# 2d Galilean Field Theories with Anisotropic Scaling

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Based on the work with Peng-xiang Hao and Zhe-fei Yu, 1906.03102 and work in progress

# Conformal symmetry in $D \geq 3$

The CFTs are the field theory with conformal symmetry quantum mechanically. They play important roles in various areas: the physics at the critical points, AdS/CFT, etc..

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The conformal group $SO(2, D)$ in $D \geq 3$ spacetime is generated by

- Translation: $P_\mu$
- Lorentzian rotation: $M_{\mu\nu}$
- Scaling transformation (Dilation): $D$
- Special conformal transformation: $K_\mu$

# 2D Conformal symmetry

In 2D, the conformal symmetry is actually

$$x^+ \to f\left(x^+\right), \qquad x^- \to g\left(x^-\right),$$

and is generated by two copies of Virasoro algebra $\{L_n, \bar{L}_n\}, n \in Z$, among which

$$\{L_{-1}, L_0, L_1\} \to SL(2,R), \qquad \{\bar{L}_{-1}, \bar{L}_0, \bar{L}_1\} \to SL(2,R).$$

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In other words, 2D conformal symmetry is much richer

$$\begin{array}{ccc} SL(2,R) \otimes SL(2,R) & \Rightarrow & Vir \otimes Vir \\ \text{Global, finite dim.} & \Rightarrow & \text{Local, infinite dim.} \end{array}$$

and more powerful:
Classification of minimal model, $\cdots$
Holographic CFT and its implications in $\text{AdS}_3/\text{CFT}_2$, including various problems in BH physics, spacetime reconstruction, $\cdots$ .

# Enhanced scaling symmetry

Scaling symmetry $\Rightarrow$ Conformal symmetry?

In 2D QFT, scaling symmetry could be enhanced to conformal symmetry

J. Polchinski (1989)

Assumption: the theory should be

- Poincare invariant
- Unitary
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In 4D, the enhancement from scaling to conformal symmetry could still be true under appropriate assumptions, but not fully proved.

In 2D, the left- and right-moving sectors are independent, there are more possibilities, if the Lorentz symmetry breaking is allowed.

# Example 1: Chiral scaling

D. Hofman and A. Strominger (2011):

chiral scaling in 2D without requiring the Lorentz invariance

$$x \to \lambda x, \quad y \to y,$$

Assumption: the dilation operator

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Two kinds of minimal theories with enhanced symmetries

- CFT: two copies of Virasoro algebra
  $\Rightarrow$ $AdS_3/CFT_2$
- Warped CFT: Virasoro-Kac-Moody algebra
  $\Rightarrow$ $AdS_3/WCFT_2$ or $WAdS_3/WCFT_2$ G. Compére, W. Song and A. Strominger (2013), ......

# Example 2: 2D GCFT

2D Galilean conformal invariant theory:

isotropic scaling + Galilean boost

$$y \to y + vx.$$

The Galilean CFT can be obtained by taking the non-relativistic limit of the conformal field theory.

It is related to the flat space holography ⇒ BMS/CFT or BMS/GCA Bagchi et.al.(2010,2012), Barnich et.al.(2001)......

# Anisotropic scaling

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$$t \to \lambda^z t, \qquad \vec{x} \to \lambda \vec{x}.$$

Such scaling is usually called Lifshitz scaling with dynamical exponent $z$.

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It happens in various systems:
Fermions at unitarity
Quantum critical points
Some statistical systems

People has tried to study some of these systems holographically D.T. Son (2008), J.McGreevy et.al. (2008), S. Kachru et.al. (2008), ...

# Setup

Consider 2D QFT with global translational symmetry, anisotropic scalings

$$x \to \lambda^c x, \quad y \to \lambda^d y,$$

and a Galilean boost symmetry

$$y \to y + vx.$$

In general

$$c \neq d$$

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Our consideration is general enough to include the WCFT and GCFT as special cases. In the following we just set $c = 1$.

# Problems

Q1: Are there enhanced symmetries in this kind of field theories?

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There are two approaches to study this problem, leading to the same conclusion.

# Approache 1

J.Polchinski(1989), D.Hofman & A. Strominger(2011):

Use the Noether current to define the conserved charges.

Due to the ambiguity in defining the Noether currents, there are modifications from local operators on the currents without destroying the canonical commutative relations.

The existence of such local operators implies the enhanced symmetries.

# Approach 2

If the field theory could be defined on a geometry in a covariant way, then the symmetries of the field theory are implied by the diffeomorphism of the underlying geometry.

