Quantum Mirror Map for Del Pezzo Geometries

arXiv:1908.11396

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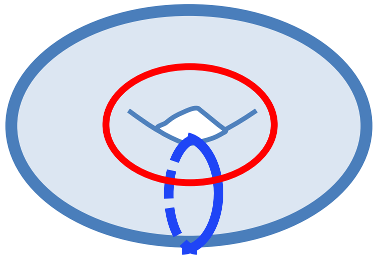
Susy. Gauge Theory



encoded in algebraic curve

e.g.) D5 del Pezzo $$0 = \left[\left(Q^{1/2} + Q^{-1/2}\right)\left(P^{1/2} + P^{-1/2}\right)\right]^2 - z/\alpha$$

A-period

Mirror Map $\Pi_A(z) \sim c_1 z + c_2 z + \ldots$

redefining the variables

not well studied

B-period

Free energy relating to BPS indices

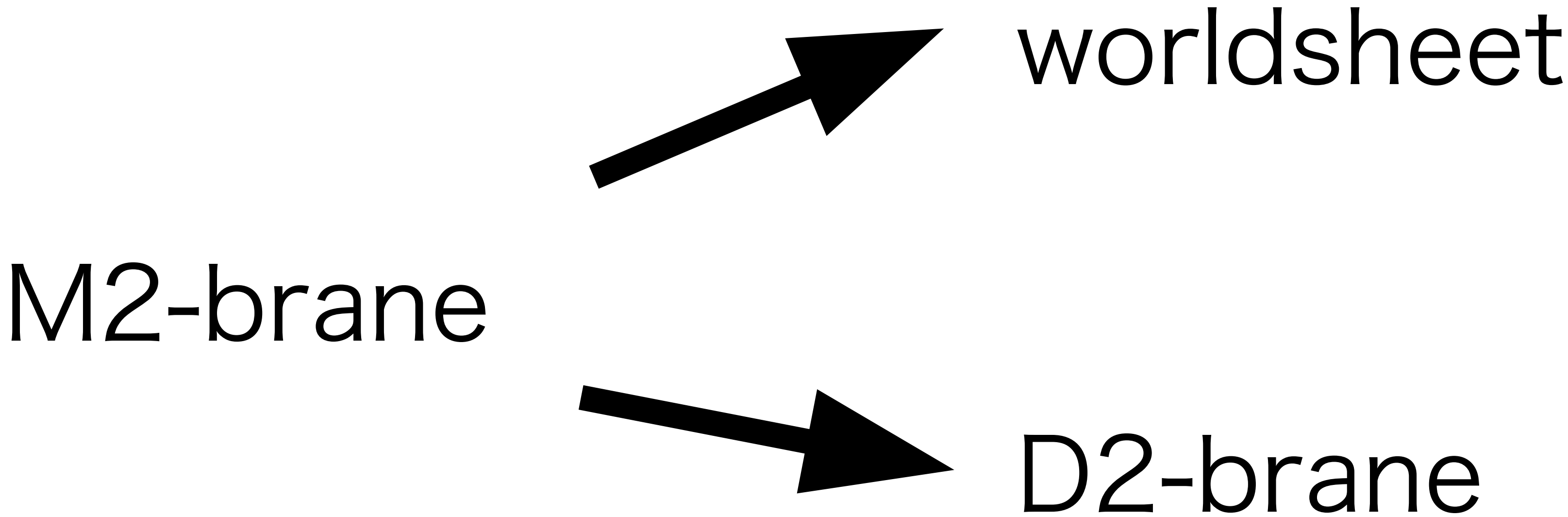
well studied

Susy. Gauge Theory

e.g. Free energy of ABJM

F = (Worldsheet inst.) + (Membrane inst.)

+ (bound state)



Susy. Gauge Theory

e.g. Free energy of ABJM

F = (Worldsheet inst.) + (Membrane inst.)

↑

A-period
(mirror map)

- they are given by (refined) topological string
  (analogous to topological string ver. of AdS/CFT)

- Well known example of AdS/CFT in this case is
  duality between Chern-Simons and topological string

Susy. Gauge Theory

e.g. Free energy of ABJM

F = (Worldsheet inst.) + (Membrane inst.)

