Entanglement Entropy of BMN strings

Minxin Huang

University of Science and Technology of China

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Based on MH, arXiv:1909.06995, arXiv:1910.02581.

AdS/CFT correspondence

- This is proposed by Maldacena in 1997. A canonical example: Type IIB superstrings on $AdS_5 \times S^5$ $\cong \mathcal{N}=4$ $SU(N)$ super-Yang-Mills on 4d

- The identification of parameters

$$R^4 = 4\pi g_s N l_s^4, \qquad 4\pi g_s = g_{YM}^2,$$

where R is the AdS (or S^5) radius, N is the rank of gauge group or Ramond flux, l_s is the string length.

- Some limits:
  1. Suppress string loop corrections $g_s \sim g_{YM}^2 \ll 1$.
  2. Suppress α' corrections (supergravity valid) $\frac{R}{l_s} \sim g_{YM}^2 N \gg 1$.

  So $AdS_5 \times S^5$ supergravity is described by strongly coupled gauge theory $N \sim \infty, g_{YM}^2 \sim 0$, $g_{YM}^2 N$ (effective t'Hooft coupling) large.

pp-wave limit

- A Penrose limit of the $AdS_5 \times S^5$ geometry.

$$ds^2 = -4dx^+dx^- - \mu^2(\vec{r}^2 + \vec{y}^2)(dx^+)^2 + d\vec{r}^2 + d\vec{y}^2,$$

  where x^+, x^- are light cone coordinates, μ is proportional to flux. Momentum in the 8 transverse directions is discretized by flux. String vacuum state $|0, p^+\rangle$, where p^+ is the light cone momentum.
  Yang-Mills side:
  Consider large charge operator Tr (Z^J) with $J \sim \sqrt{N} \sim +\infty$, where Z is one of the 3 complex scalars.
  Berenstein, Maldacena and Nastase (BMN), [arXiv:hep-th/0202021].
  Excited string states are described by BMN operators.

- The identification of parameters

$$\mu p^+ l_s^2 = \frac{J}{g_{YM}\sqrt{N}}, \quad 4\pi g_s = g_{YM}^2.$$

- It is useful to define 2 dimensionless parameters

$$\lambda' = \frac{g_{YM}^2 N}{J^2}, \qquad g = \frac{J^2}{N}.$$

  It turns out
  λ' is the effective gauge coupling constant.
  g is the effective string coupling constant.

- Two interesting situations

  1. $g = 0$, λ' finite. Free string theory = Interacting gauge theory.
     String spectrum described by planar conformal dimension of BMN operators.

  2. $\lambda' = 0$, g finite. Free gauge theory = Interacting string theory.
     This is a background with infinite negative curvature, infinite flux.
     Strings are infinitely long and tensionless.
     String spectrum completely degenerate.
     String loop amplitudes described by non-planar (higher genus) contributions.

  In this talk we focus on the second situation.

Factorization Principle

- The effective field theory approach breaks down on the string theory side. Usually we can not say much about the underlying physics. However, it seems somehow luckily the stringy physics also becomes extremely simplified.

- We proposed that the string amplitudes can be computed simply by cubic diagrams, and there is a so called "factorization" principle relating the string diagram calculations and field theory calculations, in the spirit of AdS/CFT correspondence.
  Huang, hep-th/0206248, arXiv:1009.5447.

- If our claim is valid, we can straightforwardly compute higher string loop amplitudes which are notoriously difficult.

- The general form of the factorization rule is

$$S_i = \sum_j m_{ij} F_j,$$

where S_i and F_j denote string and field theory diagram contributions, and m_{ij} are non-negative integers denoting the multiplicity of expanding the "short process" of field theory diagrams into "long process" of string diagrams.

