Generalized Uncertainty Principle and Related Cosmological Constant and Black Hole Thermodynamics

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(arXiv:1312.4118, arXiv:1407.xxxx)

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Main content

1. The GUP approach
2. Improved exponential GUP
3. The cosmological constant problem
4. Attempt to solve the cosmological constant problem
5. Black hole thermodynamics
6. Black hole evaporation
7. The ultimate case $n \to \infty$
8. Conclusion

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The GUP approach

- The Generalized Uncertainty Principle (GUP)
  - Motivation: the perturbative string theory and loop quantum gravity require the existence of the fundamental length scale that is around the Planck length $\ell_{\rm Pl} = \sqrt{G\hbar/c^3} \sim 10^{-33}\ cm$.
  (G. Veneziano, Europhys. Lett. **2** (1986) 199; L.J. Garay, Int. J. Mod. Phys. A **10** (1995) 145.)
  - Method: to deform the **Heisenberg uncertainty principle (HUP)** in such a way that it gives rise to a minimal length. $(\Delta X)_{\rm Min} \sim \ell_{\rm Pl}$ in order to present quantum gravity effects. (See two examples below)
  - Purpose: GUP provides a phenomenological description of quantum gravity effects. The generalization from HUP to GUP is in fact equivalent to the introduction of gravitational effects, i.e., G is included in $\ell_{\rm Pl}$.

The GUP approach

- UV/IR Mixing (caused by GUP)
  - Meaning: A large ΔP (UV) corresponds to a large ΔX (IR). [**Heisenberg uncertainty principle**: $(\Delta X)(\Delta P) \geq \frac{\hbar}{2}$, A large ΔP (UV) corresponds to a small ΔX (UV).]
  - Example 1: $(\Delta X)(\Delta P) \geq \frac{\hbar}{2}\{1+\beta(\Delta P)^2\}$, $(\Delta X)_{\rm Min} = \hbar\sqrt{\beta}$.
  - Example 2: $(\Delta X)(\Delta P) \geq \frac{\hbar}{2} e^{\beta(\Delta P)^2}$, $(\Delta X)_{\rm Min} = \hbar\sqrt{e\beta/2}$.
- UV/IR Mixing-related physics
  - AdS/CFT correspondence (arXiv:hep-th/9805114, ...)
  - Noncommutative field theory (arXiv:hep-th/0106048, ...)
  - Quantum gravity in asymptotically de Sitter spaces (arXiv:hep-th/9806146, ...)
  - Inflationary cosmology (arXiv:astro-ph/0009209, ...)
  - $\cdots$

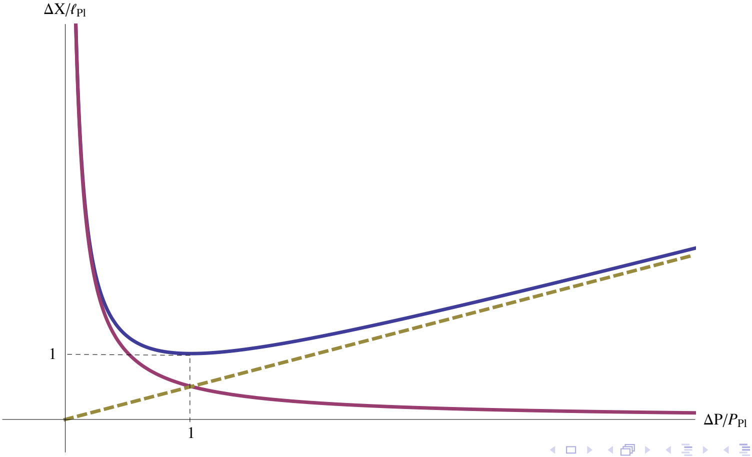
The GUP approach

The ΔP-dependence of the minimal length under the quadratic GUP:

$(\Delta X)(\Delta P) \geq \frac{\hbar}{2}\{1 + \beta(\Delta P)^2\}$, $(\Delta X)_{\rm Min} = \hbar\sqrt{\beta}$ (blue curve).

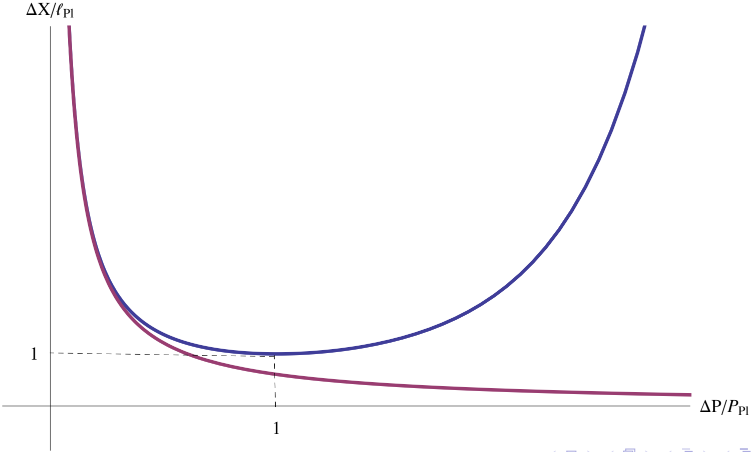
The bound for usual Heisenberg relation: $\Delta X \geq \hbar / (2\Delta P)$ (violet curve).

The asymptotic line: $\Delta X = (\hbar\beta/2)\,\Delta P$ (brown line). $(\hbar = 1, \beta = 0.01)$



The GUP approach

The ΔP-dependence of the minimal length under the exponential GUP:
$(\Delta X)(\Delta P) \geq \frac{\hbar}{2} e^{\beta(\Delta P)^2}$, $(\Delta X)_{\rm Min} = \hbar\sqrt{e\beta/2}$ (blue curve).
The bound for usual Heisenberg relation: $\Delta X \geq \hbar/(2\Delta P)$ (violet curve).
$(\hbar = 1, \beta = 0.01)$



- Example 1: A quadratic GUP

(A. Kempf, G. Mangano, and R.B. Mann, Phys. Rev. D **52**, 1108 (1995) [arXiv:hep-th/9412167])

  - A simplest generalization (1-dimensional space)

$$[\hat{X}, \hat{P}] = i\hbar(1 + \beta \hat{P}^2). \qquad (1)$$

The minimal length: $(\Delta X)_{\text{Min}} = \hbar\sqrt{\beta}$. ($(\Delta P) = 1/\sqrt{\beta}$)

  - High dimensional generalization $(i, j = 1, 2, \cdots, D)$

$$[\hat{X}_i, \hat{P}_j] = i\hbar(1 + \beta \hat{P}^2)\delta_{ij}, \qquad [\hat{P}_i, \hat{P}_j] = 0, \qquad (2)$$

$$[\hat{X}_i, \hat{X}_j] \neq 0, \quad (\textit{from the Jacobi identity}) \qquad (3)$$

where $\hat{P}^2 \equiv \sum_{i=1}^{D} \hat{P}_i^2$.

  - Invariant phase space volume under time evolution

$$\frac{1}{(1 + \beta \mathbf{p}^2)^D} d^D\mathbf{x} d^D\mathbf{p}. \qquad (4)$$

- Example 1: A quadratic GUP

(A. Kempf, G. Mangano, and R.B. Mann, Phys. Rev. D **52**, 1108 (1995) [arXiv:hep-th/9412167])

  - The density of states (3-dimensional space):

$$\frac{1}{(2\pi\hbar)^3}\frac{1}{(1+\beta\mathbf{p}^2)^3}d^3\mathbf{p}, \qquad (5)$$

  - The corresponding cosmological constant:

$$\Lambda = \frac{1}{(2\pi\hbar)^3}\int_0^\infty \frac{4\pi p^2 dp}{(1+\beta p^2)^3}\left[\frac{1}{2}pc\right] = \frac{\hbar c}{16\pi^2}\frac{1}{(\hbar\sqrt{\beta})^4}. \qquad (6)$$

If $(\Delta X)_{\mathrm{Min}} = \hbar\sqrt{\beta} \sim \ell_{\mathrm{Pl}}$,
then $\Lambda \sim \Lambda_{\mathrm{QFT}} \sim 10^{74}\ GeV^4/(\hbar^3 c^3)$.

The cosmological constant problem NOT solved, but only a natural UV cutoff ($1/\sqrt{\beta}$) provided.