↑

A-period
(mirror map)

A-period plays a key role to determine the structure of the theory.

↓

We want to understand the physical meaning of A-period.

Susy. Gauge Theory

e.g. Free energy of ABJM

F = (Worldsheet inst.) + (Membrane inst.)

⬆

A-period
(mirror map)

Recent progress

- Some kinds of ABJM has group theoretical structure coming from symmetry of curve.
(Kubo-Moriyama-Nosaka(2018))

- Free energy corresponds to B-period.
Then, what about A-period ?

Generalized ABJM theory [Honda, Moriyama, 2014]

($U(N)_k \times U(N)_{-k} \times U(N)_k \times U(N)_{-k}$)

encoded in D_5 del Pezzo with quantization

$$\widehat{H} = \left[\left(\widehat{Q}^{1/2} + \widehat{Q}^{-1/2}\right)\left(\widehat{P}^{1/2} + \widehat{P}^{-1/2}\right)\right]^2 - z/\alpha$$

$$\widehat{Q}\widehat{P} = q\widehat{P}\widehat{Q}$$

$$\Pi_A(z) \quad \Longrightarrow \quad \Pi_A(z, \hbar)$$

$$(q = \mathrm{e}^{\mathrm{i}\hbar})$$

Quantum Mirror Map

Generalized ABJM theory [Honda, Moriyama, 2014]
$(U(N)_k \times U(N)_{-k} \times U(N)_k \times U(N)_{-k})$

encoded in D_5 del Pezzo with quantization

$$\widehat{H} = \left[\left(\widehat{Q}^{1/2} + \widehat{Q}^{-1/2}\right)\left(\widehat{P}^{1/2} + \widehat{P}^{-1/2}\right)\right]^2 - z/\alpha$$

$$\widehat{Q}\widehat{P} = q\widehat{P}\widehat{Q}$$

$[SU(2)]^3$ sym.

- $[SU(2)]^3$ comes from breaking of D_5 sym.
- The sym. breaking is observed in B-period

(Kubo-Moriyama-Nosaka(2018))

⬇

We consider quantum mirror map of D_5 del Pezzo

Result

A-period is similar to B-period

- expressed by Weyl characters
- coef. of characters = Integer
- has same reps. as those in B-period

