Wall Crossing and Quivers

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Base on: H. Kim, J. Park, ZLW and P. Yi, JHEP 1109 (2011) 079
S.-J. Lee, ZLW and P. Yi, JHEP 1207 (2012) 169
S.-J. Lee, ZLW and P. Yi, JHEP 1210 (2012) 094
S.-J. Lee, ZLW and P. Yi, JHEP 1402 (2014) 047

Overview

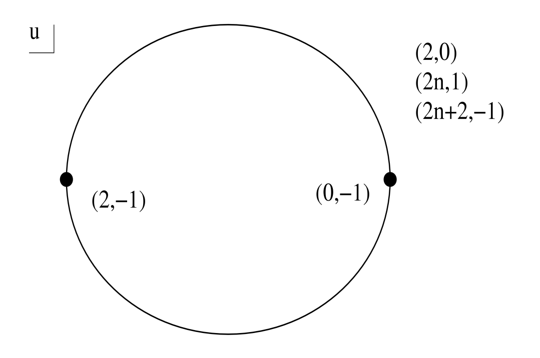
- Wall Crossing Phenomenon
- Coulomb and Higgs phase: Particles V.S. Quivers
- Intrinsic Higgs States and Quiver Invariants

Wall Crossing phenomenon

Certain quantity changes discontinuously across a codimension-one wall in a space

Wall Crossing phenomenon

BPS index changes discontinuous across the wall of marginal stability in the moduli space of a supersymmetric theory



BPS index

- 4D $\mathcal{N}$=2 SUSY:

$$Q^A_\alpha \, (A = 1,2; \alpha = 1,2)\,, \quad \left(Q^A_\alpha\right)^\dagger = \bar{Q}_{A\dot{\alpha}}\,, \quad Q^{A\alpha} = \epsilon^{\alpha\beta} Q^A_\beta\,,$$
$$\{Q^A_\alpha, \bar{Q}_{B\dot{\beta}}\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^A_B\,, \qquad \{Q^{A\alpha}, Q^B_\beta\} = 2\delta^\alpha_\beta \epsilon^{AB} Z\,.$$

Bosonic symmetries: $ISO(1,3) \times SU(2)_R \times U(1)_R$.

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  Bosonic symmetries: $ISO(1,3) \times SU(2)_R \times U(1)_R$.
- For a massive state with mass M and central charge $Z = e^{\mathrm{i}\theta}|Z|$, define

$$R^\pm_\alpha = \tfrac{1}{2}\left(e^{-\mathrm{i}\theta/2} Q^1_\alpha \pm e^{\mathrm{i}\theta/2} \sigma^\mu_{\alpha\dot{\beta}} P_\mu M^{-1} \bar{Q}^{2\dot{\beta}}\right)\,.$$

  The non-vanishing anticommutator is

$$\{R^\pm_\alpha, \bar{R}^\pm_{\dot{\beta}}\} = \sigma^\mu_{\alpha\dot{\beta}} P_\mu M^{-1} \left(M \mp |Z|\right)\,.$$

  Under $SU(2)_R$, $\left(R^\pm_\alpha, \sigma^\mu_{\alpha\dot{\beta}} P_\mu M^{-1} \bar{R}^{\pm\dot{\beta}}\right)$ transforms as a doublet

$$R^\pm_\alpha \to \cos\phi_1 \, e^{\mathrm{i}\phi_3} R^\pm_\alpha \pm \sin\phi_1 \, e^{\mathrm{i}\phi_2} \sigma^\mu_{\alpha\dot{\beta}} P_\mu M^{-1} \bar{R}^{\pm\dot{\beta}}\,.$$

BPS index

- In the rest frame

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BPS index

- The refined version: Protected spin character[D. Gaiotto, G.W. Moore and A. Neitzke, 10']

$$\Omega_\gamma(y) = \mathrm{Tr}'_{\mathcal{H}_\gamma}\left((-1)^{2J_3} y^{2(J_3+I_3)}\right)$$

  - It is an index since there is a Femionic operator $\mathcal{Q} = \epsilon_{A\alpha} Q^{A\alpha}$ which is a singlet under $J_3 + I_3$, anticommutes with $(-1)^F$, and is invertible on long-rep.
  - BPS multiplet S_j:

$$(-1)^{2j}\frac{y^{j+1} - y^{-j-1}}{y - y^{-1}}$$

  - No contributions from long-rep since $\mathcal{Q}$ is invertible on long-rep.
  - No contributions from multi-particle state

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- If no states enter or leave the Hilbert space, the structure of the spectrum changes continuously under continuous change of the parameters.
- Therefore, Ω is an invariant in the whole moduli space?

