Celestial OPE of Scattering Amplitudes

袁 野

浙江大学 浙江近代物理中心

弦论、场论与宇宙学相关专题研讨会

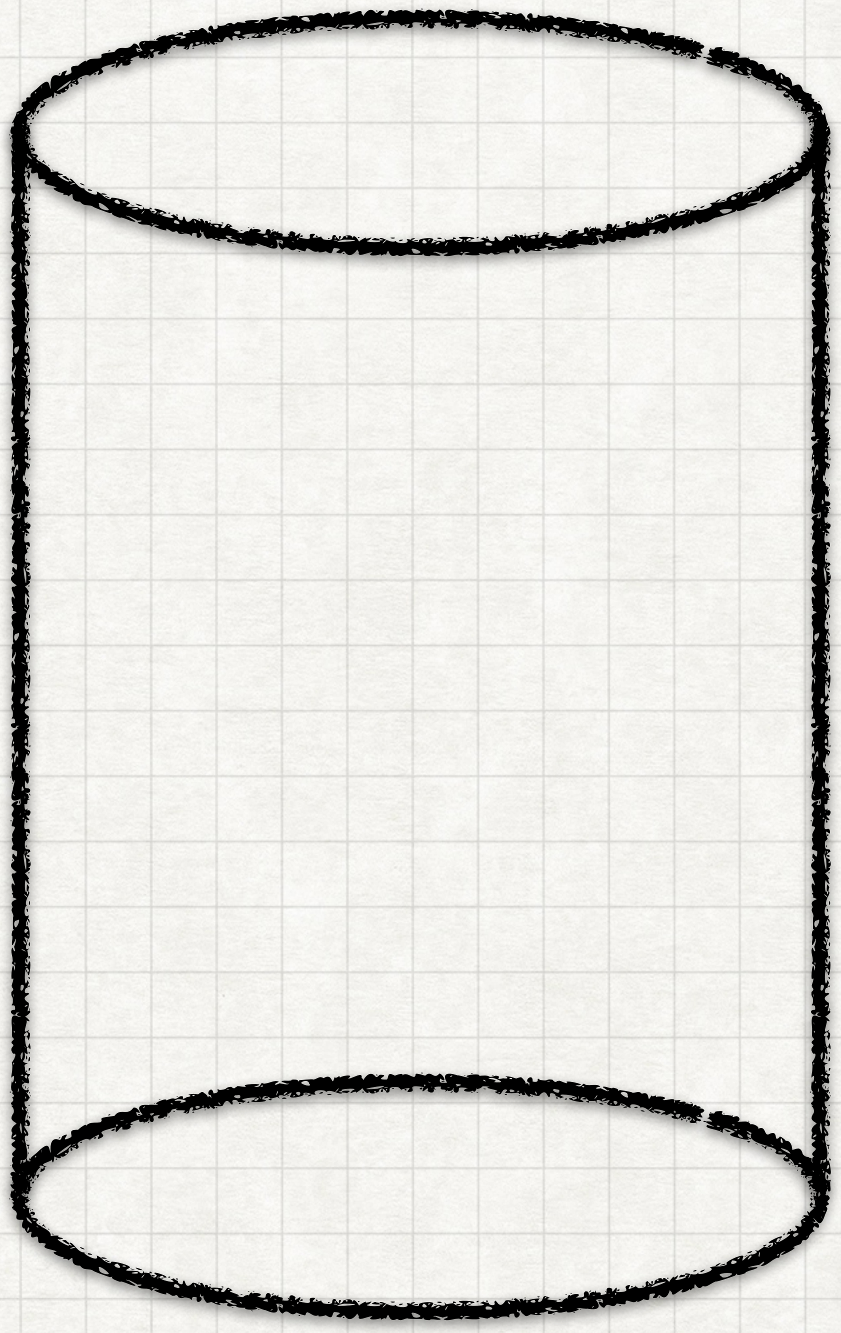
湖北宜昌，2019.11

arXiv:1910.07424, w/ A.Raclariu, M.Pate, A.Strominger

Motivation

Holography

- Asymptotic AdS
- Physics in the AdS bulk
  described by CFTs on the boundary.
- Time-like boundary,
  one extra spatial direction emerged.
- E.g., type IIB superstring in $\mathrm{AdS}_5 \times \mathrm{S}^5$
  vs $\mathcal{N} = 4$ super Yang-Mills in 4d.
- AdS isometry $\Longleftrightarrow$ conformal symmetry
- Boundary correlators



Holography?