Plan

A-period is similar to B-period

①
- expressed by Weyl characters

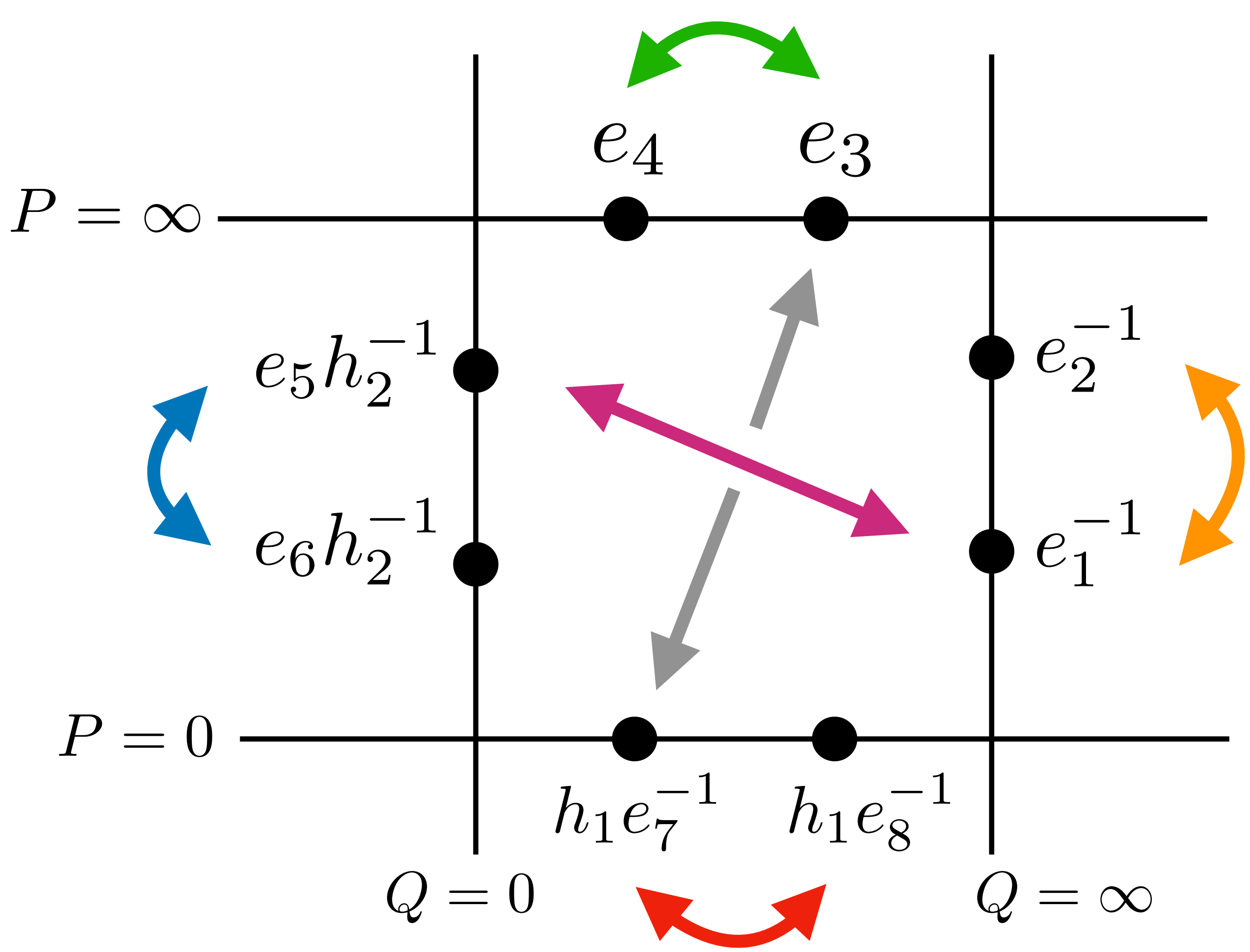
②
- coef. of characters = Integer
- has same reps. as those in B-period

D5 Del Pezzo geometry Kubo-Moriyama-Nosaka(2018)

$$0 = e_3 e_4 Q^{-1} P - (e_3+e_4)P + QP$$
$$- h_2^{-1} e_3 e_4 (e_5+e_6) Q^{-1} - \frac{z}{\alpha} - (e_1^{-1}+e_2^{-1})Q$$
$$+ h_1^2 (e_1 e_2 e_7 e_8)^{-1} Q^{-1} P^{-1} - h_1 (e_1 e_2)^{-1} (e_7^{-1}+e_8^{-1}) P^{-1} + (e_1 e_2)^{-1} Q P^{-1}$$

- $h_1^2 h_2^2 = \prod_{i=1}^{8} e_i$

classical curve



invariant under exchanging of asymptotic values

↓

D5 Weyl sym.

D_5 Del Pezzo geometry Kubo-Moriyama-Nosaka(2018)

$$\frac{\widehat{H}}{\alpha}+\frac{z}{\alpha}= e_3e_4\widehat{Q}^{-1}\widehat{P} -(e_3+e_4)\widehat{P} +\widehat{Q}\widehat{P}$$
$$-h_2^{-1}e_3e_4(e_5+e_6)\widehat{Q}^{-1} +\frac{E}{\alpha} -(e_1^{-1}+e_2^{-1})\widehat{Q}$$
$$+h_1^2(e_1e_2e_7e_8)^{-1}\widehat{Q}^{-1}\widehat{P}^{-1} -h_1(e_1e_2)^{-1}(e_7^{-1}+e_8^{-1})\widehat{P}^{-1} +(e_1e_2)^{-1}\widehat{Q}\widehat{P}^{-1}$$

quantum curve

10 parameters - (2 + 2 + 1) parameters

- 8 asymptotic values determine curve 2
- $(\widehat{P},\widehat{Q})\sim(A\widehat{P},B\widehat{Q})$ 2
- $h_1^2h_2^2=\prod_{i=1}^{8}e_i$ 1

