Photon surface in gravitational collapse

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November 24, 2019

Content

In Boyer-Linderquist coordinates, the Kerr metric has a form

$$g = -dt^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + (r^2 + a^2)\sin^2\theta d\varphi^2 + \frac{2mr}{\Sigma}(dt - a\sin^2\theta d\varphi)^2\,, \tag{1}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta\,,$$

and

$$\Delta = r^2 - 2mr + a^2\,.$$

- Killing vector fields

$$t^a = \left(\frac{\partial}{\partial t}\right)^a , \quad \varphi^a = \left(\frac{\partial}{\partial \varphi}\right)^a . \tag{2}$$

- Killing tensor
  The trivial one—metric, i.e., g_{ab}.
  The nontrivial one

$$k_{ab} = \Sigma(\ell_a n_b + n_a \ell_b) + r^2 g_{ab} , \tag{3}$$

where ℓ^a and n^a are two null vector fields

$$\ell_a = -(dt)_a + \frac{\Sigma}{\Delta}(dr)_a + a \sin^2\theta (d\varphi)_a ,$$

$$n_a = \frac{1}{2\Sigma}\Big[-\Delta(dt)_a - \Sigma(dr)_a + a \sin^2\theta \Delta(d\varphi)_a \Big] .$$

Geodesic equations and conserved quantites

In Boyer-Lindquist coordinates, the four velocity u^a of the geodesic can be expressed as

$$u^a = \frac{dx^\mu(\tau)}{d\tau}\left(\frac{\partial}{\partial x^\mu}\right)^a .$$

The conserved quantities carried by the geodesic

$$\begin{aligned} e &= -g_{ab}t^a u^b \,, \qquad & l &= g_{ab}\varphi^a u^b \,, \\ \kappa &= g_{ab}u^a u^b \,, \qquad & q &= k_{ab}u^a u^b \,, \end{aligned} \tag{4}$$

where $\kappa = -1$ for timelike geodesic curves, and $\kappa = 0$ for null geodesics.
Noted: We have 4 generalized coordinates and 4 conserved quantities. So the the system is classically integrable, and the Hamilton-Jacobi function is totally separable.

Geodesic equations and conserved quantities

From the detailed expression of the conserved quantities, we have

$$\begin{aligned}
\frac{dt}{d\tau} &= e + \frac{2mr[(r^2+a^2)e - al]}{\Sigma\Delta}\,, \\
\frac{d\varphi}{d\tau} &= \frac{1}{\Sigma\Delta\sin^2\theta}[(\Sigma - 2mr)l + (2mar\sin^2\theta)e]\,, \\
\Sigma^2\left(\frac{dr}{d\tau}\right)^2 &= \Delta(\kappa r^2 - q) + [(a^2+r^2)e - al]^2\,, \\
\Sigma^2\left(\frac{d\theta}{d\tau}\right)^2 &= q + \kappa a^2\cos^2\theta - \left(ae\sin\theta - \frac{l}{\sin\theta}\right)^2. \qquad (5)
\end{aligned}$$

The last two equations can be put into simple forms

$$\begin{aligned} \Sigma^2\left(\frac{dr}{d\tau}\right)^2 &= V(r)\,, \\ \Sigma^2\left(\frac{d\theta}{d\tau}\right)^2 &= U(\theta)\,, \end{aligned} \tag{6}$$

where

$$\begin{aligned} V(r) &= \Delta(\kappa r^2 - q) + [(a^2 + r^2)e - al]^2\,, \\ U(\theta) &= c + \cos^2\theta\left[a^2(e^2+\kappa) - \frac{l^2}{\sin^2\theta}\right]\,, \end{aligned} \tag{7}$$

where $c = q - (l - ae)^2$ is the so-called Carter constant.