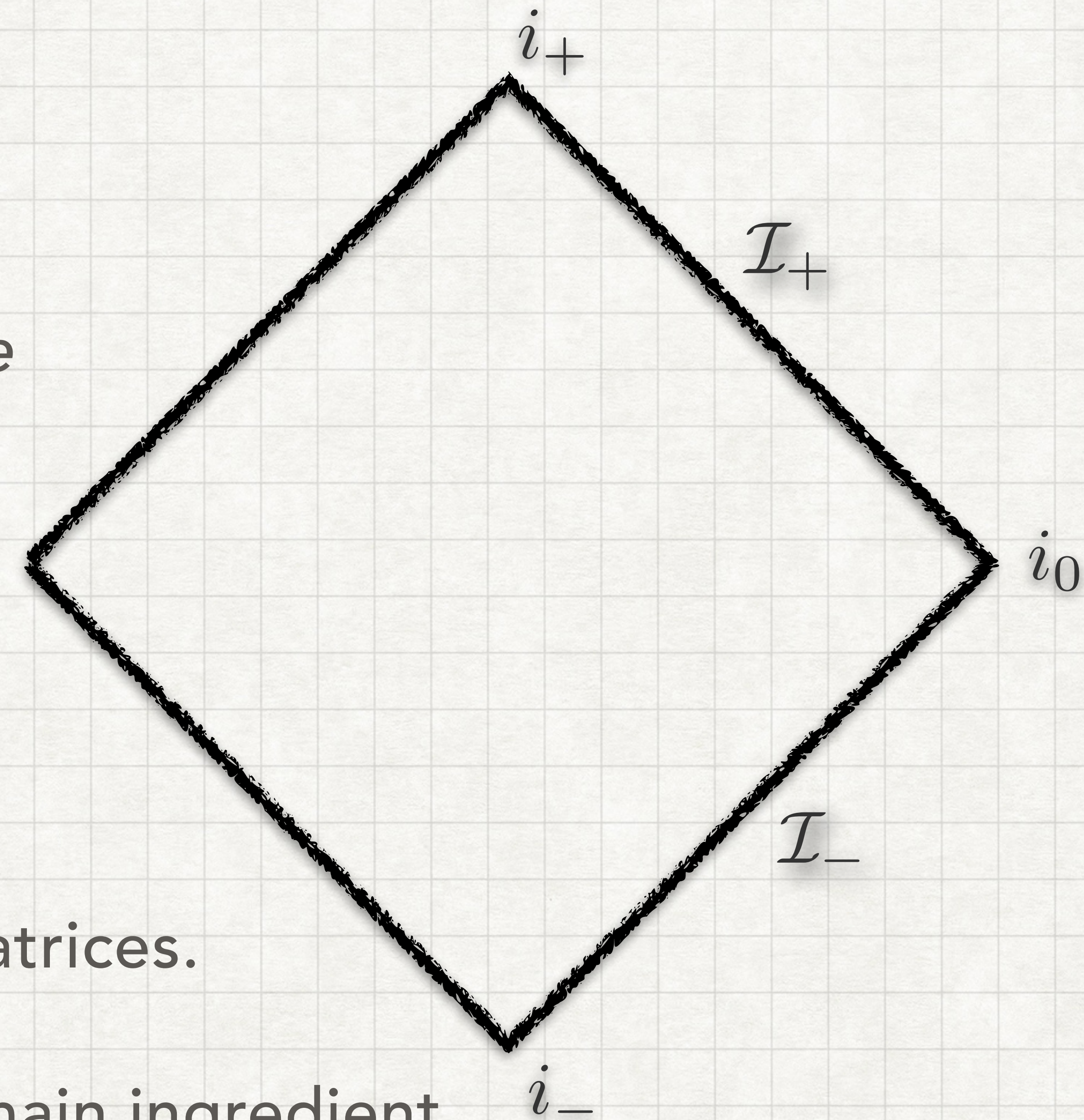
- Asymptotically Minkowski?