⇒ 5 parameters $(e_1,e_3,e_5,\bar{h}_1,\bar{h}_2)$

$(\bar{h}_1=qh_1,\ \bar{h}_2=q^{-1}h_2)$

Quantum mirror map

Aganagic, Cheng, Dijkgraaf Krefl, Vafa(2011)

$$\Pi_A(z,\hbar) \sim \oint \frac{\log P[X]}{X}\mathrm{d}X = Ez^{-1} + \left(-\frac{E^2}{2} - A_2\right) z^{-2} + \cdots$$

$$P[X] = \frac{\Psi[q^{-1}X]}{\Psi[X]},\ \widehat{Q}\Psi[X] = X\Psi[X] \qquad \left[\frac{\widehat{H}}{\alpha} + \frac{z}{\alpha}\right]\Psi[X] = 0$$

[solve Schrödinger eq. order by order]

$$\frac{A_2}{\alpha^2 q^{-1}} = \frac{e_3}{e_1} + \frac{\overline{h}_1}{e_1} + \frac{\overline{h}_1 e_3}{e_1} + \frac{e_3 e_5}{\overline{h}_2^2} + \frac{e_3 e_5}{\overline{h}_1 \overline{h}_2^2} + \frac{e_3^2 e_5}{\overline{h}_1 \overline{h}_2^2} + \frac{e_3}{\overline{h}_2} + \frac{e_3}{\overline{h}_2 e_1} + \frac{e_3 e_5}{\overline{h}_2} + \frac{e_3 e_5}{\overline{h}_2 e_1}$$

- invariant under Weyl transf.
- contains 10 terms.

Quantum mirror map

$$\Pi_A(z,\hbar) \sim \oint \frac{\log P[X]}{X}\mathrm{d}X = Ez^{-1} + \left(-\frac{E^2}{2} - A_2\right) z^{-2} + \cdots$$

$$\left[P[X] = \frac{\Psi[q^{-1}X]}{\Psi[X]},\ \widehat{Q}\Psi[X] = X\Psi[X] \qquad \left[\frac{\widehat{H}}{\alpha} + \frac{z}{\alpha}\right]\Psi[X] = 0 \right]$$

solve Schrödinger eq. order by order

$$\frac{A_2}{\alpha^2 q^{-1}} = \frac{e_3}{e_1} + \frac{\overline{h}_1}{e_1} + \frac{\overline{h}_1 e_3}{e_1} + \frac{e_3 e_5}{\overline{h}_2^2} + \frac{e_3 e_5}{\overline{h}_1 \overline{h}_2^2} + \frac{e_3^2 e_5}{\overline{h}_1 \overline{h}_2^2} + \frac{e_3}{\overline{h}_2} + \frac{e_3}{\overline{h}_2 e_1} + \frac{e_3 e_5}{\overline{h}_2} + \frac{e_3 e_5}{\overline{h}_2 e_1}$$

$$\alpha = q^{1/2}\overline{h}_2^{1/2} e_1^{1/4} e_3^{-1/2} e_5^{-1/4}$$

$$A_2 = \chi_{\mathbf{10}}$$

Plan

A-period is similar to B-period

✓① ● expressed by Weyl characters

② [
● coef. of characters = Integer
● has same reps. as those in B-period

Multi-covering & BPS

$$\Pi_A(z,\hbar) \sim \sum_{l=1}^{\infty} (-1)^{l+1} A_l z^{-l} \qquad A_4 \ni \frac{3}{2}\chi_{\mathbf{54}} + \frac{5}{2}\chi_{\mathbf{45}} + \frac{11}{2}\chi_{\mathbf{1}}$$

fractional

➡ Does this have multi-covering structure ?

- A-period of A_1 geometry has multi-covering structure. (ABJM theory)
- By this structure, coefficients are integers.