(L.N. Chang, et al., Phys. Rev. D **65**, 125028 (2002) [arXiv:hep-th/0201017])

- Example 2: An exponential GUP

(Kh. Nouicer, Phys. Lett. B **646**, 63 (2007) [arXiv:0704.1261 [gr-qc]])

  - Form in 1-dimensional space

$$[\hat{X}, \hat{P}] = i\hbar\, e^{\beta \hat{P}^2}. \tag{7}$$

The minimal length: $(\Delta X)_{\mathrm{Min}} = \hbar\sqrt{e\beta/2}$. $((\Delta P) = 1/\sqrt{2\beta})$

  - High dimensional generalization $(i, j = 1, 2, \cdots, D)$

$$[\hat{X}_i, \hat{P}_j] = i\hbar e^{\beta \hat{P}^2}\delta_{ij}, \qquad [\hat{P}_i, \hat{P}_j] = 0, \tag{8}$$

$$[\hat{X}_i, \hat{X}_j] \neq 0. \quad (\textit{from the Jacobi identity}) \tag{9}$$

  - Invariant phase space volume under time evolution

$$e^{-D\beta \mathbf{p}^2} d^D\mathbf{x} d^D\mathbf{p}. \tag{10}$$

- Example 2: An exponential GUP

(Kh. Nouicer, Phys. Lett. B **646**, 63 (2007) [arXiv:0704.1261 [gr-qc]])

  - The density of states (3-dimensional space):

$$\frac{1}{(2\pi\hbar)^3} e^{-3\beta \mathbf{p}^2} d^3\mathbf{p}, \tag{11}$$

  - The corresponding cosmological constant:

$$\Lambda = \frac{1}{(2\pi\hbar)^3} \int_0^\infty e^{-3\beta p^2} 4\pi p^2 dp \left[\frac{1}{2} pc\right] = \frac{\hbar c}{72\pi^2} \frac{1}{(\hbar\sqrt{\beta})^4}. \tag{12}$$

If $(\Delta X)_{\mathrm{Min}} = \hbar\sqrt{e\beta/2} \sim \ell_{\mathrm{Pl}}$,
then $\Lambda \sim \frac{e^2}{18}\Lambda_{\mathrm{QFT}} \sim \frac{2}{5} \times 10^{74}\, GeV^4/(\hbar^3 c^3)$.

The cosmological constant problem NOT solved, but only a natural UV cutoff $(1/\sqrt{3\beta})$ provided.

(Kh. Nouicer and M. Debbabi, Phys. Lett. A **361**, 305 (2007))

- Conclusion from the above two GUPs
  - **Neither the quadratic GUP nor the exponential GUP can solve the cosmological problem!**
  - **A new GUP is needed to provide enough suppression to the energy density of the vacuum!**

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Improved exponential GUP

- General rules to construct a GUP (D-dimensional space)

  (L.N. Chang, et al., Adv. High Energy Phys. **2011**, 493514 (2011) [arXiv:1106.0068 [hep-th]].)

  - Rotational (Lorentz) symmetry is maintained;
  - Commutators between $\hat{X}_i$ and $\hat{P}_j$ only depend only $\hat{P}_k$;
  - Commutators between $\hat{P}_i$ and $\hat{P}_j$ vanish;
  - Commutators between $\hat{X}_i$ and $\hat{X}_j$ are derived using the Jacobi identity.

- A general GUP

$$\frac{1}{i\hbar}[\hat{X}_i, \hat{P}_j] = A\left(\hat{P}^2\right)\delta_{ij} + B\left(\hat{P}^2\right)\hat{P}_i\hat{P}_j, \tag{13}$$

$$[\hat{P}_i, \hat{P}_j] = 0, \tag{14}$$

$$[\hat{X}_i, \hat{X}_j] \neq 0, \quad (\textit{from the Jacobi identity}) \tag{15}$$

where $\hat{P}^2 \equiv \sum_{i=1}^{D} \hat{P}_i^2$.

Improved exponential GUP

- Our improved exponential GUP (1-dimensional space)

$$[\hat{X}, \hat{P}] = i\hbar\, e^{\beta^n \hat{P}^{2n}}, \qquad (n: \ \text{interger}). \tag{16}$$

n is the suppressing index (to suppress the contributions of high momentum states to the energy density of the vacuum).

- The minimal length

$$(\Delta X)_{\rm Min} = \frac{\hbar\sqrt{\beta}}{2}(2ne)^{1/(2n)}, \tag{17}$$

when ΔP takes the critical value,

$$\Delta P = P_{\rm Crit} \equiv \left(\frac{1}{2n}\right)^{1/(2n)} \frac{1}{\sqrt{\beta}}. \tag{18}$$

- Property

For any n, $(\Delta X)_{\rm Min} \sim \hbar\sqrt{\beta} \sim \ell_{\rm Pl}$, and $P_{\rm Crit} \sim P_{\rm Pl} = M_{\rm Pl}\, c$.

Improved exponential GUP

- Natural generalization to D-dimensional space

$$[\hat{X}_i, \hat{P}_j] = i\hbar\, e^{\beta^n \hat{P}^{2n}} \delta_{ij}, \qquad [\hat{P}_i, \hat{P}_j] = 0, \tag{19}$$

where $\hat{P}^2 \equiv \sum_{i=1}^{D} \hat{P}_i^2$.
Using the Jacobi identity, we have

$$[\hat{X}_i, \hat{X}_j] = 2i\hbar n \beta^n \hat{P}^{2(n-1)} e^{\beta^n \hat{P}^{2n}} (\hat{P}_i \hat{X}_j - \hat{P}_j \hat{X}_i). \tag{20}$$

- The analog of the Liouville theorem

In the classical limit, $\frac{1}{i\hbar}[\hat{A}, \hat{B}] \to \{A, B\}$, we search for the invariant phase space volume under time evolution.

- The invariant phase space volume
  The analog of the Liouville theorem for the improved exponential GUP (eqs. (19) and (20)) in D dimensions gives the weighted phase space volume,

$$e^{-D\beta^n P^{2n}} \mathrm{d}^D \mathbf{X} \mathrm{d}^D \mathbf{P}. \tag{21}$$

  which is invariant under time evolution.
- The density of states in momentum space

$$\frac{1}{(2\pi\hbar)^D} e^{-D\beta^n P^{2n}} \mathrm{d}^D \mathbf{P}. \tag{22}$$

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The cosmological constant problem

- 1917 introduced by Einstein
  The cosmological constant was introduced to achieve a static universe solution (Sitzungsber. Preuss. Akad. Wiss. Phys. - Math. Kl. 142 (1917).
- 1929 abandoned by Einstein due to Hubble's Discovery
  Discovery of the expansion of the universe made it be abandoned (E.P. Hubble, Proc. Natl. Acad. Sci. **15**, 168 (1929)).
- 1998 Revived from further observations of distance-redshift relations from the Type Ia supernovae
  The accelerating expansion of the universe requires a positive cosmological constant in order to produce a negative pressure (A.G. Riess, et al., Astron. J. **116**, 1009 (1998) [arXiv:astro-ph/9805201]; S. Perlmutter, et al., Astrophys. J. **517**, 565 (1999) [arXiv:astro-ph/9812133]).

The cosmological constant problem

- 2013 The most recent observed value

  (Planck Collaboration, arXiv:1303.5076 [astro-ph.CO])

$$\Lambda_{\mathrm{Observed}} \sim 10^{-47}\, GeV^4/(\hbar^3 c^3). \tag{23}$$

- QFT calculus
  - The cosmological constant problem is regarded as the energy density of the vacuum.($\Lambda = \frac{8\pi G_N}{c^2}\rho_{vac}$)
  - The energy density of the vacuum is the sum of the zero-point energy of each oscillator over all momentum states per unit space volume. For a massless particle, one finds that

$$\begin{aligned} \Lambda_{\mathrm{QFT}} &= \frac{1}{(2\pi\hbar)^3}\int_0^{\hbar/\ell_{\mathrm{Pl}}} 4\pi p^2 dp \left[\frac{1}{2}pc\right] \\ &= \frac{\hbar c}{16\pi^2}\frac{1}{\ell_{\mathrm{Pl}}^4} \sim 10^{74}\, GeV^4/(\hbar^3 c^3). \end{aligned} \tag{24}$$

- The cosmological constant problem:

$$\frac{\Lambda_{\rm QFT}}{\Lambda_{\rm Observed}} \sim 10^{120}. \tag{25}$$

120 orders of the magnitude larger than the observed value

- Some approaches to solve the problem
  - Supersymmetry, supergravity, and superstrings
  - Anthropic consideration
  - Adjustment mechanisms
  - Changing gravity
  - Quantum cosmology
  - $\cdots$

(S. Weinberg, Rev. Mod. Phys. **61**, 1 (1989); S.M. Carroll, Living Rev. Reletiv. **3**, 1 (2001) [arXiv:astro-ph/0004075]).