Spherical orbits of the null geodesics

The spherical orbits are determined by

$$\frac{dr}{d\tau} = 0\,, \qquad \frac{d^2r}{d\tau^2} = 0\,.$$

These are equivalent to

$$V(r) = 0\,, \qquad V'(r) = 0\,,$$

and we have

$$q\Delta = [(a^2 + r^2)e - al]^2\,, \qquad q\Delta' = 4re[(a^2 + r^2)e - al]\,. \tag{8}$$

or

$$\frac{q}{e^2} = \frac{16r^2\Delta}{(\Delta')^2}\,, \qquad a\left(\frac{l}{e}\right) = a^2 + r^2 - 4r\frac{\Delta}{\Delta'}\,. \tag{9}$$

In the case of Schwarzschild, $a = 0$ and the Carter constant is vanishing, i.e., $c = 0$ or $q = l^2$, and these equations gives

$$r = 3m\,, \qquad \frac{l^2}{e^2} = 27m^2\,. \tag{10}$$

The radius of the photon orbits is not depend of the conserved quantities (the background parameter m is enough), and $l/e = 3\sqrt{3}m$. The sphere $r = 3m$ is the so-called photon sphere.
In the case of Kerr, the situation is quite different. The radius of the spherical orbits is determined by l/e and the background parameters a and m together.

Of course, l/e is not arbitrary. Since $U(\theta)$ can not be negative, we have

$$\frac{q}{e^2}\sin^2\theta - e^2(a^2\cos^2\theta - l/e)^2 \geq 0\,. \tag{11}$$

Considering (9), this implies that on the spherical orbits we have

$$(4r\Delta - \Delta'\Sigma)^2 \leq 16a^2r^2\Delta\sin^2\theta\,. \tag{12}$$

This means all of the possible spherical photon orbits have to be localised in the region by the above inequality.

Spherical orbits of the null geodesics

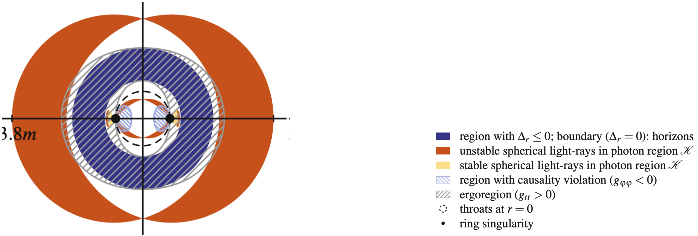


Figure: The photon region in Kerr spacetime. Figs come from the work by Grenzebach, 2016

Remarks

- The symmetries of the Kerr spacetime
- The black hole in our universe obviously has no such symmetries.
  - ⋆ The black hole in reality is always dynamical (accretion and swallow surrounding matter);
  - ⋆ Time dependence due to quantum effect such as Hawking radiation.
- Question:
  How to study the photon sphere or photon surface in general dynamical spacetimes?

Definition based on umbilical hypersurfaces

Definition (Claudel, Virbhadra, and Ellis, 2000)

A photon surface of (M, g) is an immersed, nowhere-spacelike hypersurface S of (M, g) such that, for every point $p \in S$ and every null vector $k \in T_pS$, there exists a null geodesic $\gamma : (-\epsilon, \epsilon) \to M$ of (M, g) such that $\dot{\gamma}(0) = k \, , |\gamma| \subset S$.

Theorem

Let S be a timelike hypersurface of (M, g). Let n be a unit normal field to S and let h_{ab} be the induced metric on S. Let χ_{ab} be the second fundamental form on S and let σ_{ab} be the trace-free part of χ_{ab}. Then the following are equivalent:
(i). S is a photon surface;
(ii). $\chi_{ab} k^a k^b = 0 \; \forall$ null $k \in T_pS \; \forall p \in S$;
(iii).$\sigma_{ab} = 0$;
(iv). every affine null geodesic of (S, h) is an affine null geodesic of (M, g).

Definition based on umbilical hypersurfaces

The definition implies the photon surface is an umbilical hypersurface of the spacetime, i.e.,

$$\sigma_{ab} = 0\,. \tag{13}$$

This has

$$\frac{1}{2}(n-1)n-1$$

independent conditions in general n dimensional spacetime. These conditions are very restrictive. A lot of spacetimes might have no such kind of hypersurface because that a hypersurface is determined by a scalar equation in general.
For example, in Kerr spacetime, this definition has some trouble.

Definition based on umbilical hypersurfaces

On the other hand, with this definition, the photon surface exists even for Minkowski spacetime. For example, any null hypersurface and timelike hyperboloid surface

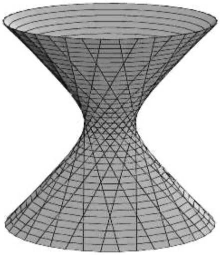


Figure: One of photon surface in Minkowski spacetime.