[Hatsuda-Marino-Moriyama-Okuyama(2013)]

multi-covering structure

$$\text{(coef. of n-th order)} = \sum_{j \leq n} \text{(coef. of j-th order)}$$

Multi-covering & BPS

$$\Pi_A(z,\hbar) \;=: \log z_{\rm eff} - \log z$$

$$\Rightarrow \quad \log z \sim -A_1 z_{\rm eff}^{-1} + \left(A_2 - A_1^2\right) z_{\rm eff}^{-2} - \left(A_3 - 3A_1A_2 + \frac{3}{2}A_1^3\right) z_{\rm eff}^{-3} + \left(A_4 - 2A_2^2 - 4A_1A_3 - 8A_1^2A_2 - \frac{8}{3}A_1^4\right) z_{\rm eff}^{-4} + \dots$$

consider inverse function

$$\left[\begin{array}{l} A_1 = 0 \\ A_2 = \chi_{\mathbf{10}} \\ A_3 = (q^{1/2} + q^{-1/2})\chi_{\mathbf{16}} \\ A_4 = \left(q^2 + q^{-2}\right)\chi_{\mathbf{1}} + \left(q^{3/2} + q^{-3/2}\right)(\chi_{\mathbf{45}} + 3\chi_{\mathbf{1}}) + \underline{\dfrac{3\chi_{\mathbf{54}} + 5\chi_{\mathbf{45}} + 11\chi_{\mathbf{1}}}{2}} \end{array}\right.$$

still fractional

Multi-covering & BPS

$$\Pi_A(z,\hbar) =: \log z_{\rm eff} - \log z$$

$$\Rightarrow \log z \sim -A_1 z_{\rm eff}^{-1} + \left(A_2 - A_1^2\right) z_{\rm eff}^{-2} - \left(A_3 - 3A_1A_2 + \frac{3}{2}A_1^3\right) z_{\rm eff}^{-3} + \left(A_4 - 2A_2^2 - 4A_1A_3 - 8A_1^2A_2 - \frac{8}{3}A_1^4\right) z_{\rm eff}^{-4} + \dots$$

(red) $= \epsilon_1(q, e_i, \bar{h}_i)$

(blue) $= -\epsilon_2(q, e_i, \bar{h}_i) + \frac{\epsilon_1(q^2, e_i^2, \bar{h}_i^2)}{2}$

(green) $= \epsilon_3(q, e_i, \bar{h}_i) + \frac{\epsilon_1(q^3, e_i^3, \bar{h}_i^3)}{3}$

(orange) $= -\epsilon_4(q, e_i, \bar{h}_i) + \frac{\epsilon_2(q^2, e_i^2, \bar{h}_i^2)}{2} + \frac{\epsilon_1(q^4, e_i^4, \bar{h}_i^4)}{4}$

multi-covering structure

$$\epsilon_1 = 0, \quad -\epsilon_2 = \chi_{\mathbf{10}},$$

$$\epsilon_3 = \left(q^{1/2} + q^{-1/2}\right)\chi_{\mathbf{16}}, \; -\epsilon_4 = \left(q^2 + q^{-2}\right)\chi_{\mathbf{1}} + \left(q + q^{-1}\right)\left(\chi_{\mathbf{45}} + 3\chi_{\mathbf{1}}\right) + 4\chi_{\mathbf{1}}$$

integer !

Multi-covering & BPS

[Huang-Klemm-Poretschkin(2013)]
[Moriyama-Nosaka-Yano(2017)]

d	$(j_{\rm L}, j_{\rm R})$	BPS	$(-1)^{d-1}\sum_{\lvert\boldsymbol{d}\rvert=1}(N^{\boldsymbol{d}}_{j_{\rm L},j_{\rm R}})_{d^+-d^-}$	representations
1	$(0;0)$	16	$8_{+1}+8_{-1}$	$\mathbf{16}$
2	$(0,\frac{1}{2})$	10	$1_{+2}+8_0+1_{-2}$	$\mathbf{10}$
3	$(0,1)$	16	$8_{+1}+8_{-1}$	$\mathbf{16}$
4	$(0,\frac{1}{2})$	1	1_0	$\mathbf{1}$
	$(0,\frac{3}{2})$	45	$8_{+2}+29_0+8_{-2}$	$\mathbf{45}$
	$(\frac{1}{2},2)$	1	1_0	$\mathbf{1}$