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Attempt to solve the cosmological constant problem

- The cosmological constant (3-dimensional space)

$$\Lambda = \frac{1}{(2\pi\hbar)^3}\int_{P_{\rm Crit}}^{\infty} 4\pi P^2 dP\, e^{-3\beta^n P^{2n}} \left[\frac{1}{2}Pc\right]. \qquad (26)$$

(Y.-G. Miao, Y.-J. Zhao, Int. J. Mod. Phys. D **23** (2014) 1450062 [arXiv:1312.4118 [hep-th]])

- To determine $P_{\rm Crit}$ by the UV/IR mixing
  - The critical momentum $P_{\rm Crit}$: the measurement precision of momentum corresponding to $(\Delta X)_{\rm Min}$.
    Sub-Planckian modes: Momentum states with $P < P_{\rm Crit}$.
    Trans-Planckian modes: Momentum states with $P > P_{\rm Crit}$.
    (L.N. Chang, et al., Adv. High Energy Phys. **2011**, 493514 (2011) [arXiv:1106.0068 [hep-th]])
  - Only Trans-Planckian modes ($P > P_{\rm Crit}$) contribute to Λ.
    - Due to the UV/IR mixing, the Sub-Planckian modes are canceled by the Trans-Planckian modes.
    - This phenomenon can be explained by introducing a dynamical connection between the two parts of momentum states. (T. Banks, Int. J. Mod. Phys. A **16**, 910 (2001))

- More detailed explanation

(L.N. Chang, et al., Adv. High Energy Phys. **2011**, 493514 (2011) [arXiv:1106.0068 [hep-th]])

  - Heisenberg Uncertainty Principle
    States at different momentum scales are independent, and their total effect on the vacuum energy is the simple sum of their individual contributions. That is, the momentum integral is from zero to cutoff.
    The usual Heisenberg relation $(\Delta X)(\Delta P) = \hbar/2$ is a simple consequence of the fact that coordinate and momentum spaces are Fourier transforms of each other. The more one wishes to localize a wave packet in coordinate space (smaller ΔX), the more momentum states one must superimpose (larger ΔP). In the usual case, there is no lower bound to ΔX one may localize the wave packet as much as one likes by simply superimposing states with ever larger momentum, and thus ever shorter wavelength, to cancel out the tails of the coordinate space distributions.

- More detailed explanation

  (L.N. Chang, et al., Adv. High Energy Phys. **2011**, 493514 (2011) [arXiv:1106.0068 [hep-th]])

  - Generalized Uncertainty Principle
    If one keeps on superimposing states with momenta beyond $P_{\rm Crit}$, then ΔX ceases to decrease and starts increasing instead. The natural interpretation of such a phenomenon would be that the trans-Planckian modes when superimposed with the sub-Planckian modes would "jam" the sub-Planckian modes and prevent them from canceling out the tails of the wave-packets effectively. The mechanism we envision here is analogous to the "jamming" behavior seen in nonequilibrium statistical physics, in which systems are found to freeze with increasing temperature. In fact, it has been argued that such "freezing by heating" could be characteristic of a background-independent quantum theory of gravity.

- The critical momentum & the cosmological constant
  - The critical momentum $P_{\rm Crit}$ In $D=3$ dimensions, our improved exponential GUP gives

$$P_{\rm Crit} \approx \left(\frac{1}{2n}\right)^{1/(2n)} \frac{1}{\sqrt{\beta}}. \tag{27}$$

  - The energy density of the vacuum is

$$\begin{aligned}\Lambda &\approx \frac{1}{(2\pi\hbar)^3}\int_{\left(\frac{1}{2n}\right)^{1/(2n)}\frac{1}{\sqrt{\beta}}}^{\infty} 4\pi P^2 dP\, e^{-3\beta^n P^{2n}} \left[\frac{1}{2}Pc\right] \\ &= \frac{\hbar c}{8\pi^2}\frac{1}{\left(\hbar\sqrt{\beta}\right)^4}\frac{1}{3^{2/n}n}\Gamma\left(\frac{2}{n},\frac{3}{2n}\right),\end{aligned} \tag{28}$$

where the upper incomplete gamma function is defined as

$$\Gamma(s,z) = \int_z^{\infty} \mathrm{d}t\, t^{s-1}\, e^{-t}. \tag{29}$$

- The coefficient & the suppressing index
  - The coefficient
    Considering $(\Delta X)_{\rm Min} \sim \hbar\sqrt{\beta} \sim \ell_{\rm Pl} = \hbar/\left(M_{\rm Pl}\, c\right)$, the coefficient

$$\frac{\hbar c}{8\pi^2}\frac{1}{(\hbar\sqrt{\beta})^4} \sim \frac{1}{8\pi^2}\frac{(M_{\rm Pl}c^2)^4}{\hbar^3 c^3} \sim 10^{74}\ GeV^4/(\hbar^3 c^3). \tag{30}$$

  - The suppressing index
    Requiring $\mathbf{\Lambda} = \mathbf{\Lambda}_{\rm Observed} \sim \mathbf{10^{-47}\ GeV^4}/(\hbar^3\mathbf{c^3})$, we obtain

$$\frac{1}{3^{2/n}n}\,\Gamma\left(\frac{2}{n}, \frac{3}{2n}\right) \sim 10^{-121}. \tag{31}$$

In the following we estimate the suppressing index n, which will be very large.

Attempt to solve the cosmological constant problem

- Estimation of the suppressing index
  - Using the asymptotic behavior of the upper incomplete gamma function,
$$\lim_{s\to 0^+} \Gamma(s,z) = E_1(z), \tag{32}$$
where $E_1(z)$ is the Expintegral function that is defined by
$$E_1(z) = \int_1^\infty \mathrm{d}t\, t^{-1}\, e^{-zt}. \tag{33}$$
  - Using the series of the Expintegral function,
$$E_1(z) = -\gamma - \ln z - \sum_{k=1}^{\infty} \frac{(-1)^k z^k}{k!\, k}, \tag{34}$$
we see $z\, E_1(z) \sim z$ when z is very small.
  - Therefore, we estimate
$$\frac{1}{3^{2/n} n} \Gamma\left(\frac{2}{n}, \frac{3}{2n}\right) \sim \frac{1}{n} \sim 10^{-121} \Longrightarrow n \sim 10^{121}. \tag{35}$$

- Final result

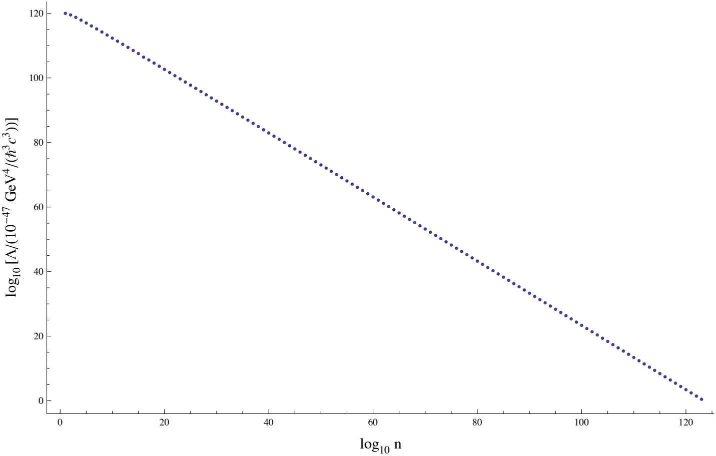
  A more precise estimation finally gives

$$n \sim 10^{123}. \tag{36}$$

Consequently, the cosmological constant problem could be solved based on the improved exponential GUP with a very large suppressing index n.

