Kinematic space of a subregion

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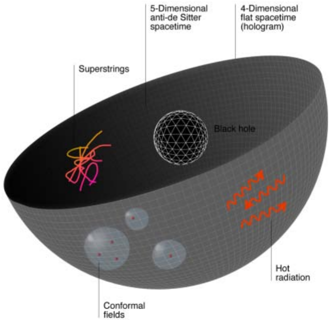
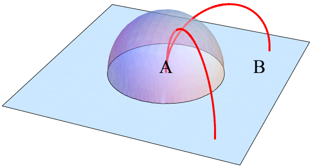
Outline

- Review of integral geometry
- Kinematic space of a subregion
- Purification and reflected entropy

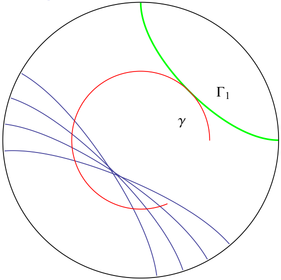


- $CFT_d \leftrightarrow$ string theory in AdS_{d+1}
- RT formula

$$S_{\mathrm{EE}}(A) = \frac{\mathrm{Area}(\gamma_A)}{4G}, \quad \partial\gamma_A = \partial A$$



Integral geometry



Czech etal. 1505.05515

$$\sigma(\gamma) = \frac{1}{4}\int_{\gamma\cap\Gamma\neq\varnothing} N(\gamma\cap\Gamma)\ \epsilon_{\mathcal{K}}$$

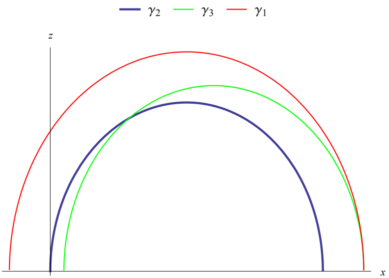
The length $\sigma(\gamma)$ of a curve γ can be expressed in terms of an integral over the geodesics Γ that have nonvanishing intersection number $N(\gamma\cap\Gamma)$ with γ

The measure $\epsilon_{\mathcal{K}}$ is given by the second derivative of the entanglement entropy

$$\epsilon_{\mathcal{K}}(u,v) = \frac{\partial^2 S(u,v)}{\partial u \partial v} \mathrm{d}u \wedge \mathrm{d}v$$

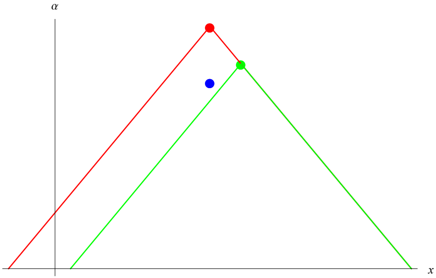
Hence we can obtain the geometry from the entanglement structure of the field theory on the boundary

Kinematic space



Hyperbolic space:

$$ds^2 = \frac{1}{z^2}\left(dz^2 + dx^2\right)$$



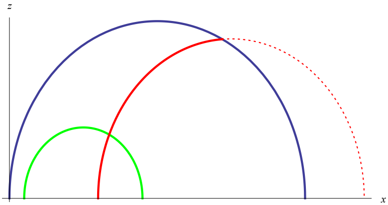
de Sitter:

$$ds^2 = \frac{1}{\alpha^2}\left(-d\alpha^2 + dx^2\right)$$

Kinematic space of a subregion

Jaffferis etal 1512.06431

Subregion-subregion duality: $S_{\rm bdy}(\rho|\sigma) \leftrightarrow S_{\rm bulk}(\rho|\sigma)$

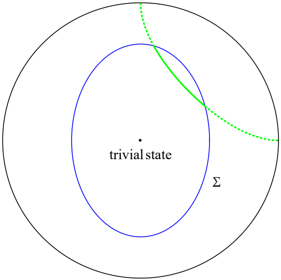


- The kinematic space shall be constructed using the information obtained within the subregion
- Some geodesics cross the RT-surface and have no clear entropic interpretation

