Scalar Black Holes

Hong Lü
Beijing Normal University

Xi-An

Wednesday, April 9, 2014

Motivations

- Try to understand the first law of black hole thermodynamics when a system involves fields with no conserved quantities.
- Constructing analytic static hairy black holes, and verify the new first law. (Always, the more solutions the better it is.)
- Try to find analytical solutions describing formation of black holes. (The more static solutions, the better chance we shall have.)

Scalars are the simplest and most versatile to address all these questions.

Talk based on

- arXiv:1401:0010, Phys. Lett. B730 (2014) 267-270, with Hai-Shan Liu.
- arXiv:1312.5374, Phys.Rev. D89 (2014) 044014, with Xing-Hui Feng and Wen Qiang
- arXiv:1402.5153, to appear in JHEP, with Hai-Shan Liu and C.N. Pope
- arXiv:1307.6243, JHEP 1311 (2013) 033, with Yi Pang and C.N. Pope
- arXiv:1403.6874, with Xue-feng Zhang

Introduction

Black holes are perhaps the most interesting objects predicted by Einstein's General Relativity.

Black holes may actually exist in our universe, and if they do, there are many of them: $> 10^{20}$.

But the motivation of studying black holes are largely theoretical.

黑洞研究的理论价值

黑洞定义：有个（有限）视界面

- 黑洞有不可避免的引力塌缩奇点，因而证明经典广义相对论是不完备的。(如果假定质量和体积成正比的话，牛顿引力就没有这方面的困境。)
- 但黑洞奇点，也就是时空撕裂，又被视界面遮住。（因此不是裸奇点。）
- 黑洞的霍金蒸发，是一个我们“唯一”的比较了解的(半经典)量子引力过程。(温度$\sim \hbar$，熵$\sim 1/\hbar$.)
- 黑洞的无毛定理，意味着黑洞是世界上最简单，最纯粹的热力学系统。
- 黑洞热力学第一定律：$dM = TdS + \Omega dJ + \cdots$.

注：黑洞的无毛定理：黑洞视界面外部的所有性质，完全由守恒量描述，比如质量，角动量，电荷等。

However

The uniqueness is known to be violated in higher dimensions, e.g. black rings in $D = 5$.

If we specify the topology, are black holes unique again once all the conserved quantities are specified? (In particular, what about spherically-symmetric and static black holes?)

We shall address this question in Einstein gravity with a minimally coupled scalar.

$$\mathcal{L}_D = \sqrt{-g}\Big(R - \tfrac{1}{2}(\partial\phi)^2 - V(\phi)\Big)\,.$$

However, when additional fields are involved, how we define “hair” ?

Scalar "charge"

In a typical black hole, all the conserved quantities participate in the first law of black hole thermodynamics, e.g. $dM = TdS + \Omega dJ + \Phi dQ$.

We would like to define the quantities appearing the first law as charges and their thermodynamical potentials. Anything not in the first law as "hair."

A scalar with a potential does not have a conserved quantity such as mass or electric/magetic charges. It satisfies a second-order differential equation

$$\Box\phi = \frac{\partial V}{\partial\phi}\,.$$

Thus there are in general two integration constants associated with the scalars. Are they viewed as "charges" or instead "hairs" of the scalar?

Will these integration constants modify the first law?

Black holes in supergravities

String and supergravities have a large number of scalars. Some of them involves in charged black hole solutions.

In ungauged supergravity, such a scalar ϕ (massless) typically have a constant global shifting symmetry under $\phi \to \phi + c$. This has a consequence that at large r, we have

$$\phi = \phi_0 + \frac{q e^{a\phi_0}}{r^{D-3}} + \cdots ,$$

One can treat ϕ_0 as a variable so that the first law is no longer valid unless one introduces a potential to $d\phi_0$ to obtain a new first law. (Gibbons,Kallosh,Kol). (This however is somewhat trivial.)

If a scalar has a scalar potential, such as in gauged supergravities, this argument is no longer valid.

All the known examples (many of them) of black holes seemed to suggest that scalar quantities do not participate in the first law.

Until recently.

KK dyonic black hole

Recently, we discovered an example for which the first law appears not valid.

Consider the KK theory from $D = 5$:

$$\mathcal{L}_5 = \sqrt{-\hat{g}}\hat{R} \quad \longrightarrow \quad \mathcal{L}_4 = \sqrt{-g}\Big(R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{4}e^{-\sqrt{3}\phi}F^2\Big).$$

It admits the a dyonic black hole with both electric and magnetic charges (Q, P). The first law $dM = TdS + \Phi_Q dQ + \Phi_P dP$ holds as one would expect.