Attempt to solve the cosmological constant problem

The relation between the cosmological constant Λ and suppressing index n



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Relationship between GUP and BH thermodynamics

- GUP

  $\longrightarrow$ Minimal length (L_{Pl})

  $\longrightarrow$ Natural UV-cutoff

  $\longrightarrow$ Corrections to BH thermodynamic parameters

  - GUP prevents BH from complete evaporation (similar to that HUP prevents the hydrogen atom from total collapse)
  - At the end of Hawking radiation, BH remnant with zero entropy, zero heat capacity, and a finite non-zero temperature (BH remenant — a possible candidate of dark matter)
  - GUP can explain sub-leading logarithmic correction of BH entropy

- Three GUP forms (1-dimensional space)
  - The quadratic GUP (GUP)

    (A. Kempf, G. Mangano, and R.B. Mann, Phys. Rev. D **52**, 1108 (1995) [arXiv:hep-th/9412167])

$$[\hat{X}, \hat{P}] = i\hbar\left(1 + \frac{\alpha^2 L_{Pl}^2}{\hbar^2}\hat{P}^2\right). \tag{37}$$

  - The exponential GUP (GUP^*)

    (Kh. Nouicer, Phys. Lett. B **646**, 63 (2007) [arXiv:0704.1261 [gr-qc]])

$$[\hat{X}, \hat{P}] = i\hbar \exp\left(\frac{\alpha^2 L_{Pl}^2}{\hbar^2}\hat{P}^2\right). \tag{38}$$

  - The improved exponential GUP ($\mathrm{GUP}^*_\mathrm{n}$)

    (Y.-G. Miao, Y.-J. Zhao, Int. J. Mod. Phys. D **23**, 1450062 (2014) [arXiv:1312.4118 [hep-th]])

$$[\hat{X}, \hat{P}] = i\hbar \exp\left(\frac{\alpha^{2n} L_{Pl}^{2n}}{\hbar^{2n}}\hat{P}^{2n}\right). \tag{39}$$

We focus on the improved exponential GUP

$$[\hat{X}, \hat{P}] = i\hbar \exp\left(\frac{\alpha^{2n} {L_{Pl}}^{2n}}{\hbar^{2n}} \hat{P}^{2n}\right),$$

α : A dimensionless constant describing the gravitational effect.
L_{Pl} : Planck length.
n : A positive integer.

- The minimal length

$$(\Delta X)_{Min} = \frac{\alpha L_{Pl}}{2}(2ne)^{\frac{1}{2n}}, \tag{40}$$

and the critical momentum

$$(\Delta P)_{Crit} \equiv \left(\frac{1}{2n}\right)^{\frac{1}{2n}} \frac{\hbar}{\alpha L_{Pl}}. \tag{41}$$

- The uncertainty relation is

$$(\Delta X)(\Delta P) \geq \frac{\hbar}{2} \exp\left(\frac{\alpha^{2n} {L_{Pl}}^{2n}}{\hbar^{2n}} (\Delta P)^{2n}\right), \tag{42}$$

where $\langle P \rangle = 0$.
Considering the saturate situation and setting $W(u) \equiv -2n \left(\frac{\alpha L_{Pl}}{\hbar}\right)^{2n} (\Delta P)^{2n}$ and $u \equiv -2n \left(\frac{\alpha L_{Pl}}{2\Delta X}\right)^{2n}$, we have

$$\Delta P = \frac{\hbar}{2\Delta X} \exp\left(-\frac{1}{2n} W\left(-2n \left(\frac{\alpha L_{Pl}}{2\Delta X}\right)^{2n}\right)\right), \tag{43}$$

where $W(u)$ is the definition of the Lambert W function,

$$W(u) \exp(W(u)) = u. \tag{44}$$

- We use Planck Units
$\hbar = c = k_B = 1$ and $L_{Pl} = M_{Pl}^{-1} = T_{Pl}^{-1} = \sqrt{G}$.

- Temperature
  - The metric of the Schwarzschild black hole
    $$ds^2 = -\left(1 - \frac{2MG}{r}\right) dt^2 + \left(1 - \frac{2MG}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \quad (45)$$
    Horizon radius: $r_h = 2MG$, satisfying $\Delta X \simeq r_h$.
  - The minimal mass (black hole remnant)
    $$M_{Min} = \frac{(\Delta X)_{Min}}{2G} = \frac{\alpha M_{Pl}}{4}(2ne)^{\frac{1}{2n}}. \quad (46)$$
  - The corrected Hawking temperature
    $$T \approx \frac{\Delta P}{2\pi} = \frac{1}{8\pi M L_{Pl}{}^2} \exp\left(-\frac{1}{2n} W\left(-\frac{1}{e}\left(\frac{M_{Min}}{M}\right)^{2n}\right)\right). \quad (47)$$
  - The maximum temperature when M decreases to black hole remnant M_{Min} :
    $$T^{Max} = \frac{T_{Pl}}{2\pi\alpha}\left(\frac{1}{2n}\right)^{\frac{1}{2n}}, \quad (48)$$

The black hole thermodynamics

Figure 1: The relations between temperature and black hole mass ($n = 2$, Hawking (black solid curve), GUP (dashed curve), GUP* (dotted curve) and $\mathrm{GUP}^*_\mathrm{n}$ (solid curve) results for $\alpha = 0.75$ (red), $\alpha = 1$ (green), $\alpha = 1.25$ (blue), respectively).

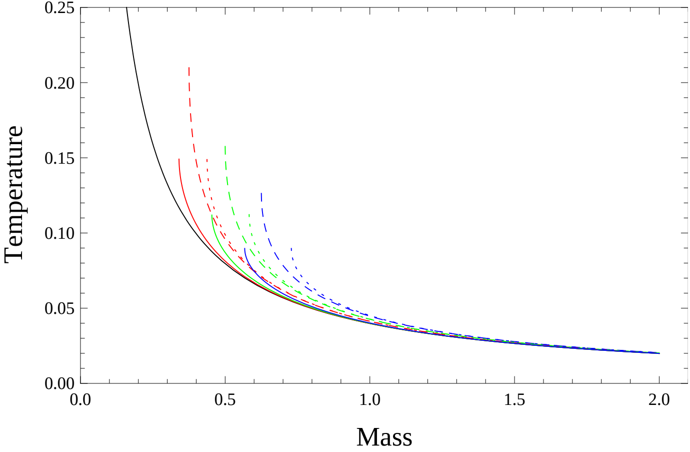
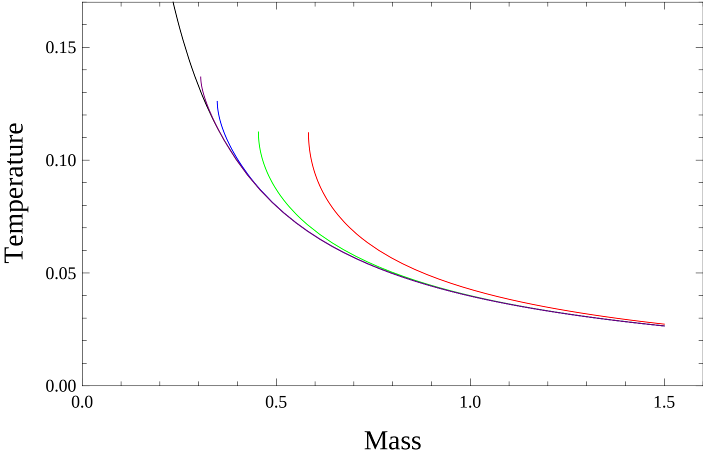


Figure 2: The relations between temperature and BH mass ($\alpha = 1$, Hawking (black) and $\mathrm{GUP}^{*}_{\mathrm{n}}$ results for n = 1 (red), n = 2 (green), n = 5 (blue), n = 10 (purple)).



- Explanation of Temperature Figures
    - Figure 1 shows that the maximum temperature decreases and the BH mass remnant increases when the parameter α grows.
    - Figure 2 shows that the maximum temperature grows and the mass-temperature relation tends to the Hawking radiation case when n is increasing, but the three GUP cases have temperature upper bounds due to the noncommutative relations.

- Entropy
  - The minimal increase of area of a black hole when absorbing a particle is: $(\Delta A)_{Min} \approx 8{L_{Pl}}^2(\ln 2)(\Delta X)(\Delta P)$.

