Boundary local SO(2,d) transformation from holography

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Outline

- Bulk Gravity as $SO(2, d)$ Gauge theory
- Bulk boundary relation
- The boundary consequence of bulk $SO(2, d)$ gauge symmetry

Motivation

- A lesson from AdS/CFT:
  Global symmetry in the boundary field theory
  $\Rightarrow$ Gauge symmetry in the bulk gravity theory

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- Thus we would expect the local $SO(2, d)$ gauge symmetry in the bulk theory.
- The $SO(2, d)$ gauge theory structure will lead to non-trivial consequence on the boundary CFT.

Einstein Gravity

- Written in terms of vielbein, the Einstein-Hilbert action is

$$S[e] = \int \epsilon_{a_1 \cdots a_D} \left[\Theta^{a_1 a_2} + \frac{(D-2)}{D\,\ell^2} e^{a_1} \wedge e^{a_2} \right] \wedge e^{a_3} \wedge \cdots \wedge e^{a_D}$$

  where the spin connection ${\omega^a}_b$ is decided by the torsion free condition

$$\mathrm{D}e^a = \mathrm{d}e^a + {\omega^a}_b \wedge e^b = 0$$

  and the curvature is decided by the spin connection

$$\Theta^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$$

- The Einstein equation is

$$\left(\Theta^{[a_1 a_2} + \ell^{-2} e^{[a_1} \wedge e^{a_2} \right) \wedge e^{a_3} \wedge \cdots \wedge e^{a_{D-1}]} = 0$$

Palatini's 1-st Order Formula

- Palatini: Treating the spin connection $\omega^a{}_b$ as independent variables

$$S[e,\omega]=\int \epsilon_{a_1\cdots a_D}\left[\Theta^{a_1a_2}+\frac{(D-2)}{D\,\ell^2}e^{a_1}\wedge e^{a_2}\right]\wedge e^{a_3}\wedge\cdots\wedge e^{a_D}$$

- The EOM's are

$$\left(\Theta^{[a_1a_2}+\ell^{-2}e^{[a_1}\wedge e^{a_2}\right)\wedge e^{a_3}\wedge\cdots\wedge e^{a_{D-1}]}=0$$

$$\mathrm{D}e^{[a_1}\wedge e^{a_2}\wedge\cdots\wedge e^{a_{D-2}]}=0$$

- If the vielbein e^a is not degenerate, the torsion free condition will be automatically implied by the second EOM!
- The 1-st order formalism is equivalent to the original second order formalism at least in the classical level.

Uplift to $SO(2, d)$

- Basic idea: $A^a{}_b \sim \omega^a{}_b\,, \qquad A^{a\bullet} \sim \ell^{-1} e^a$
- In general, we should impose an extra matter field $Y^{\hat{\alpha}}$ which is in the fundamental representation of $SO(2, d)$ and satisfies the constraints $Y^{\hat{\alpha}} Y_{\hat{\alpha}} = -1\,.$
- By the $SO(2, d)$ transformation, we can always reach the standard guage

$$Y_a = 0\,, \qquad Y_\bullet = 1\,.$$

- In this gauge

$$\mathrm{D}Y^a = \ell^{-1} e^a\,, \qquad \mathrm{D}Y^\bullet = 0\,,$$
$$\mathrm{D}\mathrm{D}Y^a = F^a{}_\bullet Y^\bullet = \ell^{-1}\,\mathrm{D}e^a\,, \qquad \mathrm{D}\mathrm{D}Y^\bullet = F^\bullet{}_b Y^b = 0\,.$$

- Now the Palatini EOM's can be unified in the $SO(2, d)$ covariant way

$$F^{[\hat{\alpha}_1 \hat{\alpha}_2} \wedge \mathrm{D}Y^{\hat{\alpha}_3} \wedge \cdots \wedge \mathrm{D}Y^{\hat{\alpha}_{D-1}]} = 0\,.$$

$SO(2,d)$ *invariant action*

- We can realize the previous uplift by a $SO(2,d)$ gauge invariant action.

$$\int \epsilon_{a_1\cdots a_D}\left[\Theta^{a_1 a_2} + \frac{(D-2)}{D\,\ell^2} e^{a_1}\wedge e^{a_2}\right]\wedge e^{a_3}\wedge\cdots\wedge e^{a_D}$$
$$\sim \int \epsilon_{\hat\alpha_1\cdots\hat\alpha_{D+1}}\left[F^{\hat\alpha_1\hat\alpha_2} - \frac{2}{D}\mathrm{D}Y^{\hat\alpha_1}\wedge \mathrm{D}Y^{\hat\alpha_2}\right]\wedge \mathrm{D}Y^{\hat\alpha_3}\wedge\cdots\wedge \mathrm{D}Y^{\hat\alpha_D}\,Y^{\hat\alpha_{D+1}}$$

- In the $D=3$ C-S formalism, the Y field is implicitly imposed when one decides the vielbein from the gauge field $e^a = A^a_L + A^a_R$.
- The metric is induced by the $Y^{\hat\alpha}$ field

$$ds^2 = \ell^2 \mathrm{D}_M Y^{\hat\alpha}\,\mathrm{D}_N Y_{\hat\alpha}\,dx^M dx^N\,.$$