For 2D CFT, it can be defined on a 2D Riemann-surface, then the infinite dimensional conformal symmetry comes from the (anti-)holomorphic mapping

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For the theories without the global Lorentz symmetry, we cannot define them in any (psuedo-)Riemannian geometry. Instead, we need to define them on Newton-Cartan geometry with scaling symmetry. C. Duval et.al. ,D.T. Son, Bagachi et.al., D. Hofman, K. Jensen,...

# Newton-Cartan geometry: flat case

It is similar to flat Euclidean geometry with the following symmetries

$$H : x \to x' = x + \delta x,$$

$$\bar{H} : y \to y' = y + \delta y,$$

$$B : y \to y' = y + vx.$$

There is a degenerate metric

$$g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

which is flat and invariant under boost transformation

$$g = BgB^{-1}.$$

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Besides, there is an antisymmetric tensor $h_{ab}$ to lower the index. It is invertible with $h^{ab}h_{bc} = \delta^a_c$, and its inverse helps us to raise the index

$$\bar{g}^{ab} = h^{ac}h^{bd}g_{cd}.$$

# Riemann geometry: curved case

The curved geometry is defined by ‘gluing flat geometry’, in the sense that the tangent space is flat with the map determined by the zweibein. One may define the covariant derivative

$$D = \partial + \omega + \Gamma$$

where $\omega$ is the spin connection to connect the points in the tangent space, while $\Gamma$ is the affine connection to connect the points in the base manifold.
In the Riemannian case, the affine connection is determined uniquely by requiring

- the metric to be compatible
- torsion free,

with zweibein postulate.

# Newton-Cartan geometry: curved case

In the Newton-Cartan geometry, the torsion-free condition cannot determine the spin connection uniquely.

To resolve this problem, one may give up the torsion-free condition. Instead one defines a scaling structure and requires it to be covariant constant. This implies that the scaling weight of the vector is invariant under parallel transport.

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Consequently one get a vanishing Riemann curvature, but non-vanishing affine connection and torsion.

# Allowed transformations

Now consider a general coordinate transformation

$$x \to f(x, y), \quad y \to g(x, y),$$

followed by anisotropic scaling

$$x \to \lambda x, \quad y \to \lambda^d y,$$

and boost transformation

$$y \to y + vx.$$

Imposing the condition that the coordinate transformation can be absorbed into the (local) scaling and boost transformation, it turns out that the allowed local transformations in the Newton-Carton geometry with scaling structure are

$$x \to f(x), \quad y \to f'(x)^d y,$$

and

$$x \to x, \quad y \to y + g(x).$$

# Enhanced symmetries

From the infinitesimal transformations

$$x \to x + \epsilon(x), \quad y \to (1 + d\epsilon'(x))y,$$

$$x \to x, \quad y \to y + \xi(x),$$

we read the generators

$$l_n = -x^{n+1}\partial_x - d(n+1)x^n y\partial_y,$$

$$m_n = x^{n+d}\partial_y,$$

which satisfy the algebra

$$[l_n, l_m] = (n-m)l_{n+m},$$
$$[l_n, m_m] = (dn-m)m_{n+m},$$
$$[m_n, m_m] = 0.$$

This algebra is called the spin-$d$ Galilean algebra. M. Henkel (2002)

If a field theory is defined covariantly on the Newton-Carton geometry with anisotropic scaling and boost symmetry, then the corresponding conservation currents and charges are exactly the ones read from Noether current method (Approach 1).

# Central extensions

The central extension is constrained by the Jacobi identity. There are various kinds of extensions, which we list here in order.

- $T$-extension is always allowable:

$$[l_n, l_m] = (n-m) l_{n+m} \Rightarrow$$
$$[L_n, L_m] = (n-m) L_{n+m} + \frac{c_T}{12} n(n^2-1)\delta_{n+m,0}.$$

  This gives the Virasoro algebra.

- $B$-extension is only allowable for $d = 1$:

$$[l_n, m_m] = (dn-m) m_{n+m} \Rightarrow$$
$$[L_n, M_m] = (n-m) M_{n+m} + \frac{c_B}{12} n(n^2-1)\delta_{n+m,0}.$$

  This gives the GCA algebra.

# Central extensions: II

$$[m_n, m_m] = 0.$$

- $M$-extension is only allowable for $d = 0$, the infinite dimensional spin-$0$ Galilean algebra

$$[M_n, M_m] = c_M n \delta_{n+m,0}.$$

This is actually the algebra for the warped CFT, with $c_M$ being the Kac-Moody level.