    (Kh. Nouicer, Phys. Lett. B **646**, 63 (2007) [arXiv:0704.1261 [gr-qc]])

    Using eq. (43), we have

$$(\Delta A)_{Min} = 4{L_{Pl}}^2 (\ln 2) \exp\left[-\frac{1}{2n} W\left(-\frac{1}{e}\left(\frac{A_{Min}}{A}\right)^n\right)\right], \quad (49)$$

    where the horizon area $A = 4\pi(\Delta X)^2$, minimum horizon area $A_{Min} = 4\pi(\Delta X)^2_{Min}$.

  - The minimal increase of the entropy is $(\Delta S)_{Min} = \ln 2$, so

$$\frac{dS}{dA} \simeq \frac{(\Delta S)_{Min}}{(\Delta A)_{Min}} = \frac{1}{4{L_{Pl}}^2} \exp\left[\frac{1}{2n} W\left(-\frac{1}{e}\left(\frac{A_{Min}}{A}\right)^n\right)\right]. \quad (50)$$

- Entropy
  - Performing the integration in eq. (50), we obtain the entropy as follows:

$$\begin{aligned} S &= \frac{1}{4{L_{Pl}}^2}\int_{A_{Min}}^{A}\exp\left[\frac{1}{2n}W\left(-\frac{1}{e}\left(\frac{A_{Min}}{A}\right)^n\right)\right]dA \\ &= \frac{(-1)^{-\frac{1}{n}}\pi\alpha^2}{4n}\left\{2n\,\Gamma\left[\frac{n-1}{n},\frac{1}{2n}W\left(-\frac{1}{e}\left(\frac{M_{Min}}{M}\right)^{2n}\right)\right]\right. \\ &\quad -\Gamma\left(-\frac{1}{n},-\frac{1}{2n}\right)+\Gamma\left[-\frac{1}{n},\frac{1}{2n}W\left(-\frac{1}{e}\left(\frac{M_{Min}}{M}\right)^{2n}\right)\right] \\ &\quad \left.-\,2n\,\Gamma\left(\frac{n-1}{n},-\frac{1}{2n}\right)\right\}, \end{aligned} \tag{51}$$

where we should choose the Cauchy principal value.

Figure 3: The relations between entropy and black hole mass ($n = 2$, Hawking (black solid curve), GUP (dashed curve), GUP^* (dotted curve) and $\mathrm{GUP}^*_{\mathrm{n}}$ (solid curve) results for α= 0.75 (red), α= 1 (green), α= 1.25 (blue), respectively).

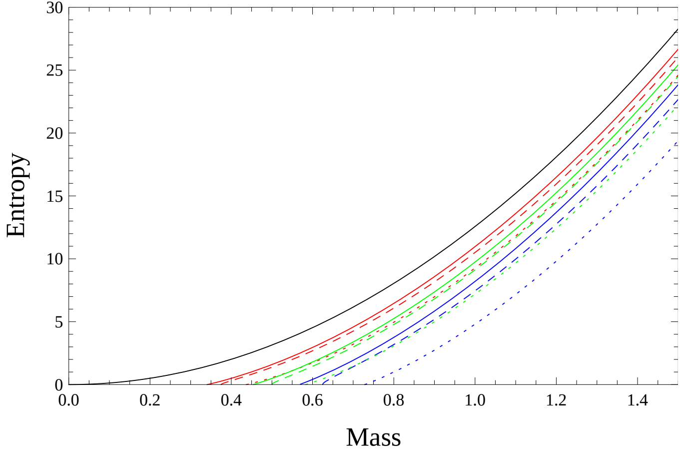
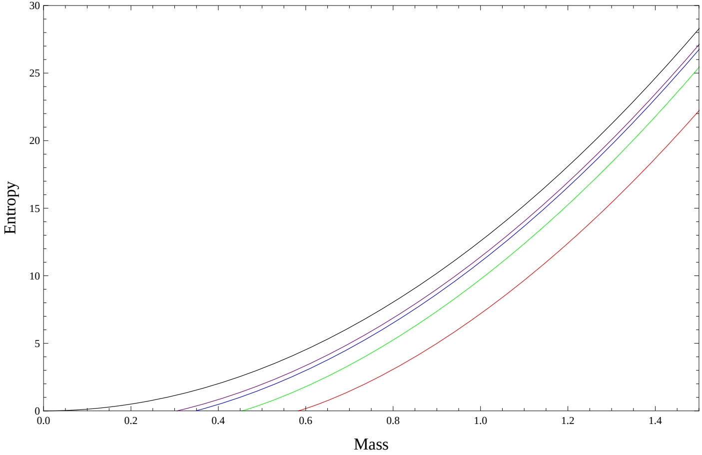


Figure 4: The relations between entropy and black hole mass ($\alpha = 1$, Hawking (black) and GUP_n^* results for n = 1 (red), n = 2 (green), n = 5 (blue), n = 10 (purple)).



- Explanation of Entropy Figures
  - Figure 3 indicates that the entropy of the black hole with the same mass is declining and the zero-entropy remnants have larger masses at the final stage of the evaporation when α is growing.
  - Figure 4 indicates that the entropy of the black hole with the same mass increases and the remnant's mass decreases when n is growing.

- Heat capacity

  The heat capacity of the black hole can be obtained from eq. (47),

$$\begin{aligned} C &= \frac{dM}{dT} \\ &= -8\pi M^2 {L_{Pl}}^2 \left[1 + W\left(-\frac{1}{e}\left(\frac{M_{Min}}{M}\right)^{2n}\right)\right] \\ &\quad \times \exp\left[\frac{1}{2n} W\left(-\frac{1}{e}\left(\frac{M_{Min}}{M}\right)^{2n}\right)\right]. \end{aligned} \tag{52}$$

  When $M = M_{Min}$, C becomes zero. Therefore, the BH remnant has zero heat capacity.

Figure 5: The relations between heat capacity and black hole mass ($n = 2$, Hawking (black solid curve), GUP (dashed curve), GUP^* (dotted curve) and $\mathrm{GUP}^*_\mathrm{n}$ (solid curve) results for α = 0.75 (red), α = 1 (green), α= 1.25 (blue), respectively).

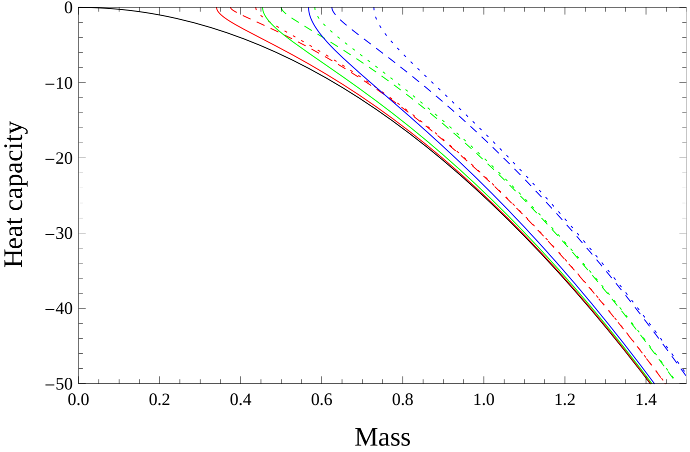
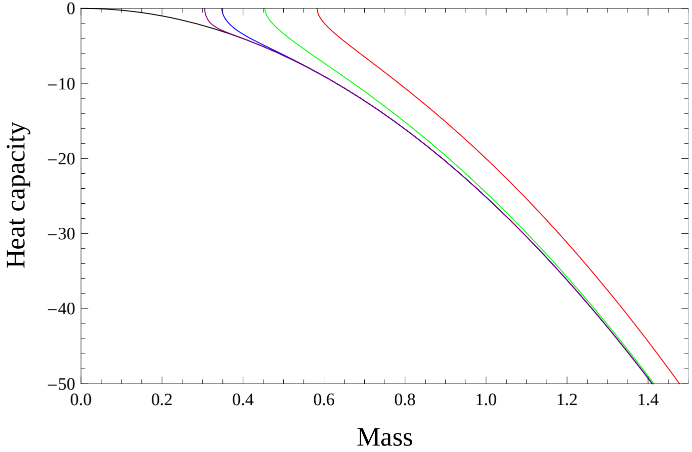


Figure 6: The relations between heat capacity and black hole mass ($\alpha = 1$, Hawking (black) and $\mathrm{GUP}_{\mathrm{n}}^{*}$ results for n = 1 (red), n = 2 (green), n = 5 (blue), n = 10 (purple)).



