Particle Collisions on Black Hole Background

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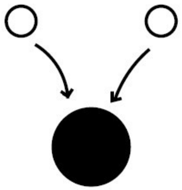
Overview

- **Flat spacetime**:

$$E_{\mathrm{CM}} \simeq 2m_0$$

- **Curved spacetime**:



$$E_{\mathrm{CM}} \gg 2m_0$$

Related work

- Banados et al, PRL [arXiv:0909.0169]

The center-of-mass (CM) Energy for the collision of two particles in the background of extremal Kerr black hole is infinite.

Related work

- Berti et al, PRL [arXiv:0911.2243]
- Jacobson et al, PRL [arXiv:0911.3363]

The CM energy is finite for the astrophysical limitations, such as the maximum spin a and back-reaction effects.

Related work

- Wei et al, PRD [arXiv:1006.1056]
  The CM energy is finite for an extremal Kerr-Newman black hole if its spin $a < 1/\sqrt{3}$.
- Zaslavskii, JETP Lett. [arXiv:1007.4598]
  The charged black holes are also found to have the similar property.

Related work

- Harada et al, PRD [arXiv:1010.0962]
  Collision of an innermost stable circular orbit particle.
- Harada et al, PRD [arXiv:1102.3316]
  Collision of two general geodesic particles.

Related work

- Zhu et al, [arXiv:1103.3848]

Duality between the frame dragging effect and electromagnetic interaction at horizon.

Overview

Plan of the talk [1]:

1. Brief review of Sen black hole
2. Equations of motion for particles
3. Radial motion and effective potential
4. CM energy for Sen black hole
    1. Extremal black hole
    2. Non-extremal black hole
    3. Near-extremal black hole
    4. Back-reaction effects
5. Conclusion

[1] Wei, Liu et al, JHEP 1012, 066 (2010)

1. Brief review of Sen black hole

- **Sen black hole metric**

$$ds^2 = -\left(\frac{\Delta - a^2\sin^2\theta}{\Sigma}\right)dt^2 + \frac{\Sigma}{\Delta}dr^2 - \frac{4aMr\sin^2\theta}{\Sigma}dtd\phi + \Sigma d\theta^2 + \frac{\Xi\sin^2\theta}{\Sigma}d\phi^2 \quad (1)$$

with the metric functions given by

$$\Delta = a^2 + \frac{r}{M}\left(Q^2 + Mr - 2M^2\right), \quad (2)$$

$$\Sigma = \frac{r}{M}\left(Q^2 + Mr\right) + a^2\cos^2\theta, \quad (3)$$

$$\Xi = \left(a^2 + \frac{r}{M}(Q^2 + Mr)\right)^2 - a^2\Delta\sin^2\theta. \quad (4)$$

M, Q and a are mass, charge and spin, respectively.

1. Brief review of Sen black hole

- **Horizon:**
  1. **Non-extremal black hole:**

$$r_{\pm} = M - \frac{Q^2}{2M} \pm \sqrt{\left(M - \frac{Q^2}{2M}\right)^2 - a^2} \tag{5}$$

  2. **Extremal black hole:**

$$r_{\rm ex} = a \tag{6}$$

- **Extremal black hole condition:**

$$Q^2 = 2M(M - a) \tag{7}$$

2. Equations of motion for particles

- **Geodesic equation for Sen black hole**

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu{}_{\nu\sigma}\frac{dx^\nu}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0 \tag{8}$$

2. Equations of motion for particles

- **Our method:**

Lagrangian $\Longrightarrow$ **Momenta** $\Longleftarrow$ **Hamilton-Jacobi equation**

$\Downarrow$

Equations of motion

2. Equations of motion for particles

On the one side,

- The Lagrangian of a free particle:

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu}. \tag{9}$$

- Normalizing condition:

$$g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu} = -\delta^2. \tag{10}$$

$\delta^2 = -1, 0, 1$ are related to the spacelike, null and timelike geodesics, respectively.