New definitions (spacetimes)

The metric of a spacetime $(\mathcal{M}, g)$ with the symmetry of a codimension-2 maximally symmetric space can be expressed as

$$g = h_{AB}(y)dy^A dy^B + r^2(y)\gamma_{ij}(z)dz^i dz^j \,, \qquad (14)$$

where γ_{ij} is the metric of a codimension-2 maximally symmetric space.

r is a function of y^A, which is a geometric quantity and can be defined by the area of a codimension-2 surface.

h_{AB} are functions of coordinates y^A, which can be viewed as the components of the metric of a two dimensional Lorentz manifold (M, h).

Definition (Photon sphere)

For the spacetime $(\mathcal{M}, g)$ with the metric (14), let $\{u, v\}$ be an orthogonal normal frame of a codimension-2 surface on which that r is a constant, and $v_A v^A = 1 = -u_A u^A$, then this surface is a photon sphere if

$$D_A\left(\frac{v^A}{r}\right) = 0\,, \tag{15}$$

and

$$u^B D_B D_A\left(\frac{v^A}{r}\right) = 0\,, \tag{16}$$

and

$$v^B D_B D_A\left(\frac{v^A}{r}\right) \neq 0\,, \tag{17}$$

where D_A is the covariant derivative (along the natural basis $\partial/\partial y^A$) which is compatible to the metric h of (M, h).

Remarks:
Two scalar equations: one for the position of the sphere, another is for the orthogonal frame.

Definition (Outer and Inner)

A photon sphere is called outer if $v^B D_B D_A(v^A/r) < 0$, *and inner if* $v^B D_B D_A(v^A/r) > 0$.

Definition (Photon surface)

An outer (inner) photon surface is a timelike hypersurface foliated by the outer (inner) photon spheres.

It is a second order differential equation.

Theorem (Preferred frame)

If $(\mathcal{M}, g)$ *is static, and in the untrapped region of the spacetime,* $\{u, v\}$ *is the orthogonal frame given by*

$$u^A = \bar{u}^A := -\epsilon^{AB}\frac{D_B r}{\|Dr\|}\,, \qquad v^A = \bar{v}^A := \frac{D^A r}{\|Dr\|}\,, \tag{18}$$

where $\|Dr\|^2 = D_A r D^A r > 0$*. Then the position of the photon sphere is determined by eq.(15), and eq.(16) is trivially satisfied.*

The equation for photon sphere can be further simplified.

Theorem (Simplified equation for photon spheres)

In the case of static, the photon sphere is determined by the following equation

$$D^A r D_A \sigma - \Box r + \frac{2\|Dr\|^2}{r} = 0\,, \qquad (19)$$

where σ is defined by $\xi^A = -e^{-\sigma}\epsilon^{AB}D_B r$, and ξ^A is the static Killing vector of the spacetime.

In the static cases, the equation for photon sphere is quite simple.

For a static spactime, by choosing coordinates, the metric can be put into a form

$$ds^2 = -h(x)dt^2 + f^{-1}(x)dx^2 + r^2(x)\gamma_{ij}dz^i dz^j \,. \tag{20}$$

By the theorem, it is easy to find that the photon sphere is determined by

$$2\frac{r_x}{r} - \frac{h_x}{h} = 0 \,. \tag{21}$$

The classification of the photon sphere is determined by the sign of

$$\frac{f}{2r}\Big[\frac{h_{xx}}{h} - 2\Big(\frac{r_x}{r}\Big)^2 - \frac{r_{xx}}{r}\Big] \,. \tag{22}$$

For example, for Reissner-Nordström de Sitter black hole, there are two photon spheres (outer and inner).

We do not know the orthogonal frame. Generally, it has a form

$$\begin{aligned} u^A &= \bar{u}^A \cosh\alpha + \bar{v}^A \sinh\alpha\,, \\ v^A &= \bar{u}^A \sinh\alpha + \bar{v}^A \cosh\alpha\,, \end{aligned} \tag{23}$$

where α is a function on the spacetime.

