An example of non-AdS holography

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1 Introduction

2 Newton-Cartan gravity
- Non-relativistic symmtries
- Constructing Newton-Cartan gravity

3 Novel non-AdS holography

4 Conclusion and future work

Introduction: holography based on symmetry

It is believed the principle for quantum gravity theory is holographic principle. In the past decades, people are trying answer the question: how general the holography principle can be? The theories dual to each other have the same symmetry group. In above case, $SO(d,2)$. In the special case

- $d = 2$, we have $SO(2,2) \sim SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$. This is AdS_3/CFT_2 duality.
- Higher spin generalization: $SL(N,\mathbb{R})$ or more complicated $hs[\lambda]$;
- p-adic holography, based on symmetry group $\mathrm{SL}(2,\mathbb{Q}_p)$
- Taking non-relativistic limit: this is found to be Newton-Cartan/NCFT duality

Why non-relativistic holography is interesting?

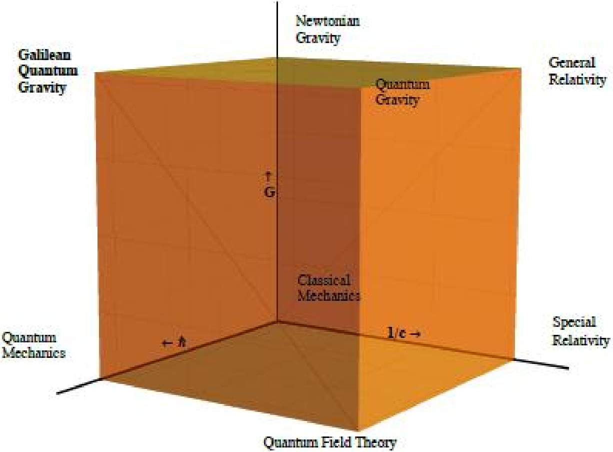
Non-relativity means spacetime scales anisotropically

$$t \to \lambda^z t, \qquad x_i \to \lambda x_i \qquad \text{for} \quad z \neq 1$$

breaking Lorentz symmetry. This scaling is usually called Lifshitz symmetry. Strongly correlated systems with this symmetry are called quantum Lifshitz fixed points.

- Many condensed matter field theory exhibits non-relativistic scaling symmetry.
- Can one define conformal symmetry in non-relativistic sense?
- Non-relativistic quantum gravity (Horava-Lifshitz) can be renormalizable (Horava, 0812.4287)
- Does holography exist in general sense? (independent of relativity)
- Near horizon geometry of naked black hole (Horowitz, Ross, 9704058)

The theory cube



Non-relativistic theory

Many body field theories describing anisotropic fixed points were proposed to be holographically dual to gravity in the background of Lifshitz geometries, where time and space scale asymptotically with the same ratio z. Non-relativistic spacetime solutions are found in Einstein gravity theory with gauge matter fields. Lifshitz spacetimes: (Kachru, Liu, Mulligan, 08)

$$ds^2 = -r^{2z}dt^2 + \frac{dr^2}{r^2} + r^2 dx_i dx_i, \quad A_t = \sqrt{\frac{2(z-1)}{z}}\, r^z$$

Schrödinger spacetime (Son, 08)

$$ds^2 = -r^{2z}dt^2 - 2r^2 dt d\xi + \frac{dr^2}{r^2} + r^2 dx_i dx_i$$

$z > 1$ generically for Null Energy condition. Schrödinger algebra contains special conformal transformation at $z = 2$.

Not enough

The construction of non-relativistic holography by Einstein gravity is far beyond satisfaction.

- We are embedding non-relativistic gravity into relativistic gravity
- There are extra matter fields in the theory (massive gauge fields/higher curvature terms)
- This does not appear to be non-relativistic limit of Einstein gravity.

1. Introduction
2. Newton-Cartan gravity
   - Non-relativistic symmtries
   - Constructing Newton-Cartan gravity
3. Novel non-AdS holography
4. Conclusion and future work

Why should we consider Newton-Cartan gravity?

The ground breaking work to drag people's attention to Newton-Cartan gravity is (Christensen, Hartong, Obers, Rollier, 1311.6471). They found the Newton-Cartan gravity appears in Lifshitz holography as one approaches the boundary of spacetime.