The solution can be described as a black hole bomb since in extremal limit

$$M = \Big(Q^{\frac{2}{3}} + P^{\frac{2}{3}}\Big)^{\frac{3}{2}} > Q + P = M_Q + M_P\,,$$

Generalization to multi-charge and higher dimensions can be found in 1307.2305 (HL, W. Yang).

Kaluza-Klein AdS dyon

Gauged supergravity:

$$\mathcal{L} = \sqrt{-g}\Big(R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{4}e^{-\sqrt{3}\,\phi}F^2 + 6g^2\cosh(\tfrac{1}{\sqrt{3}}\phi)\Big)\,,$$

admits the Kaluza-Klein AdS dyonic black hole. (1307.6243 (HL,Y.Pang,C.N.Pope))

With the scalar potential, the naively-expected first law of thermodynamics

$$dM = TdS + \Phi_Q dQ + \Phi_P dP$$

no longer works, except for four cases $P=0$ or $Q=0$ or $P=Q$ or $g=0$ for general (P,Q). We can introduce a modified first law by introducing $Z=-XdY$, such that

$$dM = TdS + \Phi_Q dQ + \Phi_P dP + XdY$$

What is the origin of (X,Y)?

Scalar charge in asymptotic AdS$_4$

For asymptotic AdS geometry, the asymptotic behavior of a massless scalar is given by

$$\phi = \frac{\phi_1}{\rho} + \frac{\phi_2}{\rho^2} + \cdots \,,$$

the boundary condition for preserving all asymptotic AdS symmetry is (1) $\phi_1 = 0$ or (2) $\phi_2 = 0$ or (3) $\phi_2 = c\phi_1^2$. (e.g. Hertog and Horowitz.) The analysis of the Wald canonical charge indicates that

$$\delta\mathcal{H}_\infty = \delta M + Z \,, \qquad Z = \tfrac{1}{12} g^2 (2\phi_2\delta\phi_1 - \phi_1\delta\phi_2) \,.$$

In the cases where $\phi_1 = 0$ or $\phi_2 = 0$ and $\phi_2 \sim \phi_1^2$, the quantity Z vanishes and the usual thermodynamics holds. (These are precisely three boundary conditions that preserve all the asymptotic AdS symmetries.) For the dyonic black hole, we find

$$Z = -X dY \,.$$

Thus the AdS dyon provides the first example of exact solutions in gauged supergravity that carries non-trivial scalar charge (that modifies the first law.)

H.L. Y. Pang and C.N. Pope, 1307.6243.

Controversy

Chow and Compére (1311.1204) generalized our dyonic AdS black hole to a more general class of solutions in the STU model. Their interpretation of the violation of the usual first law as the consequence of that the system no longer has the proper definition of mass.

We prefer to take a view that one can still give a sensible meaning of the mass, but with the first law modified with dM replaced by $dM + Z$.

At least we have demonstrated that

$$dM + Z = TdS + \Phi_Q dQ + \Phi_P dP$$

is well defined, even if the physical meaning of M is in question.

Scalar charge in asymptotic AdS$_n$

This phenomenon is more general. If we consider n dimensions:

$$e^{-1}\mathcal{L}_n = R - \tfrac{1}{2}(\partial\phi)^2 - V(\phi)\,,$$

with $V'(0) = 0$, $V(0) = -(n-1)(n-2)g^2$ and

$$V = V(0) - \tfrac{1}{2}m_0^2\phi^2 + \cdots\,, \quad m_0^2 = -\tfrac{1}{4}n(n-2)g^2$$

The scalar at asymptotic infinity have the following boundary conditions

$$\phi = \frac{\phi_1}{\rho^{(n-2)/2}} + \frac{\phi_2}{\rho^{n/2}} + \cdots$$

where ρ is the radius of the foliating sphere of the AdS. We show that the usual first law of thermodynamics no longer works, and it must be modified with dM replaced by $dM + Z$, where

$$Z = \tfrac{\omega_{n-2}g^2}{32\pi(n-1)}\Big(n\phi_2 d\phi_1 - (n-2)\phi_1 d\phi_2\Big)\,.$$

$Z = 0$ if (1) $\phi_1 = 0$; (2) $\phi_2 = 0$; (3) $\phi_2/\phi_1^{(n-2)/n}$ is fixed constant. (or (4) $g = 0$.)

Hai-Shan Liu, H.L. 1401.0010.

Modified first law by the scalar charge

$m_0^2 = -\frac{1}{4}n(n-2)g^2$ corresponds to a "conformal massless" scalar. Such a scalar can arise in $n = 4, 6$ gauged supergravities, but **not** in $n = 5, 7$.