Theorem

For the spacetime with the metric (14), the evolution of the photon sphere is described by the following equations

$$\frac{\ddot{r}_s}{r_s} - \left(\frac{\dot{r}_s}{r_s}\right)^2 = \frac{1}{r_s^2}\left[k - \frac{d-1}{d-2}\frac{8\pi G}{\Omega^{(k)}_{(d-2)}}\frac{E_s}{r_s^{d-3}}\right] - \frac{8\pi G}{d-2}p\,, \qquad (24)$$

where $p = T_{AB}v^A v^B$ *and* E_s *is the restriction of the so-called mass function*

$$E = \frac{(d-2)\Omega^{(k)}_{(d-2)}}{16\pi G} r^{d-3}\left(k - \|Dr\|^2 \mp \frac{r^2}{\ell^2}\right) \qquad (25)$$

to the photon sphere, and

$$\frac{1}{\ell^2} = \frac{2|\Lambda|}{(d-1)(d-2)}\,. \qquad (26)$$

Theorem

The evolution of α is given by

$$\dot{\vartheta} = \frac{r_s}{N_s}\left\{\left(\frac{k}{r_s^2} - \frac{d-1}{d-2}\frac{8\pi G}{\Omega_{(d-2)}^{(k)}}\frac{E_s}{r_s^{d-1}} - \frac{8\pi G}{d-2}p\right) + \frac{8\pi G}{d-2}q\sin\vartheta\right\}, \quad (27)$$

where $\sin\vartheta = \tanh\alpha$, and N_s is the restriction of

$$N = \|Dr\| = \left\{k \mp \frac{r^2}{\ell^2} - \frac{16\pi G}{(d-2)\Omega_{(d-2)}^{(k)}}\frac{E}{r^{d-3}}\right\}^{1/2} \quad (28)$$

on the photon surface.

The evolution of the energy

Theorem

The evolution of the energy inside the photon sphere is determined by

$$\dot{E}_s = \Omega^{(k)}_{(d-2)} r_s^{d-2} \left(- p\dot{r}_s + q\sqrt{\dot{r}_s^2 + N_s^2} \right) , \tag{29}$$

where $q = T_{AB} u^A v^B$.

(i). Assume we have get the the details of the photon surface in the coordinate system $\{y^A\}$, then eqs.(24), (27), and (29) form a closed system, and we can solve these equations supplied by some boundary conditions.
Conversely, starting from Eqs.(24) and (27), by eliminating α and $\dot{\alpha}$, we can get the second order equation for the photon surface. So eqs.(24) and (27) can be understood as the equation for the photon surface.

(ii).In the case of vacuum, we have $p = q = 0$ (here, we only consider the case with negative cosmological constant, and E_s now is the AMD mass)

$$E_s = \frac{(d-2)\Omega^{(k)}_{(d-2)}}{16\pi G} r_+^{d-3}\left(k + \frac{r_+^2}{\ell^2}\right), \tag{30}$$

where r_+ is the radius of the event horizon which satisfies $N(r_+) = 0$. Eq.(24) tells us

$$k r_s^{d-3} = \frac{1}{2}(d-1) r_+^{d-3}\left(k + \frac{r_+^2}{\ell^2}\right). \tag{31}$$

$k = 0$, there is no photon sphere.
In the case where $k = -1$, the photon sphere exists only when $r_+ < \ell$ (negative mass and temperature).
When $k = 1$, the photon sphere always exists.

(iii). In the case of static (matter might exist), r_s, E_s, and α do not evolve. we have

$$\frac{k}{r_s^2} - \frac{d-1}{d-2}\frac{8\pi G}{\Omega_{(d-2)}^{(k)}}\frac{E_s}{r_s^{d-1}} - \frac{8\pi G}{d-2}p = 0\,, \qquad (32)$$

and the location of the photon sphere depends on the detail content of the matter field.
$d = 4$ and $k = 1$, we have $r_s = 3E_s$ if $p = 0$, and when $p \neq 0$, we have

$$r_s = \frac{1}{\sqrt{3\pi p}}\sin\left\{\frac{1}{3}\arcsin\left[9E_s\sqrt{3\pi p}\right] + \epsilon\frac{2\pi}{3}\right\}, \qquad (33)$$

where $\epsilon = 0, \pm 1$.