- Null infinity: $\mathbb{R} \times \mathrm{S}^2$
  the latter is the *celestial sphere*
  No time direction.

- Symmetries: $\mathrm{SO}(3,1)$
  — 4d, Lorentz symmetry
  — 2d, conformal symmetry

- Boundary observables are S-matrices.

- This 2d symmetry was once a main ingredient leading to twistor strings.
  Here we'd like to explore other view points.



Goal

Ambitious:

Find a holographic dual theory of quantum gravity in asymptotically flat spacetime.

Conservative:

Find a dual description of the S-matrix

Note: In this talk we will only focus on massless particles.

PRELIMIARIES

Variables in Use

- Parametrizing the momentum (massless)

$$p_\mu \sigma^\mu_{\alpha\dot{\alpha}} = \begin{pmatrix} p^0 + p^3 & p^1 - ip^2 \\ p1 + ip^2 & p^0 - p^3 \end{pmatrix}$$

$$= \sqrt{2}\omega\xi \begin{pmatrix} 1 & \bar{z} \\ z & z\bar{z} \end{pmatrix}$$

- $(z, \bar{z})$ parametrize the position on the celestial sphere.

- ω parametrize the energy.

- $\xi = \pm 1$ keeps track of out-going/in-coming particles.

- For Minkowski space $(z, \bar{z})$ are conjugate to each other. But here we assume they are independent.

A New Set of Wave Bases

[Pasterski, Shao, `17]

- The usual scattering amplitudes are Lorentz invariant.
  Here we want some objects behaving like conformal correlators.

- A set of wave basis (e.g., spin-1) $V^{\Delta\xi}_{\mu J}(x^\mu; z, \bar{z})$, satisfying:
  — e.o.m. $(\partial_\rho \partial^\rho \delta^\mu_\nu - \partial_\nu \partial^\mu) V^{\Delta\xi}_{\mu J}(x; z, \bar{z}) = 0$
  — under Lorentz transformation, transforms as a vector in 4d and a conformal (quasi-)primary in 2d

$$V^{\Delta\xi}_{\mu J}\left(\Lambda x; \frac{az+b}{cz+d}, \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}\right) = (cz+d)^{\Delta+J}(\bar{c}\bar{z}+\bar{d})^{\Delta-J}\Lambda_\mu{}^\nu V^{\Delta\xi}_{\nu J}(x; z, \bar{z})$$

- Solution: $$V^{\Delta\xi}_{\mu J} \propto \int_0^\infty \mathrm{d}\omega\, \omega^{\Delta-1} \underbrace{\partial_J (p_\mu/\omega) e^{i\xi p\cdot x - \epsilon\omega}}_{\text{ordinary wave function}}$$

- Completeness+normalizability: $\Delta \in 1 + i\mathbb{R}$

Celestial Amplitudes

[Pasterski, Shao, Strominger, `17]

- To obtain an object transforming nicely under conformal symmetry, we simply expand the S-matrix on the new basis.

- The ordinary scattering amplitudes are fed by plane waves.

- For massless particles this is a Mellin transformation

$$\tilde{\mathcal{A}}_n(\{\Delta_a, z_a, \bar{z}_a\}) = \int_0^\infty \prod_{a=1}^n \mathrm{d}\omega_a \omega_a^{\Delta_a - 1} \mathcal{A}_n(\{\xi_a \omega_a, z_a, \bar{z}_a\})$$

- The new object is treated as a correlator on S^2, involving operators $\mathcal{O}_{\Delta_a}(z_a, \bar{z}_a)$ with conformal dimension Δ_a.