- There is no need to discuss non-relativistic gravity by starting from relativistic gravity. Newton-Cartan gravity is built on non-relativistic symmetry group.
- Horava-Lifshitz gravity allows Lifshitz spacetime as vacuum solution without matter to support.
- Dynamical Newton-Cartan gravity is equivalent to Horava-Lifshitz gravity. (Hartong, Obers, 1504.07461)

Difficulty and debate

There are a lot of different arguments on Horava-Lifshitz gravity holgraphy, and some of relevant are

- It has advantages: no need for extra massive gauge field; resolve non-renormalizable problem of gravity.
- More negative arguments: it breaks diffeomorphism; as a quantum gravity, no Einstein limit; no well-defined black hole; no string embedding yet...

Main model: 3D gravity

We are going to study the problems in 3D gravity, which is also topological.

Non-relativistic symmetry group

The gravity theory is classified with respect to its isometry group. We are interested in several examples:

- 3D Galilean/Bargmann algebra contains $\{H, G_a, P_a, J, (N)\}$

$$[J, P_a] = \epsilon_{ab} P_b, \quad [J, G_a] = \epsilon_{ab} G_b, \quad [H, G_a] = P_a,$$
$$[P_a, G_b] = N\delta_{ab} \qquad \text{(Bargmann central extension)}$$

  $a = 1, 2$. Without non-relativistic boost, Galilean algebra reduces to Lifshitz-like algebra.

- Newton-Hooke algebra, it includes Galilean algebra with cosmological constant

$$[H, P_a] = -\Lambda G_a$$

- Carrollian algebra ($c \to 0$ limit)

- Schrödinger algebra (introducing dilatation D) (Son...)

$$[H, D] = zH, \quad [P_a, D] = P_a, \quad [G_a, D] = (1-z)G_a$$
$$[D, N] = (z-2)N$$

At $z = 2$, symmetry can be enhanced to include special conformal transformation K, so that H, D, K form a $SL(2, \mathbb{R})$ and $[P_a, K] = G_a$

- Galilean conformal algebra (Bagchi, Gopakumar) (too many commutators...)
- BMS algebra , with identification BMS_3 is identical to GCA_2

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{m+n} + \frac{c_L}{12}n(n^2-1) \\
[L_n, M_m] &= (n-m)M_{m+n} + \frac{c_M}{12}n(n^2-1)
\end{aligned}$$

Einstein Gravity by gauging Poincare groups

In general dimensions, the Poincare algebra is

$$\begin{aligned} [M_{AB}, P_C] &= \eta_{AC} P_B - \eta_{BC} P_A \\ [M_{AB}, M_{CD}] &= \eta_{AC} M_{BD} - \eta_{AD} M_{BC} - \eta_{BC} M_{AD} + \eta_{BD} M_{AC} \\ A &= 0, 1, ..., d \end{aligned}$$

Then gauge field

$$A_\mu = \frac{1}{2} M_{AB} \omega_\mu^{AB} + P_A e_\mu^A$$

transforms under gauge transformation as $\delta A_\mu = \partial_\mu \Lambda + [A_\mu, \Lambda]$ where

$$\Lambda = \frac{1}{2} M_{AB} \sigma^{AB} + P_A \zeta^A$$

we can then derive

$$\delta e_\mu^A = \partial_\mu \zeta^A + e_\mu^C \sigma^A{}_C + \omega_\mu^{AC} \zeta_C$$

To identify gauge transformation ζ^a with diffeomorphism by $\zeta^A = \xi^\rho e_\rho^A$, we need to study

$$\begin{aligned} \mathcal{L}_\xi e_\mu^a &= \xi^\rho(\partial_\rho e_\mu^A - \partial_\mu e_\rho^A) + \partial_\mu(\xi^\rho e_\rho^A) \\ &= -\xi^\rho R_{\rho\mu}(P_A) + \xi^\rho(e_{\rho C}\omega_\mu^{AC} + e_{\mu C}\omega_\rho^{AC}) + \partial_\mu(\xi^\rho e_\rho^A) \end{aligned}$$

where

$$F = dA + A \wedge A = P_A R^A(P) + M_{AB} R^{AB}(M)$$

we see we can make the identification as long as curvature constraint $R_{\rho\mu}(P_A) = 0$ is imposed. This is the torsion free condition for Einstein gravity which is used to solve spin-connection uniquely from veilbein.

Invariant field

$g_{\mu\nu} = e_\mu^A e_\nu^B \eta_{AB}$ is the geometric field invariant under local Lorentz transformation.

Einstein gravity

Einstein equation is recovered by $R^{AB}(M) = 0$. Note Einstein equation is second order differential equation while gauge curvature zero condition is first order differential equation. Their equivalence is based on the fact spin-connection can be uniquely solved by vielbein.

