QCD Computations for LHC

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Contents

1. Introduction: a review of past calculations

The importance of the NLO (next-leading-order) computation:

1) Experiments (Tevatron, LHC and ILC) do need more precise computations. Most of these computations are not done. $\alpha_s(M)$

2) The difficulty with the traditional (Feynman diagrammatic) method (largely due to the complexity of the (Passarino-Veltman) tensor reduction)

What needs to be done at NLO?

Experimenters to theorists: "Please calculate the following at NLO"

Run II Monte Carlo Workshop, April 2001

Single boson	Diboson	Triboson	Heavy flavour
$W+\leq 5j$	$WW+\leq 5j$	$WWW+\leq 3j$	$t\bar{t}+\leq 3j$
$W+b\bar{b}+\leq 3j$	$WW+b\bar{b}+\leq 3j$	$WWW+b\bar{b}+\leq 3j$	$t\bar{t}+\gamma+\leq 2j$
$W+c\bar{c}+\leq 3j$	$WW+c\bar{c}+\leq 3j$	$WWW+\gamma\gamma+\leq 3j$	$t\bar{t}+W+\leq 2j$
$Z+\leq 5j$	$ZZ+\leq 5j$	$Z\gamma\gamma+\leq 3j$	$t\bar{t}+Z+\leq 2j$
$Z+b\bar{b}+\leq 3j$	$ZZ+b\bar{b}+\leq 3j$	$WZZ+\leq 3j$	$t\bar{t}+H+\leq 2j$
$Z+c\bar{c}+\leq 3j$	$ZZ+c\bar{c}+\leq 3j$	$ZZZ+\leq 3j$	$t\bar{b}+\leq 2j$
$\gamma+\leq 5j$	$\gamma\gamma+\leq 5j$		$b\bar{b}+\leq 3j$
$\gamma+b\bar{b}+\leq 3j$	$\gamma\gamma+b\bar{b}+\leq 3j$		
$\gamma+c\bar{c}+\leq 3j$	$\gamma\gamma+c\bar{c}+\leq 3j$		
	$WZ+\leq 5j$		
	$WZ+b\bar{b}+\leq 3j$		
	$WZ+c\bar{c}+\leq 3j$		
	$W\gamma+\leq 3j$		
	$Z\gamma+\leq 3j$		

Theorists to experimenters:

"In your dreams"

More Realistic Experimenter's Wish List

Les Houches 2005

process ($V \in \{Z, W, \gamma\}$)	background to
1. $pp \to V\,V$ jet	$t\bar{t}H$, new physics
2. $pp \to H + 2$ jets	H production by vector boson fusion (VBF)
3. $pp \to t\bar{t}\,b\bar{b}$	$t\bar{t}H$
4. $pp \to t\bar{t} + 2$ jets	$t\bar{t}H$
5. $pp \to V\,V\,b\bar{b}$	VBF$\to H \to VV$, $t\bar{t}H$, new physics
6. $pp \to V\,V + 2$ jets	VBF$\to H \to VV$
7. $pp \to V + 3$ jets	various new physics signatures
8. $pp \to V\,V\,V$	SUSY trilepton

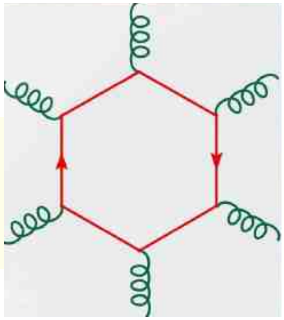
Bold action is required even for this

3) Methods developed: color decomposition, spinor helicity, (Berends-Giele) recursive relations, supersymmetric relations, string-inspired method (Bern-Kosower rules), cut method, ...

4) New development from December 2003: twistor string theory, CSW theory, BCFW recursive relations, BST method, ...

5) Complete analytic results for the 6-gluon one-loop amplitude in QCD were obtained in July 2006, due to the efforts of many people.

Status July 2006



- Status of six-gluon amplitude – progress by a lot of young people
  - analytic computation of one-loop corrections Bedford, Berger, Bern, Bidder, Bjerrum-Bohr, Brandhuber, Britto, Buchbinder, Cachazo, Dixon, Dunbar, Feng, Forde, Kosower, Mastrolia, Perkins, Spence, Travaglini, Xiao, Yang, Zhu

Amplitude	$\mathcal{N}=4$	$\mathcal{N}=1$	$\mathcal{N}=0$ cut	$\mathcal{N}=0$ rat
$--++++$	BDDK '94	BDDK '94	BDDK '94	BDK '05
$-+-+++$	BDDK '94	BDDK '94	BBST '04	BBDFK '06 XYZ '06
$-++-++$	BDDK '94	BDDK '94	BBST '04	BBDFK '06 XYZ '06
$---+++$	BDDK '94	BBDD '04	BBDI '05 BFM '06	BBDFK '06
$--+-++$	BDDK '94	BBDP '05 BBCF '05	BFM '06	XYZ '06
$-+-+-+$	BDDK '94	BBDP '05 BBCF '05	BFM '06	XYZ '06

- Numerical evaluation Ellis, Giele, Zanderighi '06

Color decomposition

Tree amplitude

$$A_n^{\rm tree}(\{k_i, h_i, a_i\}) = g^{n-2} \sum_{\sigma \in S_n/Z_n} {\rm Tr}(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}})\, A_n(k_{\sigma(1)}^{h_{\sigma(1)}}, \cdots, k_{\sigma(n)}^{h_{\sigma(n)}}),$$

S_n/Z_n: the set of non-cyclic permutations of $\{1, \cdots, n\}$.

One-loop amplitude

$$A_n^{\rm one-loop}(\{k_i, h_i, a_i\}) = g^n \sum_{J=0,\frac{1}{2},1} n_J \sum_{c=1}^{\lfloor n/2 \rfloor + 1} \sum_{\sigma \in S_n/S_{n;c}} {\rm Gr}_{n;c}\, A_{n;c}^{[J]}(\sigma),$$

$${\rm Gr}_{n;1} = N_c\, {\rm Tr}(T^{a_1} \cdots T^{a_n}),$$
$${\rm Gr}_{n;c} = {\rm Tr}(T^{a_1} \cdots T^{a_{c-1}})\, {\rm Tr}(T^{a_c} \cdots T^{a_n}).$$

Spinor helicity (Chinese magic)

Xu et. al. choose the following explicit representation for the polarization vectors:

$$\varepsilon_\mu^{(+)}(k;q) = \frac{\langle q^- | \gamma_\mu | k^- \rangle}{\sqrt{2} \langle q^- | k^+ \rangle}, \quad \varepsilon_\mu^{(-)}(k;q) = \frac{\langle q^+ | \gamma_\mu | k^+ \rangle}{\sqrt{2} \langle k^+ | q^- \rangle},$$

where k is the momentum of the polarization null vector and q is the reference null vector. In terms of the 2 component spinors:

$$k_{\alpha\dot{\alpha}} \equiv \sigma^\mu_{\alpha\dot{\alpha}} k_\mu = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}, \qquad q_{\alpha\dot{\alpha}} = \eta_\alpha \tilde{\eta}_{\dot{\alpha}},$$

$$\varepsilon^{(+)}_{\alpha\dot\beta}(k;q) \;=\; \frac{\sqrt{2}\,\eta_\alpha\tilde\lambda_{\dot\beta}}{\langle\eta\lambda\rangle}, \quad \varepsilon^{(-)}_{\alpha\dot\beta}(k;q) = \frac{\sqrt{2}\,\lambda_\alpha\tilde\eta_{\dot\beta}}{[\tilde\lambda\tilde\eta]},$$

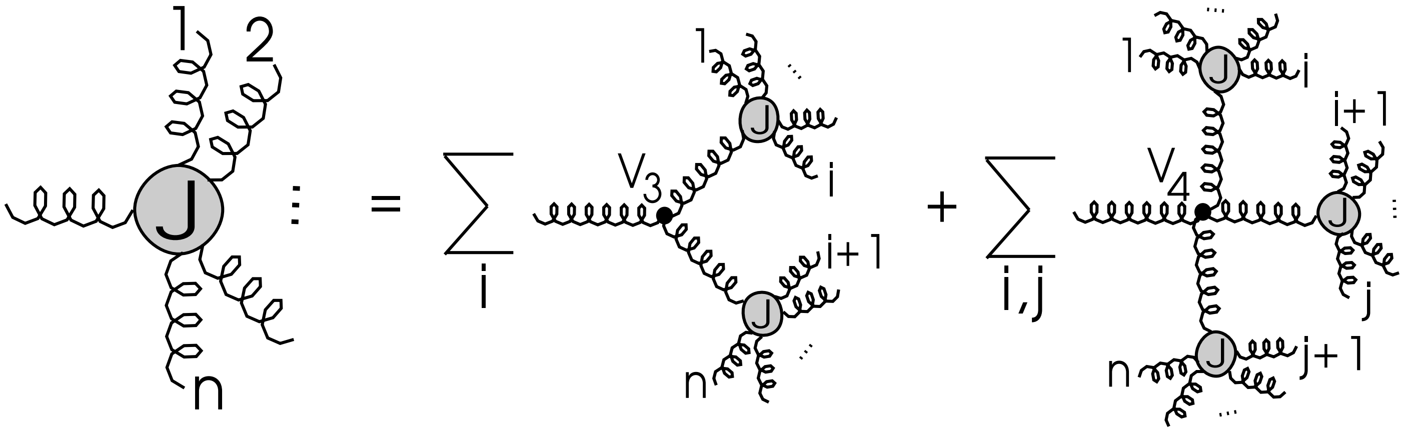
where $\langle\eta\lambda\rangle = \epsilon^{\alpha\beta}\eta_\alpha\lambda_\beta$ **and** $[\tilde\eta\tilde\lambda] = -\epsilon^{\dot\alpha\dot\beta}\tilde\eta_{\dot\alpha}\tilde\lambda_{\dot\beta}$**. Setting** $\varepsilon = \sqrt{2}\,\epsilon$**, we have:**

$$(\epsilon^-_j(k_j;q_j), \epsilon^-_l(k_l,q_l)) = \varepsilon^-_j(k_j;q_j)\cdot\varepsilon^-_l(k_l,q_l) = \frac{\langle j\,l\rangle[q_l q_j]}{[j\,q_j][l\,q_l]}.$$

Moreover,

$$p\cdot q = 2\langle\lambda\eta\rangle\,[\tilde\eta\tilde\lambda].$$

Recursive relations



$$J^\mu(1,\ldots,n) \;=\; \frac{-i}{P_{1,n}^2}\Bigg[\sum_{i=1}^{n-1} V_3^{\mu\nu\rho}(P_{1,i},P_{i+1,n})\; J_\nu(1,\ldots,i)\; J_\rho(i+1,\ldots,n)$$

$$+\sum_{j=i+1}^{n-1}\sum_{i=1}^{n-2} V_4^{\mu\nu\rho\sigma}\; J_\nu(1,\ldots,i)\; J_\rho(i+1,\ldots,j)\; J_\sigma(j+1,\ldots,n)\Bigg],$$

where the V_i are just the color-ordered gluon self-interactions,

$$V_3^{\mu\nu\rho}(P,Q) \;=\; \frac{i}{\sqrt{2}}\left(\eta^{\nu\rho}(P-Q)^\mu + 2\eta^{\rho\mu}Q^\nu - 2\eta^{\mu\nu}P^\rho\right),$$
$$V_4^{\mu\nu\rho\sigma} \;=\; \frac{i}{2}\left(2\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}\right),$$

and

$$P_{i,j} \;\equiv\; k_i + k_{i+1} + \cdots + k_j.$$

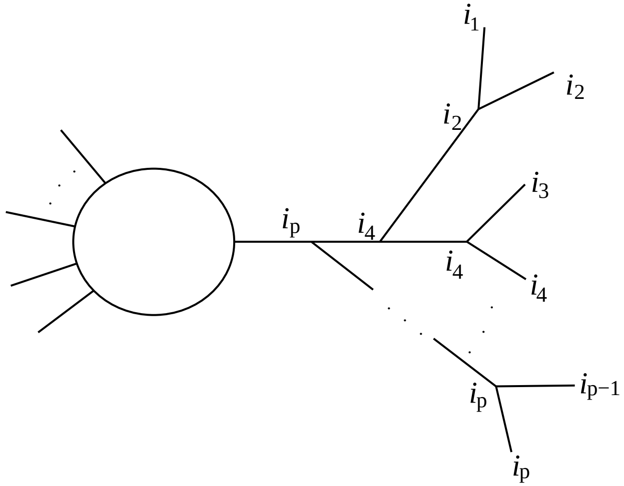
(See below in sect. 5 for more explicit rules.)