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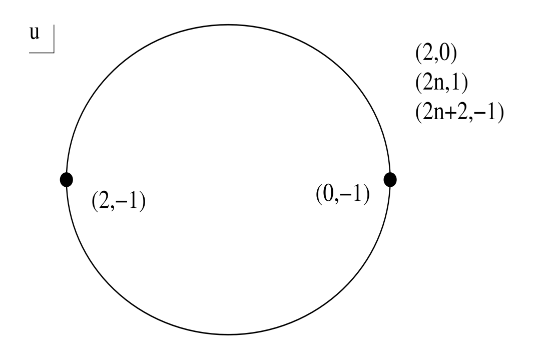
- The wall of marginal stability locates at $Z_{\gamma_1}/Z_{\gamma_2} \in \mathcal{R}^+$

Wall Crossing phenomenon

- $\mathcal{N} = 2$ pure SU(2) SYM in 4D



- How to describe the jump?

Kontsevich-Soibelman's Magic

- Twisted torus algebra

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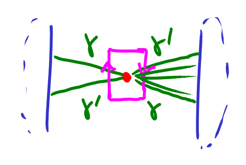

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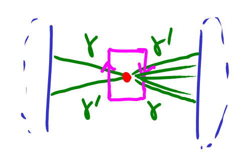

- Wall-crossing formalism [M. Kontsevich and Y. Soibelman, 08']

$$A(P) = \prod_{C_\gamma} K_\gamma^{\pm \Omega(\gamma)} = 1$$

Kontsevich-Soibelman's Magic

- Examples:
  - $\langle \gamma_1, \gamma_2 \rangle = 1$:

    $$K_{\gamma_2} K_{\gamma_1} = K_{\gamma_1} K_{\gamma_1+\gamma_2} K_{\gamma_2}$$

    Argyres-Douglas A_2 theory
  - $\langle \gamma_1, \gamma_2 \rangle = 2$:

    $$\begin{aligned} & K_{(2,-1)} \cdot K_{(0,1)} \\ = \quad & K_{(0,1)} \cdot K_{(2,1)} \cdot K_{(4,1)} \ldots K_{(2,0)}^{-2} \ldots K_{(6,-1)} \cdot K_{(4,-1)} \cdot K_{(2,-1)} \end{aligned}$$

    $\mathcal{N} = 2$ pure SU(2) SYM in 4D

A Proof by Physicists [D. Gaiotto, G.W. Moore and A. Neitzke, 09', 10']

- Adding a very heavy extenal BPS particle with $\arg Z = \theta$:

$$L_\theta = \exp\left[i \int dt \left(A_0 + e^{-i\theta}\phi + e^{i\theta}\bar{\phi}\right)\right]$$

- The Hilbert space is modified:

$$\mathcal{H} \to \mathcal{H}_{L_\theta}, \quad \Omega_\gamma(L_\theta) = \mathrm{Tr}'_{\mathcal{H}_{L_\theta,\gamma}}\left((-1)^{2J_3}\right)$$

- BPS bound: $M \geq \mathrm{Re}(e^{i\theta} Z_\gamma)$
- Any BPS state is a bound state of L_θ and BPS particles in the original theory.
- The bound states appear/disappear when θ crosses $-\arg Z_{\gamma_h}$.
- The bound state configuration is decided by the balance of classical forces

$$r_h = \frac{\langle \gamma_h, \gamma_c \rangle}{2\mathrm{Im}[e^{i\theta} Z_{\gamma_h}]}$$

A Proof by Physicists

- Let $F_{L_\theta} = \sum_\gamma \Omega_\gamma(L_\theta) X_\gamma$
- When θ crosses $-\arg Z_{\gamma_h}$, F_{L_θ} is transformed by

$$X_\gamma \to X_\gamma (1 - X_{\gamma_h})^{\pm\Omega_{\gamma_h}\langle\gamma,\gamma_h\rangle}$$

  It is $K_{\gamma_h}^{\pm\Omega_{\gamma_h}}$!
- Travel along any closed loop, F_{L_θ} must come back to itself.
- If the theory has "enough" line operators L_θ, we will have

$$\prod_{C_\gamma} K_\gamma^{\pm\Omega(\gamma)} = 1$$

Microscopic origin:

- A BPS one-particle state is generically a bound state consisting of more than one charge centers, which are spatially distributed according to balance of classical forces. [K. Lee and P. Yi, 98']