There are examples in which spin-connection cannot be uniquely solved by vielbein. This can happen in higher spin non-relativistic solutions (YL, Ross, 1504.07252). These solutions are said to be degenerate. For these solutions, one cannot trust the description by metric-like fields.

Gauging Galilean algebra

Galilean algebra in $d+1$ dimensions contains $\{H, P_a, G_a, J_{ab}\}$. Naively

$$A = H\tau + P_a e^a + G_a \omega^a + J_{ab}\omega^{ab}, \qquad a = 1, ..., d$$

But one immediately finds $R_a(P)$ only contains d components of equations. Recall Einstein gravity contains $d+1$ components of equations. This is because t component equation becomes

$$d\tau = 0$$

imposing no constraints on spin-connection. As a result, *spin-connection cannot be uniquely determined*, making the gravity theory ill-defined!

Newton-Cartan gravity by gauging Bargmann algebra

First let me explain how to understand Newton-Cartan gravity in general dimensions. By adding central extension $[P_a.G_b] = N\delta_{ab}$,

$$A = H\tau + P_a e^a + G_a\omega^a + J_{ab}\omega^{ab} + Nm$$

m is the central extended gauge field, interpreted as mass current. Then it is easy to calculate

$$\begin{aligned} F &= dA + A \wedge A \\ &= HR(H) + R^a(P)P_a + R^a(G)G_a + R^{ab}(J)J_{ab} + R(N)N \end{aligned}$$

We would like to impose the curvature constraints so that spin-connection is uniquely solved in terms of smaller set of fields. In this case τ, e^a and the gauge field m. Then we need to impose $R(H) = R^a(P) = R(N) = 0$. (See Bergshoeff et. 1011.1145)

After imposing curvature constraints, one can check

$$\begin{aligned}\delta\tau_\mu &= \partial_\mu(\xi^\rho\tau_\rho) - \xi^\rho R_{\rho\mu}(H)\\ \delta e^a_\mu &= \partial_\mu(\xi^\rho e^a_\rho) - \xi^\rho R_{\rho\mu}{}^a(P_a) + (\xi^\rho\omega^{ab}_\rho)e^b_\mu + (\xi^\rho\omega^a_\rho)\tau_\mu\\ &= \partial_\mu\xi^a + \lambda^{ab}e^b_\mu + \lambda^a\tau_\mu\end{aligned}$$

where $\xi^\rho = \{\xi^0 = \xi^\rho\tau_\rho, \xi^a = \xi^\rho e^a_\rho\}$. Therefore, gauge transformation is equivalent to diffeomorphism up to local Galilean transformation! The fields invariant under local Galilean transformations are τ_μ (clock one-form) and $h^{\mu\nu} = e^\mu_a e^\nu_b \delta^{ab}$ (spatial metric). One can always define the Lorentzian metric to be

$$g_{\mu\nu} = -\tau_\mu\tau_\nu + h_{\mu\nu}$$

Torsion constraints

One can show $d\tau$ is proportional to $\Gamma^\rho_{[\mu\nu]}$, i.e. torsion tensor. So Bargmann theory with $d\tau = 0$ is called torsionless Newton-Cartan gravity.

Classifications

Based on the features of torsion, Newton-Cartan gravity is classified into three kinds:

- Torsionless Newton-Cartan gravity: $d\tau = 0$. This corresponds to projectable Horava-Lifshitz gravity.
- Twistless torsional Newton-Cartan gravity: $\tau \wedge d\tau = 0$. This means there is a field b, such that $d\tau - zb \wedge \tau = 0$. This means b is the field coupled to scaling generator D and $[H, D] = zH$. Therefore, we should expect once we introduce scaling symmetry into Galilean algebra, (Schrödinger algebra), one should expect we obtain this kind of theory. This also corresponds to non-projectable Horava Lifshitz gravity
- Torsional Newton-Cartan gravity: no constraints on τ (Never seen examples yet..)

