanishing

CFT side

- For chiral gravity, only right moving sector of a CFT is needed. The computation could be read from known results
- For log. gravity case, we need to treat a special kind of CFT—logarithmic CFT with $c = 0$
- We introduce an extra primary field in an ordinary CFT and taking $c \to 0$ limit
- This allows us to construct the quasi-primary fields and compute OPE of twist operators as before
- Finally, we find consistent pictures in both CTMG/LCFT and CNMG/LCFT correspondence

1st law of thermodynamics

- Consider the variation of the state $|\psi>\to|\psi>+\delta|\psi>$
- It induces the variation of EE:

$$\delta S_A = -tr(\delta\rho_A \ln \rho_A) = \delta < H_A > \tag{6.1}$$

where H_A defined by $\rho_A = e^{-H_A}$ is called modular Hamiltonian or the entanglement Hamiltonian.

- The above relation could be taken as the quantum version of the 1st law of thermodynamics Blanco et.al. 1305.3182

Linearized Einstein equation

- What's the implication of this 1st law on gravity?
- Consider the CFT with AdS gravity dual
- Focus on the case that A =ball
- If the initial state is the vacuum state $|0>$, corresponding to pure AdS, then the small perturbation $|\psi>$ corresponds to the pure AdS with perturbation

$$ds^2 = \frac{l^2}{z^2}(dz^2 + dx^\mu dx_\mu + z^{d+1} H_{\mu\nu} dx^\mu dx^\nu) \qquad (6.2)$$

- Then the 1st law $\delta S_B = \delta E_B \Rightarrow$ Linearized gravitational equations Lashkari et.al. 1308.3716, Faulkner et.al. 1312.7856
- Similar story for higher curvature gravities

Conclusion

- Rényi entropy and its 1-loop quantum correction in the AdS_3 gravity shed new light on the AdS_3/CFT_2 correspondence
- We developed the short interval expansion of twist operators by considering the derivatives of the quasiprimary operators, in the ground state of CFT
- This allowed us to get the subleading contributions of Rényi entropy
- To order 8 in the short interval expansion, we reproduced exactly the classical and 1-loop quantum contributions to the Rényi entropy, even in the theory with higher spin charges
- In the context of $AdS_3/LCFT_2$ correspondence, we find consistent picture from the study of Rényi entropy
- Strong support of holographic computation of EE and RE, even with quantum correction (beyond RT formula)

Discussion

- Rényi entropy opens a new window to study the AdS_3/CFT_2 correspondence
- In the case of two disjoint intervals, the Rényi entropy S_2 is just the partition function on a torus with a modular parameter. This partition function corresponds to the 1-loop determinant of physical fluctuations around the thermal AdS space.
- The higher Rényi entropy $S_n, n > 2$ present new challenges and criterion? Our studies seem suggest that once the genus-1 partition function is in match with the 1-loop bulk partition function, so do the higher Rényi entropies $S_n(n > 2)$ at least to 1-loop. A general proof?
- What's the CFT dual of quantum AdS_3 gravity? E. Witten 1988, S. Carlip 050302, A. Maloney and E. Witten 0712.0155

Discussion

- First of all, it would be interesting to compute the Rényi entropy of a concrete CFT model, considering the limited knowledge on this issue
- Higher loop corrections around the gravitational configurations whose boundary is of genus greater than one?
- Rényi entropy in excited states or thermal case Work in progress
- Quantum quench? Cardy et.al. 2007,2014, S. Das et.al. ...
- It would be nice to generalize our study to the case with more than two intervals
- It is certainly important to generalize our prescriptions to higher dimensions

Other topics

- How the spacetime emerge? Entanglement renormalization? Swingle 2009, Raamsdonk 2009, Lee 2009, Wall et.al. Takayanagi et.al. ...
- The non-linear dynamics of gravity from thermodynamics of EE?
- The entropy of "hole" in spacetime de Boer et.al. 2013, Myers et.al. 2014
- .....

Thanks for your attention!