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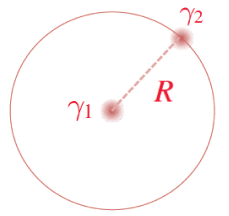
- A BPS one-particle state is generically a bound state consisting of more than one charge centers, which are spatially distributed according to balance of classical forces. [K. Lee and P. Yi, 98']
- Wall-crossing: the size of such bound states become infinitely large as a marginal stability wall is approached, e.g.,



$$R \sim \frac{\langle \gamma_1 , \gamma_2 \rangle}{2\mathrm{Im}[\bar{Z}_{\gamma_1} Z_{\gamma_2}]}$$

Low energy dynamics of BPS States

- Can we computing Ω_γ based on the bound states picture?
- Starting from a set of known BPS states with charge γ_i as a basis set
    - The charge γ of any given BPS state can be written as $\gamma = \sum_i n_i \gamma_i \, (n_i \in \mathbb{Z}^+)$
- The low energy dynamics of n_i i-type BPS states is described by a Quantum Mechanics with four super charges, with $SO(4)$ R- symmetry coming from $SO(3)$ spatial rotation and $SU(2)_R$ in the 4D $\mathcal{N} = 2$ theory.
- A BPS bound state with charge $\gamma = \sum_i n_i \gamma_i$ is related to a SUSY invariant vacuum of the system.
- The corresponding refined BPS index is

$$\Omega(y) = \mathrm{Tr}_{QM}\left((-1)^{2J_3} y^{2(J_3+I_3)}\right)$$

The Coulomb phase dynamics

- The Coulomb phase description: BPS particles interacting with Lorentz force
- Written in $\mathcal{N} = 1$ superspace

$$\Phi^{Aa} = x^{Aa} - i\theta\psi^{Aa}\ , \quad \Lambda^A = i\lambda^A + i\theta b^A$$

  then add the $\mathcal{N} = 4$ SUSY Constraints by hand
- Kinetic term

$$\mathcal{L}_1 = \int d\theta \Big(\tfrac{i}{2} g^{ab}_{AB} D\Phi^A_a \dot{\Phi}^B_b - \tfrac{1}{2} h_{AB} \Lambda^A D\Psi^B - i f^a_{AB} \dot{\Phi}^A_a \Lambda^B + \tfrac{1}{3!} c^{abc}_{ABC} D\Phi^A_a D\Phi^B_b D\Phi^C_c$$
$$+ \tfrac{1}{2!} n^{ab}_{ABC} D\Phi^A_a D\Phi^B_b \Lambda^C + \tfrac{1}{2!} m^a_{ABC} \Phi^A_a \Lambda^B \Lambda^C + \tfrac{1}{3!} l_{ABC} \Lambda^A \Lambda^B \Lambda^C \Big)$$

$\mathcal{N} = 4$ SUSY Constraints: all the couplings are decided by a single function L, e.g.

$$h_{AB} = \delta_{ab} \partial^a_A \partial^b_B L$$

The Coulomb phase dynamics

- Lorentz interaction between dyons:

$$\mathcal{L}_2 = \int d\theta \, \left(i\mathcal{K}_A(\Phi)\Lambda^A - i\mathcal{W}_{Aa}(\Phi)D\Phi^{Aa} \right)$$

$\mathcal{N} = 4$ SUSY constraints

$$\begin{aligned} \partial_{Aa}\mathcal{K}_B &= \tfrac{1}{2}\,\epsilon_{abc}\left(\partial_{Ab}\mathcal{W}_{Bc} - \partial_{Bc}\mathcal{W}_{Ab}\right) \ , \\ \epsilon_{abc}\partial_{Ab}\partial_{Bc}\mathcal{K}_C &= 0 \ , \qquad \partial_{Aa}\partial_{Ba}\mathcal{K}_C = 0 \, . \end{aligned}$$

- The $\mathcal{N} = 4$ SUSY constraints implies

$$\begin{aligned} \mathcal{W}_{Aa} &= \sum_B \frac{\langle \gamma_A, \gamma_B \rangle}{2} \mathcal{W}_a^{Dirac}(\vec{x}^A - \vec{x}^B) \\ \mathcal{K}_A &= \mathrm{Im}\left[e^{-\mathrm{i}\theta}\mathcal{Z}_A\right] = \mathrm{Im}\left[e^{-\mathrm{i}\theta}Z_A\right] - \frac{1}{2}\sum_{B\neq A} \frac{\langle \gamma_A, \gamma_B \rangle}{|\vec{x}_A - \vec{x}_B|} \end{aligned}$$

- The scalar potential $V \sim \mathcal{K}^2$
$\Rightarrow$ The moduli space $\mathcal{M}_n = \left(\{ x^{Aa} \mid \mathcal{K}_A = 0 \} - R^3 \right) / \Gamma$