MHV (maximally helicity violating) amplitude (Parke-Taylor, 1986):

$$A(1^+,\cdots,i^-,\cdots,j^-,\cdots,n^+) = i\,\frac{\langle i\,j\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle (n-1)\,n\rangle\langle n\,1\rangle},$$

and $A(1^+,\cdots,\cdots,n^+) = A(1^+,\cdots,i^-,\cdots,n^+) = 0.$

The Bern-Kosower rules



$$\mathcal{D} = \frac{(4\pi)^{\epsilon/2}}{16\pi^2}\Gamma(n_l - 2 + \epsilon/2)\int_0^1 \mathrm{d}x_{i_{n_l-1}} \int_0^{x_{i_{n_l-1}}} \mathrm{d}x_{i_{n_l-2}} \cdots \int_0^{x_{i_3}} \mathrm{d}x_{i_2} \int_0^{x_{i_2}} \mathrm{d}x_{i_1}$$

$$\times \frac{K_{\mathrm{red}}}{\left(\sum_{l<m}^{n_\ell} P_{i_l} \cdot P_{i_m} x_{i_m i_l}(1 - x_{i_m i_l})\right)^{n_l - 2 + \epsilon/2}}.$$

Referring to the last picture we obtained the reduced factor $K_{\rm red}$ as follows:

- **The master K factor:**

$$K = \prod_{i<j} \exp\{k_i \cdot k_j G^{ij} + (k_i \cdot \epsilon_j - k_j \cdot \epsilon_i)\,\dot{G}^{ij} - \epsilon_i \cdot \epsilon_j\,\ddot{G}^{ij}\}|_{\epsilon \text{ multi-linear}}$$

- **Partial integration to remove all $\ddot{G}$. This is always possible but quite tedious.**

- **Tree reduction rules:** $(\dot{G}^{ij})^n \to -\frac{\delta_{n,1}}{(K_i+K_j)^2}, \qquad i \to j$.

- **Rules for substituting G and $\dot{G}$ and adding various contributions (next page).**

Summing all the contributions and doing the Feynman parameter integration give the final result for the one-loop amplitude. 5-gluon amplitude in 1993.

The loop substitution rules for various particle contents.

- **Gluon in Loop:**

$$a): \quad \textbf{Overall } 2(1-\delta_R\epsilon/2), \qquad G^{i,j} \longrightarrow \frac{1}{2}(-\mathrm{sig}(x_{ij})+2x_{ij}),$$

$$b): \quad G^{i_1,i_2}\,G^{i_2,i_1} \longrightarrow 2 \qquad G^{i_1,i_2}\,G^{i_2,i_3}\cdots G^{i_{m-1},i_m}\,G^{i_m,i_1} \longrightarrow 1 \quad (m>2)\;;$$

- **Real Scalar in Loop:**

$$\textbf{Overall } N_s\;, \qquad G^{i,j} \longrightarrow \frac{1}{2}(-\mathrm{sig}(x_{ij})+2x_{ij})\;,$$

- **Fermion in Loop: Overall $-4N_d$ for Dirac and $-2N_w$ for Weyl,**

$$G^{i_1,i_2}\cdots\,G^{i_m,i_1} \longrightarrow \left(\frac{1}{2}\right)^m \left[\prod_{k=1}^{m}(-\mathrm{sig}(x_{i_k i_{k+1}})+2x_{i_k i_{k+1}})-(-1)^m\prod_{k=1}^{m}\mathrm{sig}(x_{i_k i_{k+1}})\right].$$

2. Witten's TST and the CSW theory

Twistor space:

a) light-like momentum p can be written in terms of two 2 component spinors:

$$p_{\alpha\dot{\alpha}} \equiv \sigma^{\mu}_{\alpha\dot{\alpha}}\, p_{\mu} = \lambda_{\alpha}\tilde{\lambda}_{\dot{\alpha}}.$$

b) Fourier transformations only for one spinors $\tilde{\lambda}$:

$$\tilde{A}(\lambda,\mu) = \int \frac{\mathrm{d}^2\tilde{\lambda}}{(2\pi)^2} \exp(i[\mu\,\tilde{\lambda}]\, A(\lambda,\tilde{\lambda}).$$

c) (λ,μ) constitute the twistor space coordinates: CP^3 and $CP^{3|4}$

$$(\lambda_1,\lambda_2,\mu^1,\mu^2) = (\xi\lambda_1,\xi\lambda_2,\xi\mu^1,\xi\mu^2).$$

Witten studied the topological string theory on this super-twistor space (type B) and established a weak-weak duality between this string theory and the N=4 SUSY Yang-Mills theory. In particular, the Yang-Mills amplitudes in twistor space have the following special property:

The n particle scattering amplitude in twistor space is nonzero only if the points P_i are all supported on an algebraic curve in twistor space. This algebraic curve has degree d given by

$$d = q - 1 + l,$$

where q is the number of positive helicity gluons in the scattering process, and l is the number of loops. It is not necessarily connected. And its genus g is bounded by the number of loops, $g \leq l$.

In particular, MHV ($q = 2$) tree ($l = 0$) amplitude is non-vanishing only on complex line: $d = 1$.

The logic of CSW theory: for tree amplitudes, the algebraic curve consists of the intersection of several complex lines. In diagrammatic language, tree amplitude can be obtained from the products of MHV tree amplitudes.

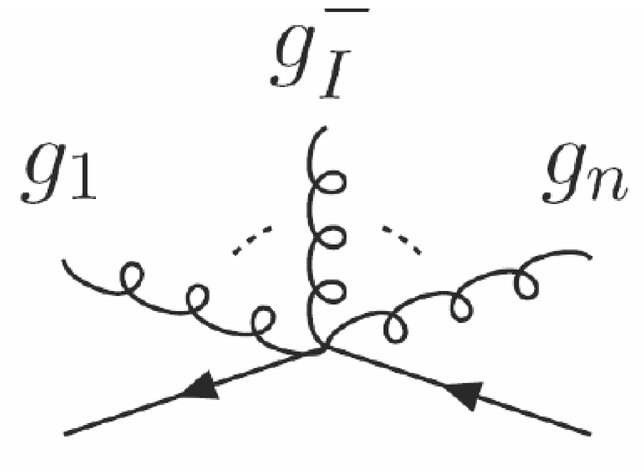
instanton sums: connected diagram vs the sums of disconnected diagram sums.

CSW theory:

- **dram all diagrams with MHV vertices**
- **the propagator is** $\frac{i}{p^2}$
- **off-shell continuation of the MHV amplitudes (**$\lambda = \sum_i \lambda_i\,[\tilde{\lambda}_i\,\tilde{\eta}]$**)**
- **sum all possible contributions**

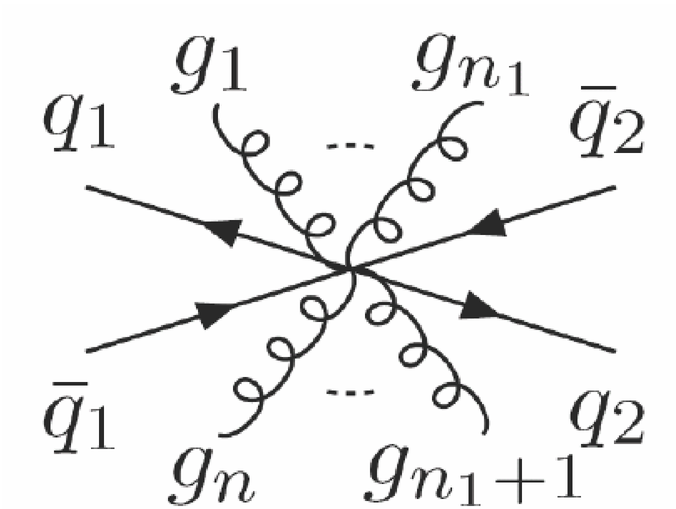
Parity conservation in QCD: It has been proved that the above rules give the correct googly (mostly minus heliicty) amplitude.

Extension to include fermions. Must introduce New MHV vertices.



$$V(\Lambda_q^+, g_1^+, \cdots, g_I^-, \cdots, g_n^+, \Lambda_{\bar q}^-) = -\frac{\langle q, I\rangle\langle \bar q, I\rangle^3}{\langle q, 1\rangle\langle 1, 2\rangle\cdots\langle n, \bar q\rangle\langle \bar q, q\rangle}$$

$$V(\Lambda_q^-, g_1^+, \cdots, g_I^-, \cdots, g_n^+, \Lambda_{\bar q}^+) = \frac{\langle q, I\rangle^3\langle \bar q, I\rangle}{\langle q, 1\rangle\langle 1, 2\rangle\cdots\langle n, \bar q\rangle\langle \bar q, q\rangle},$$



$$V(\Lambda_{q_1}^{h_1}, g_1, \cdots, g_{n_1}, \Lambda_{\bar q_2}^{-h_2}, \Lambda_{q_2}^{h_2}, g_{n_1+1}, \cdots, g_n, \Lambda_{\bar q_1}^{-h_1})$$
$$= A_0(h_1, h_2)\frac{\langle q_1, \bar q_2\rangle}{\langle q_1, 1\rangle\langle 1, 2\rangle\cdots\langle n_1, \bar q_2\rangle} \times \frac{\langle q_2, \bar q_1\rangle}{\langle q_2, n_1+1\rangle\cdots\langle n, \bar q_1\rangle}$$

3. The BCFW recursive relations

Choosing any two gluons (named k and n), and setting

$$p_k(z) = \lambda_k\,(\tilde\lambda_k - z\tilde\lambda_n), \qquad p_n(z) = (\lambda_n + z\lambda_k)\,\tilde\lambda_n.$$

The "amplitude" $A(z) = A(p_1, \cdots, p_k(z), \cdots, p_n(z))$ is a physical, on-shell amplitude for all z. (Momentum is conserved and all $p^2(z) = 0$.) $A(0)$ is the real physical amplitude.

$A(z)$ is a rational function of z and has only simple poles (and no pole at $z = \infty$). This gives ($P_{ij}(z) = P_{ij} + z\lambda_k\tilde\lambda_n$ contains p_n):

$$A(z) = \sum_{i,j} \frac{c_{ij}}{z - z_{ij}} = \sum_{i,j}\sum_h \frac{A_L^h(z_{ij})\,A_R^{-h}(z_{ij})}{(P_{ij}(z))^2},$$

and

$$A = A(0) = \sum_{ij} \sum_h \frac{A_L^h(z_{ij})\, A_R^{-h}(z_{ij})}{P_{ij}^2}, \qquad z_{ij} = -\frac{P_{ij}^2}{\langle \lambda_k | P_{ij} | \tilde{\lambda}_n \rangle}.$$

Usefulness: very compact formulas for tree amplitude.

$$A(1^+, 2^+, 3^+, 4^-, 5^-, 6^-) \;=\; i \left[\frac{\langle 6^- | (1+2) | 3^- \rangle^3}{\langle 6\,1\rangle \langle 1\,2\rangle [3\,4] [4\,5]\, s_{612} \langle 2^- | (6+1) | 5^- \rangle} + \frac{\langle 4^- | (5+6) | 1^- \rangle^3}{\langle 2\,3\rangle \langle 3\,4\rangle [5\,6] [6\,1]\, s_{561} \langle 2^- | (6+1) | 5^- \rangle} \right].$$

For more gluons, BCFW gives more compact expressions.

4. The computation of the cut part

By Passarino-Veltman reduction, a general one-loop amplitude can be written as follows:

$$A^{\text{one-loop}} = \sum_i \sum_{j=2,4} c_i^{(j)}(D)\, I_j^{D(i)}[1].$$

where $I_j^{D(i)}[1]$'s are scalar integrals. Expanding in $D = 4 - 2\epsilon$ gives

$$A^{\text{one-loop}} = \sum_i \sum_{j=2,4} c_i^{(j)}(4)\, I_j^{D(i)}[1] + (\textbf{rational part}).$$

The coefficients $C_i^{(j)}$ can be computed by cut-method.

The rational part is identically 0 for SUSY theory.

For $N = 4$ SUSY Yang-Mills theory, only box integrals ($j = 4$) appear. The coefficient can be easily computed by quadruple cut: cutting all the four lines in the box integral. The equations

$$p^2 = (p - K_1)^2 = (p - K_1 - K_2)^2 = (p + K_4)^2 = 0$$

lead to solutions: p_+ and p_-. The coefficient $c_i^{(4)}$ are given as a sum over the products of 4 tree-amplitude.

BCF also showed that this is still true if some of the external lines are massless.

The computation of the coefficients of the triangle and bubble integrals is more involved but systematic method is developed by using spinor integration. The method was applied to compute all the triangle and bubble coefficient for the 6-gluon QCD amplitude: Britto-Feng-Marstrlia, hep-ph/0602178.

5. The computation of the rational part

- **try and error, check by numerical computation of Feynman diagrams (BDK, obsolete)**
- **D-dimensional unitarity (BDDK, Brandhuber-McNamara-Spence-Travaglini, hep-th/0506068 up to $A_5(1^+2^+3^+4^+5^+)$)**
- **bootstrap recursive (inspired from tree recursive relation of BCFW) (Bern-Dixon-Kosower)**
- **directly computing Feynman diagrams, keeping only term contributing to rational part. Quite old-fashion but with tricks from string theory. (Xiao-Yang-Zhu)**

Why it is feasible to compute the rational part directly from Feynman integral?