The phenomenon implies that the dual field theories depend on the boundary conditions of the conformal massless scalar.

Black holes: $dM + Z = TdS$

Solitons: $dM + Z = 0$.

Further comments on the first law

The first law is not modified in asymptotic flat spacetimes. The reason is simple. A field with mass m behave in asymptotic spacetimes as

$$\phi \sim \frac{1}{r^{D-3}}\Big(c_1 e^{-mr} + c_2 e^{mr}\Big)\,.$$

Thus one of the c_i can be allowed in the solution, and hence it *cannot* contribute to the first law.

In asymptotic AdS, on the other hand, one has

$$\phi \sim \frac{c_1}{r^\alpha} + \frac{c_2}{r^\beta}\,.$$

both (c_1, c_2) can exist in the solution, and form a thermodynamical conjugate pair.

Einstein-Proca theory

$$\mathcal{L} = R\,{*\mathbf{1}} + (n-1)(n-2)\ell^{-2}\,{*\mathbf{1}} - 2{*F}\wedge F - 2m_0^2{*A}\wedge A\,.$$

where

$$\sigma = \sqrt{4m_0^2\ell^2 + (n-3)^2}\,.$$

Black hole asymptotics:

$$A_0 \sim \frac{q_1}{r^{(n-3-\sigma)/2}} + \frac{q_2}{r^{(n-3+\sigma)/2}}\,.$$

We show that the first law is given by

$$dM = TdS - \frac{\omega_{n-2}}{4\pi}\sigma\, q_1 dq_2\,.$$

ArXiv:1402.5153, with Hai-Shan Liu and C.N. Pope

A general conjecture

A general conjecture is that all the integration constants in a second-derivative theories, when existing in pairs, can contribute the first law. $1 \times dM - p_i dq_i = TdS$.

Interestingly, the integration constants of the graviton give mass and a scaling factor that scales the asymptotic vacuum time, which is chosen to be 1.

Exact scalar black holes

Two types of scalar black holes

- Scalar black hole with scalar charges that enter the first law
- Scalar black hole without scalar charge: hairy black hole

The task is not as simple.

Scalar might be the simplest field in QFT, but its scalar potential V can be written with an infinite possibility.

Thus for a random given scalar potential, the chance of finding an exact solution is more or less zero.

We shall focus on the second type. (Because we could not find an analytic example of the first type, although numerical solutions can be constructed easily.)

Recent Progress

- A. Anabalon, arXiv:1204.2720.
- A. Anabalon, D. Astefanesei, R. Mann, 1308.1693.
- A. Anabalon, D. Astefanesei, 1309.5863.
- P.A. Gonzalez, E. Papantonopoulos, J. Saavedra, Y. Vasquez, 1309.2161.
- A. Acena, A. Anabalon, D. Astefanesei, R. Mann, 1311.6065.

By and large in four dimensions.

The results are given in a very unapparent coordinate system.

Our approach is motivated by p-branes

In a typical construction of isotropic black p-branes, the metric ansatz is given by (hep-th/9604052,hep-th/9606033)

$$ds^2 = e^{2A}(-f dt^2 + dx^i dx^i) + e^{2B}(f^{-1} dr^2 + r^2 d\Omega^2_{D-d-1}),$$

where $d = p+1$ and the metric functions A, B, f and the involved scalars ϕ_i are functions of r only. In order for the solution to have a regular horizon, one needs to impose a relation

$$(p+1)A + (D-p-3)B = 0$$

that is consistent with the equations of motion. The vanishing of f at certain $r = r_0$ gives rise to the horizon whilst the functions A and B run smoothly from the horizon to the asymptotic infinity. It turns out the dilaton scalars ϕ_i depend on the function A but it is independent of the function f. In the case of charged AdS black holes in gauged supergravities, the contribution from the scalar potentials only modifies the function f, but leaves the relation between A and ϕ_i unchanged. This suggests that the relation between A and ϕ_i can be determined independent of a scalar potential. This observation allows us to construct a class of scalar hairy black holes.