The Coulomb phase dynamics

- The R symmetry is $SO(4) = SU(2)_L \times SU(2)_R$. $SU(2)_L$ is the rotation group, while $SU(2)_R$ is descendant of $SU(2)$ R-symmetry of the underlying 4D $\mathcal{N} = 2$ theory.
  - The generators:

$$
\begin{aligned}
J_a &= L_a + \sum_A \left(-\frac{i}{8}\, \epsilon_{abc} [\hat{\psi}^{Ab}, \hat{\psi}^{Ac}] - \frac{i}{4} [\hat{\psi}^{Aa}, \hat{\lambda}^A] \right) \\
I_a &= \sum_A \left(-\frac{i}{8}\, \epsilon_{abc} [\hat{\psi}^{Ab}, \hat{\psi}^{Ac}] + \frac{i}{4} [\hat{\psi}^{Aa}, \hat{\lambda}^A] \right)
\end{aligned}
$$

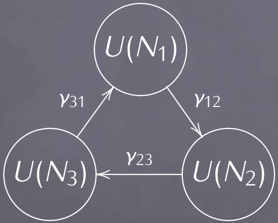
  - x^{Aa} : $(3,1)$; $\psi^{Am} = \{\psi^{Aa}, \lambda^A\}$ and super charge Q_m: $(2,2)$.
- MPS formula:[J. Manschot, B. Pioline and A. Sen, 10′]
  The index is given by a sum of fixed point contributions. Due to the y^{2J_3} factor in the index, the fixed point configurations are the solutions of $\mathcal{K}_A = 0$ with all particles aligning on the z-axis.
- An observation: all the known Coulomb branch BPS states are $SU(2)_R$ singlet.

The Higgs phase dynamics: Quiver Theory

- The Higgs phase description: Quiver quantum mechanics

  [F. Denef, 02']

- Quiver: Nodes+Arrows



$$W = c_{mnp}\Phi_m^{a_1\bar{a}_2}\Phi_n^{a_2\bar{a}_3}\Phi_p^{a_3\bar{a}_1}$$

- Node v: a $U(N_v)$ vector multiplet $(A_v, X_v^i, \lambda_v, D_v)$; a FI parameter ζ_v;
- Arrow $s(v \to w)$:a bifundamental chiral multiplet (ϕ^s, ψ^s, F^s), in the $(\bar{\mathbf{N}}_v, \mathbf{N}_w)$ of $U(N_v) \times U(N_w)$; Number of arrows$=\langle \gamma_v, \gamma_w \rangle$
- Closed loop: a superpotential

The Higgs phase dynamics: Brane picture

- The $N = 2$ theory: Type II on a CY_3
- The BPS particles: The D-branes wrap on a supersymmetric cycle
- $\langle \gamma_1, \gamma_2 \rangle$: the intersecting number between two cycles
- $U(N_i)$ nodes: a basis of the cycles wrapped with N_i D-branes
- Bifundamental fields: open-strings attached between two D-branes on different cycles

The Higgs phase dynamics

- The R symmetry is $SO(4) = SU(2)_L \times SU(2)_R$. (Q_1, Q_2) transforms as a doublet under the rotation group $SU(2)_L$, and $(Q_1, \bar{Q}^{\dot{1}})$ transforms as a doublet under the $SU(2)_R$ which is descendant of $SU(2)$ R-symmetry of the underlying 4D $\mathcal{N} = 2$ theory. Especially, I_3 can be identified as the overall U(1) on Q_α.
- The moduli space: a Kähler manifold

$$M = \{\phi^a \mid \frac{\partial W}{\partial \phi^a} = 0, \sum_{a:\to v} \phi^{a\dagger}\phi^a - \sum_{a:v\to} \phi^a \phi^{a\dagger} = \zeta_v\} / \prod_v U(N_v)$$

- On the moduli space, the rotation $SU(2)$ is identified as the $SU(2)_{\text{Lefschetz}}$

$$L_3 = (l-d)/2 \ , \quad L_+ = K\wedge \ , \quad L_- = K\lrcorner$$

It is acting on the cohomology $H(M) = \bigoplus_l H^l(M)$ and $d = \dim_{\mathbb{C}} M$.