- OS model, 1930s;
- Cosmic Censorship, 1969;
- LTB, shell-crossing singularity, 1973, Yodzis, Seifert, Muller Zum Hagen;
- LTB, shell-focusing singularity, Eardley and Samrr, 1979;
- massless scalar field, Christodoulou, Choptuik, 1980s-1990s;
- ......

Naked singularity is possible for some initial deta.

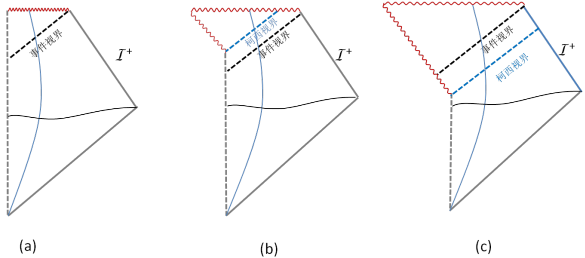


Figure: (a). OS model. (b). local naked singularity. (c). global naked singularity.

The matter field is dust

$$T_{ab} = \epsilon U_a U_b \,, \tag{34}$$

The metric of the spacetime can be put into the so-called LTB form

$$ds^2 = -dt^2 + \frac{[r_x(t,x)]^2}{1+\kappa(x)} dx^2 + r^2(t,x)\gamma_{ij} dz^i dz^j \,, \tag{35}$$

The Einstein equations reduce to

$$E_t = 0 \,, \qquad E_x = 4\pi r^2 r_x \epsilon \,, \tag{36}$$

and

$$r_t^2 = \kappa + \frac{2E}{r} \,. \tag{37}$$

The square of Ricci curvature has a form

$$R_{ab}R^{ab} = 64\pi^2\epsilon^2 \,, \tag{38}$$

and the Kretschmann scalar can be written as

$$R_{abcd}R^{abcd} = \frac{48E^2}{r^6} - \frac{32EE'}{r^5 r'} + \frac{12E'^2}{r^4 r'^2} \,. \tag{39}$$

Possible singularity happens at $r = 0$ or $r' = 0$.
$r' = 0$—shell-crossing singularity;
$r = 0$—shell-focusing singularity.

One can integrate eq.(37), and get

$$t = \pm\left\{\sqrt{\frac{r^3}{2E}}F\Big(-\frac{\kappa r}{2E}\Big) - \sqrt{\frac{r_0^3}{2E}}F\Big(-\frac{\kappa r_0}{2E}\Big)\right\}, \qquad (40)$$

where $r_0 = r(0, x)$, and

$$F(x) = \begin{cases} x^{-3/2}\arcsin\left[x^{1/2}\right] - x^{-1}(1-x)^{1/2}\,, & 0 < x \leq 1\,, \\ 2/3\,, & x = 0\,, \\ -(-x)^{-3/2}\mathrm{arcsinh}[(-x)^{1/2}] - x^{-1}(1-x)^{1/2}\,, & x < 0\,. \end{cases} \qquad (41)$$

The function F is positive, smooth, ...

Function F

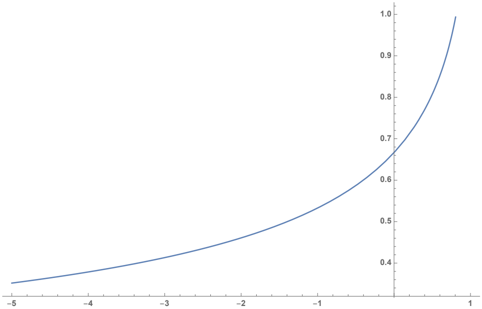


Figure: The function F is smooth, positive, monotonically increasing ......

Marginal bounded collapse solutions to the LTB model

For marginally bounded collapse, i.e., $\kappa = 0$, one has solution (Eardley and Smarr, 1979)

$$r(t,x) = \left\{ \frac{9}{2} E(x) \left[t_0(x) - t \right]^2 \right\}^{1/3}, \qquad (42)$$

where
$t_0(x)$ — the location of the singularity.
$(t_0)_x > 0$—no shell-crossing singularity.
The range of coodinates—$t \in (-\infty, t_0(x))$ and $x \in [0, +\infty)$.

Why gravitational collapse ?