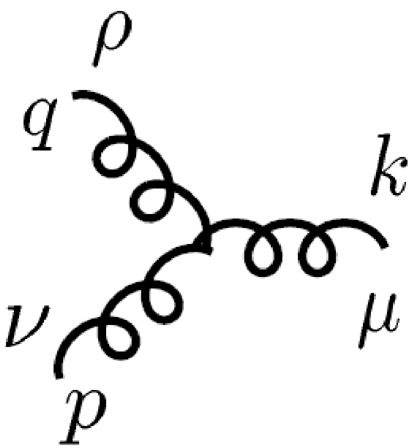
The Bern-Dunbar-Dixon-Kosower theorem:

$$I_m^D[f(p)] = \int \frac{\mathrm{d}^D p}{i\,\pi^{D/2}} \frac{f(p)}{p^2(p-K_1)^2\cdots(p+K_m)^2},$$

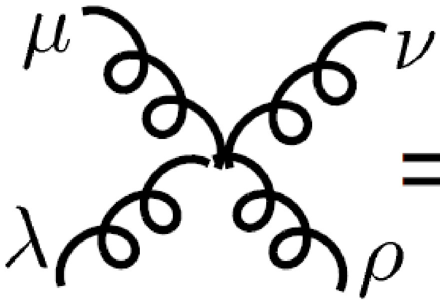
The rational part is 0 if $f(p)$ is a polynomial function of the internal momentum p of degree $m-2$ or less. For phenomenologically interesting models and by choosing a suitable gauge, the degree of $f(p)$ is always not greater than m.

Our strategy: do calculation while keeping only the leading and sub-leading polynomial terms in p.

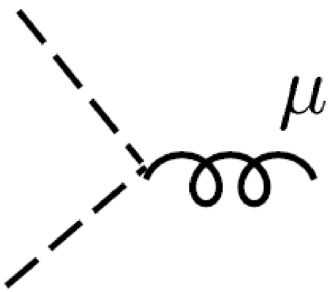
Feynman diagrams and Feynman rules



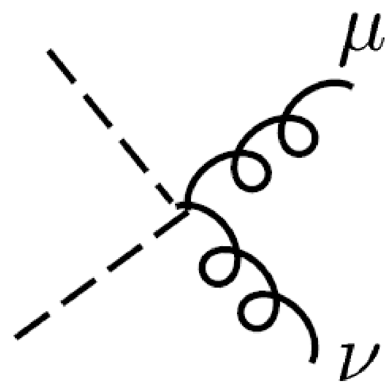
$$= \frac{i}{\sqrt{2}} \Big(\eta_{\nu\rho}(p-q)_\mu + \eta_{\rho\mu}(q-k)_\nu + \eta_{\mu\nu}(k-p)_\rho$$



$$= i\eta_{\mu\rho}\eta_{\nu\lambda} - \frac{i}{2}(\eta_{\mu\nu}\eta_{\rho\lambda} + \eta_{\mu\lambda}\eta_{\nu\rho})$$

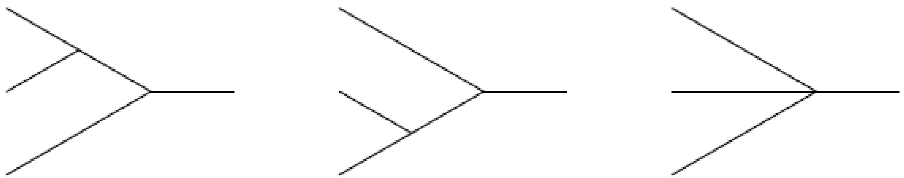


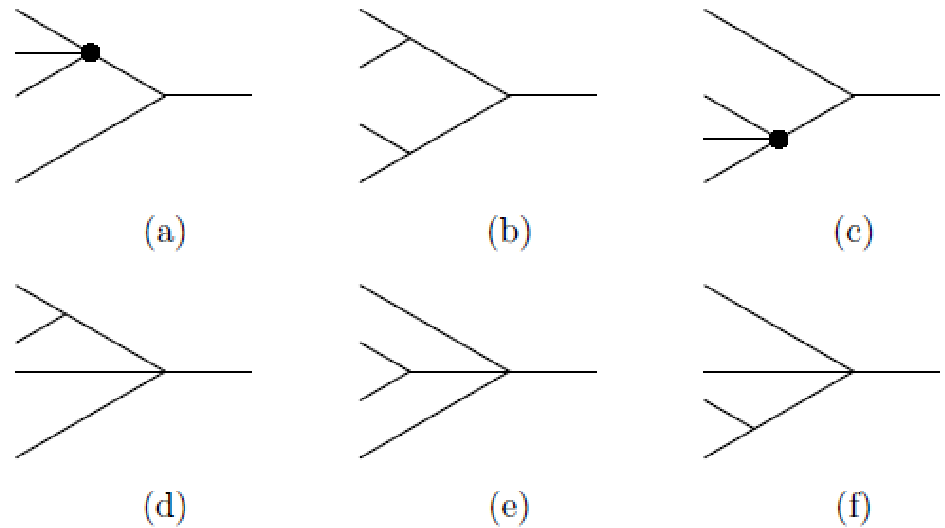
$$= \frac{i}{\sqrt{2}}(p-q)_\mu$$



$$= -\frac{i}{2}\eta_{\mu\nu}$$

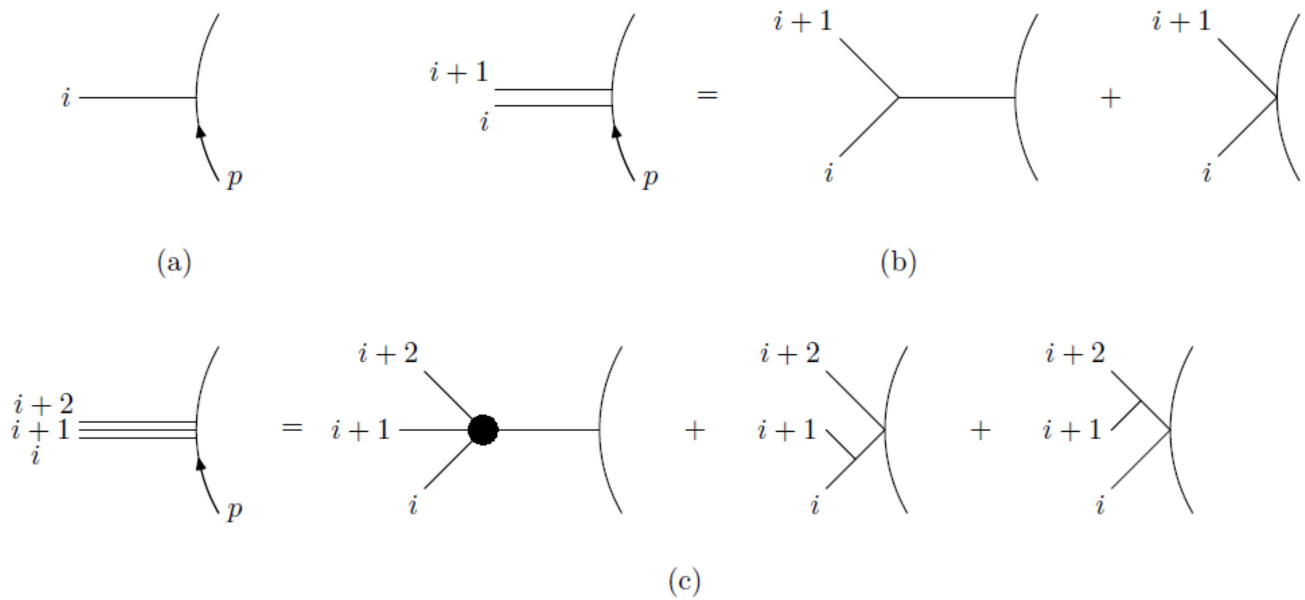
Feynman diagrams and Feynman rules (I): Trees





$$
\begin{aligned}
\epsilon_{i(i+1)} &= P(\epsilon_i, k_i; \epsilon_{i+1}, k_{i+1}) \equiv \frac{1}{(k_i + k_{i+1})^2}\Big((\epsilon_i, k_{i+1})\,\epsilon_{i+1} \\
&- (\epsilon_{i+1}, k_i)\,\epsilon_i + \frac{1}{2}(\epsilon_i, \epsilon_{i+1})\,(k_i - k_{i+1})\Big), \\
\epsilon_{i(i+1)(i+2)} &= P(\epsilon_{i(i+1)}, k_{i(i+1)}; \epsilon_{i+2}, k_{i+2}) + P(\epsilon_i, k_i; \epsilon_{(i+1)(i+2)}, k_{(i+1)(i+2)}) \\
&+ \frac{1}{s_{i(i+1)(i+2)}}\left((\epsilon_i, \epsilon_{i+2})\,\epsilon_{i+1} - \frac{1}{2}\,(\epsilon_i, \epsilon_{i+1})\,\epsilon_{i+2} - \frac{1}{2}\,(\epsilon_{i+1}, \epsilon_{i+2})\,\epsilon_i\right)
\end{aligned}
$$

Feynman diagrams and Feynman rules (II): Tree to loop

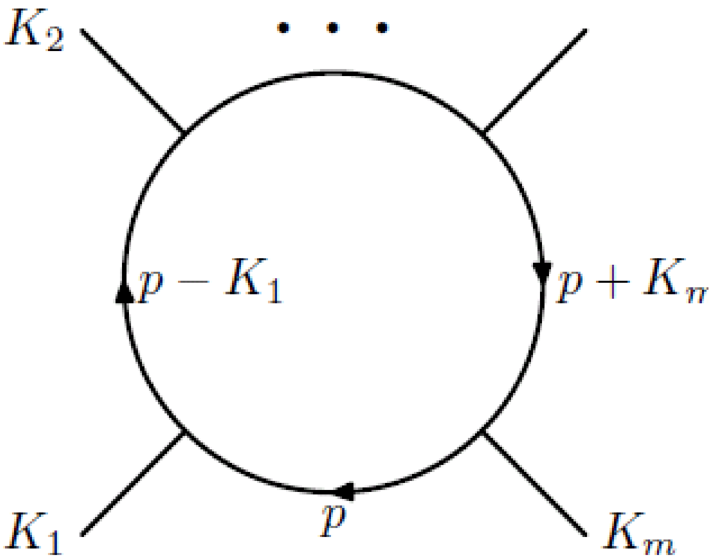


$$
\begin{aligned}
P_i(p) &= (\epsilon_i, p) = (\epsilon_i, p - k_i), \\
P_{i(i+1)}(p) &= (\epsilon_{i(i+1)}, p) - \frac{1}{2}(\epsilon_i, \epsilon_{i+1}), \\
P_{i(i+1)(i+2)}(p) &= (\epsilon_{i(i+1)(i+2)}, p) - \frac{1}{2}\left((\epsilon_{i(i+1)}, \epsilon_{i+2}) + (\epsilon_i, \epsilon_{(i+1)(i+2)})\right)
\end{aligned}
$$

Deriving the rational part (I): Feynman parametrization

By using Feynman parametrization we have

$$
\begin{aligned}
I_n^D[1] &\equiv \int \frac{\mathrm{d}^D p}{i\pi^{D/2}} \frac{1}{p^2(p-k_1)^2\cdots(p+k_n)^2} \\
&= (-1)^n\,\Gamma(n-D/2)\int \mathrm{d}^n a\,\frac{\delta(1-\sum_i a_i)}{(a\cdot S\cdot a)^{n-\frac{D}{2}}},
\end{aligned}
$$



where $$a\cdot S\cdot a = \sum_{i,j=1}^{n} a_i\,a_j\,S_{ij}$$

$$
S = -\frac{1}{2}\begin{pmatrix}
0 & k_1^2 & (k_1+k_2)^2 & \cdots & (k_1+k_2+\cdots k_{n-1})^2 \\
* & 0 & k_2^2 & \cdots & (k_2+k_3+\cdots k_{n-1})^2 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
* & * & * & 0 & k_{n-1}^2 \\
* & * & * & * & 0
\end{pmatrix}
$$

An $n\times n$ symmetric matrix of external kinematic variables.