- A nice example

$$\langle \bar{O}^J_{-m,m} O^J_{-n,n} \rangle_{\text{torus}} = \frac{1}{2}\Big(\sum_{J_1=1}^{J-1} \langle \bar{O}^J_{-m,m} O^{J_1}_0 O^{J-J_1}_0 \rangle \langle \bar{O}^{J_1}_0 \bar{O}^{J-J_1}_0 O^J_{-n,n} \rangle$$
$$+ \sum_{J_1=1}^{J-1} \sum_{k=-\infty}^{+\infty} \langle \bar{O}^J_{-m,m} O^{J_1}_{-k,k} O^{J-J_1} \rangle \langle \bar{O}^{J_1}_{-k,k} \bar{O}^{J-J_1} O^J_{-n,n} \rangle \Big).$$

Factor of 2: there are 2 ways to expand the short process

$$(1234) \to (12)(34) \to (2143),$$
$$(1234) \to (23)(41) \to (3214).$$

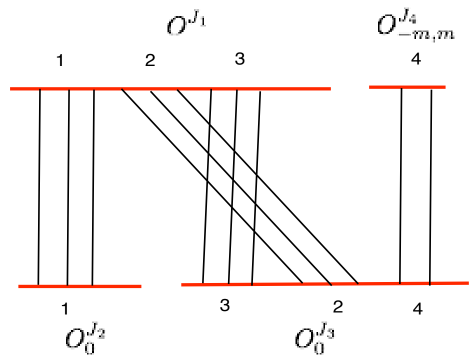
- The $2 \to 2$ scattering process: There was a puzzle that the factorization seems to break down for S-channel in our previous paper Huang, arXiv:1009.5447.

- In the recent paper Huang arXiv:1909.06995, we resolve this puzzle, by including some missing diagrams in our previous paper.

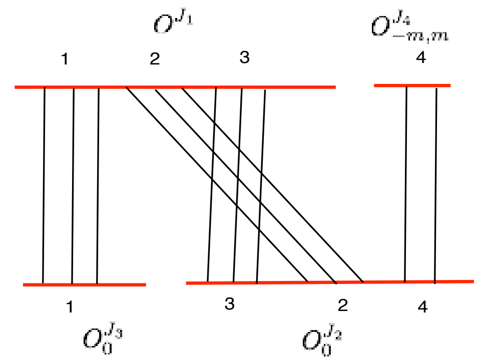
- We consider again several examples:
  $\langle \bar{O}^{J_1} \bar{O}^{J_4} O^{J_2} O^{J_3} \rangle$,
  $\langle \bar{O}^{J_1} \bar{O}^{J_4}_{-m,m} O^{J_2}_0 O^{J_3}_0 \rangle$,
  $\langle \bar{O}^{J_1}_{-m,m} \bar{O}^{J_4} O^{J_2}_{-n,n} O^{J_3} \rangle$.
  $(J_1 > J_2 > J_3 > J_4$, with $J = J_1 + J_4 = J_2 + J_3)$.

- For illustration we consider the second example.

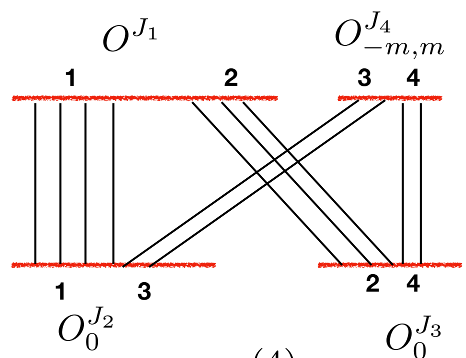
- Some field theory diagrams for $\langle \bar{O}^{J_1} \bar{O}^{J_4}_{-m,m} O^{J_2}_0 O^{J_3}_0 \rangle$. These previous diagrams have vanishing contributions $F^{(4)}_1 = 0, F^{(4)}_2 = 0$. The new diagram $F^{(4)}_3$ is non-vanishing.



$F^{(4)}_1$



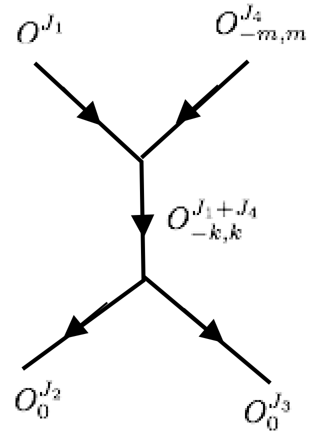
$F^{(4)}_2$



$F^{(4)}_3$

- On the string side, there is no T and U channel contribution. We check the S-channel factorization

$S^{(4)}$:



$$
\begin{aligned}
S^{(4)} &= \sum_{k=-\infty}^{+\infty} \langle \bar{O}^{J_1} \bar{O}^{J_4}_{-m,m} O^{J_1+J_4}_{-k,k} \rangle \langle \bar{O}^{J_1+J_4}_{-k,k} O^{J_2}_0 O^{J_3}_0 \rangle \\
&= 2F_3^{(4)}
\end{aligned}
$$

A Probability Interpretation

- Analogous to quantum mechanics, we would like to interpret the two single-string correlator $\langle \bar{O}^J_{-m,m} O^J_{-n,n} \rangle_h$ of genus h as the physical h-loop probability amplitude of preparing the initial as $O^J_{-m,m}$ and observing the final state $O^J_{-n,n}$.