- Explanation of Heat Capacity Figures
  - Figure 5 indicates that the heat capacity of black holes with the same mass increases when α increases, and that the heat capacity vanishes for Hawking and GUP cases when the evaporation process terminates.
  - Figure 6 shows that the heat capacity tends to the Hawking situation except for the existence of mass remnants when n increases.

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- The black hole evaporation
  Every black hole whose temperature is higher than $2.725\ K$ will emit photons and other particles.
  - For simplification we assume that the black hole only radiates photons.
  - We give the evaporation rate and the decay time without and with the consideration of the UV/IR mixing effect, respectively.
  - These expressions are too complicated to be represented as any analytical functions, therefore the numerical results are provided.

- Evaporation rate without UV/IR mixing effect
  - Considering the invariant phase space volume of our improved exponential GUP ($\mathrm{GUP}_\mathrm{n}^*$), see eq. (21), we have the average energy density at temperature T,

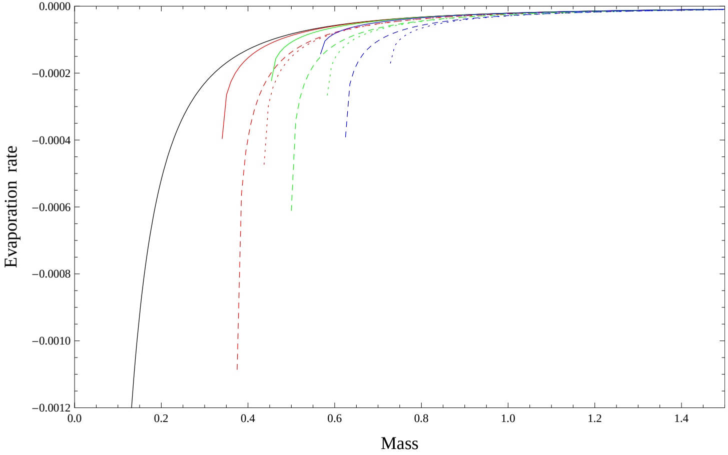
$$\mathcal{E}_\gamma(T) = \frac{2}{(2\pi)^3} \int e^{-3\alpha^{2n} {L_{Pl}}^{2n} P^{2n}} \frac{P d^3 P}{e^{\frac{P}{T}} - 1}. \tag{53}$$

  - According to the Stefan-Boltzmann law, we get the evaporation rate

$$\begin{aligned} \frac{dM}{dt} &= -\frac{A}{4} \mathcal{E}_\gamma(T) \\ &= -\frac{4G^2M^2}{\pi} \int_0^\infty e^{-3\alpha^{2n} {L_{Pl}}^{2n} P^{2n}} \frac{P^3 dP}{e^{\frac{P}{T}} - 1}, \end{aligned} \tag{54}$$

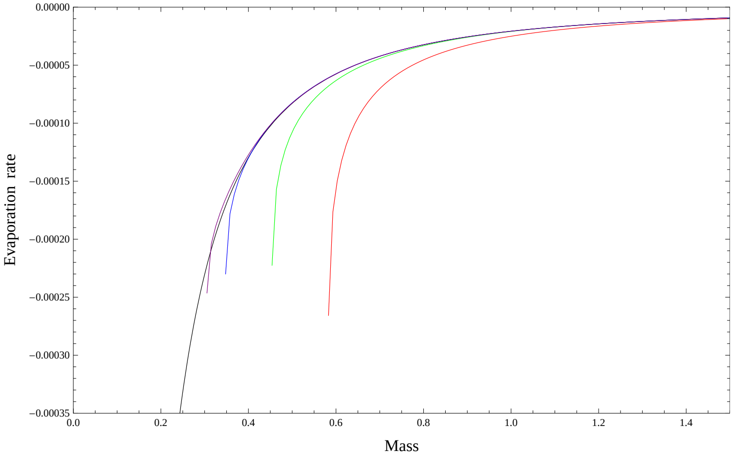
    where the zero lower limit of integral means all momentum states from zero to infinity have contributions to the energy density. This case corresponds to without the consideration of the UV/IR mixing effect.

Figure 7: The relations between evaporation rate and black hole mass ($n = 2$, Hawking (black solid curve), GUP (dashed curve), GUP^* (dotted curve) and $\mathrm{GUP}^*_{\mathrm{n}}$ (solid curve) results for $\alpha = 0.75$ (red), $\alpha = 1$ (green), $\alpha = 1.25$ (blue), respectively).



Black hole evaporation

Figure 8: The relations between evaporation rate and black hole mass ($\alpha = 1$, Hawking (black) and $\mathrm{GUP}_{\mathrm{n}}^{*}$ results for n $= 1$ (red), n $= 2$ (green), n $= 5$ (blue), n $=$ 10 (purple)).



Black hole evaporation

- Explanation of Evaporation Rate Figures (Without UV/IR Mixing)
  - Figure 7 indicates that α is positive correlation with black hole masses when the evaporation rates are equal, and at the final stage of the evaporation the mass remnants satisfy the inequality: $M_{Min}^{GUP^*} > M_{Min}^{GUP} > M_{Min}^{GUP_n^*}$.
    The evaporation rate increases for the same BH mass when α grows.
  - Figure 8 indicates that the evaporation rate declines when n increases, and it tends to the Hawking case for a large n.

Black hole evaporation

- Decay time without UV/IR mixing effect

The decay time of the evaporation process from mass M to M_{Min} is

$$t = \int_{M}^{M_{Min}} \frac{-\pi dM}{4G^2M^2 \int_0^\infty e^{-3\alpha^{2n}{L_{Pl}}^{2n}P^{2n}} \frac{P^3 dP}{e^{\frac{P}{T}}-1}}. \qquad (55)$$

Table 1: Hawking, GUP and GUP^*-corrected decay time with black hole mass $M = 1 \sim 5$ ($\alpha = 1$, in Planck units).

Hawking, GUP, GUP^*–Corrected Decay Time					
Form/M	1	2	3	4	5
Hawking	16085	128680	434294	1.02944×10^6	2.01062×10^6
GUP	9838.03	114511	412314	999674	1.97308×10^6
GUP^*	9070.31	113601	411361	998701	1.9721×10^6

Table 2: $\mathrm{GUP}^*_\mathrm{n}$-corrected decay time with $n = 2 \sim 10$, ∞ and black hole mass $M = 1 \sim 5$ ($\alpha = 1$, in Planck units).

$\mathrm{GUP}^*_\mathrm{n}$–Corrected Decay Time					
n/M	1	2	3	4	5
2	13868.8	126225	431761	1.02687×10^6	2.00802×10^6
3	14911.3	127502	433115	1.02826×10^6	2.00944×10^6
4	15248.1	127843	433457	1.0286×10^6	2.00978×10^6
5	15405	128000	433614	1.02876×10^6	2.00994×10^6
6	15495.8	128091	433705	1.02885×10^6	2.01003×10^6
7	15555.3	128150	433764	1.02891×10^6	2.01009×10^6
8	15597.5	128192	433806	1.02895×10^6	2.01013×10^6
9	15629.1	128224	433838	1.02898×10^6	2.01016×10^6
10	15653.7	128248	433863	1.02901×10^6	2.01019×10^6
∞	15868.8	128463	434078	1.02922×10^6	2.0104×10^6

Black hole evaporation

- Evaporation rate with UV/IR mixing effect

Only Trans-Planckian modes ($P > (\Delta P)_{Crit}$) contribute to $\mathcal{E}_\gamma(T)$, therefore the evaporation rate becomes

(Y.-G. Miao, Y.-J. Zhao, Int. J. Mod. Phys. D **23** (2014) 1450062 [arXiv:1312.4118 [hep-th]])

$$\frac{dM}{dt} = -\frac{4G^2M^2}{\pi}\int_{\left(\frac{1}{2n}\right)^{\frac{1}{2n}}\frac{1}{\alpha L_{Pl}}}^{\infty} e^{-3\alpha^{2n}{L_{Pl}}^{2n}P^{2n}}\frac{P^3 dP}{e^{\frac{P}{T}}-1}, \tag{56}$$

Black hole evaporation

Figure 9: The relations between evaporation rate and black hole mass with UV/IR mixing effect ($n = 2$, Hawking (black solid curve), GUP (dashed curve), GUP^* (dotted curve) and $\mathrm{GUP}^*_\mathrm{n}$ (solid curve) results for $\alpha = 0.75$ (red), $\alpha = 1$ (green), $\alpha = 1.25$ (blue), respectively).