A novel construction of the kinematic space is needed for the subregion

Surface/state correspondence

Miyaji&Takayanagi 1503.03542 Surface in bulk $\Sigma \leftrightarrow$ State $|\Psi(\Sigma)\rangle$ in the CFT



- $|\Psi(\Sigma)\rangle$ can be defined using PI on the (optimized) surface $M(\Sigma)$ with $\partial M(\Sigma) = \Sigma$ Takayanagi 1808.09072
- The kinematic space can be established for every surface Σ using the geodesics ending on it

Entanglement of Purification

Takayanagi&Umemoto 1708.09393

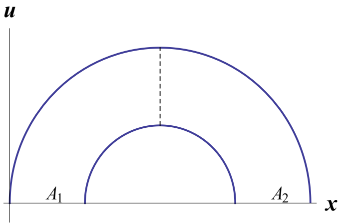
ρ_A can be purified by A' so that

$$\mathrm{Tr}_{A'}|\psi_{AA'}\rangle\langle\psi_{AA'}| = \rho_A$$

Entanglement of purification is defined as the minimization of the entanglement entropy between two "complemented" subsystems ($A = A_1 \cup A_2$)

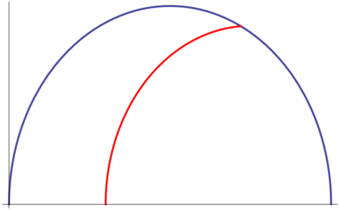
$$E_P(A_1 : A_2) = \min S(\rho_{A_1 A_1'})$$

The conjectured dual of E_P is the entanglement wedge cross section $E_W(A_1 : A_2)$



Purification and RT surface

- The optimized purification $A_1'A_2'$ is given by the RT surface, which is obtained by RG flow of $\overline{A_1A_2}$ to the IR according to SS duality
- The optimized $\psi_{A_1A_2A_1'A_2'}$ should have no extra entanglement within the DoFs in $A_1'A_2'$



$S(\rho_{A_1A_1'})$ corresponds to a geodesic ending on the RT surface and the kinematic space can be constructed Espíndola etal 1804.05855

No concrete description for the subsystem $A_1'A_2'$

Reflected entropy

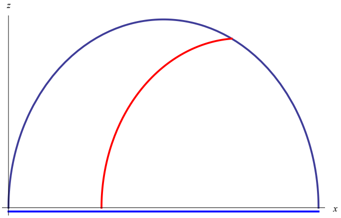
Dutta&Faulkner 1905.00577

A different interpretation of $E_W(A_1 : A_2)$

$$S_R(A_1 : A_2) = S(A_1 A_1')_{|\sqrt{\rho_{A_1 A_2}}\rangle} = 2E_W(A_1 : A_2)$$

$\rho_{A_1 A_2}$ can be turned into a pure state by switching bras to kets

$$|\sqrt{\rho_{A_1 A_2}}\rangle \in \mathrm{End}\mathcal{H}_{A_1} \otimes \mathrm{End}\mathcal{H}_{A_2} = (\mathcal{H}_{A_1} \otimes \mathcal{H}^*_{A_1}) \otimes (\mathcal{H}_{A_2} \otimes \mathcal{H}^*_{A_2})$$



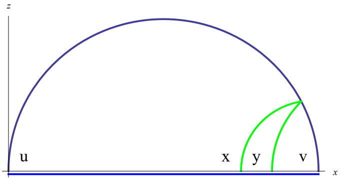
kinematic space of reflected geodesics

XH, work in progress

For a single interval $A = [u, v]$, $A_1 = [u, y], A_1' = [u, x]$ corresponds to a reflected geodesics (X_A picked to make $\ell + \ell'$ minimized)

$$S(A_1, A_1')_{|\sqrt{\rho_A}\rangle} = \ell(y, X_A(\xi)) + \ell'(X_A(\xi), x)\,,$$