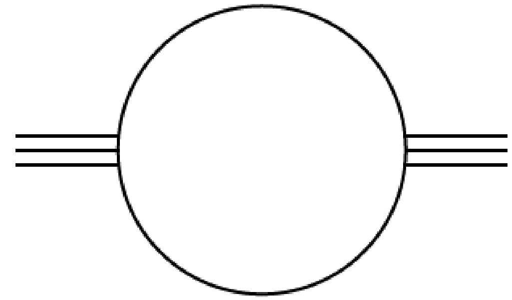
Deriving the rational part (II): recursive relations

$$
\begin{aligned}
I_n^D[g_l(a)\,a_i] &= \frac{1}{2}\,(n-1-l-D)\,\gamma_i\,I_n^{D+2}[g_l(a)] \\
&+ \frac{1}{2}\sum_j S_{ij}^{-1}\,I_{n-1}^{D(j)}[g_l(a)] + \frac{1}{2}\sum_j S_{ij}^{-1}\,I_n^{D+2}[\partial_j g_l(a)], \\
I_n^D[a_i f(a)] &= P_{ij}\,\left(I_n^{D(j)}[f(a)] + I_n^{D+2}[\partial_j f(a)]\right) + \frac{\gamma_i}{\Delta}\,I_n^D[f(a)], \\
\gamma_i &= \sum_j S_{ij}^{-1}, \qquad \Delta = \sum_i \gamma_i, \\
P_{ij} &= \frac{1}{2}\left(S_{ij}^{-1} - \frac{\gamma_i\,\gamma_j}{\Delta}\right).
\end{aligned}
$$

Deriving the rational part (III): the bubble integral

Bubble integrals:

$$I_2^D[f(p)] \equiv \int \frac{\mathrm{d}^D p}{i(\pi)^{D/2}} \frac{f(p)}{p^2\,(p+K)^2}.$$



K is the sum of momenta of consecutive external gluons on one side of the bubble diagram. For $K^2 \neq 0$ we have ($D = 4 - 2\epsilon$):

$$\begin{aligned}
I_2^D[1] &= \frac{\gamma_\Gamma}{\epsilon(1-2\epsilon)}\,(-K^2)^{-\epsilon},\\
I_2^D[p^\mu] &= -\frac{K^\mu}{2}\, I_2^D[1],\\
I_2^D[a_1^2] = I_2^D[a_2^2] &= \frac{2-\epsilon}{2(3-2\epsilon)}\, I_2^D[1]\\
&= \frac{1}{3}\, I_2^D[1] + \frac{1}{18} + O(\epsilon).
\end{aligned}$$

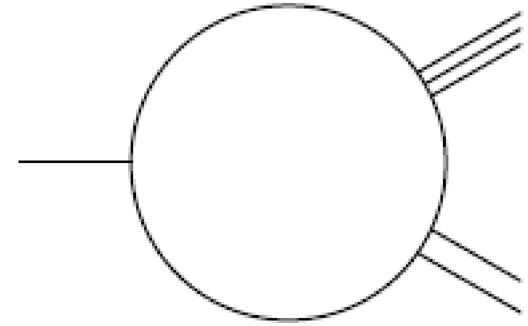
Deriving the rational part (IV): higher dim. scalar integral

$$I_n^{D+2}[1] = \frac{1}{(n-1-D)\,\Delta}\left[2\,I_n^{D}[1] - \sum_j \gamma_j\, I_{n-1}^{D(j)}[1]\right]$$

Rational part arises because I_n^D is divergent and the pre-factor depends on $D = 4 - 2\,\epsilon$.

$$\begin{aligned}
I_2^{D+2}[1] &= \frac{K^2}{9}, \qquad I_3^{D+2}[1] = \frac{1}{2}, \qquad I_4^{D+2}[1] = I_4^{D+2}[a_i] = 0, \\
I_4^{D+4}[1] &= \frac{1}{3}\times\frac{1}{2} + \frac{1}{3^2}\times 2\times\frac{1}{2} = \frac{5}{18}, \\
I_5^{D+2}[1] &= I_5^{D+2}[a_i] = I_5^{D+2}[a_i\,a_j] = I_5^{D+4}[1] = I_5^{D+4}[a_i] = 0, \\
I_5^{D+6}[1] &= = \frac{1}{4}\times\frac{5}{18} + \frac{1}{4^2}\times 2\times\frac{1}{6} = \frac{13}{144}.
\end{aligned}$$

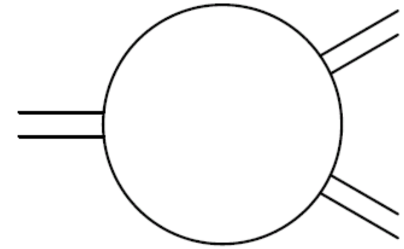
Deriving the rational part (V): Triangle integral



$$
\begin{aligned}
I_3^D(\epsilon_1,\epsilon_2) &\equiv -\int \frac{\mathrm{d}^D p}{i\pi^{D/2}} \frac{(\epsilon_1,p)\,(\epsilon_2,p)}{p^2(p-k_1)^2(p+K_3)^2} \\
&= \textbf{cut part} - \frac{(K_2^2+K_3^2)}{2(K_2^2-K_3^2)^2}\,(\epsilon_1,k_1)\,(\epsilon_2,k_1) - \frac{1}{2}\,(\epsilon_1,\epsilon_2) \\
&- \frac{((\epsilon_1,k_1)(\epsilon_2,k_1)+(\epsilon_1,k_1)(\epsilon_2,K_2)+(\epsilon_1,K_2)\,(\epsilon_2,k_1))}{2\,(K_2^2-K_3^2)}, \\
I_3^{D,R}(\epsilon_1,\epsilon_2) &= -\frac{(\epsilon_1,K_2)\,(\epsilon_2,k_1)}{2(K_2^2-K_3^2)} - \frac{1}{2}\,(\epsilon_1,\epsilon_2), \qquad (\epsilon_1,k_1)=0,
\end{aligned}
$$

$$
\begin{aligned}
I_3(\epsilon_i) &\equiv \int \frac{\mathrm{d}^D p}{i\pi^{D/2}}\ \frac{(\epsilon_1,p)\,(\epsilon_2,p-k_1)\,(\epsilon_3,p)}{p^2\,(p-k_1)^2\,(p+K_3)^2} \\
&= \frac{1}{36}\Big((\epsilon_2,4\,K_2-7\,k_1)\,(\epsilon_1,\epsilon_3)-(2\leftrightarrow 3)+4(\epsilon_1,K_2)\,(\epsilon_2,\epsilon_3)\Big) \\
&- \frac{(K_2^2+K_3^2)}{6\,(K_2^2-K_3^2)^2}\,(\epsilon_1,K_2)\,(\epsilon_2,k_1)\,(\epsilon_3,k_1) \\
&- \frac{(\epsilon_1,K_2)\,((\epsilon_2,k_1)\,(\epsilon_3,K_3)-(\epsilon_2,K_2)\,(\epsilon_3,k_1))}{6\,(K_2^2-K_3^2)} \\
&- \frac{(K_2^2+K_3^2)}{12\,(K_2^2-K_3^2)}\,((\epsilon_1,\epsilon_2)\,(\epsilon_3,k_1)+(\epsilon_1,\epsilon_3)\,(\epsilon_2,k_1))
\end{aligned}
$$

Very complicated Feynman-like rules. And indeed the 3 mass triangle is much more complicated.



$$
\begin{aligned}
I_3(\epsilon_1,\epsilon_2,\epsilon_3) &\equiv \int \frac{\mathrm{d}^D p}{i\pi^{D/2}} \frac{(\epsilon_1,p)(\epsilon_2,p-K_1)(\epsilon_3,p+K_3)}{p^2(p-K_1)^2(p+K_3)^2}, \\
&= -F_0(s_1,s_2,s_3)\big((\epsilon_1,K_1)(\epsilon_2,K_1)(\epsilon_3,K_2)+(\epsilon_1,K_3)(\epsilon_2,K_2)(\epsilon_3,K_2) \\
&\quad + (\epsilon_1,K_3)(\epsilon_2,K_1)(\epsilon_3,K_3)+(\epsilon_1,K_3)(\epsilon_2,K_1)(\epsilon_3,K_2) \\
&\quad - (\epsilon_1,K_1)(\epsilon_2,K_2)(\epsilon_3,K_3)\big) - \sum_{i=1}^{3}(\epsilon_1,K_i)(\epsilon_2,K_i)(\epsilon_3,K_i)F_i(s_1,s_2,s_3) \\
&\quad - \frac{1}{2\Delta}\big((s_1-s_2-s_3)(\epsilon_1,K_1)(\epsilon_2,K_1)(\epsilon_3,K_3) \\
&\quad + (s_2-s_3-s_1)(\epsilon_1,K_1)(\epsilon_2,K_2)(\epsilon_3,K_2) \\
&\quad + (s_3-s_1-s_2)(\epsilon_1,K_3)(\epsilon_2,K_2)(\epsilon_3,K_3)\big) \\
&\quad + \frac{7}{36}\big((\epsilon_1,\epsilon_2)(\epsilon_3,K_3-K_2)+(\epsilon_2,\epsilon_3)(\epsilon_1,K_1-K_3) \\
&\quad + (\epsilon_3,\epsilon_1)(\epsilon_2,K_2-K_1)\big) \\
&\quad + \frac{1}{12\Delta}\big((\epsilon_1,\epsilon_2)(\epsilon_3,K_3-K_2)s_1(s_2+s_3-s_1) \\
&\quad + (\epsilon_2,\epsilon_3)(\epsilon_1,K_1-K_3)s_2(s_3+s_1-s_2) \\
&\quad + (\epsilon_3,\epsilon_1)(\epsilon_2,K_2-K_1)s_3(s_1+s_2-s_3)\big) \\
&\quad + \frac{1}{12\Delta}\big((\epsilon_1,\epsilon_2)(\epsilon_3,K_1)(s_3-s_2)(s_2+s_3-s_1) \\
&\quad + (\epsilon_2,\epsilon_3)(\epsilon_1,K_2)(s_1-s_3)(s_3+s_1-s_2) \\
&\quad + (\epsilon_3,\epsilon_1)(\epsilon_2,K_3)(s_2-s_1)(s_1+s_2-s_3)\big),
\end{aligned}
$$

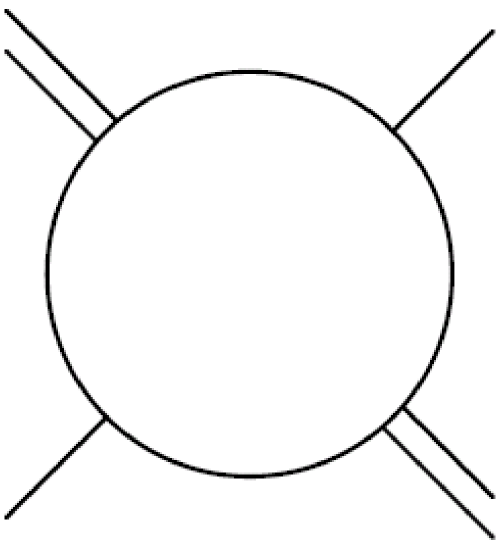
$$
\begin{aligned}
s_i = K_i^2,\ F_0(s_1,s_2,s_3) &= \frac{10 s_1 s_2 s_3}{3\Delta^2} + \frac{(s_1+s_2+s_3)}{6\Delta}, \\
F_1(s_1,s_2,s_3) &= \frac{5(s_1+s_2-s_3)s_2 s_3}{3\Delta^2} + \frac{(s_1-s_3)}{3\Delta}, \\
F_2(s_1,s_2,s_3) &= \frac{5(s_2+s_3-s_1)s_3 s_1}{3\Delta^2} + \frac{(s_2-s_1)}{3\Delta}, \\
F_3(s_1,s_2,s_3) &= \frac{5(s_3+s_1-s_2)s_1 s_2}{3\Delta^2} + \frac{(s_3-s_2)}{3\Delta}, \\
\Delta &= s_1^2+s_2^2+s_3^2-2(s_1 s_2+s_2 s_3+s_3 s_1),
\end{aligned}
$$

Deriving the rational part (VI): 2 mass easy box integral

The basic strategy: decomposition into simple ones (equivalent to tensor reduction)

For 2 mass easy box with $\epsilon_1 = \eta_1 \tilde{\lambda}_1$ and $\epsilon_3 = \eta_3 \tilde{\lambda}_3$, we have:

$$\begin{aligned} I_4^{D,R}(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) &= -\frac{\langle \eta_1\, 1\rangle \langle \eta_3\, 3\rangle}{\langle 1\, 3\rangle^2} I_4^{D,R}(\lambda_3 \tilde{\lambda}_1, \epsilon_2, \lambda_1 \tilde{\lambda}_3, \epsilon_4) \\ &+ \frac{\langle \eta_1\, 3\rangle \langle \eta_3\, 1\rangle}{\langle 1\, 3\rangle^2} I_4^{D,R}(k_1, \epsilon_2, k_3, \epsilon_4,) \\ &+ \frac{\langle \eta_1\, 3\rangle}{\langle 1\, 3\rangle} I_4^{D,R}(k_1, \epsilon_2, \epsilon_3, \epsilon_4) + \frac{\langle \eta_3\, 1\rangle}{\langle 3\, 1\rangle} I_4^{D,R}(\epsilon_1, \epsilon_2, k_3, \epsilon_4). \end{aligned}$$



$$
\begin{aligned}
I_4^{D,R}(\epsilon_1,\epsilon_2,k_3,\epsilon_4) &= \frac{K_2^2+s}{6(K_2^2-s)^2}(\epsilon_1,K_2)(\epsilon_2,k_1)(\epsilon_4,k_1) \\
&+ \quad \frac{K_4^2+t}{6(K_4^2-t)^2}(\epsilon_1,K_4)(\epsilon_2,k_1)(\epsilon_4,k_1) \\
&+ \quad \frac{1}{12}\left[\frac{K_2^2+s}{K_2^2-s}+\frac{K_4^2+t}{K_4^2-t}\right]((\epsilon_1,\epsilon_2)(\epsilon_4,k_1)+(\epsilon_1,\epsilon_4)(\epsilon_2,k_1)) \\
&+ \quad \frac{(\epsilon_1,K_2)}{6(K_2^2-s)}(\epsilon_2,k_1)(\epsilon_4,k_3)+\frac{(\epsilon_1,K_4)}{6(K_4^2-t)}(\epsilon_2,k_3)(\epsilon_4,k_1) \\
&+ \quad \frac{1}{9}((\epsilon_1,\epsilon_2)\,\epsilon_4+(\epsilon_1,\epsilon_4)\,\epsilon_2+(\epsilon_2,\epsilon_4)\,\epsilon_1,k_3)
\end{aligned}
$$

where $s=(k_1+K_2)^2$ **and** $t=(K_2+k_3)^2$**. Invariant under the interchange** $2\leftrightarrow 4$ **(**$s\leftrightarrow t$**).**