- The dimensions are restricted to *unitary principal series*.

Examples

- Gluon amplitudes, after color decomposition

$$\mathcal{A}_n = \sum_{\rho\in S_n} \mathrm{tr}(T^{\rho(1)}\cdots T^{\rho(n)})A_n[\rho]$$

- MHV sector

$$A_n[1^+ \ldots i^- \ldots j^- \ldots n^+] = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}\delta^4$$

$$= \frac{\omega_i\omega_j}{\prod_{a\neq i,j}\omega_a}\frac{z_{ij}^4}{z_{12}z_{23}\cdots z_{n1}}\delta^4$$

where the momentum conservation

$$\delta^4 \equiv \delta^4(\sum_a p_a)$$

$$\equiv \delta(\sum_a \xi_a\omega_a)\delta(\sum_a \xi_a z_a)\delta(\sum_a \xi_a \bar{z}_a)\delta(\sum_a \xi_a z_a\bar{z}_a)$$

Examples

- Assume all are out-going; denote $\Delta = 1 + i\lambda$
- 3-point (turn into (2,2) signature)

$$\tilde{A}_3[1^-2^-3^+] = \delta(\sum_a \lambda_a)\frac{\text{sgn}(z_{12}z_{23}z_{31})\delta(\bar{z}_{12})\delta(\bar{z}_{13})}{|z_{12}|^{-1-i\lambda_3}|z_{23}|^{1-i\lambda_1}|z_{13}|^{1-i\lambda_2}}$$

In terms of $(h, \bar{h})$ we have

$$- \text{ helicity} \qquad (h, \bar{h}) = (\frac{i}{2}\lambda, 1 + \frac{i}{2}\lambda)$$

$$+ \text{ helicity} \qquad (h, \bar{h}) = (1 + \frac{i}{2}\lambda, \frac{i}{2}\lambda)$$

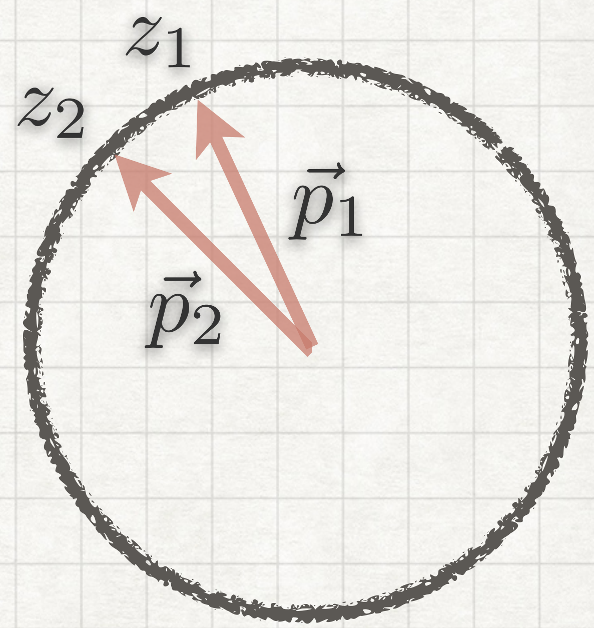
- The constraint $\delta(\sum_a \lambda_a)$ is always present.
This comes from the overall energy integral.

Spectrum & OPE

Possibility of a CFT?

- Any chance that these 2d correlators arise from a CFT?
- To specify a CFT, we need:
  — spectrum, species of conformal families w/ dimension;
  — collection of all OPE coefficients;
  — crossing symmetry;
  — unitarity bound (for unitary CFTs), etc.
- It is natural to think that $\tilde{A}_n = \langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle$,
  where $\mathcal{O}$ is the "gluon operator".
- Crossing symmetry guaranteed by construction.
- *What are the OPE coefficients?*