General formalism

We consider Einstein gravity in general dimensions coupled to a scalar ϕ, with a scalar potential:

$$e^{-1}\mathcal{L} = R - \tfrac{1}{2}(\partial\phi)^2 - V(\phi)\,.$$

The equations of motion are

$$\Box\phi = \frac{dV}{d\phi}\,, \qquad E_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2}\partial_\mu\phi\partial_\nu\phi - \tfrac{1}{D-2}V g_{\mu\nu} = 0\,.$$

Inspired by the black p-brane ansatz, we consider the following static and spherically symmetric black hole ansatz ($p = 0$):

$$ds^2 = -H^{-1} f dt^2 + H^{\frac{1}{D-3}}\Big(\frac{dr^2}{f} + r^2 d\Omega_{D-2}^2\Big)\,.$$

We find that the specific combination $E^t{}_t - E^r{}_r = 0$ of the Einstein equations of motion implies

$$\frac{2(D-3)}{D-2}\phi'^2 = \frac{H'^2}{H^2} - \frac{2H''}{H} - \frac{2(D-2)H'}{rH}\,.$$

Thus we see indeed that H and ϕ can be determined from each other, independent of the function f and the scalar potential.

The function f can be solved from the combination $E^t{}_t - E^i{}_i = 0$, where the index i denotes any specific sphere direction. We find

$$\frac{H''}{H} + \frac{H'}{H}\Big(\frac{f'}{f} - \frac{H'}{H} + \frac{D-2}{r}\Big)$$

$$-\frac{(D-3)}{(D-2)r^2 f}\Big(r^2 f'' + (D-4)rf' - 2(D-3)(f-1)\Big) = 0\,,$$

This is a second-order linear equation with a source for f and can be solved in a variety of situations. The scalar potential can be determined by the remaining Einstein equations, given by

$$V = -\frac{r^2 f'' + (3D-8)rf' + 2(D-3)^2(f-1)}{2r^2 H^{\frac{1}{D-3}}}\,.$$

Thus provided with an ansatz for ϕ or H, the remaining functions and the scalar potential can be solved consecutively.

Summary of philosophy

In the usual construction of black holes, one starts a specific theory and then proposes some ansatz and proceeds to solve the equations of motion. In this approach, instead of considering some specific choice of the scalar potentials, we start with some reasonable simple assumption for the solution of the scalar. We can then determine H and f and finally V consecutively.

Caution: The scalar potential obtained from eom is a function of the radial variable r. We must then convert the radial dependence from $\phi(r)$ to obtain $V(\phi)$. Thus, if we start with a specific function $\phi(r)$ involving a constant q, the conversion may give rise to a potential $V(\phi)$ that also depends on q explicitly. If this happens, we cannot regard the parameter q as an integration constant in the solution, but rather it is a pre-fixed parameter in the Lagrangian. Such a parameter cannot participate in the black hole first law of thermodynamics. In our construction, we are mainly interested in solutions with parameters that do not explicitly appear in the scalar potential and hence they are true integration constants.

A p-brane inspired scalar

In supergravities with flat spacetime vacua, a linearized scalar satisfies the massless equation $\Box\phi = 0$. It falls off as

$$\phi = \alpha + \frac{\beta}{r^{D-3}} + \cdots .$$

In gauged supergravities with AdS vacua, a linearized scalar satisfies instead

$$(\Box + 2(D-3)g^2)\phi = 0 \,,$$

It falls off as

$$\phi = \frac{\alpha}{r^2} + \frac{\beta}{r^{D-3}} + \cdots .$$

We call such a scalar massless since when embedded in (gauged) supergravities, the scalar may belong to the same super-multiplet as the graviton.

We are now in the position to construct explicit scalar hairy black holes. To make contact with (gauged) supergravity solutions, we are interested in a scalar that is massless. Furthermore, we require that the solution have a smooth $g = 0$ limit. The most general solution of ϕ at the large r expansion must then take the form

$$e^{\phi} = 1 + \frac{q}{r^{D-3}} + \cdots ,$$

Inspired by the p-brane construction, we consider the ansatz

$$\phi = \sqrt{\frac{D-2}{2(D-3)}}\,\nu \log \frac{H_1}{H_2} \qquad \text{with} \qquad H_i = 1 + \frac{q_i}{r^{D-3}}\,,$$

where ν is a constant that parameterizes some degrees of choices.

The remains are then some simple details, and we can find H and f and finally V respectively.

Note that this procedure ensures that the black hole, if exist, must be hairy, since there can be no scalar charge that could contribute the first law.

Once the horizon condition is specified, in a generic theory, both falloff modes in the asymptotic infinity would be turned on. However, we have lots of freedom to tune the scalar potential such that the slower falloff mode does not get turned on.

Local solution

$$
\begin{aligned}
H &= H_1^{1+\mu} H_2^{1-\mu}, \qquad \phi = \sqrt{\frac{D-2}{2(D-3)}}\, \nu \log \frac{H_1}{H_2}, \\
f &= H_1 H_2 + g^2 r^2 \left(H_1^{1+\mu} H_2^{1-\mu}\right)^{\frac{D-2}{D-3}} \\
&\quad -\alpha\, r^2 H_2 (H_1 - H_2)^{\frac{D-1}{D-3}}\, {}_2F_1[1, \tfrac{D-2}{D-3}(1+\mu); \tfrac{2(D-2)}{D-3}; 1 - \tfrac{H_2}{H_1}]\,.
\end{aligned}
$$

$$
\mu^2 + \nu^2 = 1, \qquad H_i = 1 + q_i / r^{D-3}
$$

The function f has two new integration constants g and α.