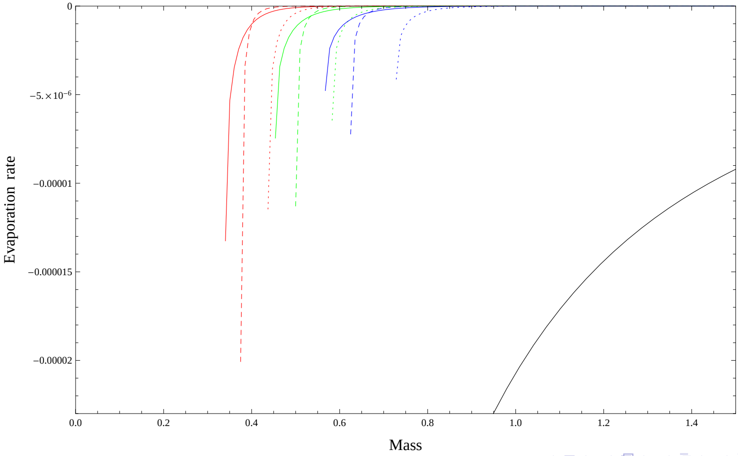
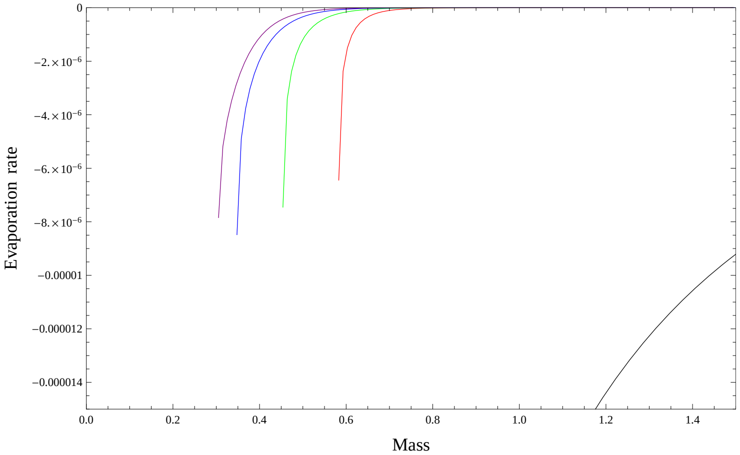


Figure 10: The relations between evaporation rate and black hole mass with UV/IR mixing effect ($\alpha = 1$, Hawking (black) and GUP_n^* results for n = 1 (red), n = 2 (green), n = 5 (blue), n = 10 (purple)).



- Explanation of Evaporation Rate Figures (with UV/IR Mixing)
  - Figure 9 indicates that the evaporation rate increases for the same BH mass when α grows, and it is much smaller than that without UV/IR mixing. That is, when the UV/IR mixing is considered, the BH radiates very slowly.
  - Figure 10 indicates that the evaporation rate declines when n increases, and it is much smaller than that without UV/IR mixing. That is, when the UV/IR mixing is considered, the BH almost stops radiation.

- Decay time with UV/IR mixing effect

  The decay time of the evaporation process from mass M to M_{Min} is

$$t = \int_{M}^{M_{Min}} \frac{-\pi dM}{4G^2M^2 \int_{\left(\frac{1}{2n}\right)^{\frac{1}{2n}} \frac{1}{\alpha L_{Pl}}}^{\infty} e^{-3\alpha^{2n} {L_{Pl}}^{2n} P^{2n}} \frac{P^3 dP}{e^{\frac{P}{T}} - 1}}. \quad (57)$$

Black hole evaporation

Table 3: Hawking, GUP and GUP*–UV/IR mixing effect corrected decay time with black hole mass $M = 1 \sim 5$ ($\alpha = 1$, in Planck units).

Hawking, GUP, GUP*–Corrected Decay Time with UV/IR Mixing Effect					
Form/M	1	2	3	4	5
Hawking	16085	128680	434294	1.02944×10^6	2.01062×10^6
GUP	8.55644×10^{10}	9.44548×10^{21}	6.86861×10^{32}	4.85234×10^{43}	3.45809×10^{54}
GUP*	1.9191×10^8	1.08681×10^{16}	4.63509×10^{23}	1.99868×10^{31}	8.83542×10^{38}

Black hole evaporation

Table 4: GUP_n^*–UV/IR mixing effect corrected decay time with $n = 2 \sim 10, \infty$ and black hole mass $M = 1 \sim 5$ ($\alpha = 1$, in Planck units).

GUP_n^*–Corrected Decay Time with UV/IR Mixing Effect					
n/M	1	2	3	4	5
2	3.51206×10^{8}	9.35241×10^{15}	3.22915×10^{23}	1.25875×10^{31}	5.24436×10^{38}
3	5.97713×10^{8}	3.48826×10^{16}	2.85769×10^{24}	2.66066×10^{32}	2.6514×10^{40}
4	9.62743×10^{8}	1.15183×10^{17}	1.96492×10^{25}	3.81531×10^{33}	7.93206×10^{41}
5	1.44373×10^{9}	3.05941×10^{17}	9.33582×10^{25}	3.2474×10^{34}	1.20992×10^{43}
6	2.02961×10^{9}	6.79022×10^{17}	3.3019×10^{26}	1.83312×10^{35}	1.09052×10^{44}
7	2.70789×10^{9}	1.31352×10^{18}	9.34557×10^{26}	7.60396×10^{35}	6.63259×10^{44}
8	3.46654×10^{9}	2.2875×10^{18}	2.2338×10^{27}	2.49888×10^{36}	2.99818×10^{45}
9	4.29459×10^{9}	3.67207×10^{18}	4.68656×10^{27}	6.86424×10^{36}	1.07885×10^{46}
10	5.18231×10^{9}	5.52793×10^{18}	8.87472×10^{27}	1.63812×10^{37}	3.24633×10^{46}
∞	∞				

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The ultimate case $n \to \infty$

In the ultimate case $n \to \infty$:

- Absolutely minimal length

$$M_{Min}^{\infty} = \frac{\alpha M_{Pl}}{4}. \tag{58}$$

- Temperature

$$T = \frac{1}{8\pi M {L_{Pl}}^2}. \tag{59}$$

The maximal temperature: $T_{\infty}^{Max} = \frac{T_{Pl}}{2\pi\alpha}$.

- Entropy

$$S = \frac{A - A_{Min}}{4L_{Pl}^2} = \frac{\pi\alpha^2}{4}\left[\left(\frac{M}{M_{Min}^{\infty}}\right)^2 - 1\right]. \tag{60}$$

- Heat capacity

$$C = -8\pi M^2 L_{Pl}^2, \tag{61}$$

which cannot be larger than $-\frac{\pi\alpha^2}{2}$ due to the presence of the absolutely minimal length.

- Evaporation rate & decay time without UV/IR mixing effect
  - Setting $z \equiv P/T$ the evaporation rate becomes

$$\frac{dM}{dt} = -\frac{4G^2M^2T^4}{\pi}\int_0^\infty e^{-3\left(\frac{\alpha Tz}{T_{Pl}}\right)^{2n}} \frac{z^3 dz}{e^z - 1}. \tag{62}$$

- Evaporation rate & decay time without UV/IR mixing effect
  - Only $z \leq \frac{T_{Pl}}{\alpha T} = \frac{8\pi M}{\alpha T_{Pl}}$ the integration does not vanish,

$$\begin{aligned}\frac{dM}{dt} &= -\frac{1}{1024\pi^5 M^2 G^2}\int_0^{\frac{8\pi M}{\alpha T_{Pl}}}\frac{z^3 dz}{e^z - 1} \\ &= \frac{1}{1024\pi^5 M^2 G^2}\left[\left(\frac{8\pi M}{\alpha T_{Pl}}\right)^4 - \left(\frac{8\pi M}{\alpha T_{Pl}}\right)^3 \ln\left(e^{\frac{8\pi M}{\alpha T_{Pl}}} - 1\right)\right. \\ &\quad + \left(\frac{8\sqrt{3}\pi M}{\alpha T_{Pl}}\right)^2 Li_2\left(e^{-\frac{8\pi M}{\alpha T_{Pl}}}\right) + \left(\frac{48\pi M}{\alpha T_{Pl}}\right) Li_3\left(e^{-\frac{8\pi M}{\alpha T_{Pl}}}\right) \\ &\quad \left. + 6 Li_4\left(e^{-\frac{8\pi M}{\alpha T_{Pl}}}\right) - \frac{\pi^4}{15}\right], \qquad (63)\end{aligned}$$

where $Li_n(x)$ is the polylogarithm, $Li_n(x) \equiv \sum_{k=1}^{\infty} \frac{x^k}{k^n}$.