The Higgs phase dynamics

- Acting on $H^{p,q}(M)$, the overall $U(1)$ generator I_3 is identified as
$$I_3 = (p-q)/2$$
Obviously, $[I_3, L_{1,2,3}] = 0$.
- The protected spin character is computed in the Higgs phase as

$$\begin{aligned}\Omega_{\text{Higgs}}(y) &= \text{tr}\,(-1)^{2L_3} y^{2L_3+2I_3} \\ &= \text{tr}\,(-1)^{l-d} y^{l-d+p-q} \\ &= \text{tr}\,(-1)^{p+q-d} y^{2p-d}\end{aligned}$$

Quivers with oriented closed loops

- For quivers without oriented closed loop, $\Omega_C = \Omega_H$
- For quivers with oriented closed loops,
  - $\Omega_C \neq \Omega_H$ in general. [F. Denef, G.W. Moore, 07']
  - The scaling solutions in Coulomb phase make the moduli space non-compact. The naive fixed point formulae is divergent at $y = 1$. The MPS formula with a minimal subtraction scheme which is consistent with wall-crossing was proposed.[J. Manschot, B. Pioline and A. Sen, 10']
  - Superpotential appears in Higgs phase.
  - Both phases share the D term data $\zeta_v = \mathrm{Im}(e^{-i\theta} Z_v)$, but only Higgs phase contains the data of superpotential. Coulomb phase index is related to the ambient space

$$X = \{\phi^a \mid \sum_{a:\to v} \phi^{a\dagger}\phi^a - \sum_{a:v\to} \phi^a\phi^{a\dagger} = \zeta_v\}/\prod_v U(N_v)$$

    of Higgs phase moduli space M?
  - In the $\mathcal{N} = 2$ supergravity, $\Omega_H - \Omega_c$ is related to the index of single centered BPS black holes which are always angular momentum singlet.[J. Manschot, B. Pioline and A. Sen, 10']

Conjectures

- Conjecture I: In the k-th branch of the moduli space

$$\Omega^{(k)}_{\text{Coulomb}}(y) = (-y)^{-d_k} D_k(-y)$$

$i^*_{M_k}(H(X_k))$: pull-back of the ambient cohomology;
d_k: thecomplex dimension of M_k;
$D_k(x)$: the reduced Poincaré polynomial

$$D_k(x) \equiv \sum_l x^l \dim \left[i^*_{M_k}(H^l(X_k)) \right]$$

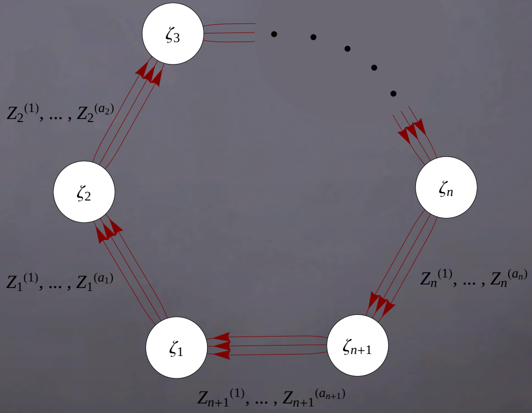
- Conjecture II: The Intrinsic Higgs states in $H(M_k) - i^*_{M_k}(H(X_k))$ are essentially depend on the middle cohomology. The corresponding index

$$(-y)^{-d_k} \chi_{\xi=-y^2}(M_k) - (-y)^{-d_k} D_k(-y)$$

is a branch independent invariant of the quiver.
$\chi_\xi = \sum_p \sum_q (-1)^q h^{p,q} \ \xi^p$: the refined Euler character

Abelian Cyclic Quivers

- Cyclic $(n+1)$-Gon:



Abelian Cyclic Quivers

- D-term conditions

$$
\begin{aligned}
|Z_{n+1}|^2 - |Z_1|^2 &= \zeta_1 \,, \\
|Z_1|^2 - |Z_2|^2 &= \zeta_2 \,, \\
&\vdots \\
|Z_n|^2 - |Z_{n+1}|^2 &= \zeta_{n+1} \,,
\end{aligned}
$$

- Superpotential

$$
W = \sum_{\beta_1=1}^{a_1} \cdots \sum_{\beta_{n+1}=1}^{a_{n+1}} c_{\beta_1 \beta_2 \cdots \beta_{n+1}} Z_1^{(\beta_1)} Z_2^{(\beta_2)} \cdots Z_{n+1}^{(\beta_{n+1})} \,,
$$

- Branches: One of the Z_i vanishing
  - 1. Generic $c_{\beta_1 \beta_2 \cdots \beta_{n+1}} \Rightarrow$ generic F-term algebraic equations.
  - 2. F-term conditions have scaling symmetries $Z_i \to \lambda_i Z_i$.
    $\Rightarrow$ No solution to F-term conditions with all Z_i nontrivial.