The solutions are asymptotic to AdS in global coordinates with the cosmological constant

$$
\Lambda = -(D-1)(D-2) g^2
$$

where $\ell = 1/g$ is the AdS radius.

The scalar potential that is responsible for such a solution given by

$$
\begin{aligned}
V \;=\; & -\tfrac{1}{2}(D-2)g^2 e^{\frac{\mu-1}{\nu}\Phi}\Big[(\mu-1)((D-2)\mu-1)e^{\frac{2}{\nu}\Phi} \\
& -2(D-2)(\mu^2-1)e^{\frac{1}{\nu}\Phi}+(\mu+1)((D-2)\mu+1)\Big] \\
& -\tfrac{(D-3)^2}{2(3D-7)}(\mu+1)\alpha\, e^{-\frac{1}{\nu}(4+\frac{\mu+1}{D-3})\Phi}(e^{\frac{1}{\nu}\Phi}-1)^{3+\frac{2}{D-3}} \\
& \times\Big[(3D-7)e^{\frac{1}{\nu}\Phi}\,{}_2F_1[2,1+\tfrac{(D-2)(\mu+1)}{D-3};3+\tfrac{2}{D-2};1-e^{\frac{1}{\nu}\Phi}]+ \\
& -\Big((3D-7)+(D-2)(\mu-1)\Big)\times \\
& \times\,{}_2F_1[3,2+\tfrac{(D-2)(\mu+1)}{D-3};4+\tfrac{2}{D-2};1-e^{\frac{1}{\nu}\Phi}]\Big]\,,
\end{aligned}
$$

where

$$\Phi=\sqrt{\tfrac{2(D-3)}{D-2}}\,\phi\,.$$

It is important to note that the parameters (q_1, q_2) disappear in the scalar potential, and hence they are integration constants from solving the equations of motion. On the other hand, the constants $\mu^2+\nu^2=1$, g^2 and α in the function f also appear in the scalar potential. These parameters specify a class of theories and they do not participate in the first law of thermodynamics.

Comments

- When $\alpha = 0$, the scalar potential for appropriate μ can arise in gauged supergravities in low-lying dimensions.

- For $D = 4$ and $D = 5$, the hypergeometric function can be expressed in terms of polynomials and potential becomes much simpler.

- In $D = 4$, the α-deformed potential can reproduce those in ω-deformed gauged supergravities. A. Anabalon and D. Astefanesei, 1311.7459.

Explicit examples in $D=4$

Scalar potential

$$V=(1-\alpha)V_0(\phi)+\alpha V_0(-\phi)$$

$$V_0(\phi)=g^2e^{-\frac{1-\mu}{\nu}\phi}\Big((\mu-1)(2\mu-1)e^{\frac{2}{\nu}\phi}-4(\mu^2-1)e^{\frac{1}{\nu}\phi}+(\mu+1)(2\mu+1)\Big)\,.$$

The solution

$$ds^2=-H^{-1}fdt^2+H(\frac{dr^2}{f}+r^2d\Omega_2^2)\qquad H=H_1^{1+\mu}H_2^{1-\mu}.$$

$$f=H_1H_2+(1-\alpha)g^2r^2H_1^{2(1+\mu)}H_2^{2(1-\mu)}$$

$$+\alpha g^2r^2H_1H_2\Big(\mu(2\mu+1)H_1^2-(4\mu^2-1)H_1H_2+\mu(2\mu-1)H_2^2\Big)\,,$$

The system can be embedded in ω-dependent $N=8$ gauged supergravity when $\mu=0,\frac{1}{4},\frac{1}{2},\frac{3}{4}$. Let $\alpha=\sin^2\omega$, then $\mu=0\to N=4$ truncation, $\mu=\frac{1}{2}\to$ KK truncation, both are independent of ω. $\mu=\frac{1}{4},\frac{3}{4}$ gives the ω-dependent $N=8$ solutions.

Global analysis

We can show that the local solutions can describe a black hole. For a given mass M, our theory admits two black holes: one is the Schwarzschild, and another is the new scalar one. They have different temperature and entropy, but satisfy the same first law of thermodynamics

$$dM = TdS\,.$$

The uniqueness theorem is broken (in both asymptotic flat and AdS) and the black holes are hairy indeed.