Chern-Simons theory

In $d = 3$ dimensional spacetime, one usually uses $J_a = \frac{1}{2}\epsilon_{abc}M^{bc}$ as the algebra generator. For

$$A = P_a e^a + J_a \omega^a$$

we can write down Chern-Simons action

$$S_{cs} = \mathrm{Tr}_R \int \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

Tr is actually an invariant bilinear product on Lie algebra, which maps two generators to a number. In case of $ISO(2,1)$, this bilinear product is (Witten 88')

$$< P_a, J_b >= \eta_{ab}$$

resulting in Einstein-Hilbert action

$$S = \int e^a \wedge R_a(\omega)$$

Einstein AdS Chern-Simons

In terms of AdS gravity, gauge field A takes values in $so(2,2)$ algebra. This algebra allows a second parameters of bilinear product:

$$< J_a, J_b >= \eta_{ab}, \quad < P_a, P_b >= -\Lambda\eta_{ab}$$

The presence of two parameters of bilinear products is the fact of isomorphism

$$so(2,2) = SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$$

The bilinear product of such semisimple Lie algebra is given by Cartan-Killing metric. This gives familiar AdS gravity

$$S_{EH} = S_{CS}[A] - S_{CS}[\bar{A}]$$

Chern-Simons Newton-Cartan gravity

The Galilean/Bargmann algebra are not semi-simple. The Cartan-Killing metric is degenerate. It is pointed out by Witten that for non-semisimple Lie algebra, non-degenerate invariant bilinear product can exist sometimes. For Lie algebra $[T_A, T_B] = f_{AB}{}^C T_C$, such invariant bilinear product is defined by solving

$$f^D_{AB}\Omega_{CD} + f^D_{AC}\Omega_{BD} = 0$$

In (Witten, Nappi, 9310112), they consider algebra $T_A = \{J, P_1, P_2, T\}$,

$$[J, P_a] = \epsilon_{ab}P_b, \quad [P_a, P_b] = \epsilon_{ab}T,$$

which is known as central extended Poincare group P^c_2. They find bilinear product

$$< P_a, P_b >=< J, T > \delta_{ab}$$

However, even by solving the equations above, one cannot have non-degenerate bilinear product (NDBP) for Bargmann algebra. This makes it hard to build Chern-Simons theory. A resolution suggested in (Papageorgiou, Schroers, 0907.2880) is to introduce more generators in the Lie algebra so that bilinear product can be non-degenerate. They find we only need to extend

$$[G_a, G_b] = S\epsilon_{ab}$$

Then $\{H, P_a, G_a, J, N, S\}$ form an NDBP, which is given by

$$< H, S >= - < J, N >= - < P_1, G_2 >=< P_2, G_1 >$$

Therefore, we can write down a Chern-Simons gravity action! Note the whole algebra is Galilean with two $u(1)$ extensions (Bergshoeff, Rosseel, 1604.08042).

Chern-Simons action

Let's take $A = H\tau + P_a e^a + G_a \Omega^a + J\Omega + Nm + S\zeta$,, then

$$\mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right) =$$
$$-\epsilon_{ab} R^a(G) \wedge e^b + \frac{1}{2}\epsilon_{ab}\tau \wedge \Omega^a \wedge \Omega^b - \Omega \wedge dm$$
$$+\zeta \wedge d\tau$$

where

$$R^a(G) = d\Omega^a - \epsilon^{ab}\Omega \wedge \Omega^b$$

As one can see ζ works as Lagrangian multiplier, used to impose torsionless condition $d\tau = 0$! The extra generators are necessary.

Extended Bargmann gravity

The action above can be written in terms of gauge invariant metric-like fields

$$\mathcal{L} = e\left(h^{\mu\rho}h^{\nu\sigma}K_{\mu\nu}K_{\rho\sigma} - (h^{\mu\nu}K_{\mu\nu})^2 - \tilde{\Phi}\mathcal{R}\right) ,$$

where

$$\tilde{\Phi} = -v^{\mu}m_{\mu} + \frac{1}{2}h^{\mu\nu}m_{\mu}m_{\nu} .$$

is the Newton Potential. $K_{\mu\nu}$ is the extrinsic curvature, reminding us the decomposition of Einstein gravity in terms of extrinsic curvature. $\mathcal{R}$ is the Ricci curvature on spatial directions.

$u(1)^2$ extensions

N and S are two $u(1)$ extensions. It has been proved the extended Bargmann algebra are non-relativistic limit of $iso(2,1) \oplus u(1)^2$