By setting $\epsilon_1 = k_1$ we get

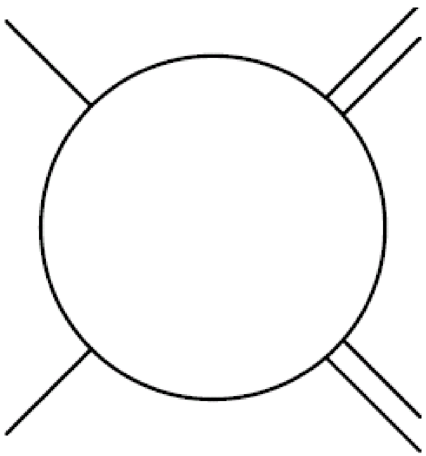
$$\begin{aligned} & I_4^{D,R}(k_1, \epsilon_2, k_3, \epsilon_4) \\ & \quad = \quad \frac{1}{18}(2(k_1, k_3)(\epsilon_2, \epsilon_4) - ((\epsilon_2, k_1)(\epsilon_4, k_3) + (\epsilon_2, k_3)(\epsilon_4, k_1))). \end{aligned}$$

One more:

$$\begin{aligned} I_4^{D,R}(\lambda_3\tilde{\lambda}_1, \epsilon_2, \lambda_1\tilde{\lambda}_3, \epsilon_4) = & \frac{4}{9}\Big((\epsilon_2, k_1)(\epsilon_4, k_3) + (\epsilon_2, k_3)(\epsilon_4, k_1)\Big) \\ - & \frac{5}{9}(k_1, k_3)(\epsilon_2, \epsilon_4) - \frac{1}{4}\left(\frac{K_2^2 + s}{K_2^2 - s} + \frac{K_4^2 + t}{K_4^2 - t}\right)(\epsilon_2, k_1)(\epsilon_4, k_1) \\ - & \frac{1}{4}\left(\frac{K_2^2 + t}{K_2^2 - t} + \frac{K_4^2 + s}{K_4^2 - s}\right)(\epsilon_2, k_3)(\epsilon_4, k_3). \end{aligned}$$

Deriving the rational part (VII): 2 mass hard box integral

$$I_4^{2mh}(\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4;c_3,c_4)$$
$$\equiv I_4\big[(\epsilon_1,p)(\epsilon_2,p-k_1)\big((\epsilon_3,p+K_4)+c_3\big)\big((\epsilon_4,p+K_4)+c_4\big)\big]$$
$$=\int\frac{\mathrm{d}^Dp}{i\pi^{D/2}}\frac{(\epsilon_1,p)(\epsilon_2,p-k_1)((\epsilon_3,p+K_4)+c_3)((\epsilon_4,p+K_4)+c_4)}{p^2(p-k_1)^2(p-k_{12})^2(p+K_4)^2},$$



$$I_4^{2mh}(\lambda_1\tilde\eta_1,\eta_2\tilde\lambda_2,\epsilon_3,\epsilon_4;c_3,c_4)$$
$$=\frac{1}{\langle 2|K_3|1]}\Big[-\frac{1}{6}\langle\eta_2|k_2K_3k_1|\tilde\eta_1\rangle(\epsilon_3,\epsilon_4)-t\langle\eta_2\,2\rangle[\tilde\eta_1\,1]I_4^{2mh}(\lambda_1\tilde\lambda_2,\epsilon_3,\epsilon_4)$$
$$+t\langle\eta_2|k_2|\tilde\eta_1\Big]I_3^{3m}(\epsilon_3,\epsilon_4)+\langle\eta_2\,2\rangle[\tilde\eta_1\,1]\Big(\frac{1}{2}\big(\langle 1|\epsilon_3|2]c_4+\langle 1|\epsilon_4|2]c_3\big)$$
$$+\frac{1}{18}\big(\langle 1|\epsilon_3|2]\epsilon_4+\langle 1|\epsilon_4|2]\epsilon_3,7k_1+2k_2+9K_4\big)\Big)$$
$$+\frac{1}{18}\langle\eta_2\,2\rangle[\tilde\eta_1\,2]\big((\epsilon_3,k_{12})(\epsilon_4,k_{12})-2s_{12}(\epsilon_3,\epsilon_4)\big)$$
$$+\frac{1}{18}\langle\eta_2|(k_2+K_3)|\tilde\eta_1]\big((\epsilon_3,k_2+K_3)(\epsilon_4,k_2+K_3)-2t(\epsilon_3,\epsilon_4)\big)$$
$$-\frac{1}{18}\langle\eta_2|K_3|\tilde\eta_1]\big((\epsilon_3,K_3)(\epsilon_4,K_3)-2K_3^2(\epsilon_3,\epsilon_4)+\langle 2|K_3|\tilde\eta_1\rangle\big(I_3^{2m}(\eta_2\tilde\lambda_2,\epsilon_3,\epsilon_4)$$
$$+I_3^{2m}\big(\eta_2\tilde\lambda_2,\big(c_3-(\epsilon_3,K_3)\big)\epsilon_4+\big(c_4+(\epsilon_4,K_4+k_1)\big)\epsilon_3\big)\big)$$
$$+I_3^{3m}(v,\epsilon_3,\epsilon_4)+I_3^{3m}\big(v,\big(c_3-(\epsilon_3,K_3)\big)\epsilon_4+c_4\epsilon_3\big),$$
$$+\langle\eta_2|K_4|1]\big(\tilde I_3^{2m}(\lambda_1\tilde\eta_1,\epsilon_3,\epsilon_4)$$
$$+\tilde I_3^{2m}\big(\lambda_1\tilde\eta_1,\big(c_3-(\epsilon_3,k_2+K_3)\big)\epsilon_4+\big(c_4+(\epsilon_4,K_4)\big)\epsilon_3\big)\Big],$$

$$v=\langle\eta_2|K_3|1]\lambda_1\tilde\eta_1+\langle\eta_2|K_3|2]\lambda_2\tilde\eta_1-(k_2,K_3)\eta_2\tilde\eta_1.$$

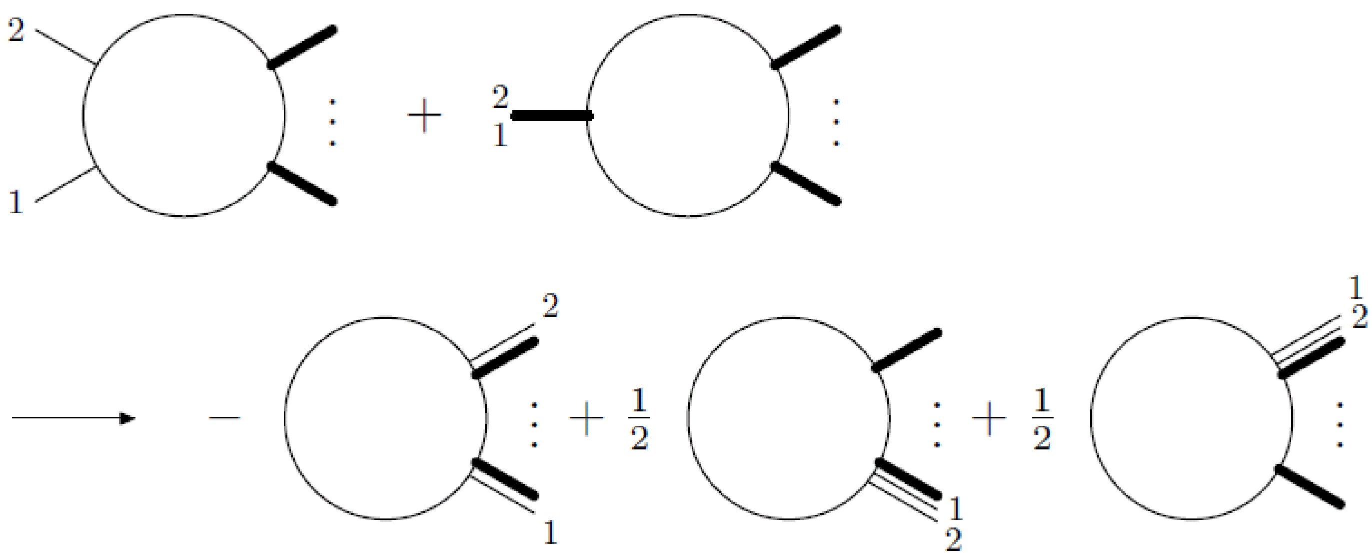
The rational part (VIII): Pentagon, hexagon and higher point

Basic method: tensor reduction

For 2^+3^+, $\epsilon_2 = \lambda_3\tilde{\lambda}_2$ **and** $\epsilon_3 = \lambda_2\tilde{\lambda}_3$, **we have**

$$\frac{(\epsilon_2, p)(\epsilon_3, p - k_2)}{p^2(p-k_2)^2(p-k_{23})^2)} + \frac{(\epsilon_{23}, p) - (\epsilon_2, \epsilon_3)/2}{p^2(p-k_{23})^2}$$

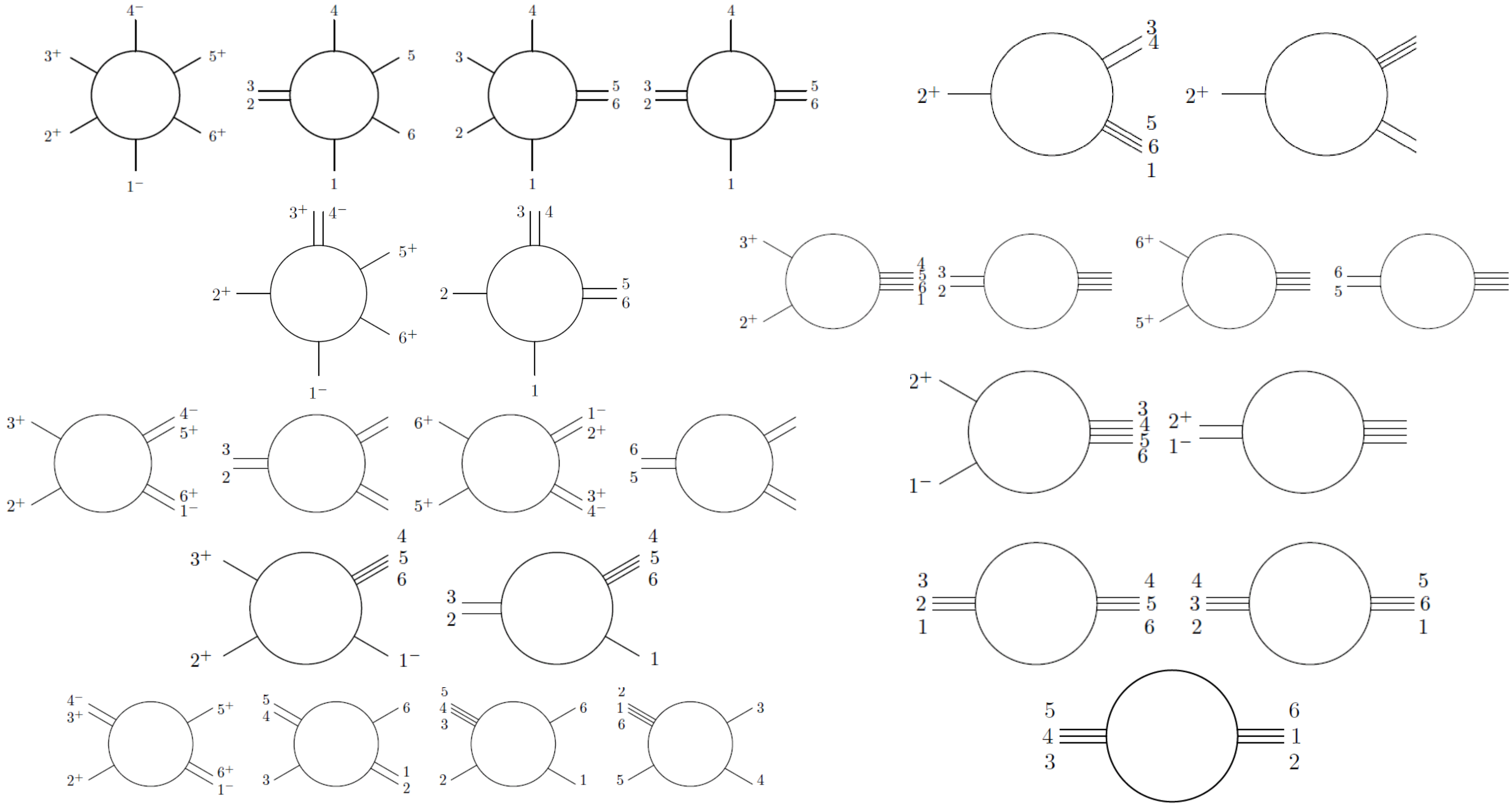
$$= \quad -\frac{1}{(p-k_2)^2} + \frac{1/2}{p^2} + \frac{1/2}{(p-k_{23})^2}.$$



For $2^+3^-4^+$, **much more complicated.**

MHV: $A_6(1^-2^+3^+4^-5^+6^+)$

Total of 51 diagrams. Can be classified into the following 10 sets. Computed them in sets.