However

All the exact solutions of scalar black holes, ours or in the literature, have vanishing Z.

Kaluza-Klein AdS dyon (and its STU generalizations) remain the only example with non-vanishing Z.

Charged system

We can construct exact solutions of hairy charged black holes.

Exact formation of black holes

There hitherto exists no exact solution (from a Lagrangian) describing gravitational collapsing to a black hole.

A necessary condition for such a solution: Existing a static (or stationary) black hole.

Before our constructions: not too many static black holes known. The simplest is the Schwarzschild.

Now we have an infinite number of static scalar black holes, surely at least one of them allow us to generalize to time-dependent solution?

The answer is YES!

黑洞是如何形成的呢？

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega_2^2, \qquad f = 1 - \frac{2GM}{c^2 r}$$

Schwarzschild半径：$r_0 = 2M$. (Using natural units $G = 1 = c$.)

（均匀）物质分布质量 $M \propto r^3$

因此即使密度不高的物体，只要足够大，也可形成视界面，成为黑洞。

一旦视界面形成，视界面内的奇点就不可避免地会产生（在爱因斯坦经典引力框架下）。Cosmic censorship hypothesis.

但黑洞具体的形成过程还没有一个精确解来描述。

AdS/CFT: a new motivation

引力-场论对偶，也就是AdS时空对偶一个在AdS边界上的强耦合场论，为强耦合场论研究提供了一个有效理论。

AdS黑洞对应于一个有温度的场论对偶时空。

AdS中的引力塌缩及黑洞形成，为场论非平衡热力学系统的研究提供了一个新方法。

Vaidya metric

Kerr-Schild form

$$
\begin{aligned}
ds^2 &= -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega_2^2 = -f(dt^2 - \frac{dr^2}{f^2}) + \cdots \\
&= -f(dt + \frac{dr}{f})(dt - \frac{dr}{f}) + \cdots .
\end{aligned}
$$

Define $dt + \frac{dr}{f} = du$, one has

$$
ds^2 = -f du(du - 2\frac{dr}{f}) + \cdots = 2dudr - f du^2 + r^2 d\Omega_2^2 .
$$

Vaidya metric

$$
ds^2 = 2dudr - \left(1 - \frac{2M(u)}{r}\right) du^2 + r^2 d\Omega_2^2 .
$$

Energy-Momentum tensor $T_{uu} = \frac{2M'(u)}{r^2}$. $T^{\mu\nu}T_{\mu\nu} = 0$ and $T^{\mu}{}_{\mu} = 0$. Pure radiation energy absorbtion.

New static black holes

It is clear that a necessary condition to have an exact solution of black hole formation is that there exists an exact solution of static (or stationary) black hole.

The study of the Schwarzschild black hole formation has been done over decades, and we do not expect that we can do better.

What about the formation of some new static black hole?

Scalar-hairy black holes

Recently, we have obtained an infinite number of scalar theories that admits scalar-hairy black holes. Here is one of them

$$\mathcal{L} = \sqrt{-g}\Big(R - \tfrac{1}{2}(\partial\phi)^2 - V(\phi)\Big)\,,$$
$$V = -2g^2(\cosh\phi + 2) - 2\alpha^2(2\phi + \phi\cosh\phi - 3\sinh\phi)\,,$$

Two parameters (g^2, α^2), both of which can be positive, 0, or negative. $\alpha = 0$ can be embedded in $\mathcal{N} = 4$, $D = 4$ gauged supergravity.

The potential has a stationary point $\phi = 0$, with

$$V(0) = -6g^2 = 2\Lambda\,, \qquad \Lambda = -3g^2\,.$$

This Lagrangian was first introduced by Zloshchastiev, Phys. Rev. Lett. **94**, 121101 (2005).

Scalar-hairy black holes

Static black hole

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r(r+q) d\Omega^2_{2,k}\,, \qquad e^\phi = 1 + \frac{q}{r}\,,$$
$$f(r) = g^2 r^2 + k - \tfrac{1}{2}\alpha^2 q^2 + (g^2 - \alpha^2) q r + \alpha^2 r^2 \left(1 + \frac{q}{r}\right) \log\left(1 + \frac{q}{r}\right).$$

One integration constant q, the scalar "charge." Curvature singularity $r = 0$ and $r = -q$, Event-Horizon condition

$$f(r=0) = 1 - \tfrac{1}{2}\alpha^2 q^2 < 0\,, \qquad q > \frac{\sqrt{2}}{\alpha}$$

Mass of the black hole $M_0 = \alpha^2 q^3/12$. First law $dM = TdS$

It is worth remarking that for a given mass, the theory also admits a Schwarzschild black hole satisfying the same first law of thermodynamics but with different temperature and entropy, therefore making black holes hairy.