Aside

For $z = 2$ Schrödinger algebra, we need to introduce three more generators to make Jacobi identity held and bilinear product non-degenerate. (Niels Obers, Jelle Hartong, YL, 1604.08054)

$$\begin{aligned} A &= H\tau + P_a e^a + G_a \omega^a + J\omega + Nm + Db + Kf \\ &\quad + S\zeta + Y\alpha + Z\beta\,. \end{aligned}$$

where $[P_a, P_b] = \epsilon_{ab} Z$ and $[P_a, G_b] = N\delta_{ab} - Y\epsilon_{ab}$. The action is

$$\begin{aligned} \mathcal{L} &= 2c_1 \Big[\tilde{R}^2(G) \wedge e^1 - \tilde{R}^1(G) \wedge e^2 + \tau \wedge \omega^1 \wedge \omega^2 \\ &\quad - m \wedge d\omega - f \wedge e^1 \wedge e^2 + \zeta \wedge (d\tau - 2b \wedge \tau) \\ &\quad + \alpha \wedge (db - f \wedge \tau) + \beta \wedge (df + 2b \wedge f) \Big] \end{aligned}$$

The extended Schrödinger Algebra

The $z = 2$ algebra we are considering is

$$
\begin{aligned}
&[J, P_a] = \epsilon_{ab} P_b, \quad [J, G_a] = \epsilon_{ab} G_b, \quad [H, G_a] = P_a, \\
&[P_a, G_b] = N\delta_{ab} - \epsilon_{ab} Y \\
&[G_a, G_b] = S\epsilon_{ab}, \quad [P_a, P_b] = \epsilon_{ab} Z \\
&[H, D] = 2H, \quad [P_a, D] = P_a, \quad [G_a, D] = -G_a \\
&[P_a, K] = G_a, \quad [D, K] = 2K, \quad [D, K] = 2K \\
&[H\,, Y] = -Z\,, \quad [H\,, S] = -2Y\,, \quad [K\,, Y] = S\,, \\
&[K\,, Z] = 2Y\,, \quad [D\,, S] = 2S\,, \quad [D\,, Z] = -2Z\,.
\end{aligned}
$$

The bilinear product is

$$
\begin{aligned}
& < H, S >= - < J, N >= - < P_1, G_2 >=< P_2, G_1 > \\
= \quad & < D, Y >=< K, Z >
\end{aligned}
$$

Asymptotic symmetry

Take $\hat{L}_m = \{H, D, K\}$, $\hat{M}_m = \{S, Y, Z\}$, $Y^i_r = \{P_a, G_a\}$

$$
\begin{aligned}
[\hat{L}_m, \hat{L}_n] &= (m-n)\hat{L}_{m+n} + \frac{c_L}{2}(m^3 - m)\delta_{m+n,0} \\
[\hat{L}_m, \hat{M}_n] &= (m-n)\hat{M}_{m+n} + \frac{c}{2}(m^3 - m)\delta_{m+n,0} \\
[\hat{L}_m, Y^i_r] &= (\frac{m}{2} - r)Y^i_{m+r}, \qquad [\hat{L}_m, J_n] = -nJ_{m+n}, \\
[\hat{L}_m, N_n] &= -nN_{m+n}, \quad [Y^i_r, Y^i_s] = (r-s)N_{s+r}, \\
[Y^1_r, Y^2_s] &= -\hat{M}_{r+s} + c\left(s^2 - \frac{1}{4}\right)\delta_{r+s,0}, \qquad [J_n, Y^i_r] = Y^j_{r+n}\epsilon_{ij} \\
[J_m, N_n] &= cn\delta_{m+n,0} \qquad [J_n, \hat{M}_m] = -2nN_{m+n}, \\
[J_m, J_n] &= c_J n\delta_{m+n,0}
\end{aligned}
$$

Super GCA algebra (Bagchi, Gopakumar) with central extensions

1. Introduction
2. Newton-Cartan gravity
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   - Constructing Newton-Cartan gravity
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Newton-Hooke gravity

To get full analogy to $\mathrm{AdS}_3/\mathrm{CFT}_2$, we need cosmological constant to be introduced. Another feature we know of AdS_3 in terms of Chern-Simons theory is that the Einstein-Hilbert action is difference of two copies of Chern-Simons theory, as a result of isometry group

$$SO(2,2) = SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$$

What's important to us is Newton-Hooke algebra can also be written in the similar product form.

$$pNH_3 = P_2^c \times P_2^c$$

Simple observation shows P_2^c is isomorphic to NH_2.