- Infinite $M$-extensions, in which there are infinite $c_M$ charges

$$[M_n, M_m] = (n - m)(c_{\mathcal{M}})_{n+m}, \quad [M_n, (c_M)_m] = -m(c_{\mathcal{M}})_{n+m}.$$

The familiar case is the algebra for the Schrödinger symmetry, in which $d = 1/2$. Here we show that for arbitrary spin $d$, there could be similar algebraic structure.

We have shown that there are enhanced symmetries in the anisotropic GCFT.

Q2: Properties?

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Q2: Properties?

Quantization, state-operator correspondence, correlation functions, modular properties, $\cdots$

# Reference plane

Let us focus on the case that $d$ is an integer, in which case the Cartan subalgebra is generated by $(L_0, M_0)$.

It is better to study the theories on a cylinder

$$(\phi, t) \sim (\phi + 2\pi, t),$$

which are related to $x, y$ by

$$x = t + \phi, \quad y = t - \phi.$$

Consider the following complex transformation which maps the canonical cylinder to the reference plane

$$z = e^{ix} = e^{t_E - i\phi}, \quad \tilde{y} = (iz)^d y,$$

where $t_E = -it$ is the Wick-rotated time.

# Radial quantization

The Hilbert space are defined on the equal imaginary time slices.

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On the reference plane, these states are defined at the origin and the radial infinity, which leads to the radial quantization.

# Primary operators

The subgroup keeping the origin invariant is

$$L_0, \ L_{n>0}, \quad M_{-d+1}, \ M_{-d+2}, \ \cdots .$$

The local operators can be labelled by the eigenvalues $(h_{\mathcal{O}}, \xi_{\mathcal{O}})$ of the generators $L_0, M_0$ of the Cartan subalgebra

$$[L_0, \mathcal{O}(0,0)] = h_{\mathcal{O}}\mathcal{O}(0,0), \quad [M_0, \mathcal{O}(0,0)] = \xi_{\mathcal{O}}\mathcal{O}(0,0).$$

Requiring $h_{\mathcal{O}}$ to be bounded below, one arrives at the highest weight representations

$$[L_n, \mathcal{O}(0,0)] = 0, \quad [M_n, \mathcal{O}(0,0)] = 0, \quad \text{for } n > 0.$$

This defines the primary operator. One can get the tower of descendant operators by acting $L_{-n}, \ M_{-n}$ with $n > 0$ on $\mathcal{O}$.

# State-operator correspondence

The operators inserting at the origin give the states,

$$\mathcal{O}(0,0)|0\rangle \to |h_{\mathcal{O}}, \xi_{\mathcal{O}}\rangle.$$

This gives a bijection between the states in the Hilbert space at infinitely past and the operators insertion at the origin on the reference plane.

# Primary state

$$L_0|h,\xi\rangle = h|h,\xi\rangle, \quad M_0|h,\xi\rangle = \xi|h,\xi\rangle,$$
$$L_n|h,\xi\rangle = 0, \quad M_n|h,\xi\rangle = 0, \quad n > 0.$$

By acting the generators $L_n,\ M_n$ with $n < 0$, one gets the descendant states, which are labelled by two vectors $\vec{I},\ \vec{J}$,

$$|\vec{I},\vec{J},h,\xi\rangle = L_{-1}^{I_1}\cdots M_{-1}^{J_1}\cdots|h,\xi\rangle.$$

A state is either a primary state or a descendant state, and the Hilbert space is spanned by the modules

$$H = \oplus\sum V_{h,\xi},$$

where $V_{h,\xi}$ is the module consisting of a primary state and the tower of all its descendants.

# Two-point functions

The correlation functions are invariant under the global transformations

$$\langle 0|GO(x_1, y_1)O(x_2, y_2)|0\rangle = 0$$

where

$$G \in \{L_{-1},\ L_0,\ L_1,\ M_{-d},\ \cdots,\ M_d\}.$$

Moving $G$ from the left to the right gives the constraints on the two-point functions. For example, the translation symmetries require that the correlation functions must depend only on $x = x_1 - x_2$ and $y = y_1 - y_2$.