- — One of important way to form a black hole (the formation of supermassive black hole is still unclear up to date).
- — The spacetime is dynamical, it is not clear how the photon surface emerge in gravitational collapse.
- — Naked singularity is possible. How about the photon surface for the naked singularity in gravitational collapse? absent (like event horizon)? or still present?
- — The role of photon surface in the observation is important. The shadow can be viewed as the photo of the photon sphere. So it is a succedaneum of the event horizon in some sense.
  Is this right in dynamical process?
- – A lot of questions ......

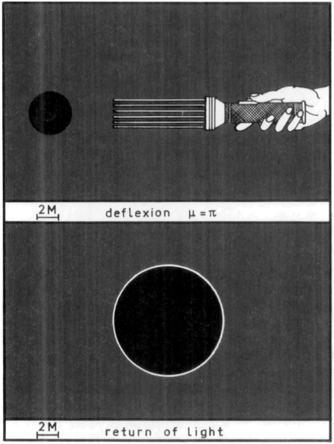


Fig. 2. Return of light deflected by 180° from a bare black hole

Figure: J.-P. Luminet (1979)

The idea of shadow

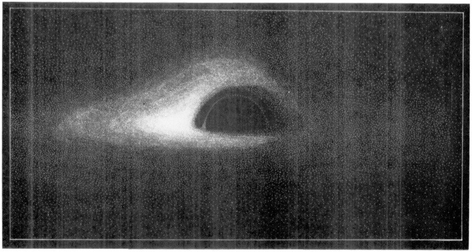


Figure: The shadow of Schwarzschild black hole with thin accretion disk. J.-P. Luminet (1979)

By the definition (15), or from eqs. (24) and (27), in the general LTB model, we get the equation for the photon surface (denoted by $x = x(t)$)

$$\ddot{x} = \frac{1+\kappa}{r r_x} + \left(\frac{r_t}{r} - \frac{2r_{tx}}{r_x}\right)\dot{x} - \left(\frac{r_x}{r} + \frac{r_{xx}}{r_x} - \frac{1}{2}\frac{\kappa_x}{1+\kappa}\right)\dot{x}^2 + \frac{r_x^2}{1+\kappa}\left(\frac{r_{xt}}{r_x} - \frac{r_t}{r}\right)\dot{x}^3 \,, \qquad (43)$$

where "$\dot{}$" denotes the total derivative with respect to the coordinate t.

The boundary conditions

(i). Outside the dust ball, the solution is the standard Schwarzschild spacetime.
(ii). The change of the energy E, i.e., $\dot{E}$ is vanishing on the boundary.

In OS model, the dust is homogenous in space, the metric is just the one in FLRW universe, i.e., we have

$$r = a(t)x\,, \qquad \kappa = -Kx^2$$

where $K = 0\,, \pm 1$. Here, we only consider the case with $K = 1$. Let $x = \sin\chi$, the metric can be expressed as (Rezzolla)

$$ds^2 = -dt^2 + a^2(t)(d\chi^2 + \sin^2\chi d\Omega_2^2)\,, \tag{44}$$

and the photon surface equation (43) now becomes

$$\ddot{\chi} = \frac{-a\dot{a}\dot{\chi} + \cot\chi - a^2\cot\chi\dot{\chi}^2}{a^2}\,. \tag{45}$$

By the boundary conditions, we have an analytic solution to the photon surface equation

$$\cos\chi = c_1\cos\eta + c_2\sin\eta\,, \tag{46}$$

where c_1 and c_2 are two integral constants

$$c_1 = -\left(1-\frac{6m}{r_0}\right)\sqrt{1-\frac{2m}{r_0}} + \sqrt{\frac{2m}{r_0}}\sqrt{\frac{12m}{r_0}\left(1-\frac{3m}{r_0}\right)}\,, \tag{47}$$

and

$$c_2 = \left(1-\frac{6m}{r_0}\right)\sqrt{\frac{2m}{r_0}} + \sqrt{\frac{12m}{r_0}\left(1-\frac{2m}{r_0}\right)\left(1-\frac{3m}{r_0}\right)}\,. \tag{48}$$

Here, r_0 is the initial radius of the dust ball.