2. Equations of motion for particles

- **Conjugate momenta:**

$$P_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = g_{\mu\nu}\dot{x}^\nu. \tag{11}$$

- **Hamiltonian:**

$$\mathcal{H} = P_\mu \dot{x}^\mu - \mathcal{L} = \frac{1}{2} g^{\mu\nu} P_\mu P_\nu. \tag{12}$$

The conjugate momenta from (11)

$$P_t = -\frac{\Delta - a^2 \sin^2\theta}{\Sigma}\dot{t} - \frac{2aMr\sin^2\theta}{\Sigma}\dot{\phi}, \tag{13}$$

$$P_r = \frac{\Sigma}{\Delta}\dot{r}, \tag{14}$$

$$P_\theta = \Sigma\dot{\theta}, \tag{15}$$

$$P_\phi = -\frac{2aMr\sin^2\theta}{\Sigma}\dot{t} + \frac{\Xi\sin^2\theta}{\Sigma}\dot{\phi}. \tag{16}$$

2. Equations of motion for particles

On the other side,

- **Hamilton-Jacobi equation:**

$$\frac{\partial S}{\partial \lambda} = -\mathcal{H} = -\frac{1}{2} g^{\mu\nu}(\partial_\mu S)(\partial_\nu S). \tag{17}$$

- **Separation of the Hamilton-Jacobi function:**

$$S = \frac{1}{2}\delta^2\lambda - Et + l\phi + S_r(r) + S_\theta(\theta). \tag{18}$$

2. Equations of motion for particles

- **Momenta from (18):**

$$
\begin{aligned}
P_t &= \frac{\partial S}{\partial t} = -E, & (19)\\
P_r &= \frac{\partial S_r}{\partial r}, & (20)\\
P_\theta &= \frac{\partial S_\theta}{\partial \theta}, & (21)\\
P_\phi &= \frac{\partial S}{\partial \phi} = l. & (22)
\end{aligned}
$$

2. Equations of motion for particles

Inserting (18) in (17),

$$\Delta P_r^2 = \frac{1}{\Delta}\left[\left(r(r+Q^2/M)+a^2\right)E - al\right]^2 - (\delta^2 r(r+Q^2/M)+\mathcal{K}), \quad (23)$$

$$P_\theta^2 = -\left[\delta^2 a^2 \cos^2\theta + (l\csc\theta - Ea\sin\theta)^2\right] + \mathcal{K} \quad (24)$$

with $\mathcal{K}$ the Carter constant.

2. Equations of motion for particles

The four first-order geodesic equations for a free particle:

$$\frac{dt}{d\tau} = \frac{1}{\Delta\Sigma}(\Xi E - 2Marl), \tag{25}$$

$$\frac{dr}{d\tau} = \pm\frac{\sqrt{\Re}}{\Sigma}, \tag{26}$$

$$\frac{d\theta}{d\tau} = \pm\frac{\sqrt{\Theta}}{\Sigma}, \tag{27}$$

$$\frac{d\phi}{d\tau} = \frac{1}{\Delta\Sigma}\left(2MarE + l\csc^2\theta(\Sigma - 2Mr)\right). \tag{28}$$

with $\Re$ and Θ given by

$$\Re = \left[\left(r(r + Q^2/M) + a^2\right)E - al\right]^2 - \Delta(\delta^2 r(r + Q^2/M) + \mathcal{K}),$$

$$\Theta = \mathcal{K} - (l - aE)^2 - \left[a^2(\delta^2 - E^2) + l^2\csc^2\theta\right]\cos^2\theta.$$

3. Radial motion and effective potential

Plan of this section:

- **Purpose**: determine the range of angular momentum for particle to fall into the black hole
- **Model**: particle moves on the equatorial plane ($\theta = \pi/2$) and $E = \delta^2 = 1$, $M = 1$
- **Method**: analyze the effective potential

3. Radial motion and effective potential

- **Rewrite the radial equation (26) as**

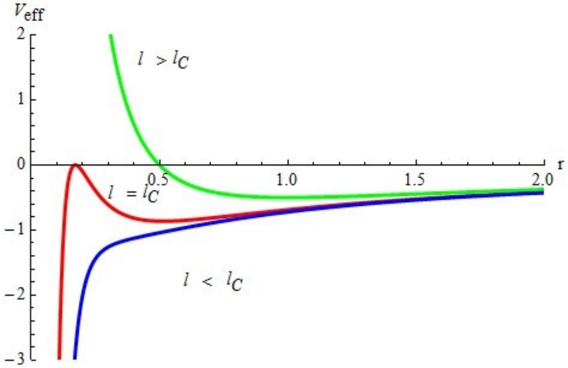
$$\frac{\dot{r}^2}{2} + V_{\text{eff}} = 0 \tag{29}$$

with

$$\begin{aligned} V_{\text{eff}} &= -\frac{\Re}{2\Sigma^2} \\ &= -\frac{2r^2 + (2Q^2 - l^2)r + (2a^2 - 4al - l^2(Q^2 - 2))}{2r(Q^2 + r)^2}. \end{aligned} \tag{30}$$