The results for 7 sets (31 diagrams) are:

$$R_{10}(1) = -\frac{1}{36}\, s_{345}(\epsilon_{345}, \epsilon_{612}), \qquad R_9(1) = -\frac{1}{18}\, s_{123}(\epsilon_{123}, \epsilon_{456}),$$

$$R_8(1) = -\frac{1}{9}\, s_{12}\,(\epsilon_{12}, \epsilon_{3456}),$$

$$R_7(1) = \frac{1}{8}\, s_{23}\,((\epsilon_{45}, \epsilon_{61}) + (\epsilon_4, \epsilon_{561}) + (\epsilon_{456}, \epsilon_1)) - \frac{1}{12}\, s_{23}(\epsilon_{4561}, k_2),$$

$$R_1(1) = -\frac{1}{36}\left((\epsilon_1, k_2)\,(\epsilon_4, k_{12}) + \frac{1}{2}(\epsilon_1, k_{612})\,(\epsilon_4, k_{345}) + \frac{1}{4}(\epsilon_1, k_{123})\,(\epsilon_4, k_{456})\right) + \frac{1}{72}(4s_{12} - 2s_{345} - s_{123})\,(\epsilon_1, \epsilon_4),$$

$$R_3(1) = \frac{1}{18}\,(\epsilon_1, k_2)\,(\epsilon_{456}, k_3) + \frac{1}{18}\,(2s_{12} - s_{123} - 3s_{23})\,(\epsilon_1, \epsilon_{456}),$$

$$R_4(1) = \frac{1}{36}\,(\epsilon_{45}, k_3)\,(\epsilon_{61}, k_2) + \frac{1}{36}\,(2s_{345} - s_{45} - s_{61} - 3s_{23})\,(\epsilon_{45}, \epsilon_{61}).$$

The pure rational part is

$$
\begin{aligned}
R_0 \;=\;& -\frac{1}{36}\, s_{345}(\epsilon_{345},\epsilon_{612}) - \frac{1}{18}\, s_{123}(\epsilon_{123},\epsilon_{456}) - \frac{1}{9}\, s_{12}\,(\epsilon_{12},\epsilon_{3456}) \\
&+ \frac{1}{8}\, s_{23}\,((\epsilon_{45},\epsilon_{61}) + (\epsilon_4,\epsilon_{561}) + (\epsilon_{456},\epsilon_1)) - \frac{1}{12}\, s_{23}(\epsilon_{4561},k_2) \\
&- \frac{1}{36}\Big((\epsilon_1,k_2)\,(\epsilon_4,k_{12}) + \frac{1}{2}(\epsilon_1,k_{612})\,(\epsilon_4,k_{345}) \\
&+ \frac{1}{4}(\epsilon_1,k_{123})\,(\epsilon_4,k_{456})\Big) + \frac{1}{72}(4s_{12} - 2s_{345} - s_{123})\,(\epsilon_1,\epsilon_4) \\
&+ \frac{1}{18}\,(\epsilon_1,k_2)\,(\epsilon_{456},k_3) + \frac{1}{18}\,(2s_{12} - s_{123} - 3s_{23})\,(\epsilon_1,\epsilon_{456}) \\
&+ \frac{1}{36}\,(\epsilon_{45},k_3)\,(\epsilon_{61},k_2) + \frac{1}{36}\,(2s_{345} - s_{45} - s_{61} - 3s_{23})\,(\epsilon_{45},\epsilon_{61}) \\
&- \frac{1}{18}((\epsilon_2,\epsilon_{34})\epsilon_1 + (\epsilon_2,\epsilon_1)\epsilon_{34} + (\epsilon_1,\epsilon_{34})\epsilon_2, k_5 - k_6) \\
&- \frac{1}{24}\,((\epsilon_2,\epsilon_{34})(\epsilon_1,k_2) + (\epsilon_2,\epsilon_1)(\epsilon_{34},k_2)) + \frac{1}{12 s_{12}}(\epsilon_2,k_1)(\epsilon_{34},k_1 + 2k_2)(\epsilon_1,k_2) \\
&+ \frac{1}{8s_{12}}\,(\epsilon_3,\epsilon_4)(\epsilon_1,k_2)(\epsilon_2,k_1) + \frac{\langle 2\,5\rangle}{4\langle 2\,6\rangle}\frac{(\epsilon_2,k_1)}{s_{12}}(\epsilon_1,k_2)((\epsilon_3,\epsilon_{45}) + (\epsilon_{34},\epsilon_5)) \\
&+ \frac{\langle 2\,6\rangle}{9\langle 2\,5\rangle}((\epsilon_2,\epsilon_{34})\epsilon_{61} + (\epsilon_2,\epsilon_{61})\epsilon_{34} + (\epsilon_{34},\epsilon_{61})\epsilon_2, k_5) \\
&+ \frac{\langle 2\,5\rangle}{\langle 2\,6\rangle}\Big[\frac{1}{9}\,((\epsilon_2,\epsilon_{345})\epsilon_1 + (\epsilon_2,\epsilon_1)\epsilon_{345} + (\epsilon_{345},\epsilon_1)\epsilon_2, k_6) \\
&- \frac{1}{12}\,((\epsilon_2,\epsilon_{345})\epsilon_1 + (\epsilon_2,\epsilon_1)\epsilon_{345}, k_2) + \frac{1}{6\, s_{12}}\,(\epsilon_1,k_2)(\epsilon_2,k_1)(\epsilon_{345},k_2 - k_6)\Big] \\
&+ (k_2,k_5)\,(\epsilon_{34},\epsilon_{61}\,\left[\frac{5}{18}\frac{\langle 2\,3\rangle\langle 5\,6\rangle}{\langle 2\,5\rangle^2} - \frac{1}{18}\frac{\langle 3\,5\rangle\langle 2\,6\rangle}{\langle 2\,5\rangle^2}\right] \\
&+ ((\epsilon_{34},k_2)(\epsilon_{61},k_5) + (\epsilon_{34},k_5)(\epsilon_{61},k_2))\,\left[-\frac{2}{9}\frac{\langle 2\,3\rangle\langle 5\,6\rangle}{\langle 2\,5\rangle^2} + \frac{1}{36}\frac{\langle 3\,5\rangle\langle 2\,6\rangle}{\langle 2\,5\rangle^2}\right] \\
&+ (k_2,k_6)(\epsilon_1,\epsilon_{345})\,\left[-\frac{5}{18}\frac{\langle 2\,3\rangle\langle 5\,6\rangle}{\langle 2\,6\rangle^2} - \frac{1}{18}\frac{\langle 3\,6\rangle\langle 2\,5\rangle}{\langle 2\,6\rangle^2}\right] \\
&+ (\epsilon_1,k_2)(\epsilon_{345},k_6)\,\left[\frac{4}{9}\frac{\langle 2\,3\rangle\langle 5\,6\rangle}{\langle 2\,6\rangle^2} + \frac{1}{18}\frac{\langle 3\,6\rangle\langle 2\,5\rangle}{\langle 2\,6\rangle^2}\right] \\
&+ \frac{1}{36}\,(7(\epsilon_2,\epsilon_{34})(\epsilon_{561},k_2) - 7(\epsilon_2,\epsilon_{561})(\epsilon_{34},k_2) + 4(\epsilon_{34},\epsilon_{561})(\epsilon_2,k_{34})) \\
&- \frac{1}{4}(\epsilon_3,\epsilon_4)\,(\epsilon_2,\epsilon_{561}) - \frac{1}{4}((\epsilon_5,\epsilon_{61}) + (\epsilon_{56},\epsilon_1))\,(\epsilon_2,\epsilon_{34}) \\
&+ \frac{1}{36}\,(7(\epsilon_2,\epsilon_{345})(\epsilon_{61},k_2) - 7(\epsilon_2,\epsilon_{61})(\epsilon_{345},k_2) - 4(\epsilon_{345},\epsilon_{61})(\epsilon_2,k_{61})) \\
&- \frac{1}{4}(\epsilon_6,\epsilon_1)(\epsilon_2,\epsilon_{345}) - \frac{1}{4}((\epsilon_3,\epsilon_{45}) + (\epsilon_{34},\epsilon_5))(\epsilon_2,\epsilon_{61}) \\
&+ \frac{1}{18}\,(2(\epsilon_2,\epsilon_{3456})(\epsilon_1,k_2) - 5(\epsilon_2,\epsilon_1)(\epsilon_{3456},k_2) - 2(\epsilon_{3456},\epsilon_1)(\epsilon_2,k_1)) \\
&- \frac{1}{4}((\epsilon_3,\epsilon_{456}) + (\epsilon_{345},\epsilon_6) + (\epsilon_{34},\epsilon_{56}))\,\left[(\epsilon_2,\epsilon_1) - \frac{(\epsilon_2,k_1)(\epsilon_1,k_2)}{s_{12}}\right] \\
&+ \frac{1}{6\, s_{12}}\,(\epsilon_1,k_2)(\epsilon_2,k_1)(\epsilon_{3456},k_2) + (\epsilon_1,k_2)(\epsilon_{345},k_2)\,\frac{1}{4}\,\frac{\langle 2\,3\rangle\langle 5\,6\rangle}{\langle 2\,6\rangle^2}.
\end{aligned}
$$

The simple pole terms are:

$$
\begin{aligned}
R_1 \;=\;& \frac{(\epsilon_2,k_{61})}{6(s_{61} - s_{345})}\,((\epsilon_{34},k_2)(\epsilon_1,k_6) - (\epsilon_{34},k_5)(\epsilon_1,k_2)) \\
&- \frac{(\epsilon_2,k_{34})}{12(s_{34} - s_{234})}\,(\epsilon_{34},k_2)(\epsilon_1,k_{56}) \\
&- \frac{s_{61} + s_{345}}{12(s_{61} - s_{345})}\,((\epsilon_2,\epsilon_{34})(\epsilon_1,k_2) + (\epsilon_2,\epsilon_1)(\epsilon_{34},k_2)) \\
&- \frac{s_{34} + s_{234}}{24(s_{34} - s_{234})}\,((\epsilon_2,\epsilon_{34})(\epsilon_1,k_2) + (\epsilon_2,\epsilon_1)(\epsilon_{34},k_2)) \\
&+ \frac{(\epsilon_3,\epsilon_4)(\epsilon_1,k_2)}{8}\left[\frac{2(\epsilon_2,k_{61})}{s_{61} - s_{345}} - \frac{(\epsilon_2,k_{34})}{s_{34} - s_{234}}\right] \\
&+ \frac{\langle 2\,6\rangle}{4\langle 2\,5\rangle}\left[\frac{(\epsilon_2,k_{34})}{s_{34} - s_{234}} - \frac{(\epsilon_2,k_{61})}{s_{61} - s_{345}}\right](\epsilon_3,\epsilon_4)(\epsilon_{61},k_2) \\
&+ \frac{\langle 2\,6\rangle}{4\langle 2\,5\rangle}\left[\frac{(\epsilon_2,k_{34})}{s_{34} - s_{234}} - \frac{(\epsilon_2,k_{61})}{s_{61} - s_{345}}\right](\epsilon_1,\epsilon_6)(\epsilon_{34},k_2) \\
&+ \frac{\langle 2\,5\rangle}{4\langle 2\,6\rangle}\left[\frac{(\epsilon_2,k_{61})}{s_{61} - s_{345}}\right](\epsilon_1,k_2)((\epsilon_3,\epsilon_{45}) + (\epsilon_{34},\epsilon_5)) \\
&+ \frac{\langle 2\,6\rangle}{\langle 2\,5\rangle}\Bigg[\frac{1}{12}\left[\frac{s_{34} + s_{234}}{s_{34} - s_{234}} + \frac{s_{61} + s_{345}}{s_{61} - s_{345}}\right]((\epsilon_2,\epsilon_{34})\epsilon_{61} + (\epsilon_2,\epsilon_{61})\epsilon_{34}, k_2) \\
&+ \left[\frac{(\epsilon_2,k_{34})(\epsilon_{34},k_2)(\epsilon_{61},k_5)}{6(s_{34} - s_{234})} + \frac{(\epsilon_2,k_{61})(\epsilon_{34},k_5)(\epsilon_{61},k_2)}{6(s_{61} - s_{345})}\right]\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
- & \frac{\langle 2\,5\rangle}{\langle 2\,6\rangle}\left[\frac{s_{61}+s_{345}}{12(s_{61}-s_{345})}((\epsilon_2,\epsilon_{345})\epsilon_1+(\epsilon_2,\epsilon_1)\epsilon_{345},k_2)\right.\\
+ & \left.(\epsilon_2,k_{61})(\epsilon_{345},k_2)\frac{(\epsilon_1,k_6)}{6(s_{61}-s_{345})}\right]\\
+ & \frac{s_{61}+s_{345}}{s_{61}-s_{345}}\left[\frac{1}{4}\,\frac{\langle 2\,3\rangle\langle 5\,6\rangle}{\langle 2\,5\rangle^2}\,(\epsilon_{34},k_2)(\epsilon_{61},k_2)+\frac{1}{4}\,\frac{\langle 2\,3\rangle\langle 5\,6\rangle}{\langle 2\,6\rangle^2}\,(\epsilon_{345},k_2)(\epsilon_1,k_2)\right]\\
+ & \frac{s_{34}+s_{234}}{s_{34}-s_{234}}\,\frac{1}{4}\,\frac{\langle 2\,3\rangle\langle 5\,6\rangle}{\langle 2\,5\rangle^2}\,(\epsilon_{34},k_2)(\epsilon_{61},k_2)\\
- & \frac{s_{34}+s_{234}}{12(s_{34}-s_{234})}\,((\epsilon_2,\epsilon_{34})(\epsilon_{561},k_2)+(\epsilon_2,\epsilon_{561})(\epsilon_{34},k_2))\\
- & \frac{1}{4}(\epsilon_3,\epsilon_4)\frac{(\epsilon_2,k_{34})(\epsilon_{561},k_2)}{s_{34}-s_{234}}-\frac{1}{4}((\epsilon_5,\epsilon_{61})+(\epsilon_{56},\epsilon_1))\frac{(\epsilon_2,k_{34})(\epsilon_{34},k_2)}{s_{34}-s_{234}}\\
+ & \frac{s_{61}+s_{345}}{12(s_{61}-s_{345})}\,((\epsilon_2,\epsilon_{345})(\epsilon_{61},k_2)+(\epsilon_2,\epsilon_{61})(\epsilon_{345},k_2))\\
- & \frac{1}{4}(\epsilon_6,\epsilon_1)\frac{(\epsilon_2,k_{61})(\epsilon_{345},k_2)}{s_{61}-s_{345}}-\frac{1}{4}((\epsilon_3,\epsilon_{45})+(\epsilon_{34},\epsilon_5))\frac{(\epsilon_2,k_{61})(\epsilon_{61},k_2)}{s_{61}-s_{345}}.
\end{aligned}
$$