The Newton-Hooke Algebra

The algebra we are considering is

$$\begin{aligned}
&[J, P_a] = \epsilon_{ab} P_b, \quad [J, G_a] = \epsilon_{ab} G_b, \quad [H, G_a] = P_a, \\
&[P_a, G_b] = N \delta_{ab} \\
&[G_a, G_b] = S \epsilon_{ab} \\
&[H, P_a] = -\Lambda G_a, \quad [P_a, P_b] = \Lambda S \epsilon_{ab}
\end{aligned}$$

Bilinear product

$$\begin{array}{ccccccccc}
 & H & P_1 & P_2 & G_1 & G_2 & J & N & S \\
H & c_5 & 0 & 0 & 0 & 0 & c_4 & \Lambda_c c_1 & -c_2 \\
P_1 & 0 & -\Lambda_c c_1 & 0 & 0 & -c_2 & 0 & 0 & 0 \\
P_2 & 0 & 0 & -\Lambda_c c_1 & c_2 & 0 & 0 & 0 & 0 \\
G_1 & 0 & 0 & c_2 & c_1 & 0 & 0 & 0 & 0 \\
G_2 & 0 & -c_2 & 0 & 0 & c_1 & 0 & 0 & 0 \\
J & c_4 & 0 & 0 & 0 & 0 & c_3 & c_2 & c_1 \\
N & \Lambda_c c_1 & 0 & 0 & 0 & 0 & c_2 & 0 & 0 \\
S & -c_2 & 0 & 0 & 0 & 0 & c_1 & 0 & 0
\end{array}$$

In fact, based on the extension in Bargmann case, we should consider again the $u(1)^2$ extension of AdS isometry algebra. For each copy of $SL(2,R) \times U(1)$, we consider their generators as L_m, N_0. Then

$$L_{-1} = c\mathcal{L}_{-1}\,, \quad L_0 = \frac{1}{2}\mathcal{L}_0 + \frac{c^2}{2}\mathcal{N}_0\,,$$
$$L_1 = c\mathcal{N}_1\,, \quad N_0 = -\frac{1}{2}\mathcal{L}_0 + \frac{c^2}{2}\mathcal{N}_0\,,$$

by taking $c \to \infty$, we can see

$$[\mathcal{L}_{-1}\,,\mathcal{L}_0] = -i\mathcal{L}_{-1}\,, \qquad [\mathcal{L}_{-1}\,,\mathcal{N}_1] = -i\mathcal{N}_0\,, \qquad [\mathcal{L}_0\,,\mathcal{N}_1] = -i\mathcal{N}_1\,.$$

which is exactly P_2^c

One can write down Chern-Simons gravity action on $P_2^c \times P_2^c$

$$\mathrm{Tr}\left(A\wedge dA+\frac{2}{3}A\wedge A\wedge A\right)=$$
$$2c_1\Big[-\epsilon_{ab}R^a(G)\wedge e^b+\frac{1}{2}\epsilon_{ab}\tau\wedge\Omega^a\wedge\Omega^b-\Omega\wedge dm$$
$$+\zeta\wedge d\tau+\Lambda_c\tau\wedge e^1\wedge e^2\Big]+c_2\big[\Omega^a\wedge R^a(G)+2\zeta\wedge d\Omega$$
$$+\Lambda_c e^a\wedge R^a(P)-2\Lambda_c\tau\wedge R(N)+\Lambda_c e^a\wedge\Omega^a\wedge\tau\big]$$

or equivalently

$$\begin{aligned}\mathcal{L}_{NEH}[c_1] &= \tau\wedge d\zeta-\omega\wedge dm+\omega^2\wedge de^1-e^2\wedge d\omega^1-\omega\wedge e^a\wedge\omega^a\\ &\quad +\Lambda\tau\wedge e^1\wedge e^2+\tau\wedge\omega^1\wedge\omega^2\\ \mathcal{L}_{TMG}[c_2] &= 2\zeta\wedge d\omega+\omega^a\wedge d\omega^a+2\omega\wedge\omega_1\wedge\omega_2\end{aligned}$$

We can write $\mathcal{L}_{NEH}[c_1]=\mathcal{L}_{CS}[A]-\mathcal{L}_{CS}[\bar{A}]$, for A takes values in P_2^c

Reduction from Einstein gravity

We can obtain the theory above by starting from Einstein gravity with two u(1) fields, and take non-relativistic limit.