A-period

B-period

$$\begin{cases} \epsilon_1 = 0, \\ -\epsilon_2 = \chi_{\mathbf{10}}, \\ \epsilon_3 = (q^{\frac{1}{2}}+q^{-\frac{1}{2}})\chi_{\mathbf{16}}, \\ -\epsilon_4 = (q^2+q^{-2})\chi_{\mathbf{1}} + (q+q^{-1})(\chi_{\mathbf{45}}+3\chi_{\mathbf{1}}) + 4\chi_{\mathbf{1}}, \end{cases}$$

same reps. as B-period for each order

Physical meaning ?? ➡ Future work

Summary

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A-period is similar to B-period

- expressed by Weyl characters
- coef. of characters = Integer
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Appendix

before using SU(2) character

$$\epsilon_7 = (q^{\frac{11}{2}} + q^{-\frac{11}{2}})\chi_{\mathbf{16}} + (q^{\frac{9}{2}} + q^{-\frac{9}{2}})(\chi_{\mathbf{144}} + 4\chi_{\mathbf{16}}) + (q^{\frac{7}{2}} + q^{-\frac{7}{2}})(\chi_{\mathbf{560}} + 4\chi_{\mathbf{144}} + 13\chi_{\mathbf{16}})$$
$$+ (q^{\frac{5}{2}} + q^{-\frac{5}{2}})(\chi_{\mathbf{720}} + 4\chi_{\mathbf{560}} + 9\chi_{\mathbf{144}} + 25\chi_{\mathbf{16}}) + (q^{\frac{3}{2}} + q^{-\frac{3}{2}})(3\chi_{\mathbf{560}} + 8\chi_{\mathbf{144}} + 27\chi_{\mathbf{16}})$$
$$+ (q^{\frac{1}{2}} + q^{-\frac{1}{2}})(\chi_{\mathbf{720}} + 3\chi_{\mathbf{560}} + 9\chi_{\mathbf{144}} + 27\chi_{\mathbf{16}}),$$

---

after using SU(2) character

$$\epsilon_7 = \chi_{\frac{11}{2}}\chi_{\mathbf{16}} + \chi_{\frac{9}{2}}(\chi_{\mathbf{144}} + 3\chi_{\mathbf{16}}) + \chi_{\frac{7}{2}}(\chi_{\mathbf{560}} + 3\chi_{\mathbf{144}} + 9\chi_{\mathbf{16}})$$
$$+ \chi_{\frac{5}{2}}(\chi_{\mathbf{720}} + 3\chi_{\mathbf{560}} + 5\chi_{\mathbf{144}} + 12\chi_{\mathbf{16}}) + \chi_{\frac{3}{2}}(-\chi_{\mathbf{720}} - \chi_{\mathbf{560}} - \chi_{\mathbf{144}} + 2\chi_{\mathbf{16}})$$
$$+ \chi_{\frac{1}{2}}(\chi_{\mathbf{720}} + \chi_{\mathbf{144}}),$$

χ_n SU(2) character

much simpler for q-dependent term & coef.

but negative coef,

without overall minus sign do not appear

$$\epsilon_4 = (q^2+q^{-2})\,\chi_{\mathbf{1}} + (q+q^{-1})\,(\chi_{\mathbf{45}}+3\chi_{\mathbf{1}}) + (-\chi_{\mathbf{54}}+\chi_{\mathbf{45}}+3\chi_{\mathbf{1}})$$

with overall minus sign

$$-\epsilon_4 = (q^2+q^{-2})\chi_{\mathbf{1}} + (q+q^{-1})(\chi_{\mathbf{45}}+3\chi_{\mathbf{1}}) + 4\chi_{\mathbf{1}}$$