Abelian Cyclic Quivers: $i^*_{M_k}(H(X_k))$

- k-th Branch: $\sum_{i=l}^{k} \zeta_i > 0$, $\quad \sum_{i=k+1}^{l} \zeta_i < 0$
- Ambient space

$$X_k = \mathbb{CP}^{a_1-1} \times \cdots \times \mathbb{CP}^{a_{k-1}-1} \times \mathbb{CP}^{a_{k+1}-1} \times \cdots \times \mathbb{CP}^{a_{n+1}-1}$$

- The a_k F-terms $\partial_{Z_k} W = 0$ define a complete intersecting. The complex dimension $d_k = \sum_{i=1}^{n+1} a_i - 2a_k - n$.
- For the ambient space, $H^{p,q}(X_k)$ with $p \neq q$ are null, and

$$P[X_k](x) = \frac{\prod_{i \neq k}(1 - x^{2a_i})}{(1 - x^2)^n} = \sum b_{2l}(X_k) \cdot x^{2l}$$

- Lefschetz hyperplane theorem: $H^{p,q}(M_k)$ with $p \neq q, p + q < d_k$ are null

$$D_k(x) = b_{d_k}(X_k) \cdot x^{d_k} + \sum_{0 \leq 2l < d_k} b_{2l}(X_k) \cdot (x^{2l} + x^{2d_k - 2l})$$

Abelian Cyclic Quivers: $\Omega^{(k)}_{\text{Coulomb}}(y)$

- MPS Formula

$$\Omega^{(k)}_{\text{Coulomb}}(y) \;=\; \frac{(-1)^{\sum_{i=1}^{n+1} a_i - n}}{(y-y^{-1})^n}\left[G_k(y) + (-1)^n G_k(y^{-1}) + H_k(y) + (-1)^n H_k(y^{-1})\right]$$

$$G_k(y) \;+\; (-1)^n G_k(y^{-1}) = \sum_p s(p)\; y^{\sum_{i=1}^{n+1} a_i\, \text{sign}[z_i - z_{i+1}]}\,, \quad s(p) = \text{sign}[\det M]\,,$$

$$M_{i,i} \;=\; a_i \frac{z_i - z_{i+1}}{|z_i - z_{i+1}|^3} + a_{i+1}\frac{z_{i+1} - z_{i+2}}{|z_{i+1} - z_{i+2}|^3}\,,$$

$$M_{i,i+1} \;=\; M_{i+1,i} = -a_{i+1}\frac{z_{i+1} - z_{i+2}}{|z_{i+1} - z_{i+2}|^3}\,.$$

- The subtraction polynomial

$$H_k(y) = \sum_{\substack{0 \le l < n \\ l - \sum_{i=1}^{n+1} a_i \in 2\mathbb{Z}}} \lambda_l\, y^l\,,$$

the coefficients λ_l are decided uniquely by requiring that $\Omega^{(k)}_{\text{Coulomb}}(y)$ is finite when $y = 1$.

Abelian Cyclic Quivers: $\Omega^{(k)}_{\text{Coulomb}}(y)$

- The index is invariant within each branch, so that we may pick a particularly convenient set of FI constants and simplify the problem.
- At $\zeta_k = -\zeta_{k+1} > 0,\ \zeta_i = 0 (i \neq k, k+1)$ the fixed points are

$$|z_k - z_{k+1}| = \frac{a_k}{\rho}, \quad |z_i - z_{i+1}| = \frac{a_i}{\rho + \zeta_k} \quad (i \neq k),$$
$$\sum_{i \neq k} \text{sign}[z_i - z_{i+1}] \frac{a_i}{\rho + \zeta_k} + \text{sign}[z_k - z_{k+1}] \frac{a_k}{\rho} = \sum_i (z_i - z_{i+1}) = 0$$

- The fixed points contribution is

$$G_k(y) = \sum_{\{t_{i \neq k} = \pm 1\}} \Big[\prod_{i \neq k} t_i \Big] \cdot \Theta\Big(\sum_{i \neq k} a_i t_i - a_k \Big)\, y^{\sum_{i \neq k} a_i t_i - a_k},$$
$$\Theta(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

Abelian Cyclic Quivers: Proof of conjecture I

- Uniqueness of the H_k was guaranteed by three requirements: regularity of index at $y = 1$, definite parity of G_k, and parity of H_k coinciding with that of G_k.
- The first conjecture is equivalent to

$$y^n G_k(y^{-1})\Big|_{\text{nonpositive}} = y^n \tilde{G}_k(y^{-1}) \equiv \left(y^{-d_k} \prod_{i \neq k} \left(1 - y^{2a_i}\right) \right)\Bigg|_{\text{nonpositive}}$$