Various geometric objects in OS model

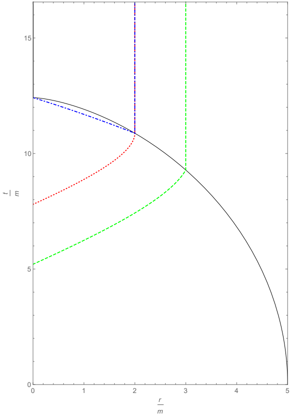


Figure: OS model for the gravitational collapse of homogenous dust. The green dashed line corresponds to the photon surface, the red dotted line is the event horizon, the blue dotdashed line is the apparent horizon.

From the solution, we can get the occurrence time of the photon surface

$$\eta_s = \begin{cases} -\arccos(c_1)\,, & 3m \le r \le (18/5)m\,, \\ +\arccos(c_1)\,, & (18/5)m \le r < \infty\,. \end{cases} \tag{49}$$

The time for the occurrence of the event horizon is given by (Rezzolla)

$$\eta_e = 2\arccos\left(\sqrt{\frac{2m}{r_0}}\right) - \arcsin\left(\sqrt{\frac{2m}{r_0}}\right). \tag{50}$$

Some calculation show that $\eta_e - \eta_s$ is always negative. So the photon sphere always appear before the event horizon.

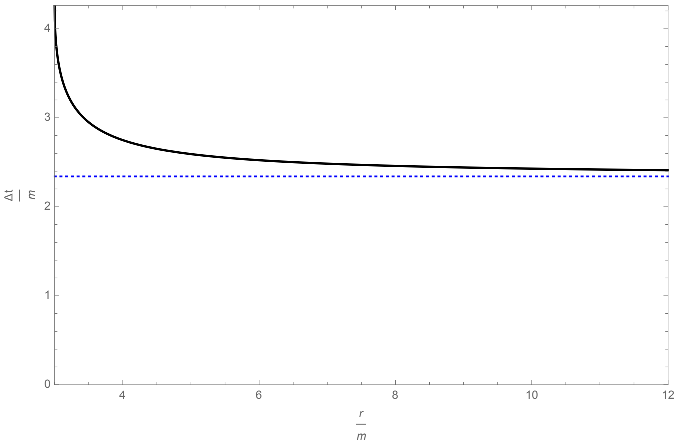


Figure: The time difference between the occurrence of the event horizon and photon surface. The limit value is $((9\sqrt{6}-8)/6)m \approx 2.3409m$.

- To our knowledge, this is the first analytic photon surface solution in a time-dependent spacetime (or dynamical spacetime).
- For a fixed total mass m, Δt has a limit value $2.3409m$. This tells us that Δt does not increase when the size of the dust ball becomes very large once the total mass is fixed. The reason behind this limit value is still unclear to us.

- The event horizon still tightly follows the photon surface even for a system with very large size and small density.
  - ⋆ For a system with several Solar masses, Δt is about $10^{-5} \sim 10^{-4}s$.
  - ⋆ For a galaxy with 10^9 Solar masses, Δt is about $10^4 s$, i.e., about 3.4 hours.
- This phenomenon also implies Δt is not so sensitive to the strength of gravitational field of the system.

The observation of the photon sphere or the photon surface is a very good approximation of the event horizon.

Consistence with the definition

The photon surface solution here is consist to the our definition.

The equation for the photon surface is quite complicated even in the case of marginal bounded collapse. Here, we give the numerical results. As in the paper by Eardley and Smarr, we assume

$$E(x) = x^3\,, \qquad t_0(x) = \zeta x^\nu\,, \qquad 0 \le x < 1\,,$$
$$E(x) = 1\,, \qquad t_0(x) = x^2 - 1 + \zeta\,, \qquad 1 < x < \infty\,, \quad (51)$$

where $\zeta \ge 0$, and $\nu \ge 1$ is an integer. The boundary of the dust ball corresponds to $r = 1$. Outside the dust ball, the spacetime is the Schwarzschild solution which has a unit ADM mass.