Scalar-hairy black holes: Kerr-Schild form

X. Zhang and HL, arXiv: 1403.6874

$$\begin{aligned} ds^2 &= 2dudr - fdt^2 + r(r+q)d\Omega_{2,k}^2\,, \qquad e^{\phi} = 1 + \frac{q}{r}\,,\\ f(r) &= g^2r^2 + k - \tfrac{1}{2}\alpha^2 q^2 + (g^2-\alpha^2)qr \\ &\qquad + \alpha^2 r^2 \Big(1+\frac{q}{r}\Big)\log\Big(1+\frac{q}{r}\Big)\,. \end{aligned}$$

Time-dependent ansatz:

$$ds^2 = 2dudr - H(r,u)du^2 + r(r+Q(u))d\Omega_{2,k}^2\,, \quad e^{\phi} = 1 + \frac{Q(u)}{r}\,,$$

Equations of motion imply

$$\begin{aligned} Q(u) &= q\tanh(\tfrac{1}{2}\alpha^2 q\, u)\,,\\ H &= g^2r^2 + k - \tfrac{1}{2}\alpha^2 q^2 + (g^2-\alpha^2)rQ(u) \\ &\qquad + \alpha^2 r^2\Big(1+\frac{Q(u)}{r}\Big)\log\Big(1+\frac{Q(u)}{r}\Big)\,. \end{aligned}$$

Note $\lim_{u\to\infty} Q(u) = q$, and in this limit, the solution becomes the static hairy black hole.

Time takes to be static

Let

$$u_0 = \tfrac{1}{2}\alpha^2 q \approx \frac{1}{(\alpha^4 M_0)^{\frac{1}{3}}}$$

we have

$$\tanh(\tfrac{1}{2}\alpha^2 q u) \to 1 - e^{-u/u_0}\,,$$

i.e. an exponential behavior with the relaxation time u_0.

The bigger energy or mass M_0 results a shorter u_0.

Global properties

Let R be the radius of $d\Omega^2_{2,k}$, we have

$$r = \tfrac{1}{2}\left(2\tilde{R} - q\tanh(\tfrac{1}{2}\alpha^2 q\, u)\right), \qquad \tilde{R} = \sqrt{R^2 + \tfrac{1}{4}q^2\tanh^2(\tfrac{1}{2}\alpha^2 q\, u)}$$

we can rewrite the solution as

$$\begin{aligned} ds^2 &= 2h\, du\, dR - \tilde{H} du^2 + R^2 d\Omega^2_{2,k}\,, \qquad h = \frac{R}{\tilde{R}}\,, \\ \tilde{H} &= g^2 R^2 + k - \frac{\alpha^2 q(4R^2 + q^2)\tanh(\tfrac{1}{2}\alpha^2 q\, u)}{4\tilde{R}} \\ &\quad + \alpha^2 R^2 \log\left(\frac{2\tilde{R} + q\tanh(\tfrac{1}{2}\alpha^2 q\, u)}{2\tilde{R} - q\tanh(\tfrac{1}{2}\alpha^2 q\, u)}\right)\,, \\ e^{\phi} &= \frac{2\tilde{R} + q\tanh(\tfrac{1}{2}\alpha^2 q\, u)}{2\tilde{R} - q\tanh(\tfrac{1}{2}\alpha^2 q\, u)}\,. \end{aligned}$$

For large R, the metric functions behave as

$$\begin{aligned} h &= 1 - \frac{q^2\tanh^2(\tfrac{1}{2}\alpha^2 q\, u)}{8R^2} + \mathcal{O}(R^{-4})\,, \\ \tilde{H} &= g^2 R^2 + k - \frac{\alpha^2 q^3 \tanh(\tfrac{1}{2}\alpha^2 q\, u)(3 - \tanh^2(\tfrac{1}{2}\alpha^2 q\, u))}{12R} + \mathcal{O}(R^{-3})\,. \end{aligned}$$

Asymptotic structure and Vaidya mass

Thus we see that the metric is asymptotic $R \to \infty$ flat or (A)dS depends on the sign of the cosmological constant $\Lambda = -3g^2$.