- Proof:

$$\begin{aligned}
& y^n G_k(y^{-1})\Big|_{\text{nonpositive}} \\
= \quad & y^n \sum_{\{t_{i\neq k}=\pm 1\}} \left[\prod_{i\neq k} t_i\right] \cdot \Theta\Big(\sum_{i\neq k} a_i t_i - a_k\Big)\, y^{-\sum_{i\neq k} a_i t_i + a_k}\Bigg|_{\text{nonpositive}} \\
= \quad & \sum_{\{t_{i\neq k}=\pm 1\}} \left[\prod_{i\neq k} t_i\right] \cdot \Theta\Big(\sum_{i\neq k} a_i t_i - a_k - n\Big)\, y^{-\sum_{i\neq k} a_i t_i + a_k + n}\Bigg|_{\text{nonpositive}} \\
= \quad & y^n \tilde{G}_k(y^{-1})\ .
\end{aligned}$$

Abelian Cyclic Quivers: $H(X_k)$

- The Adjunction formula:

$$\begin{aligned}
\mathrm{td}(\mathcal{T}M_k) &= \left[\prod_{i\neq k}\left(\frac{J_i}{1-e^{J_i}}\right)^{a_i}\right]\cdot\left(\frac{1-e^{-\sum_{i\neq k}J_i}}{\sum_{i\neq k}J_i}\right)^{a_k} \\
\mathrm{ch}_\xi(\mathcal{T}^*M_k) &= \sum_p \mathrm{ch}(\wedge^p\mathcal{T}^*M_k)\,\xi^p = \left[\prod_{i\neq k}\frac{(1+\xi e^{-J_i})^{a_i}}{1+\xi}\right]\cdot\left(\frac{1}{1+\xi e^{-\sum_{i\neq k}J_i}}\right)^{a_k}
\end{aligned}$$

where J_i is the Kähler form from each $\mathbb{CP}^{a_i-1}$ factor in X_k.

- Applying the Hirzebruch-Riemann-Roch formula

$$\begin{aligned}
\chi_\xi(M_k) &= \int_{M_k}\mathrm{td}(\mathcal{T}M_k)\cdot\mathrm{ch}_\xi(\mathcal{T}^*M_k) \\
&= \int_{X_k}\mathrm{td}(\mathcal{T}M_k)\cdot\mathrm{ch}_\xi(\mathcal{T}^*M_k)\cdot\Big(\sum_{i\neq k}J_i\Big)^{a_k} \\
&= \frac{1}{(1+\xi)^n}\int_{X_k}\left[\prod_{i\neq k}\left(J_i\frac{1+\xi e^{-J_i}}{1-e^{-J_i}}\right)^{a_i}\right]\cdot\left(\frac{1-e^{-\sum_{i\neq k}J_i}}{1+\xi e^{-\sum_{i\neq k}J_i}}\right)^{a_k}
\end{aligned}$$

Abelian Cyclic Quivers: Proof of conjecture II

- Let $\omega_i \equiv e^{-J_i}$,

$$\begin{aligned}
\Omega^{(k)}_{\text{Higgs}}(y) &= (-y)^{-d_k}\chi_{\xi=-y^2}(M_k) \\
&= (-1)^{d_k} y^{a_k} \prod_{i\neq k} \frac{y^{a_i}-y^{-a_i}}{y-y^{-1}} \\
&\quad + \frac{(-y)^{n+2-\sum_i a_i}}{(y^2-1)^n} \prod_i \oint_{\omega_i=1} \frac{d\omega_i}{2\pi i} \left[\prod_i \left(\frac{1-y^2\omega_i}{1-\omega_i}\right)^{a_i}\right] \cdot \frac{1}{1-y^2\prod_i \omega_i}
\end{aligned}$$

$$\Rightarrow \quad \Omega^{(k)}_{\text{Higgs}}(y) - \Omega^{(k')}_{\text{Higgs}}(y) = (-1)^{d_k-1} \frac{y^{a_k-a_{k'}}-y^{-a_k+a_{k'}}}{y-y^{-1}} \prod_{i\neq k,k'} \frac{y^{a_i}-y^{-a_i}}{y-y^{-1}}$$

- From the expression of $G_k(y)$, we get

$$\begin{aligned}
&\frac{G_k(y)+(-1)^n G_k(y^{-1}) - G_{k'}(y) - (-1)^n G_{k'}(y^{-1})}{(y-y^{-1})^n} \\
&= (-1)^{d_k-1} \frac{y^{a_k-a_{k'}}-y^{-a_k+a_{k'}}}{y-y^{-1}} \prod_{i\neq k,k'} \frac{y^{a_i}-y^{-a_i}}{y-y^{-1}} \\
&= \Omega^{(k)}_{\text{Coulomb}}(y) - \Omega^{(k')}_{\text{Coulomb}}(y)
\end{aligned}$$

- $\Omega^{(k)}_{\text{Higgs}}(y) - \Omega^{(k)}_{\text{Coulomb}}(y)$ is independent of k.