The double pole terms are:

$$
\begin{aligned}
R_2 \;=\; & \frac{s_{61}+s_{345}}{6(s_{61}-s_{345})^2}\Big[(\epsilon_2,k_{61})(\epsilon_{345},k_2)(\epsilon_{61},k_2)-(\epsilon_2,k_{61})(\epsilon_{34},k_2)(\epsilon_1,k_2)\\
- & \frac{\langle 2\,5\rangle}{\langle 2\,6\rangle}(\epsilon_2,k_{61})(\epsilon_{345},k_2)+\frac{\langle 2\,6\rangle}{\langle 2\,5\rangle}\,(\epsilon_{34},k_2)(\epsilon_{61},k_2)\Big]\\
+ & \frac{(s_{34}+s_{234})}{6(s_{34}-s_{234})^2}\Big[\frac{\langle 2\,6\rangle}{\langle 2\,5\rangle}\,(\epsilon_{34},k_2)(\epsilon_{61},k_2)\,(\epsilon_2,k_{34})\\
- & \frac{1}{2}\,(\epsilon_2,k_{34})(\epsilon_{34},k_2)(\epsilon_1,k_2)-(\epsilon_2,k_{34})(\epsilon_{34},k_2)(\epsilon_{561},k_2)\Big]
\end{aligned}
$$

$$\times \frac{1}{2\,t_{345}}\left(\frac{\langle 1^-|\,(3+4)\,|5^-\rangle\,\langle 1\,3\rangle\langle 2\,6\rangle\langle 3\,5\rangle}{\langle 3\,6\rangle} - \frac{\langle 1^-|\,(2+6)(3+5)\,|2^+\rangle\,\langle 1\,2\rangle\langle 3\,6\rangle}{\langle 2\,6\rangle}\right) \qquad \text{(B4)}$$

For $A_{n;1}^{\mathcal{N}=0}(1^-,2^+,3^+,4^-,5^+,6^+)$ we can make use of the symmetry to write the rational remainder as

$$\widehat{R}_6(1,4) = \widehat{R}_6^a(1,4) + \widehat{R}_6^a(1,4)\Big|_{flip} \qquad \text{(B2)}$$

where

$$X(1,2,3,4,5,6)\Big|_{\rm flip} \equiv X(1,6,5,4,3,2) \qquad \text{(B3)}$$

and *Beware of and make sure the* $2/9A^{tree}$ *is right. Also make sure subtraction of boundary of cut is properly included!!!*

$$\begin{aligned}
&\widehat{R}_6^a(1,4)\\
&= -\frac{8\langle 1\,3\rangle\langle 1\,4\rangle^3}{9\langle 1\,2\rangle\langle 1\,6\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 3\,5\rangle\langle 5\,6\rangle} - \frac{(\langle 1\,4\rangle[4\,3]+2\langle 1\,5\rangle[5\,3])\,\langle 1^-|\,(4+5)\,|3^-\rangle\,\langle 1\,5\rangle^2[3\,5]}{6\langle 1\,2\rangle\langle 1\,6\rangle\langle 2\,5\rangle\langle 3\,5\rangle\langle 5\,6\rangle[3\,4]^2 t_{345}}\\
&+\frac{\langle 2^-|\,(1+4)\,|6^-\rangle\,\langle 5^-|\,(1+4)\,|6^-\rangle\,\langle 1\,5\rangle^2[5\,6]t_{146}}{2s_{16}\,\langle 3^-|\,(1+4)\,|6^-\rangle\,\langle 5^-|\,(2+3)\,|4^-\rangle\,\langle 2\,3\rangle\langle 2\,5\rangle^2\langle 5\,6\rangle[4\,6]}\\
&-\frac{[3\,5]^3[4\,6]^3}{3\,\langle 2^-|\,(3+5)\,|4^-\rangle^2\,[1\,4][1\,6][3\,4][4\,5]} + \frac{\langle 1^-|\,(2+4)\,|3^-\rangle^3\,[2\,4]}{3\,\langle 5^-|\,(2+3)\,|4^-\rangle\,\langle 1\,6\rangle\langle 2\,3\rangle\langle 5\,6\rangle[3\,4]^2 t_{234}}\\
&+\frac{\langle 1^-|\,(2+4)\,|3^-\rangle\,\langle 1\,2\rangle[2\,3](\langle 1^-|\,(2+4)\,|3^-\rangle+\langle 1\,2\rangle[2\,3])}{6\langle 1\,6\rangle\langle 2\,3\rangle\langle 2\,5\rangle\langle 5\,6\rangle[3\,4]^2 t_{234}}\\
&+\frac{\langle 1^-|\,(2+4)\,|3^-\rangle^2\,\langle 1\,2\rangle\langle 1\,5\rangle[2\,3]}{2\,\langle 1^-|\,(2+3)\,|4^-\rangle\,\langle 1\,6\rangle\langle 2\,5\rangle^2\langle 5\,6\rangle[3\,4]t_{234}}\\
&-\frac{\langle 5^-|\,(1+4)\,|6^-\rangle\,\langle 1\,4\rangle\langle 1\,5\rangle[5\,6]t_{146}}{6s_{16}\,\langle 3^-|\,(1+4)\,|6^-\rangle\,\langle 5^-|\,(2+3)\,|4^-\rangle\,\langle 2\,3\rangle\langle 2\,5\rangle\langle 5\,6\rangle}\\
&-\frac{\langle 1\,2\rangle\langle 1\,4\rangle^2[2\,3]}{6\langle 1\,6\rangle\langle 2\,3\rangle\langle 2\,5\rangle\langle 3\,4\rangle\langle 5\,6\rangle[3\,4]}\left(1+\frac{3\langle 1\,5\rangle\langle 2\,3\rangle}{\langle 1\,3\rangle\langle 2\,5\rangle}\right)\\
&+\frac{\langle 2^-|\,(1+4)\,|6^-\rangle^3\,\langle 3^-|\,(2+5)\,|4^-\rangle\,[2\,3][4\,6]}{3\,\langle 2^-|\,(3+5)\,|4^-\rangle^2\,\langle 5^-|\,(1+4)\,|6^-\rangle\,\langle 5^-|\,(2+3)\,|4^-\rangle\,\langle 2\,3\rangle^2[1\,4][1\,6]}\\
&+\frac{\langle 1\,4\rangle^2\langle 1\,5\rangle^2[3\,5]}{6\langle 1\,2\rangle\langle 1\,6\rangle\langle 2\,5\rangle\langle 3\,4\rangle\langle 3\,5\rangle\langle 5\,6\rangle[3\,4]}\left(1+\frac{3\langle 1\,2\rangle\langle 3\,5\rangle}{\langle 1\,3\rangle\langle 2\,5\rangle}\right)\\
&+\frac{\langle 1\,4\rangle\langle 1\,5\rangle[2\,3][5\,6]}{6\,\langle 1^-|\,(3+4)\,|2^-\rangle\,\langle 2\,5\rangle\langle 5\,6\rangle t_{234}}\left(\frac{\langle 1\,4\rangle}{\langle 2\,3\rangle}+\frac{3\,\langle 1^-|\,(2+4)\,|3^-\rangle\,\langle 1\,5\rangle}{\langle 1^-|\,(2+3)\,|4^-\rangle\,\langle 2\,5\rangle}\right)\\
&-\frac{\langle 1^-|\,(3+5)\,|4^-\rangle^4\,[3\,5]^3}{6\,\langle 2^-|\,(3+5)\,|4^-\rangle\,\langle 6^-|\,(3+5)\,|4^-\rangle\,\langle 1\,2\rangle\langle 1\,6\rangle\langle 3\,5\rangle[3\,4]^2[4\,5]^2 t_{345}}\\
&+\frac{\langle 1^-|\,(3+5)\,|4^-\rangle^4\,\langle 2\,6\rangle[2\,6][3\,5]^4}{6\,\langle 1^-|\,(3+4)\,|5^-\rangle\,\langle 1^-|\,(4+5)\,|3^-\rangle\,\langle 2^-|\,(3+5)\,|4^-\rangle^2\,\langle 6^-|\,(3+5)\,|4^-\rangle^2\,[3\,4][4\,5]t_{345}}\\
&+\frac{\langle 1^-|\,(2+6)(3+5)\,|6^+\rangle^2\,\langle 1\,6\rangle[2\,6][3\,5]}{3\,\langle 1^-|\,(3+4)\,|5^-\rangle\,\langle 1^-|\,(4+5)\,|3^-\rangle\,\langle 6^-|\,(3+5)\,|4^-\rangle\,\langle 2\,6\rangle\langle 3\,6\rangle\langle 5\,6\rangle t_{345}}\\
&\quad\times\left(\frac{\langle 1\,4\rangle}{2\langle 3\,5\rangle}-\frac{\langle 1\,6\rangle[3\,5]}{\langle 6^-|\,(3+5)\,|4^-\rangle}\right)\\
&-\frac{\langle 5^-|\,(2+3)\,|6^-\rangle\,[5\,6]t_{146}^2([1\,4]\langle 1\,2\rangle\langle 1\,5\rangle+\langle 2\,5\rangle\langle 1\,6\rangle[4\,6])}{3\,\langle 2^-|\,(3+5)\,|4^-\rangle\,\langle 3^-|\,(1+4)\,|6^-\rangle\,\langle 5^-|\,(2+3)\,|4^-\rangle\,\langle 1\,6\rangle\langle 2\,3\rangle\langle 5\,6\rangle[1\,6]\langle 2\,5\rangle[1\,4]}\\
&-\frac{\langle 1\,5\rangle^2[2\,3]^2[5\,6]}{6\,\langle 1^-|\,(3+4)\,|2^-\rangle\,\langle 5^-|\,(2+3)\,|4^-\rangle\,\langle 2\,5\rangle\langle 5\,6\rangle t_{234}}\\
&\quad\times\left(3\langle 1\,4\rangle+\frac{3\,\langle 1^-|\,(2+4)\,|3^-\rangle\,\langle 1\,5\rangle\langle 2\,3\rangle}{\langle 1^-|\,(2+3)\,|4^-\rangle\,\langle 2\,5\rangle}-\frac{2\langle 1\,5\rangle\langle 2\,6\rangle[2\,3]}{\langle 5\,6\rangle[3\,4]}\right)\\
&+\frac{\langle 1\,4\rangle^3\langle 1\,5\rangle[2\,5]}{6\,\langle 1^-|\,(3+4)\,|2^-\rangle\,\langle 3^-|\,(1+4)\,|6^-\rangle\,\langle 1\,3\rangle\langle 1\,6\rangle^2\langle 2\,3\rangle\langle 2\,5\rangle^2\langle 3\,4\rangle\langle 5\,6\rangle}\\
&\quad\times\Big((\langle 1^-|\,6(2+5)\,|3^+\rangle+\langle 1^-|\,(5+6)2\,|3^+\rangle)(\langle 1\,3\rangle\langle 2\,5\rangle-3\langle 1\,5\rangle\langle 2\,3\rangle)\\
&\qquad -3\langle 1\,2\rangle\langle 1\,5\rangle\langle 2\,3\rangle\langle 3\,5\rangle[2\,5]\Big)\\
&+\frac{\langle 1\,4\rangle^3\langle 3\,6\rangle[2\,6]}{6\,\langle 1^-|\,(3+4)\,|5^-\rangle\,\langle 3^-|\,(1+4)\,|2^-\rangle\,\langle 1\,2\rangle\langle 1\,3\rangle\langle 2\,6\rangle^2\langle 3\,4\rangle\langle 3\,5\rangle^2\langle 5\,6\rangle}\\
&\quad\times\Big(2(\langle 1^-|\,2(5+6)\,|3^+\rangle+\langle 1^-|\,(2+6)5\,|3^+\rangle)\langle 1\,3\rangle\langle 2\,6\rangle\\
&\qquad -3(2\,\langle 1^-|\,2(5+6)\,|3^+\rangle+\langle 1^-|\,65\,|3^+\rangle)\langle 1\,6\rangle\langle 2\,3\rangle\Big)\\
&-\frac{\langle 1^-|\,(4+5)\,|3^-\rangle\,\langle 1\,5\rangle^2[3\,5]}{2\,\langle 1^-|\,(3+5)\,|4^-\rangle\,\langle 1\,2\rangle\langle 1\,6\rangle\langle 2\,3\rangle\langle 2\,5\rangle\langle 3\,5\rangle\langle 5\,6\rangle[3\,4]t_{345}}\\
&\quad\times\left(\frac{\langle 1^-|\,(2+6)(3+5)\,|2^+\rangle\,\langle 1\,2\rangle\langle 3\,5\rangle}{\langle 2\,5\rangle}-\langle 1^-|\,(3+4)\,|5^-\rangle\,\langle 1\,3\rangle\langle 2\,5\rangle\right)\\
&+\frac{\langle 2^-|\,(1+4)\,|6^-\rangle\,t_{146}^2}{3s_{16}\,\langle 2^-|\,(3+5)\,|4^-\rangle^2\,\langle 3^-|\,(1+4)\,|6^-\rangle\,\langle 2\,3\rangle\langle 2\,5\rangle\langle 2\,6\rangle[1\,4]}\\
&\quad\times\left(\frac{\langle 1\,6\rangle\langle 2\,5\rangle\langle 2\,6\rangle[4\,6]^2[5\,6]}{\langle 5^-|\,(2+3)\,|4^-\rangle}-\frac{\langle 2^-|\,(1+4)\,|6^-\rangle\,\langle 1\,2\rangle^2[1\,4][2\,6]}{\langle 5^-|\,(2+3)\,|6^-\rangle}\right)\\
&+\frac{\langle 2^-|\,(1+4)\,|6^-\rangle^2\,\langle 1\,2\rangle[2\,6]t_{146}}{2s_{16}\,\langle 2^-|\,(3+5)\,|4^-\rangle\,\langle 3^-|\,(1+4)\,|6^-\rangle\,\langle 5^-|\,(1+4)\,|6^-\rangle\,\langle 2\,3\rangle\langle 2\,5\rangle\langle 2\,6\rangle\langle 5\,6\rangle[4\,6]}\\
&\quad\times\left(\frac{\langle 5^-|\,(1+4)\,|6^-\rangle\,\langle 1\,5\rangle\langle 2\,6\rangle}{\langle 2\,5\rangle}-\frac{\langle 6^-|\,(1+4)\,|6^-\rangle\,\langle 1\,6\rangle\langle 2\,5\rangle}{\langle 2\,6\rangle}+\frac{\langle 1\,4\rangle\langle 5\,6\rangle[4\,6]}{3}\right)\\
&+\frac{\langle 1^-|\,(2+6)(3+5)\,|6^+\rangle^2\,\langle 1\,6\rangle[2\,6][3\,5]}{\langle 1^-|\,(3+4)\,|5^-\rangle\,\langle 1^-|\,(3+5)\,|4^-\rangle\,\langle 1^-|\,(4+5)\,|3^-\rangle\,\langle 6^-|\,(3+5)\,|4^-\rangle\,\langle 2\,3\rangle\langle 2\,6\rangle\langle 3\,5\rangle\langle 3\,6\rangle\langle 5\,6\rangle}
\end{aligned}$$