$$\begin{aligned} I_{EH}[c_1] &= -E^0 \wedge d\Omega^0 + E^a \wedge d\Omega^a + E^0 \wedge \Omega^1 \wedge \Omega^2 + E^1 \wedge \Omega^2 \wedge \Omega^0 \\ &\quad + E^2 \wedge \Omega^0 \wedge \Omega^1 + \Lambda E^0 \wedge E^1 \wedge E^2 + Z_1 \wedge dZ_2 \\ I_{TMG}[c_2] &= -\Omega^0 \wedge d\Omega^0 + \Omega^a \wedge d\Omega^a + 2\Omega^0 \wedge \Omega^1 \wedge \Omega^2 + Z_2 \wedge dZ_2 \end{aligned}$$

$$E^0 = \frac{1}{2}\left(2c\tau + \frac{m}{c}\right) \quad Z_1 = \frac{1}{2}\left(2c\tau - \frac{m}{c}\right) \quad \Omega^1 = \frac{1}{c}\omega^2$$

$$\Omega^2 = -\frac{1}{c}\omega^1, \quad \Omega^0 = \frac{1}{2}\left(2\omega - \frac{\zeta}{c^2}\right), \qquad Z_2 = \frac{1}{2}\left(2\omega + \frac{\zeta}{c^2}\right)$$

take $c \to \infty$ limit, one can obtain NH gravity.

Asymptotic symmetry

The story is even valid for Brown-Henneaux boundary condition. We can write down asymptotic solution:

$$a_+ = \mathcal{N}_+ + \mathcal{F}^0 \mathcal{L}_0 + \mathcal{F}^- \mathcal{L}_- + \mathcal{F}^N \mathcal{N}_0.$$

By calculating asymptotic symmetry algebra $\delta a = d\epsilon + [a, \epsilon]$,

$$\begin{aligned}
[\mathcal{L}_m, \mathcal{L}_n] &= (m-n)\mathcal{L}_{m+n} + 2\pi\gamma_1 m^3 \delta_{m+n,0}, \\
[\mathcal{L}_m, \mathcal{N}_n] &= -n\mathcal{N}_{m+n} - 2\pi i\, m^2 \left(2\gamma_2 - \frac{\gamma_1}{c^2}\right)\delta_{m+n,0}, \\
[\mathcal{N}_m, \mathcal{N}_n] &= -\frac{4\pi m}{c^2}\left(2\gamma_2 - \frac{\gamma_1}{c^2}\right)\delta_{m+n,0}.
\end{aligned}$$

where γ_1, γ_2 are two parametres in non-degenerate bilinear product.

Asymptotic symmetry

For any finite c, the algebra is equivalent to Virasoro with $u(1)$ affine by redefinition

$$\mathcal{L}'_n = \mathcal{L}_n + a\mathcal{N}_n + b\sum_m \mathcal{N}_{n-m}\mathcal{N}_m$$

But for $c \to \infty$, which is case we are interested in, we should have twisted $u(1)$ Virasoro algebra

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n} + 2\pi\gamma_1 m(m^2-1)\delta_{m+n},$$
$$[\mathcal{L}_m, \mathcal{N}_n] = -n\mathcal{N}_{m+n} - 4\pi i\gamma_2 m(m+1)\delta_{m+n}.$$

up to a shift in zero mode.

Therefore, we have analogy with AdS/CFT.

Relativistic	Non-relativistic
AdS_3	pNH 3D
isometry $SL(2,R) \times SL(2,R)$	isometry $P_2^c \times P_2^c$
Vir $\times$ Vir	Vir_t with $U(1)$ $\times$ Vir_t with $U(1)$
Coset $\frac{SO(2,2)}{SO(2,1)}$	Coset $\frac{P_2^c \times P_2^c}{P_2^c \times U(1)}$
Topological massive gravity	NH gravity with dynamics
CFT	twisted WCFT (maybe)
higher dimension so(d+1,2)	with extension $so(d,1) \oplus u(1)$
WZW Exist, Liouville theory	WZW Exist, (Nappi-Witten)
BTZ black hole	???
Higher spin generalization $hs[\lambda]$	???

Higher dimensional generalizations

Consider the conformal algebra $so(d+1,2)$ in $d+1$ dimensions,

$$[D, P_a] = iP_a, \quad [D, K_a] = -iK_a, \quad [P_a, K_b] = -2iD\eta_{ab} - 2iM_{ab}$$
$$[M_{ab}, K_c] = i\left(\eta_{ac}K_b - \eta_{bc}K_a\right), \quad [M_{ab}, P_c] = i\left(\eta_{ac}P_b - \eta_{bc}P_a\right)$$
$$[M_{ab}, M_{cd}] = i\left(\eta_{ac}M_{bd} + \eta_{bd}M_{ac} - \eta_{ad}M_{bc} - \eta_{bc}M_{ad}\right)\,,$$

We add to this a $u(1)$ generator Q. We can also introduce another $so(d,1)$ algebra generated by Z_{ab} whose commutation relations are

$$[Z_{ab}, Z_{cd}] = i\left(\eta_{ac}Z_{bd} + \eta_{bd}Z_{ac} - \eta_{ad}Z_{bc} - \eta_{bc}Z_{ad}\right)\,.$$