OPE Limit & Collinear Limit



- To extract the OPE coefficients we probe the limit $z_{12} \to 0$, keeping $\bar{z}_1, \bar{z}_2$ fixed.
- This is the (holomorphic) collinear limit in the bulk.
- Parametrize the energies

$$\omega_1 = x\omega_0, \quad \omega_2 = (1-x)\omega_0$$

- The amplitude factorize (e.g., for two + helicity gluons)

$$A_n = \frac{1}{x(1-x)\omega_0 z_{12}} \underbrace{\frac{\omega_i \omega_j}{\omega_0 \prod_{a \neq 1,2,i,j} \omega_a} \frac{z_{ij}^4}{z_{23} \cdots z_{n2}} \delta^4}_{A_{n-1}} + O(z_{12}^0)$$

- In particular, $\delta^4(p_1 + p_2 + \cdots) \approx \delta^4(p_0 + \cdots)$

OPE From the Mellin Transformation

- The info in the remaining amplitude is not relevant for the leading order computation, so

$$\begin{aligned}\tilde{A}_n &= \int_0^\infty \mathrm{d}\omega_1 \mathrm{d}\omega_2 \cdots \frac{\omega_1^{\Delta_1-1}\omega_2^{\Delta_2-1}}{x(1-x)\omega_0 z_{12}} A_{n-1} + O(z_{12}^0) \\ &= \frac{\int_0^1 \mathrm{d}x\, x^{\Delta_1-2}(1-x)^{\Delta_2-2}}{z_{12}} \int_0^\infty \mathrm{d}\omega_0 \omega_0^{\Delta_1+\Delta_2-2} \cdots A_{n-1} + O(z_{12}^0) \\ &= \frac{\mathrm{B}(\Delta_1-1,\Delta_2-1)}{z_{12}} \langle \mathcal{O}^+_{\Delta_1+\Delta_2-1} \cdots \rangle + O(z_{12}^0)\end{aligned}$$

- The new operator from the OPE is also a + helicity gluon operator, whose dimension is $\Delta_1+\Delta_2-1$.

- Take the color factor into consideration, we have

$$\mathcal{O}^{+\,a_1}_{\Delta_1}\mathcal{O}^{+\,a_2}_{\Delta_2} \sim \frac{f^{a_1 a_2}_{a_3}\,\mathrm{B}(\Delta_1-1,\Delta_2-1)}{z_{12}} \mathcal{O}^{+\,a_3}_{\Delta_1+\Delta_2-1} + \cdots$$

Constraints from Symmetries

The Idea

- The kind of CFTs we are looking for are not arbitrary.
  They come with extra ingredients.

- Poincare symmetry also contains bulk *translations*, which is not part of the celestial conformal group

$$P \int_0^\infty \mathrm{d}\omega \omega^{\Delta-1} \cdots \mathcal{A} = \int_0^\infty \mathrm{d}\omega \omega^{\Delta} \cdots \mathcal{A}$$

  indicating that $P\mathcal{O}_\Delta = \mathcal{O}_{\Delta+1}$.

- Soft theorems are tied to residual symmetries at infinity.
  — Gauge theories: large gauge transformations.
  — Gravity: BMS symmetry.

Ansatz

- For the pure gluon theory we assume

$$\mathcal{O}^{+a}_{\Delta_1}(z_1,\bar{z}_1)\mathcal{O}^{+b}_{\Delta_2}(z_2,\bar{z}_2) \sim \frac{if^{ab}_c}{z_{12}} C(\Delta_1,\Delta_2)\mathcal{O}_{\Delta_1+\Delta_2-1}(z_2,\bar{z}_2)$$

- The need of z_{12} is due to the need of the $\frac{1}{(p_1+p_2)^2}$ propagator.

- The three gluon vertex has the form $AA\partial A$. The leading behavior comes from this. Counting energy weights determines the dimension of the new operator.

- Bose statistics requires $C(\Delta_1,\Delta_2) = C(\Delta_2,\Delta_1)$

- We are then left with the coefficient $C(\Delta_1,\Delta_2)$ to solve.