Outside the dust, the spacetime is the standard Schwarzschild, so on the surface of the dust ball, we have $r(1) = 3$. This means the photon surface has to be anchored at the point with coordinates

$$t\big|_{x=1} = \zeta - \sqrt{6}\,, \tag{52}$$

on the boundary, $\dot{E} = 0$ gives another boundary condition

$$\left.\frac{dt}{dx}\right|_{x=1} = 3 + \frac{\sqrt{6}\,\nu\zeta}{3}\,. \tag{53}$$

The boundary conditions (52) and (53) are enough to determine the photon surface.

Numerical results

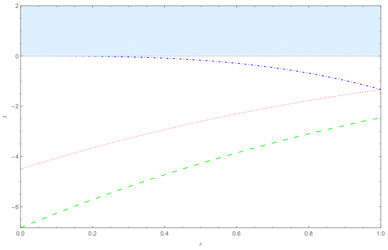
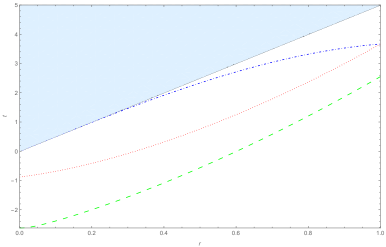
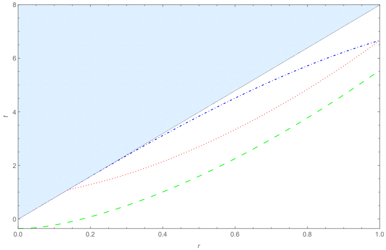


Figure: $\zeta = 0$, OS model with $K = 0$. The green dashed line corresponds to the photon surface, the red dotted line is the event horizon, and the blue dotdashed line depicts the apparent horizon.

Numerical results

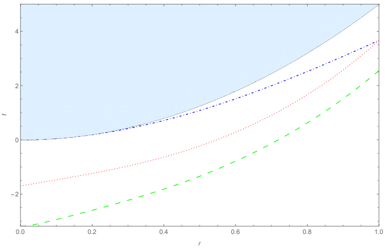


(a). $\zeta = 5.0$, no globally naked singularity.

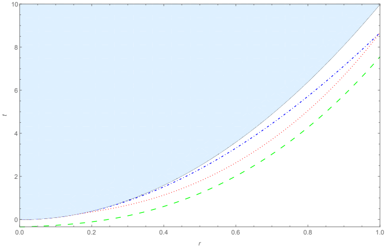


(b). $\zeta = 8.0$, globally naked singularity.

Figure: ν=1

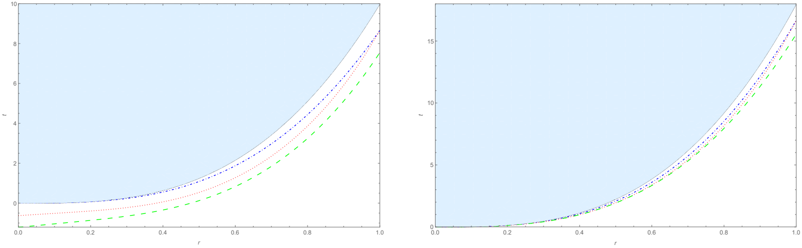


(a). $\zeta = 5.0$, no globally naked singularity.



(b). $\zeta = 10.0$, globally naked singularity.

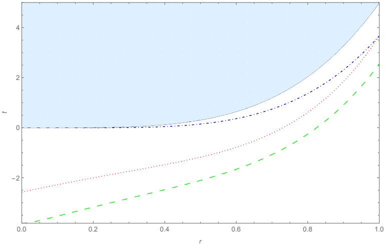
Figure: ν=2



(a). $\zeta = 10.0$, no globally naked singularity.

(b). $\zeta = 18.0$, globally naked singularity.

Figure: ν=3



$\zeta = 5.0$, similar to the OS model, no globally naked singularity.

Figure: ν=4

- The photon surface always emerges regardless that the final state is a black hole or a globally naked singularity,
- and the photon surface always precedes to the event horizon.
- Another point is that the event horizon always follows closely after the photon surface.

Conclusion and discussion

- New definition based on the geometry of codimension-2 surface
- Analytic solution of the photon surface equation in OS model
- The photon surface has some good behavior to mimic the event horizon.
- The definition based on codimension-2 geometry for general spacetime is under consideration
- The photon surface in gravitational collapse in other gravity theories.
- ....

Thanks !