The Vaidya "mass" is

$$\begin{aligned} M(u) &= \tfrac{1}{24}\alpha^2 q^3 \tanh(\tfrac{1}{2}\alpha^2 q\, u)\Big(3 - \tanh^2(\tfrac{1}{2}\alpha^2 q\, u)\Big) \geq 0\,, \\ &\approx M_0(1 - 6e^{-u/u_0})\,. \end{aligned}$$

Singularity and apparent horizon

There is a power-law curvature singularity at $R = 0$. It is "covered" by the apparent horizon

$$0 = g^{\mu\nu}\partial_\mu R\,\partial_\nu R = \frac{\tilde{H}}{h^2}\,,$$

for some $R = R_0(\mu)$.

Birth of the black hole

If we set $u = 0$, the solution becomes flat and (A)dS. Naively, this might suggest that the black hole appears suddenly from "nothing", the vacuum. This however is misleading. At small u, we have

$$\phi = \frac{\alpha^2 q^2}{2R} u + \mathcal{O}(u^3)\,, \qquad \tilde{H} = g^2 R^2 + k - \frac{\alpha^4 q^4}{8R} u + \mathcal{O}(u^3)\,,$$
$$h = 1 - \frac{\alpha^4 q^4}{32 R^2} u^2 + \mathcal{O}(u^4)\,.$$

The polynomial curvature invariants are

$$R^\mu{}_\mu = -12 g^2 - \frac{\alpha^4 q^4}{4 R^3} u + \mathcal{O}(u^2)\,,$$
$$R_{\mu\nu} R^{\mu\nu} = 36 g^4 + \frac{3 g^2 \alpha^4 q^4}{2 R^3} u + \mathcal{O}(u^2)\,,$$
$$R_{\kappa\lambda\mu\nu} R^{\kappa\lambda\mu\nu} = 24 g^4 + \frac{g^2 \alpha^4 q^4}{R^3} u + \mathcal{O}(u^2)\,.$$

Early evolution

At $u = 0$ the spacetime becomes AdS, flat or dS except for a singular point at $R = 0$ where the scalar polynomials are not well defined (their values depending on the path to $(0,0)$ in the (u,R) plane). Nonetheless we can still connect the spacetime to an AdS, flat or dS vacuum for $u < 0$ with $M(u) = 0$. Up to the linear order in u, the energy-momentum tensor $T_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}(R + 6g^2)g_{\mu\nu}$ (vacuum background subtracted) is given by

$$T_{uu} = \frac{\alpha^4 q^4}{8R^3} - \frac{\alpha^4 q^4 (g^2 R^2 + k)}{8R^3} u + \mathcal{O}(u^2)\,, \quad T_{ij} = \tilde{g}_{ij} \frac{\alpha^4 q^4}{8R} u + \mathcal{O}(u^2)\,.$$

(Note that $T^{\mu}{}_{\mu} = \alpha^4 q^4\, u/(4R^3) + \mathcal{O}(u^2)$, $T_{\mu\nu}T^{\mu\nu} = \alpha^8 q^8 u^2/(16R^3) + \mathcal{O}(u^3)$, both of which vanish for $u = 0$ and $R \neq 0$.) Thus the birth of our evolving black hole can be understood as a vacuum being kick-started by a singular energy-momentum tensor at $u = 0$. In this process, the singularity at $(0,0)$ may become globally naked if the initial mass accumulation is not fast enough.

Mending the naked singularity

To be concrete, let us consider $k = 1$, and the existence of the event horizon requires that $q > \sqrt{2}/\alpha$, all black holes must have $M_0 \gtrsim 1/\alpha$. On the other hand, the event horizon of the static black hole is located at $r_+ \sim M_0$. For the black hole formation, we therefore have $u_0 \lesssim r_+$ and the information of singularity will be trapped.

For $M_0 \lesssim 1/\alpha$, a black hole cannot be formed and the naked singularity persists. However, if $\alpha \sim 1/\ell_p$, where ℓ_p is the Planck length, then the corresponding Compton wavelength $\ell_{\mathrm{comp}} = \ell_p^2/M_0$ will be larger than r_+ and hence the singularity will be smeared out by the quantum effect.

e.g. for an electron: $r_s = 10^{-57}m$, $r_q = 10^{-37m}$. an electron at $10^6 Gev$, $\lambda_{\mathrm{compton}} = 10^{-24}m$.

Conclusions

- Scalar black holes can be hairy. We have many examples of exact solutions to prove this.

- Black holes with conformally-massless scalar turned on contain a new Z term in their first law of thermodyamics. (No-known such an example of exact solutions involving only scalars.)

- Conformally-massless scalars arise in gauged supergravities in 4 and 6 dimensions, but not in 5 and 7.

- A more general conjecture is that all the integration constants in a second-derivative theories, when existing in pairs, can contribute the first law. $1 * dM - p_i dq_i = TdS$.

- We obtained an exact solution describing time evolution and formation of a black hole.