4	$(0,\frac{1}{2})$	1	1_0		$\mathbf{1}$	
	$(0,\frac{3}{2})$	45	$8_{+2}+29_0+8_{-2}$		$\mathbf{45}$	
	$(\frac{1}{2},2)$	1	1_0		$\mathbf{1}$	

multi-covering structure

(coef. of n-th order) = $\sum_{j\leqq n}$ (coef. of j-th order)

- $(e_1, e_3, e_5, \bar{h}_1, \bar{h}_2) = (1, 1, 1, q, q^{-1})$ (2,2) model

  $= (q^{-1/2}, q^{1/2}, q^{-1/2}, q, q^{-1})$ (1,1,1,1) model

$$\mu = \log z, \quad \mu_{\text{eff}} = \log z_{\text{eff}}$$

(effective) chemical pot.

= log [(effective) complex modulus]

$$\Xi_k(z) = \sum_{N=0}^{\infty} z^N \underline{Z_k(N)} = \text{Det}\left[1 + z\widehat{H}^{-1}\right],$$

Partition fn. of N M2-branes

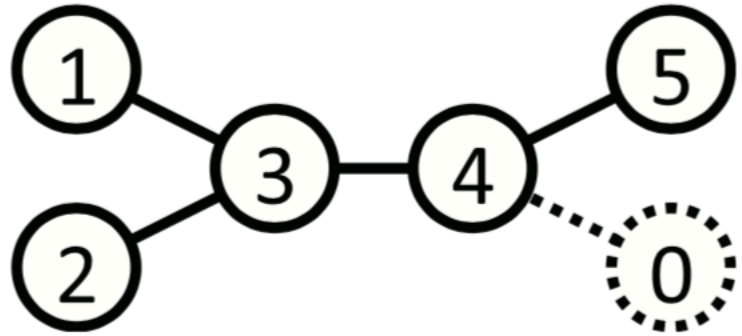
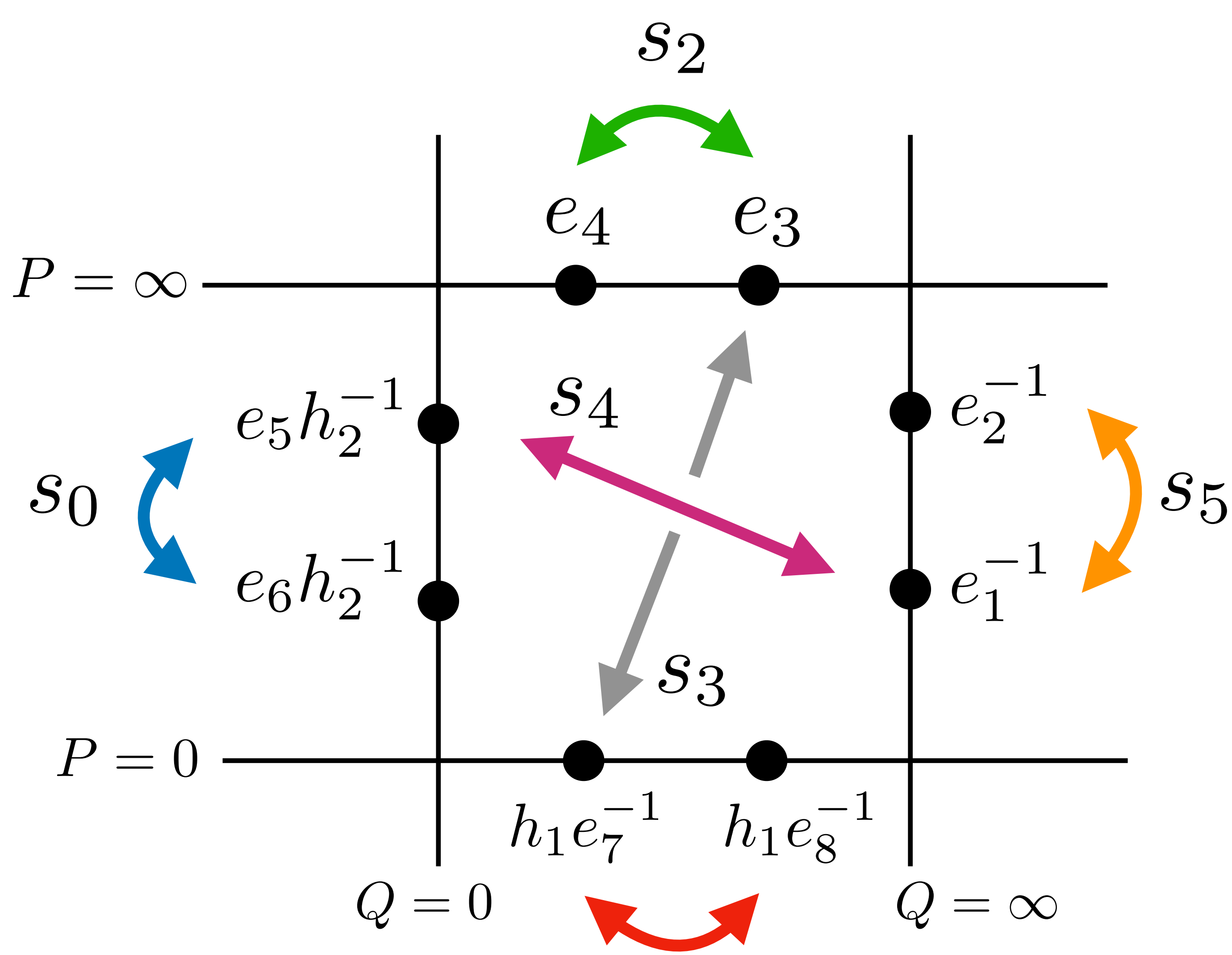