Translation

- LHS

$$P\mathcal{O}^{+a}_{\Delta_1}\mathcal{O}^{+b}_{\Delta_2} = \mathcal{O}^{+a}_{\Delta_1+1}\mathcal{O}^{+b}_{\Delta_2} + \mathcal{O}^{+a}_{\Delta_1}\mathcal{O}^{+b}_{\Delta_2+1}$$

$$= \frac{if^{ab}{}_c}{z_{12}}\big(C(\Delta_1+1,\Delta_2) + C(\Delta_1,\Delta_2+1)\big)\mathcal{O}^{+c}_{\Delta_1+\Delta_2}$$

- RHS

$$P\mathcal{O}^{+c}_{\Delta_1+\Delta_2-1} = \mathcal{O}^{+c}_{\Delta_1+\Delta_2}$$

- When the symmetry is not broken, the two sides have to equal

$$C(\Delta_1+1,\Delta_2) + C(\Delta_1,\Delta_2+1) = C(\Delta_1,\Delta_2)$$

Soft Theorems

- Soft theorems governs the universal factorizing structure of scattering amplitudes when a massless external particle has vanishing momentum.

- For gluons there are the leading and the subleading theorems.

- The leading conformally soft theorem indicates

[Lysov, Pasterski, Strominger, `14]

$$\lim_{\Delta_1 \to 1} \mathcal{O}^{+a}_{\Delta_1}(z_1, \bar{z}_1) \mathcal{O}^{+b}_{\Delta_2}(z_2, \bar{z}_2) \sim \frac{i f^{ab}{}_c}{(\Delta_1 - 1) z_{12}} \mathcal{O}^{+c}_{\Delta_2}(z_2, \bar{z}_2)$$

i.e.,

$$\lim_{\Delta_1 \to 1} (\Delta_1 - 1) C(\Delta_1, \Delta_2) = 1$$

Soft Theorems

- Subleading soft theorem implies that the celestial amplitude is invariant under the action of

$$\bar{\delta}_b \mathcal{O}^{\pm a}_{\Delta}(z,\bar{z}) = -(\Delta - 1 \mp 1 + \bar{z}\bar{\partial}) i f^a{}_{bc} \mathcal{O}^{\pm}_{\Delta-1}{}^c(z,\bar{z})$$

- No need to worry about the $\bar{z}\bar{\partial}$ part, since it leads to descendants that are to be compared with terms we omit in the ansatz.

- Applying on both sides of the ansatz we have

$$(\Delta_1 - 2)C(\Delta_1 - 1, \Delta_2) f^a{}_{dc} f^{cb}{}_e + (\Delta_2 - 2)C(\Delta_1, \Delta_2 - 1) f^b{}_{dc} f^{ac}{}_e$$
$$= (\Delta_1 + \Delta_2 - 3)C(\Delta_1, \Delta_2) f^{ab}_c f^c{}_{de}$$

- A further application of Jacobi identity gives

$$(\Delta_1 - 2)C(\Delta_1 - 1, \Delta_2) = (\Delta_1 + \Delta_2 - 3)C(\Delta_1, \Delta_2)$$

Determining the Coefficient

- Collect the constraints derived so far

$$\begin{aligned} \text{ansatz} \quad & C(\Delta_1, \Delta_2) = C(\Delta_2, \Delta_1) \\ \text{translation} \quad & C(\Delta_1 + 1, \Delta_2) + C(\Delta_1, \Delta_2 + 1) = C(\Delta_1, \Delta_2) \\ \text{leading soft} \quad & \lim_{\Delta_1 \to 1} (\Delta_1 - 1) C(\Delta_1, \Delta_2) = 1 \\ \text{subleading soft} \quad & (\Delta_1 - 2) C(\Delta_1 - 1, \Delta_2) = (\Delta_1 + \Delta_2 - 3) C(\Delta_1, \Delta_2) \end{aligned}$$