- **Conditions for the critical radius and angular momentum:**

$$V_{\text{eff}} = 0 \quad \text{and} \quad \partial_r V_{\text{eff}} = 0. \tag{31}$$



3. Radial motion and effective potential

- **Critical values:**

$$L_{1,2} = -2 \pm \sqrt{4 - 2Q^2 + 4a}, \; r_{1,2} = 2 - Q^2 + a \mp \sqrt{4 - 2Q^2 + 4a};$$
$$L_{3,4} = 2 \mp \sqrt{4 - 2Q^2 - 4a}, \; r_{3,4} = 2 - Q^2 - a \mp \sqrt{4 - 2Q^2 - 4a}.$$

- **Numerical results:**

1. $r_1 \leq r_3 < r_+ \leq r_4 \leq r_2$
2. $L_2 < 0$, $L_4 > 0$ **and** $|L_2| \geq |L_4|$
3. $|L_2| = |L_4| = 4$ **for** $a = Q = 0$ (**Schwarzschild black hole**).

Range of angular momentum for particle to fall into the black hole:

1. **Non-extremal black hole:** $(L_2,\ L_4)$
2. **Extremal black hole:** $(L_2^{\mathbf{ex}},\ L_4^{\mathbf{ex}})$

CM energy formula

- **Particle 1**: $\left(m_0, l_1, P^{\mu}_{(1)} = (P^0_{(1)},\ P^i_{(1)})\right)$
- **Particle 2**: $\left(m_0, l_2, P^{\mu}_{(2)} = (P^0_{(2)},\ P^i_{(2)})\right)$
- **Center of mass system**: $P^{\mu} = (P^0,\ 0)$

CM energy

$$P^0 P_0 \Rightarrow E^2_{\rm CM} \Leftarrow g_{\mu\nu}(P^{\mu}_{(1)} + P^{\mu}_{(2)})(P^{\nu}_{(1)} + P^{\nu}_{(2)}) \tag{32}$$

$$E_{\rm CM} = \sqrt{2}m_0\sqrt{1 - g_{\mu\nu}u^{\mu}_{(1)}u^{\nu}_{(2)}} \tag{33}$$

4.1 Extremal black hole

- **CM energy of collision for an extremal black hole:**

$$\left(\frac{E_{\rm CM}}{\sqrt{2}m_0}\right)^2 = \frac{K}{(r-2a+2)(r-a)^2}, \qquad (34)$$

with K is given by

$$\begin{aligned} K = \quad & 4r + 2(r-2a+3)(a-r)^2 - l_1 l_2(r-2a) - 2a(l_1+l_2) \\ & - \sqrt{l_1^2(2a-r) + 4(r-al_1) + 2(r-a)^2} \\ & \times \sqrt{l_2^2(2a-r) + 4(r-al_2) + 2(r-a)^2}. \end{aligned} \qquad (35)$$

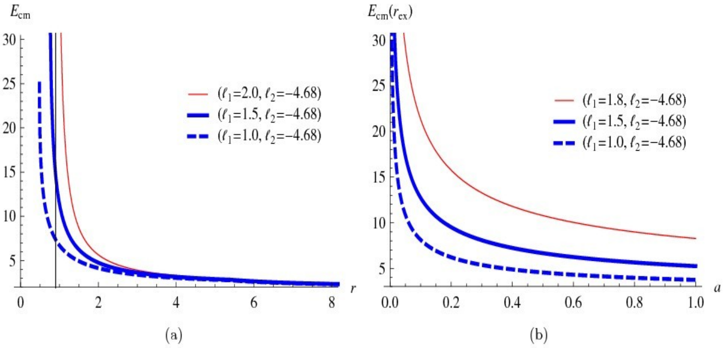
4.1 Extremal black hole

- **Limiting value of CM energy at horizon:**

$$E_{CM}(r = r_{ex}) = 2m_0\sqrt{1 + \frac{(l_1 - l_2)^2}{2a(l_1 - 2)(l_2 - 2)}}. \qquad (36)$$

- **Summary:**
  1. $l_1 = l_2 \Rightarrow E_{\mathbf{CM}} = 2m_0$
  2. $l_1 = 2$, **or** $l_2 = 2 \Rightarrow E_{\mathbf{CM}} = \infty$
  3. $E_{\mathbf{CM}} = \infty$ **can occur for** $l = 2 \in (L_2^{\mathbf{ex}},\ L_4^{\mathbf{ex}})$
  4. $a = 0 \Rightarrow E_{\mathbf{CM}} = \infty$ **(naked singularity case)**

4.1 Extremal black hole



The behavior of CM energy for an extremal Sen black hole.