The ultimate case $n \to \infty$

- Evaporation rate & decay time without UV/IR mixing effect
  - The decay time

$$t = \int_{M}^{M_{Min}^{\infty}} \frac{dM}{(dM/dt)} \tag{64}$$

can be calculated via numerical methods, see Table 2.

- Evaporation rate & decay time with UV/IR mixing effect

$$\frac{dM}{dt} = -\frac{4G^2M^2T^4}{\pi}\int_{\frac{T_{Pl}}{\alpha T}\left(\frac{1}{2n}\right)^{\frac{1}{2n}}}^{\infty} e^{-3\left(\frac{\alpha Tz}{T_{Pl}}\right)^{2n}} \frac{z^3 dz}{e^z - 1}. \qquad (65)$$

The lower bound of the integration $\frac{T_{Pl}}{\alpha T}\left(\frac{1}{2n}\right)^{\frac{1}{2n}} \to \frac{T_{Pl}}{\alpha T}$. Only $z \leq \frac{T_{Pl}}{\alpha T}$ the integration does not vanish, so the evaporation rate reduces to zero and thus the decay time inclines to infinity. In other words, the black hole has no radiation.

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Conclusion

- Propose the improved exponential GUP by introducing a suppressing index n in D-dimensional space.
- Give the invariant volume of phase space under time evolution in the classical limit. (The analog of the Liouville theorem)
- Point out only the trans-Planckian modes contribute the energy density of the vacuum in accordance with to the effect of UV/IR mixing.
- Determine the suitable suppressing index, $n \sim 10^{123}$, and calculate the cosmological constant consistent with the observed small value.
- Use the improved exponential GUP to modify the thermodynamical parameters of the Schwarzschild black hole and compare them with that of other GUPs.
- Analyze these parameters in the ultimate case $n \to \infty$.

Thank you!

Appendix

In the classical limit the commutation relations of operators reduce to the Poisson brackets via $\frac{1}{i\hbar}[\hat{A}, \hat{B}] \to \{A, B\}$,

$$\begin{aligned} \{X_i, P_j\} &= e^{\beta^n P^{2n}} \delta_{ij}, \\ \{P_i, P_j\} &= 0, \\ \{X_i, X_j\} &= 2n\beta^n P^{2(n-1)} e^{\beta^n P^{2n}} (P_i X_j - P_j X_i), \end{aligned}$$

where $i, j = 1, 2, \cdots, D$, $P^2 = \sum_{i=1}^{D} {P_i}^2$.

Appendix

Using the generic Poisson bracket of arbitrary functions,

$$\{F,G\} = \left(\frac{\partial F}{\partial X_i}\frac{\partial G}{\partial P_j} - \frac{\partial F}{\partial P_i}\frac{\partial G}{\partial X_j}\right)\{X_i,P_j\} + \frac{\partial F}{\partial X_i}\frac{\partial G}{\partial X_j}\{X_i,X_j\},$$

where the Einstein summation convention has been used, we get the equations of motion of X_i and P_i,

$$\begin{aligned}\dot{X}_i &= \{X_i,H\} = \{X_i,P_j\}\frac{\partial H}{\partial P_j} + \{X_i,X_j\}\frac{\partial H}{\partial X_j},\\ \dot{P}_i &= \{P_i,H\} = -\{X_j,P_i\}\frac{\partial H}{\partial X_j}.\end{aligned}$$

Appendix

During an infinitesimal time interval δt, the evolutions of X_i and P_i take the forms,

$$X_i{}' = X_i + \dot{X}_i\,\delta t = X_i + \left(\{X_i, P_j\}\frac{\partial H}{\partial P_j} + \{X_i, X_j\}\frac{\partial H}{\partial X_j}\right)\delta t,$$

$$P_i{}' = P_i + \dot{P}_i\,\delta t = P_i - \{X_j, P_i\}\frac{\partial H}{\partial X_j}\delta t.$$

and the infinitesimal phase space volume changes to be

$$\mathrm{d}^D\mathbf{X}'\mathrm{d}^D\mathbf{P}' = \left|\frac{\partial\left(X_1{}', \cdots, X_D{}'; P_1{}', \cdots, P_D{}'\right)}{\partial\left(X_1, \cdots, X_D; P_1, \cdots, P_D\right)}\right|\mathrm{d}^D\mathbf{X}\mathrm{d}^D\mathbf{P},$$

where the elements of the Jacobian are as follows,

$$\frac{\partial X_i{}'}{\partial X_j} = \delta_{ij} + \frac{\partial \delta X_i}{\partial X_j}, \qquad \frac{\partial X_i{}'}{\partial P_j} = \frac{\partial \delta X_i}{\partial P_j},$$

$$\frac{\partial P_i{}'}{\partial X_j} = \frac{\partial \delta P_i}{\partial X_j}, \qquad \frac{\partial P_i{}'}{\partial P_j} = \delta_{ij} + \frac{\partial \delta P_i}{\partial P_j}.$$

To the first order in δt, this Jacobian equals

$$\left|\frac{\partial\left(X_1{}',\cdots,X_D{}',P_1{}',\cdots,P_D{}'\right)}{\partial\left(X_1,\cdots,X_D,P_1,\cdots,P_D\right)}\right| = 1 + \frac{\partial\delta X_i}{\partial X_i} + \frac{\partial\delta P_i}{\partial P_i},$$

and then we have

$$\begin{aligned}\left(\frac{\partial\delta X_i}{\partial X_i} + \frac{\partial\delta P_i}{\partial P_i}\right)\frac{1}{\delta t} &= \frac{\partial}{\partial X_i}\left(\{X_i,P_j\}\frac{\partial H}{\partial P_j} + \{X_i,X_j\}\frac{\partial H}{\partial X_j}\right)\\ &- \frac{\partial}{\partial P_i}\left(\{X_j,P_i\}\frac{\partial H}{\partial X_j}\right)\\ &= -2Dn\beta^n P^{2(n-1)}e^{\beta^n P^{2n}}P_i\frac{\partial H}{\partial X_i}.\end{aligned}$$

Moreover, to the first order in δt the weight factor changes to be

$$\begin{aligned} e^{-D\beta^n P'^{2n}} &= e^{-D\beta^n (P_i+\delta P_i)^{2n}} \\ &= e^{-D\beta^n \left(P^2 - P_i\left\{X_j,P_i\right\}\frac{\partial H}{\partial X_j}\delta t - \left\{X_j,P_i\right\}\frac{\partial H}{\partial X_j}P_i\delta t\right)^n} \\ &= e^{-D\beta^n P^{2n}}\left(1 + 2Dn\beta^n P^{2(n-1)} e^{\beta^n P^{2n}} P_i \frac{\partial H}{\partial X_i}\delta t\right). \end{aligned}$$

As a consequence, we arrive at the result that the weighted phase space volume is invariant under time evolution, i.e.,

$$\begin{aligned} & e^{-D\beta^n P'^{2n}} \mathrm{d}^D \mathbf{X}' \mathrm{d}^D \mathbf{P}' \\ = \; & e^{-D\beta^n P^{2n}}\left(1 + 2Dn\beta^n P^{2(n-1)} e^{\beta^n P^{2n}} P_i \frac{\partial H}{\partial X_i}\delta t\right) \\ & \times \left(1 - 2Dn\beta^n P^{2(n-1)} e^{\beta^n P^{2n}} P_j \frac{\partial H}{\partial X_j}\delta t\right) \mathrm{d}^D \mathbf{X} \mathrm{d}^D \mathbf{P} \\ = \; & e^{-D\beta^n P^{2n}} \mathrm{d}^D \mathbf{X} \mathrm{d}^D \mathbf{P}. \end{aligned}$$