- These together determines that

$$C(\Delta_1, \Delta_2) = \mathrm{B}(\Delta_1 - 1, \Delta_2 - 1)$$

- Similar analysis also shows that

$$\mathcal{O}^{+a}_{\Delta_1}(z_1, \bar{z}_1) \mathcal{O}^{-b}_{\Delta_2}(z_2, \bar{z}_2) \sim \frac{i f^{ab}_c}{z_{12}} B(\Delta_1 - 1, \Delta_2 + 1) \mathcal{O}^{-c}_{\Delta_1 + \Delta_2 - 1}(z_2, \bar{z}_2)$$

Case Of Einstein Gravity

- The ansatz (two + helicity gravitons)

$$\mathcal{G}^{+}_{\Delta_1}(z_1,\bar{z}_1)\mathcal{G}^{\pm}_{\Delta_2}(z_2,\bar{z}_2) \sim \frac{\bar{z}_{12}}{z_{12}} E_{\pm}(\Delta_1,\Delta_2)\mathcal{G}^{\pm}_{\Delta_1+\Delta_2}(z_2,\bar{z}_2)$$

  Analysis works similarly as the gluon case.

- Subleading soft symmetry corresponds to 2d conformal transformations. They do not generate new constraints.

- Subsubleading soft theorem implies invariance under the action

$$\bar{\delta}\mathcal{G}^{\pm}_{\Delta}(z,\bar{z}) = -\frac{\kappa}{4}\big((\Delta \mp 2)(\Delta \mp 2 - 1) + 4(\Delta \mp 2)\bar{z}\bar{\partial} + 3\bar{z}^2\bar{\partial}^2\big)\mathcal{G}^{\pm}_{\Delta-1}(z,\bar{z})$$

- The constraints determines that

$$E_{\pm}(\Delta_1,\Delta_2) = -\frac{\kappa}{2}\mathrm{B}(\Delta_1 - 1, \Delta_2 \mp 2 + 1)$$

More Examples

- We also studied gluons minimally coupled to gravitons

$$\mathcal{G}^{+}_{\Delta_1}(z_1,\bar{z}_1)\mathcal{O}^{\pm a}_{\Delta_2}(z_2,\bar{z}_2) \sim -\frac{\kappa}{2}\mathrm{B}(\Delta_1-1,\Delta_2\mp 1+1)\frac{\bar{z}_{12}}{z_{12}}\mathcal{O}^{\pm a}_{\Delta_1+\Delta_2}(z_2,\bar{z}_2)$$

$$\mathcal{O}^{+a}_{\Delta_1}(z_1,\bar{z}_1)\mathcal{O}^{-b}_{\Delta_2}(z_2,\bar{z}_2) \sim \frac{if^{ab}_{c}}{z_{12}}\mathrm{B}(\Delta_1-1,\Delta_2+1)\mathcal{O}^{-c}_{\Delta_1+\Delta_2-1}(z_2,\bar{z}_2) + \delta^{ab}\frac{\kappa}{2}\mathrm{B}(\Delta_1,\Delta_2+2)\frac{\bar{z}_{12}}{z_{12}}\mathcal{G}^{-}_{\Delta_1+\Delta_2}(z_2,\bar{z}_2)$$

- We checked all these results against the computation from the collinear limits. The above is consistent with the hF^2 coupling.

Conclusion

- Scattering amplitudes receives a representation on the celestial sphere, which looks like conformal correlators.
- To see whether this can be promoted to a correspondence between theories in the Minkowski bulk and CFTs on the celestial sphere we should understand the resulting spectrum and interactions.
- On the one hand, we worked out the leading OPE between primary operators using Mellin transformation of the collinear limit of scattering amplitudes.
- On the other hand, we determined the same coefficients using constrains from symmetries and soft theorems.

Outlook

- Other operators in the OPE.
  In particular, what plays the role of stress tensor?
  Analog of double-trace, triple-trace operators, etc?

- Analog of Sugawara construction in the gluon case. Connection with the double copy construction of gravity amplitudes.

- A proper understanding of the conformal dimensions.
  What is the unitary principal series doing?

- Role of the case with massive particles.
  Especially string theory amplitudes.

Thank You Very Much!