- We find the sum of the probability amplitude over final states is actually quite simple and independent of the initial state mode

$$\sum_{n=-\infty}^{\infty} \langle \bar{O}^J_{-m,m} O^J_{-n,n} \rangle_h = \frac{(4h-1)!!}{(2h+1)(4h)!} g^{2h}.$$

The result can be better derived using the integral formula or factorization formula. We skip the details here.

- We can prove $\langle \bar{O}^J_{-m,m} O^J_{-n,n} \rangle_h \geq 0$, both from field theory and string theory perspectives.

- Putting together the results, we can write the total probability $p_{m,n} \geq 0$ including all string loop contributions

$$p_{m,n} = \frac{g}{2\sinh(g/2)} \sum_{h=0}^{\infty} \langle \bar{O}^J_{-m,m} O^J_{-n,n} \rangle_h.$$

This is properly normalized by the vacuum correlator

$$\langle \bar{O}^J O^J \rangle_{\text{all genera}} = \frac{2\sinh(g/2)}{g} = \sum_{h=0}^{\infty} \frac{(4h-1)!!}{(2h+1)(4h)!} g^{2h}.$$

For any initial mode m, we have

$$\sum_{n=-\infty}^{\infty} p_{m,n} = 1$$

- Some discussions about multi-strings (as virtual states).

Formulation in Quantum Mechanics

- We denote the orthonormal BMN states of free string theory by $|n\rangle$. Let us assume the transition amplitude between BMN states can be described by a unitary operator $e^{i\widehat{H}(g)}$, where $\widehat{H}(g)$ is a Hermitian operator corresponding to the time integral of Hamiltonian in a usual quantum mechanics system. Here $\widehat{H}(0) = 0$ for free string theory and the operator $\widehat{H}(g)$ models string interactions at finite coupling g.

- Our proposal
  The matrix element $p_{m,n}$ does not correspond to the usual transition amplitude $\langle m|e^{i\widehat{H}(g)}|n\rangle$.
  Instead, we have the same normalization relation

$$\sum_{n=-\infty}^{\infty} p_{m,n} = \sum_{n=-\infty}^{\infty} |\langle m|e^{i\widehat{H}(g)}|n\rangle|^2 = 1.$$

  This strongly suggests $p_{m,n} = |\langle m|e^{i\widehat{H}(g)}|n\rangle|^2$.

Unitarity

- Taking $p_{m,n}$ as the transition amplitude $\langle m|e^{i\widehat{H}(g)}|n\rangle$ would be inconsistent with unitarity.
  Allow BMN operator $O^J_{-m,m}$ normalized by a function $f_m(g)$
  The unitarity condition
  $$\sum_{n=-\infty}^{\infty} f_m(g)^* f_{m'}(g)|f_n(g)|^2 p_{m,n}p^*_{m',n} = \delta_{m,m'}.$$
  Consider two cases

  1. $m = m'$ and $g = 0$. We deduce $|f_m(0)| = 1$ for any mode m.

  2. $m \neq m'$ and both non-zero. Expand for small g
  $$\begin{aligned} & \sum_{n=-\infty}^{\infty} f_m(g)^* f_{m'}(g)|f_n(g)|^2 p_{m,n}p^*_{m',n} \\ = \ & 2f_m(0)^* f_{m'}(0)\langle \bar{O}^J_{-m,m} O^J_{-m',m'}\rangle_{\text{torus}} + \mathcal{O}(g^3). \end{aligned}$$
     This is clearly non-zero, violating unitarity.