In[72]:= `p00 = %70`

Out[72]=

```
25022118756678932138273556865452679497402676781206862792805356439697884213437958113615
 /
1168075627225659898464419664108806338586752881199234982762109893273398374565839287 4
```

In[73]:= `p01 =`

```
18835424398718928352706461196384522932616669167255766282841628434267891582913180333 55
  /
 44925985662525380710169987081107936099490495430739807029311918972053783637147664 9 -
2379093105536000 0 / (518397975010773 * e)
```

Out[73]=

$$\frac{1883542439871892835270646119638452293261666916725576628284162843426789158291318033355}{4492598566252538071016998708110793609949049543073980702931191897205378363714766 49} - \frac{23790931055360000}{518397975010773\, e}$$

In[74]:= `pp = p01 - 2 p00`

Out[74]=

$$-\frac{47581862110720000}{518397975010773} - \frac{23790931055360000}{518397975010773\, e}$$

In[75]:= `Factor[%]`

Out[75]=

$$-\frac{23790931055360000\,(1 + 2\, e)}{518397975010773\, e}$$

NMHV: $A_6(1^-2^-3^+4^-5^+6^+)$

Some terms are:

$$
\begin{aligned}
R_1 \;&=\; I_3^{3m}(k_1,\epsilon_3,\epsilon_4) + (\epsilon_4,k_5)\, I_3^{3m}(k_1,\epsilon_3) \\
&-\; \frac{1}{2}\left[\tilde{I}_3^{2m(4)}(\epsilon_4,k_1,\epsilon_3) + (s_{56}-s_{234})\,\tilde{I}_3^{2m(4)}(\epsilon_4,\epsilon_3)\right] \\
&-\; \frac{1}{2}\left[I_3^{2m(3)}(\epsilon_3,\epsilon_4,k_{12}) + s_{12}\, I_3^{2m(3)}(\epsilon_3,\epsilon_4)\right] \\
&+\; \frac{1}{36}\, s_{34}\,(\epsilon_{34},k_{12}) - \frac{1}{36}((\epsilon_3,k_2)(\epsilon_4,k_3) - 2s_{23}(\epsilon_3,\epsilon_4)) \\
&-\; \frac{1}{72}((\epsilon_3,k_{12})(\epsilon_4,k_{56}) + 2s_{123}(\epsilon_3,\epsilon_4))
\end{aligned}
$$

$$
\begin{aligned}
R_2 \;=\;& -\langle 4\,3\rangle[3\,2]\big(I_4^{2mh(2)}(\lambda_2\tilde\lambda_1,\lambda_6\tilde\lambda_3,\epsilon_{61}) - I_4^{1m(2)}(\lambda_2\tilde\lambda_1,\lambda_6\tilde\lambda_3,\epsilon_{61})\big)\\
+\;& \langle 5\,4\rangle[4\,3]\,\big(I_4^{1m(3)}(\lambda_4\tilde\lambda_1,\lambda_6\tilde\lambda_5,\epsilon_{61}) - I_4^{2mh(4)}(\lambda_4\tilde\lambda_1,\lambda_6\tilde\lambda_5,\epsilon_{61})\\
+\;& I_3^{2m(3)}(\lambda_6\tilde\lambda_3,\epsilon_{61},\lambda_4\tilde\lambda_1) - \langle 6|k_{45}|3]\,I_3^{2m(3)}(\lambda_4\tilde\lambda_1,\epsilon_{61})\\
-\;& \big((\epsilon_{61},k_{45}) + \frac{1}{2}(\epsilon_6,\epsilon_1)\big)\,I_3^{2m(3)}(\lambda_6\tilde\lambda_3,\lambda_4\tilde\lambda_1)\\
+\;& \tilde I_3^{2m(4)}(\lambda_4\tilde\lambda_1,\lambda_6\tilde\lambda_3,\epsilon_{61}) + \langle 4|k_{23}|1]\,\tilde I_3^{2m(4)}(\lambda_6\tilde\lambda_3,\epsilon_{61})\\
+\;& \big((\epsilon_{61},k_{23}) - \frac{1}{2}(\epsilon_6,\epsilon_1)\big)\,\tilde I_3^{2m(4)}(\lambda_6\tilde\lambda_3,\lambda_4\tilde\lambda_1)\\
-\;& I_3^{3m}(\lambda_4\tilde\lambda_1,\epsilon_{61},\lambda_6\tilde\lambda_3) - \big((\epsilon_{61},k_{23}) - \frac{1}{2}(\epsilon_6,\epsilon_1)\big)\,I_3^{3m}(\lambda_4\tilde\lambda_1,\lambda_6\tilde\lambda_3)\\
+\;& \frac{7}{18}\,s_{34}(\tilde\epsilon_{34},\epsilon_{61}) + \frac{1}{4}\,\langle 6\,4\rangle\,[1\,3]\,(\epsilon_{61},k_3-k_4)
\end{aligned}
$$

(Compact) formula for rational part of tensor box integral – 3m and 4m cases

$$I_{4,m}^D = \int \frac{\mathrm{d}^D p}{i\,\pi^{D/2}} \, \frac{\prod_{i=1}^m (\epsilon_i, p)}{p^2 (p+K_1)^2 (p+K_2)^2 (p+K_3)^2}$$

$K_{1,2,3}$ **and** $K \equiv \epsilon_{\mu\nu\rho\sigma} K_1^\nu K_2^\rho K_3^\sigma$**.** $\mathrm{det} K_i \cdot K_j \propto K^2$.

Expanding ϵ_i **in terms of** K_i **and** K**:**

$$\epsilon_i = \sum_{j=1}^{3} a_{ij}\, K_j + c_i K$$

$$(K_i, p) = (p + K_i)^2 - p^2 - K_i^2$$

An example: linear box integral

$$I_{4,1}^D = \int \frac{\mathrm{d}^D p}{i\,\pi^{D/2}} \frac{(\epsilon, p)}{p^2(p+K_1)^2(p+K_2)^2(p+K_3)^2}$$

$$\to \left(-\sum_{i=1}^{3} a_i K_i^2 \right) I_4^D[1] + \sum_{i=1}^{3} a_i\, I_3^{D(i)}[1] - (a_1 + a_2 + a_3)\, I_3^{D(0)}[1]$$

We note:

$$-\sum_{i=1}^{3} a_i K_i^2 = \frac{1}{2}\left((\epsilon, p^+) + (\epsilon, p^-)\right)$$

with $p^2 = 0$ **and** $(p + K_i)^2 = 0$. $(K, p^+) = -(K, p^-)$

(Quadruple cuts to compute box coefficients: BCF, hep-th/0412103).

$$\int \frac{\mathrm{d}^D p}{i\,\pi^{D/2}}\,\frac{(K,p)^2}{p^2(p+K_1)^2(p+K_2)^2(p+K_3)^2}$$
$$= -\frac{1}{2}\,K^2\,I_4^{D+2}[1] \to 0.$$

$$\int \frac{\mathrm{d}^D p}{i\,\pi^{D/2}}\,\frac{(K,p)^4}{p^2(p+K_1)^2(p+K_2)^2(p+K_3)^2}$$
$$= \frac{3}{4}\,(K^2)^2\,I_4^{D+4}[1] \to \frac{3}{4} \times \frac{5}{18}\,(K^2)^2.$$

Compact formula for triangle integral?

$$I_{3,m}^{D} = \int \frac{\mathrm{d}^D p}{i\,\pi^{D/2}} \, \frac{\prod_{i=1}^{m}(\epsilon_i, p)}{p^2(p+K_1)^2(p+K_2)^2}$$

Expanding ϵ_i in terms of:

$$K_1, K_2, l, \bar{l},$$

$$(K_i, l) = 0, \quad l^2 = 0, \quad l = \lambda\,\bar{\mu},$$

$$(K_i, \bar{l}) = 0, \quad \bar{l}^2 = 0, \quad \bar{l} = \mu\,\bar{\lambda}.$$

An example: linear triangle integral

$$\begin{aligned} I^D_{3,1} &= -(a_1 K_1^2 + a_2 K_2^2)\, I_3^D[1] - (a_1 + a_2)\, I_2^{D(0)}[1] \\ &\quad + a_1\, I_2^{D(1)}[1] + a_2\, I_2^{D(2)}[1], \end{aligned}$$

by expanding $\epsilon = a_1 K_1 + a_2 K_2 + \bar{c}\, l + c\bar{l}$.

See recent paper: G. Ossola, C. Papadopoulos and R. Pittau, hep-ph/0609007, eq. (2.4).

6. Perspective

- **Seems feasible for $n = 7, 6, \cdots$: both cut-part (triangle and bubble coefficients) and rational part**
  - **high-point tensor integrals: direct reduction by inserting "spinor-string" (BDK 98) and keeping only $n, \cdots, n-3$ tensors (BDDK theorem)**
  - **box and triangle integrals: red. by expanding ϵ_i. (See also: Binoth et. al., hep-th/0609054)**

- **Attacking the wish lists (proposal already submitted).**