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- The $d = 0$ case is special, since the representation is special.

$$\langle \mathcal{O}_1(x, y)\mathcal{O}_2(0, 0)\rangle = d_{\mathcal{O}}\delta_{h_1,h_2}\delta_{\xi_1,-\xi_2}\frac{1}{x^{2h_1}}e^{\xi y}.$$

- The $d = 1$ case, there are no descendant operators involved when doing the local transformations on the primary operators.

$$\langle \mathcal{O}_1(x, y)\mathcal{O}_2(0, 0)\rangle = d_{\mathcal{O}}\delta_{h_1,h_2}\delta_{\xi_1,\xi_2}\frac{1}{x^{2h_1}}e^{2\xi y/x}.$$

# Two-point function: $d \geq 2$

For $d \geq 2$, the correlation functions become much more involved. The correlation functions of the descendant operators with the primary operators are not vanishing in such cases. Namely one has to consider the following correlation functions

$$f(n, d) = \langle (M_n \mathcal{O}_1)(x, y) \mathcal{O}_2(0, 0) \rangle.$$

Solving the constraints from the invariance of the two-point functions under the global transformations, one gets

$$f(-d+1, d) = -\frac{1}{2} x f(-d, d),$$

$$f(n, d) = \frac{(d-1)!(d-n)!}{2(2d-1)!(-n)!} (-1)^{n+d} x^{n+d} f(-d, d), \quad \text{for } n \in [-d+2, 0].$$

In the end, one finds

$$\langle \mathcal{O}_1(x, y) \mathcal{O}_2(0, 0) \rangle = d_{\mathcal{O}} \delta_{h_1, h_2} \delta_{\xi_1, (-1)^{d+1} \xi_2} \frac{1}{x^{2h_1}} e^{2 C^d_{2d-1} (-1)^{d+1} \xi y / x^d},$$

where $C^n_m$ is the binomial coefficient.

# Modular transformation

The theories can be defined on a torus

$$(\phi, t) \sim (\phi + \alpha, t + \beta).$$

Consider the symmetry transformation of the theory

$$\phi \to f(\phi), \quad t \to f'(\phi)^d t,$$

and

$$t \to t + g(\phi).$$

After doing the transformation, the spatial circle and thermal circle exchange with each other. This requires that

$$f(\phi) = \lambda\phi, \quad g(\phi) = k\phi.$$

with

$$\lambda = \frac{2\pi}{\alpha}, \qquad k = -(\frac{2\pi}{\alpha})^d \frac{\beta}{\alpha}.$$

The new torus is

$$(\phi, t) \sim (\phi + \alpha', t + \beta')$$

where

$$\alpha' = \frac{4\pi^2}{\alpha}, \qquad \beta' = -(\frac{2\pi}{\alpha})^{d+1}\beta.$$

# Modular invariant theories

For $d \neq 0$, the partition function is invariant under the modular transformation

$$Z(\alpha', \beta') = Z(\alpha, \beta).$$

This leads to the Cardy-like formula

$$S(\Delta, \xi) = 2\pi\Delta(\frac{H_v}{\xi})^{\frac{1}{d+1}} + 2\pi M_v(\frac{\xi}{H_v})^{\frac{1}{d+1}},$$

for

$$\xi >> H_v, \quad \Delta >> M_v,$$

where $M_v$ and $H_v$ are the translation charges of the vacuum.

For $d = 1$, it matches with the result in GCA theories Bagchi et.al. 2012,2013

# Modular covariant theories: $d = 0$

In the case $d = 0$, there exists a nonvanishing $M$-extension,

$$[M_n, M_m] = c_M n \delta_{n+m,0}.$$

the torus partition function is covariant under the modular transformation

$$Z(\alpha, \beta) = e^{\frac{\beta^2 c_M}{2\alpha}} Z(\alpha', \beta')$$

There is an anomaly due to the $M$ central charges.
For

$$\xi >> H_v, \qquad \Delta >> M_v,$$

the microcanonical ensemble entropy is

$$S(\Delta, \xi) = -2\pi \frac{H_v \xi}{c_M} - \frac{2\pi}{c_M} \sqrt{(H_v^2 + 2c_M M_v)(-2c_M \Delta + \xi^2)},$$

which has been obtained in the warped CFT Detournay et.al. 2012.

# Conclusion

Under the assumption that the dilation operator is diagonalizable, and has a discrete, non-negative spectrum, we showed in two different ways that the field theories with global translation, Galilean boost and anisotropic scaling could have enhanced symmetries.

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Under the assumption that the dilation operator is diagonalizable, and has a discrete, non-negative spectrum, we showed in two different ways that the field theories with global translation, Galilean boost and anisotropic scaling could have enhanced symmetries.

In particular, we define covariantly the field theories in a Newton-Cartan geometry with anisotropic scaling.

This paves the way to study the properties of the theories, including the quantization, state-operator correspondence, two-point functions and modular properties.

# Other questions

An explicit example?

Other properties of the theory?

Scaling anomaly?

Bootstrapping?

Holographic duals?

# Thank you for your patience!