- The correct proposal

$$\langle m|e^{i\hat{H}(g)}|n\rangle = e^{i\theta_{m,n}(g)}\sqrt{p_{m,n}}.$$

For $g=0$ this is the identity matrix $\delta_{m,n}$, so the phase angle $\theta_{m,m}(0) = 0$. For $m \neq n$, the phase angle $\theta_{m,n}(0)$ is not determined.
The unitarity condition is

$$\sum_{n=-\infty}^{+\infty} e^{i[\theta_{m,n}(g)-\theta_{m',n}(g)]}\sqrt{p_{m,n}p_{m',n}} = \delta_{m,m'}.$$

For $m = m'$ this is already satisfied, while the cases of $m \neq m'$ may provide some constrains for the phase angles.

- Again we consider the non-trivial case of $m \neq m'$ and both non-zero. Expanding for small g

$$\begin{aligned} &\sum_{n=-\infty}^{+\infty} e^{i[\theta_{m,n}(g)-\theta_{m',n}(g)]}\sqrt{p_{m,n}p_{m',n}} \\ = \;& [e^{i\theta_{m,m'}(0)} + e^{-i\theta_{m',m}(0)}]\sqrt{\langle \bar{O}^J_{-m,m} O^J_{-m',m'}\rangle_{\text{torus}}} + \mathcal{O}(g^2). \end{aligned}$$

An interesting relation $\theta_{m,m'}(0) + \theta_{m',m}(0) = \pi$.

Entanglement Entropy

- Entanglement entropy appears in quantum information theory as a measure of information for entangled states. It is also popular in condensed matter physics as a new type of order parameters to understand quantum phases of matters and critical phenomena.

- Ryu and Takayanagi proposed to compute entanglement entropy in conformal field theory holographically in terms of minimal surfaces in AdS space.
  Ryu and Takayanagi, Phys. Rev. Lett. **96**, 181602 (2006)

- Since we have a probability distribution for a BMN string, we can define an entanglement entropy

$$S_m(g) = -\sum_{n=-\infty}^{\infty} p_{m,n} \log(p_{m,n}).$$

- Some discussions.
  The quantum evolution is unitary.
  The entanglement appears between an observer and the BMN strings.
  The BMN basis is the preferred basis of entanglement.
  Tracing out the observer, we get a mixed state of BMN strings.
  This is like a decoherence process.

- Some special cases

  1. Free string theory $g = 0$, we have $S_m(0) = 0$.

  2. The zero mode decouples, so we have $S_0(g) = 0$.

- For a quantum system with Hilbert space of finite dimension D, the maximal entanglement entropy $\log(D)$ is achieved by a mixed state with uniformly distributed probability over an orthogonal basis. We shall prove an upper bound for the entanglement entropy.

- A bound for the correlator.

  For genus h, the field theory calculations of the two-point amplitude $\langle \bar{O}^J_{-m,m} O^J_{-n,n} \rangle_h$ consist of $\frac{(4h-1)!!}{2h+1}$ cyclically inequivalent diagrams of dividing the long string into $4h$ segments.

$$\begin{aligned} & \langle \bar{O}^J_{-m,m} O^J_{-n,n} \rangle_h \\ \leq \quad & \frac{(4h-1)!!}{2h+1} \frac{4hg^{2h}}{\pi^2(m-n)^2} \int_0^1 dx_1 \cdots dx_{4h} \delta(\sum_{i=1}^{4h} x_i - 1) \\ = \quad & \frac{16h^2 g^{2h}}{2^{2h}(2h+1)! \pi^2 (m-n)^2}. \end{aligned}$$

  The strength of BMN string interactions are bounded by an inverse square law.

- Summing over all genera, we have an estimate of the probability matrix element as

$$p_{m,n} \leq \frac{f(g)}{\pi^2(m-n)^2},$$

  where $f(g) := \frac{2g}{\sinh(g/2)}[\frac{g^2+4}{2g}\sinh(\frac{g}{2}) - \cosh(\frac{g}{2})] \sim g^2$.

- First notice for $0 < p < 1$, the function $-p\log(p)$ achieved maximum at $p = e^{-1}$, and it is monotonic in $p \in (0, e^{-1})$. We can choose an integer

$$n_0 \geq \max(\frac{\sqrt{e \cdot f(g)}}{\pi}, 2),$$

and evaluate the sum in 3 parts for $n \leq m - n_0$, $m - n_0 < n < m + n_0$, and $n \geq m + n_0$.