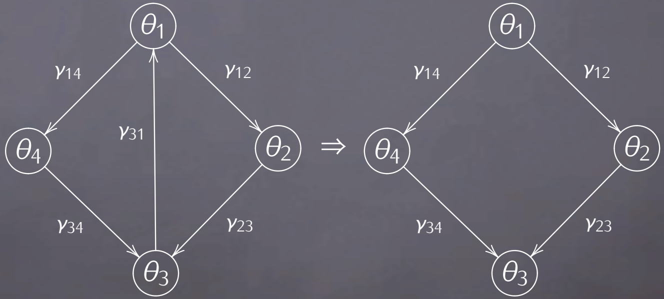
Numerical illustration

- 3-gon with $a_1 = 4, a_2 = 5, a_3 = 6$

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & 0 & & 2 & & 0 & & & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & 0 & & 0 & & 3 & & 0 & & 0 & \\
0 & & 0 & & 26 & & 26 & & 0 & & 0 \\
 & 0 & & 0 & & 3 & & 0 & & 0 & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & & 0 & & 2 & & 0 & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & & & 1 & & & & &
\end{array}
\;,\qquad
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 2 & & 0 & \\
0 & & 26 & & 26 & & 0 \\
 & 0 & & 2 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
\;,\qquad
\begin{array}{ccc}
 & 1 & \\
26 & & 26 \\
 & 1 &
\end{array}
\;.
$$

More general quivers

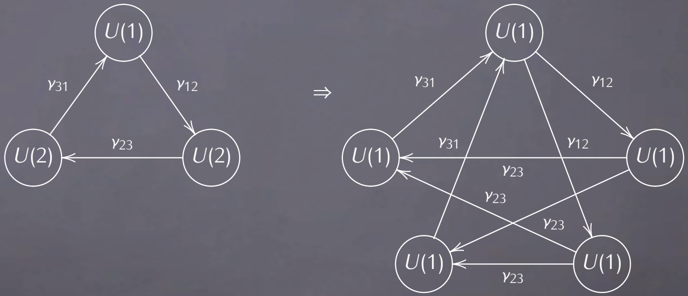
- Ambient space: maximal reduced quiver without loop
  The branches for a multi-loop quiver are described by the non-empty branches of its maximal reduced quiver without loop, e.g.



  - Vanishing of the edge 31: $\theta_3 > 0\,, \quad \theta_1 < 0\,.$
  - Four different branches depending on the sign of θ_2 and θ_4

More general quivers

- Abelian quiver: Toric data, easy to deal with
- How to deal with nonabelian gauge group?[Bumsig Kim et.al]
  - Fully Abelianize varieties $\tilde{X}$ of the nonabelian varieties X



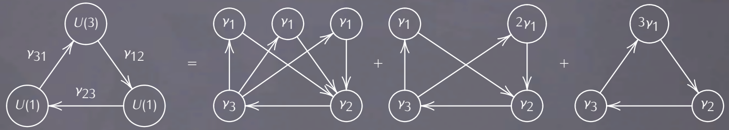
  - Additional insertion:

$$\int_X a = \frac{1}{|W|}\int_{\tilde{X}} \hat{a} \wedge \frac{e(\Delta)}{c(\Delta)}$$

e.g. for a single $U(n)$: $\prod_{i<j} \frac{-(J_i-J_j)^2}{1-(J_i-J_j)^2}$

More general quivers

- Coulomb phase computation: MPS type of partition sum



- A one to one map for cases without intrinsic Higgs:

$$\frac{-(J_i - J_j)^2}{1-(J_i - J_j)^2} = 1 - \frac{1}{1-(J_i - J_j)^2} = 1 + \delta_{ij}$$

$$\frac{e(\Delta)}{c(\Delta)} = 1+(\delta_{12}+\delta_{13}+\delta_{23})+(\delta_{12}\delta_{13}+\delta_{12}\delta_{23}+\delta_{13}\delta_{23}+\delta_{12}\delta_{13}\delta_{23})$$

Summary

- The relation between PSC in 4D $\mathcal{N}=2$ theory and refined index in $\mathcal{N}=4$ QM.
- Proof of the two conjecture about the Coulomb phase and the Higgs phase indices for cyclic quiver.
- More general quivers:
  - The basic way of computing Higgs cohomology is established
  - Certain smart improvement is needed
- More future directions:
  - Can we directly compute $\Omega_{Intrinsic}$, e.g., by localization...?
  - The relation between $\Omega_{Intrinsic}$ and black hole entropy
  - Whether and how Kontsevich-Soibelman algebra know about the quiver invariants?

Thank You!