By making combination

$$P_a = c\mathcal{P}_a, \quad K_a = c\mathcal{K}_a, \quad D = \frac{\mathcal{D}}{2} + c^2\mathcal{N}, \quad Q = c^2\mathcal{N} - \frac{\mathcal{D}}{2}$$
$$M_{ab} = \frac{\mathcal{M}_{ab}}{2} + c^2 S_{ab}, \quad Z_{ab} = \frac{\mathcal{M}_{ab}}{2} - c^2\mathcal{S}_{ab}\,.$$

Higher dimensional generalizations

The higher dimensional generalization of Galilean algebra with $u(1) \oplus so(d,1)$ extension is

$$
\begin{aligned}
&[\mathcal{P}_a, \mathcal{K}_b] = -2i\mathcal{N}\eta_{ab} - 2i\mathcal{S}_{ab}, \quad [\mathcal{D}, \mathcal{P}_a] = i\mathcal{P}_a, \quad [\mathcal{D}, \mathcal{K}_a] = -i\mathcal{K}_a, \\
&[\mathcal{M}_{ab}, \mathcal{K}_c] = i\left(\eta_{ac}\mathcal{K}_b - \eta_{bc}\mathcal{K}_a\right), \quad [\mathcal{M}_{ab}, \mathcal{P}_c] = i\left(\eta_{ac}\mathcal{P}_b - \eta_{bc}\mathcal{P}_a\right) \\
&[\mathcal{M}_{ab}, \mathcal{M}_{cd}] = i\left(\eta_{ac}\mathcal{M}_{bd} + \eta_{bd}\mathcal{M}_{ac} - \eta_{ad}\mathcal{M}_{bc} - \eta_{bc}\mathcal{M}_{ad}\right) \\
&[\mathcal{M}_{ab}, \mathcal{S}_{cd}] = i\left(\eta_{ac}\mathcal{S}_{bd} + \eta_{bd}\mathcal{S}_{ac} - \eta_{ad}\mathcal{S}_{bc} - \eta_{bc}\mathcal{S}_{ad}\right) .
\end{aligned}
$$

In $d = 3$, this theory is possibly related to spin matrix theory – the non-relativistic limit of $\mathcal{N} = 4$ super Yang-Mills theory (Harmark, Orselli, 1409.4417).

$$\lambda \to 0, \qquad E - \vec{\Omega} \cdot \vec{J} \to 0, \qquad \frac{E - \vec{\Omega} \cdot \vec{J}}{\lambda} \quad \text{fixed}$$

String theory embedding

- What is the string embedding of this gravity? The question is closely related to the origin of two $u(1)$ gauge fields. One possible interpretation is that the $u(1)$ gauge fields are R-symmetry generators of supergravity. The enlightens us to consider $\mathcal{N} = (2,2)$ supergravity.
- What happens to conformal algebra $so(2,2)$ if we take $c \to \infty$? The truth is by defining $g = c^{-2}$,

$$E = \frac{D}{c^2} = \mathcal{N} + \frac{g}{2}\mathcal{D}, \quad J = -\frac{Q}{c^2} = \mathcal{N} - \frac{g}{2}\mathcal{D}$$

  E is getting close to R-charge, so this is near BPS state.

Near BPS limit

We should think of this limit to be zooming in states near the BPS bound.

1 Introduction

2 Newton-Cartan gravity

3 Novel non-AdS holography

4 Conclusion and future work

Conclusion

- We pave the way to build up a non-relativistic gravity theory which is completely analogous to AdS_3 gravity. It is interesting to talk about any generalizations about it.
- We showed the whole phase space of WZW model of AdS can be mapped to its non-relativistic limit.
- We also derive the asymptotic symmetry of this non-AdS solution, which is twisted $u(1)$ Virasoro algebra.
- We hope this work shed light upon a non-relativistic version of holography, independent of Einstein gravity.
- We hope this model could provide a simpler model of holography, and the time is the holographic direction. Holographic direction decouples and is emergent.

Future work

We have many future work can do:

- NH supergravity/ NH torsional gravity
- String embedding from AdS reduction
- Higher spin
- Generalization of conformal symmetry
- BTZ black hole analogue
- Applications to cosmology
- Non-relativistic string theory and dualities
- Generalization of entanglement entropy in non-relativistic CFT
- Everything else you did in AdS_3/CFT_2