- The two parts that extends to $\pm\infty$ are symmetric with the same contributions, and in the middle part the entropy is maximal with a uniformly distributed probability ensemble. We find

$$S_m(g) \leq 2 \sum_{n=n_0}^{+\infty} \frac{f(g)}{\pi^2 n^2} \log(\frac{\pi^2 n^2}{f(g)}) + \log(2n_0 - 1).$$

- The sum is convergent, we choose an integer n_0 for optimal bound.

Strong coupling limit

- String perturbation series is actually convergent. (Some discussions) We can extrapolate to strong coupling $g \to \infty$.

- The optimal bound $n_0 \sim g^{2+\epsilon}$, we have

$$S_m(g) < (2+\epsilon)\log(g), \quad g \sim \infty.$$

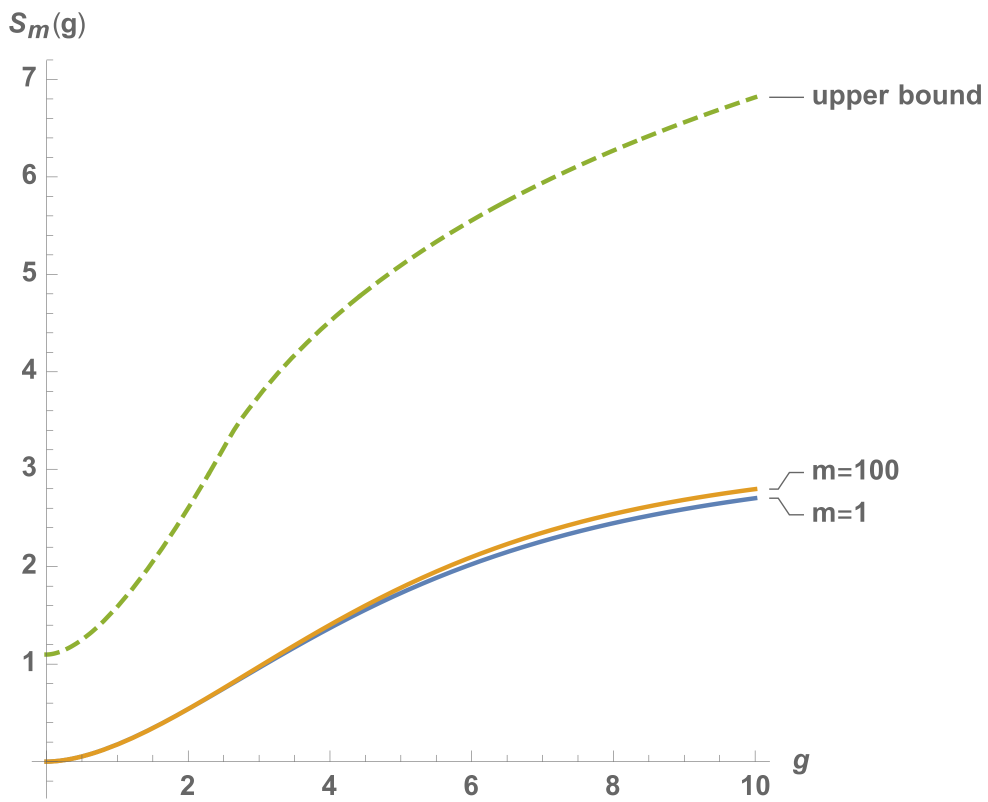
  Interpretation as "effective dimension" of the Hilbert space.

- The logarithmic bound is universal, e.g. for BMN operator with 3 stringy modes $O^J_{(m_1,m_2,m_3)}$ with closed string level matching condition $m_1+m_2+m_3=0$, we have

$$S_{(m_1,m_2,m_3)}(g) < (6+\epsilon)\log(g), \quad g \sim \infty.$$

Some numerical analysis

- As an illustration example we plot the entanglement entropy $S_m(g)$ for for two case $m = 1, 100$ and $0 < g < 10$. We use the data up to genus 3 and also truncate the sum at $|n| < 10000$. The numerical accuracy is sufficient for our purpose.



- Some salient features.

  1. $S_m(g)$ appears to be a monotonically increasing function of g.

  2. $S_m(g)$ seems to depends very weakly on the string mode m.

- Some further questions

  1. Can our bound be further improved? Is there a finite bound at strong coupling limit?

  2. Connections to (entanglement) entropy in other contexts? Geometric interpretations?

Thank You