Figure 3: The Dynkin diagram of the D_5 algebra.

from [Kubo,Moriyama,Nosaka(2018)]



$$s_1 : (\overline{h}_1, \overline{h}_2, e_1, e_3, e_5; \alpha) \mapsto \left(\frac{e_1 e_3 e_5}{\overline{h}_1 \overline{h}_2^2}, \overline{h}_2, e_1, e_3, e_5; \alpha \right),$$

$$s_2 : (\overline{h}_1, \overline{h}_2, e_1, e_3, e_5; \alpha) \mapsto \left(\frac{\overline{h}_1}{e_3}, h_2, e_1, \frac{1}{e_3}, e_5; e_3 \alpha \right),$$

$$s_3 : (\overline{h}_1, \overline{h}_2, e_1, e_3, e_5; \alpha) \mapsto \left(\overline{h}_1, \frac{e_1 e_5}{\overline{h}_1 \overline{h}_2}, e_1, \frac{e_1 e_3 e_5}{\overline{h}_1 \overline{h}_2^2}, e_5; \alpha \right),$$

$$s_4 : (\overline{h}_1, \overline{h}_2, e_1, e_3, e_5; \alpha) \mapsto \left(\frac{\overline{h}_1 \overline{h}_2}{e_1 e_5}, \overline{h}_2, \frac{\overline{h}_2}{e_5}, e_3, \frac{\overline{h}_2}{e_1}; \alpha \right),$$

$$s_5 : (\overline{h}_1, \overline{h}_2, e_1, e_3, e_5; \alpha) \mapsto \left(\overline{h}_1, \frac{\overline{h}_2}{e_1}, \frac{1}{e_1}, e_3, e_5; \frac{\alpha}{e_1} \right),$$

Susy. Gauge Theory

e.g. Free energy of ABJM

F = (Worldsheet inst.) + (Membrane inst.)

poles poles

+ (bound state)

poles

No pole

e.g. Quantum dilogarithm

$$\mathrm{i}\log\Phi \sim \sum_{d=1}^{\infty} \frac{(-1)^d Q^d}{2d\sin\left(\frac{dk}{2}\right)} + \sum_{d=1}^{\infty} \frac{(-1)^d Q^{2\pi d/k}}{2d\sin\left(\frac{2\pi^2 d}{k}\right)}$$

poles poles

No pole