which gives the kinematic space of the subregion A



Length of a reflected geodesic

The length of the reflected geodesic reads

$$\ell(x,y) = 2\log\left(-\frac{x(-2y+u+v)+y(u+v)-2uv}{u-v}\right)$$

For convenience, we turn it into a form (after a subtraction of $2\log(x-y)$)

$$2\log\frac{z+1}{z-1}$$

that depends only on the cross-ratio

$$z = \frac{(x-u)(y-v)}{(y-u)(x-v)}$$

Holographic computation of $\langle O(x)\Delta_A^{is}O(y)\rangle$

Faulkner etal 1806.10560

Modular operator Δ_A

$$S_A a|\Psi\rangle = a^\dagger|\Psi\rangle, \quad S_A = J_A {\Delta_A}^{1/2}$$

$\langle O(x)\Delta_A^{is}O(y)\rangle$ can be computed holographically using two geodesics that meet somewhere the RT surface γ_A

$$\langle O(x)\Delta^{is}O(y)\rangle \simeq \exp\left(-m[\ell(x,X_A(\xi)) + \ell'(X_A(\xi),y)]\right)$$

The tangent vectors are related by a boost at the bulk point $X_A(\xi)$

$$n'_i = n_i, \quad (n'_+, n'_-) = (e^{-2\pi s}n_+, e^{+2\pi s}n_-)$$

In the case of $s = -\frac{i}{2}$, this reflected boundary condition implies $\delta\ell + \delta\ell' = 0$.

Field theory computation

$$\langle O(x)\Delta^{\frac{1}{2}}O(y)\rangle = \lim_{n\to 1} \mathrm{Tr}_A[\rho_A^{n/2}O(x)\rho_A^{n/2}O(y)]$$

The rhs is precisely what we want ($O'(x)$ is acting on the space $\mathcal{H}^*_{A_1}\otimes\mathcal{H}^*_{A_2}$)

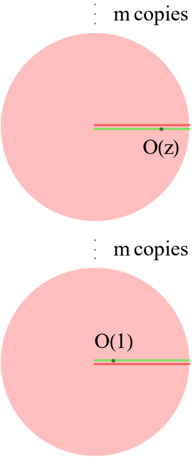
$$\langle\sqrt{\rho_{A_1A_2}}|O'(x)O(y)|\sqrt{\rho_{A_1A_2}}\rangle\,,$$

which can be obtained by $n = 2m$ then $m \to 1/2$

$$\lim_{m\to 1} \mathrm{Tr}_A[\rho_A^m O(x)\rho_A^m O(y)] = \lim_{m\to 1}\langle O(e^{2m\pi i}x)O(y)\rangle$$

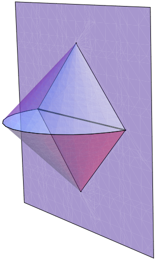
A phase shift $x \to e^{2\pi i}x$ takes the point to the next sheet. $m = 1/2$ then gives

$$\log(z-1)^2 \to \log(z+1)^2$$



Further developments

1. Kinematic space of the entanglement wedge, *i.e.*, x, y not on the same time slice as A. $r O(x)\Delta_A^{is} O(y)\rangle$ is not restricted
2. Higher dimensions



Further developments

3. Multi-interval case

4. Construction of bulk operators within the entanglement wedge. In global AdS, the HKLL formula can be obtained from the inverse Radon transform. Every OPE block corresponds to a bulk field smearred over the geodesic

OPE block Czech etal. 1604.03110

$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) \equiv |x_1 - x_2|^{-\Delta_i - \Delta_j} \sum_{\mathcal{O}} C_{ij\mathcal{O}} B^{ij}_{\mathcal{O}}(x_1, x_2),$$

Summary

- The kinematic space of a subregion can be computed using the reflected geodesic. The measure is again given by the second derivative of the length of the geodesic
- The length can be obtained from the correlator $\langle O(x)\Delta_A^{1/2}O(y)\rangle$, which is the generalized version reflected entropy. The point is that it can be computed entirely using information encoded in the subregion