4.2 Non-extremal black hole

- **CM energy of collision for a non-extremal black hole:**

$$\left(\frac{E_{\rm CM}}{\sqrt{2}m_0}\right)^2 = \frac{H}{(r+Q^2)(r^2-r(2-Q^2)+a^2)} \qquad (37)$$

with

$$\begin{aligned} H = \quad & 2r^3 + 2a^2(r+Q^2+1) - 2r^2(1-2Q^2) \\ & -2rQ^2(1-Q^2) - 2a(l_1+l_2) - (r+Q^2-2)l_1 l_2 \\ & -\sqrt{2(a-l_1)^2 + (2r-l_1^2)(r+Q^2)} \\ & \times\sqrt{2(a-l_2)^2 + (2r-l_2^2)(r+Q^2)}. \end{aligned} \qquad (38)$$

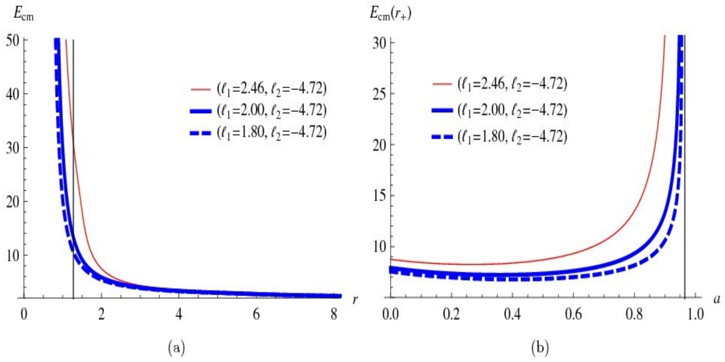
- **Limiting value of CM energy at outer horizon:**

$$\frac{E_{\rm CM}(r=r_+)}{2m_0}=\sqrt{1+\frac{(l_1-l_2)^2}{2r_-(l_1-l_c)(l_2-l_c)}},\qquad (39)$$

where $l_c=2r_+/a$.

- **Summary (outer horizon):**
  1. $l_1=l_2\Rightarrow E_{\rm CM}(r=r_+)=2m_0$
  2. $l_1=l_c$, **or** $l_2=l_c\Rightarrow E_{\rm CM}=\infty$
  3. $E_{\rm CM}(r=r_+)=\infty$ **could not occur for** l_c **not in** $(L_2,\ L_4)$

4.2 Non-extremal black hole



The behavior of CM energy for a non-extremal Sen black hole.

4.3 Near-extremal black hole

- **CM energy at outer horizon of a near-extremal black hole:**

$$\frac{E_{CM}^{max}}{m_0} \sim 11.66\epsilon^{-1/2} + \mathcal{O}(\epsilon^{1/2}) \tag{40}$$

with $\epsilon = a_{\text{max}} - a \ll 1$ **and** $a_{\text{max}} = 1 - \frac{Q^2}{2}$.

- **A Planck-scale energy collision needs (**$m_0 \sim 1$ **Gev):**

$$\epsilon \sim 10^{-36}. \tag{41}$$

- **It is very hard for the near-extremal black hole to be a particle accelerator to Planck-scale energy.**

4.4 Back-reaction effects

Note on back-reaction effects:

$$E_{\rm CM} \lesssim 10^{28} \cdot \left(\frac{m_0}{1\rm MeV}\right)^{1/2} \left(\frac{M}{100\rm M_\odot}\right)^{1/2} \rm GeV. \qquad (42)$$

Example $m_0 \sim 1$ **MeV and** $M \sim 100 M_\odot$**,**

- **Kerr black hole:** $E_{\rm CM} \sim 10^{12}$ GeV[2]
- **Sen black hole:** $E_{\rm CM} \sim 10^{28}$ GeV

Back-reaction effects has a weak effect on the CM energy for the Sen black hole geometry.

[2]E. Berti et al, PRL, arXiv:0911.2243[gr-qc]

4. Conclusion

Conclusion (without gravitational radiation):

1. Arbitrarily high energy can be approached for the extremal black hole.
2. CM energy is limited for a non-extremal black hole.
3. Near-extremal black hole could not be a particle accelerator to Planck-scale energy.
4. Back-reaction effects has a weak effect on the CM energy for the Sen black hole geometry.

Thank you!