$SO(2,d)$ *invariant action*

- The EOM's of the previous action are

$$F^{[\hat\alpha_1\hat\alpha_2}\wedge \mathrm{D}Y^{\hat\alpha_3}\wedge\cdots\wedge \mathrm{D}Y^{\hat\alpha_{D-1}]}=0\,,$$

  as well as the EOM from δY

$$\epsilon_{\hat\alpha_1\cdots\hat\alpha_D\hat\alpha}G^{\hat\alpha_1\cdots\hat\alpha_D}+\epsilon_{\hat\alpha_1\cdots\hat\alpha_D\hat\beta}G^{\hat\alpha_1\cdots\hat\alpha_D}Y^{\hat\beta}Y_{\hat\alpha}=0\,,$$

  where

$$\begin{aligned}
&G^{\hat\alpha_1\cdots\hat\alpha_D}\\
&=(D-2)(D-3)F^{[\hat\alpha_1\hat\alpha_2}\wedge F^{\hat\alpha_3}{}_{\hat\beta}Y^{|\hat\beta|}Y^{\hat\alpha_4}\wedge \mathrm{D}Y^{\hat\alpha_5}\wedge\cdots\wedge \mathrm{D}Y^{\hat\alpha_D]}\\
&\quad -2(D-1)F^{[\hat\alpha_1}{}_{\hat\beta}Y^{|\hat\beta|}Y^{\hat\alpha_2}\wedge \mathrm{D}Y^{\hat\alpha_3}\wedge\cdots\wedge \mathrm{D}Y^{\hat\alpha_D]}\\
&\quad -(D-1)F^{[\hat\alpha_1\hat\alpha_2}\wedge \mathrm{D}Y^{\hat\alpha_3}\wedge\cdots\wedge \mathrm{D}Y^{\hat\alpha_D]}+\tfrac{2(D+1)}{D}\mathrm{D}Y^{\hat\alpha_1}\wedge\cdots\wedge \mathrm{D}Y^{\hat\alpha_D}
\end{aligned}$$

- Providing the 1st EOM, the 2nd EOM will be automatically satisfied.
- Thus the new EOM will not introduce any further constraints, and the above system is equivalent to the original Einstein gravity classically.

Embedding gauge

- Einstein gauge

$$Y^{\hat{M}} = 0\,, \qquad Y^{\hat{\bullet}} = 1\,.$$

  In this gauge, it comes back to the Palatini action
- Another natural gauge choice which is more relevant for CFT construction is the embedding gauge:

$$\ell Y^{\hat{\mu}}(x,z) \;=\; \frac{x^\mu}{z}\,, \quad \ell Y^{\hat{d}}(x,z) = \frac{1 - x^\mu x_\mu - \alpha z^2}{2z}\,, \quad \ell Y^{\hat{\bullet}}(x,z) = \frac{1 + x^\mu x_\mu + \alpha z^2}{2z}\,.$$

- In the embedding gauge, $A = 0$ gives rise to the pure AdS vacuum.
- Fixing in the embedding gauge, any coordinate transformation can be mapped to a gauge transformation up to the orthogonal $SO(1,d)$. The isometry of pure AdS is mapped to its rigid part.

Holographic dual

- The usual holographic dictionary related the bulk metric g_{MN} and the boundary E-M tensor $T^{\mu\nu}$.
- In the $SO(2,d)$ gauge theory formalism, the duality is between the bulk gauge field A_M and the boundary $SO(2,d)$ conserved currents $\mathbb{J}^\mu$.
- Usually, for the flat background, the $SO(2,d)$ conformal currents is given by

$$(\mathbb{J}^{\hat{\alpha}\hat{\beta}})^\mu = 2\mathbb{Y}^{[\hat{\alpha}}\partial_\nu\mathbb{Y}^{\hat{\beta}]}T^{\mu\nu}\,,$$

  where the $SO(2,d)$ null-vector $\mathbb{Y}^{\hat{\alpha}}$ is

$$\mathbb{Y}^{\hat{\mu}} = x^\mu\,,\qquad \mathbb{Y}^{\hat{d}} = \tfrac{1}{2}(1-\eta_{\mu\nu}x^\mu x^\nu)\,,\qquad \mathbb{Y}^{\hat{\bullet}} = \tfrac{1}{2}(1+\eta_{\mu\nu}x^\mu x^\nu)\,.$$

- The conservation law $\partial_\mu\mathbb{J}^\mu = 0$ is equivalent to

$$\partial_\mu T^{\mu\nu} = 0\,,\qquad T^{[\mu\nu]} = 0\,,\qquad T^\mu{}_\mu = 0\,.$$

- The boundary metric is consistently given by

$$\mathbb{g}_{\mu\nu} = \partial_\mu\mathbb{Y}^{\hat{\alpha}}\partial_\nu\mathbb{Y}_{\hat{\alpha}} = \eta_{\mu\nu}.$$

The structure of $SO(2,d)$ currents

- The $\mathbb{Y}^{\hat{\alpha}}$ can be viewed as a boundary background field with $\Delta = -1$. In the present case, its bulk dual exact reproduce the $Y^{\hat{\alpha}}(x,z)$ in the embedding gauge

$$\ell Y^{\hat{\alpha}}(x,z) \;=\; z^{-1}{}_0F_1\left(;\Delta - \tfrac{d}{2}+1; -\tfrac{\alpha z^2}{4}\Box\right)\mathbb{Y}^{\hat{\alpha}}(x)$$

- A basis for $so(2,d)$ algebra

$$\begin{aligned}(\mho_\mu)^{\hat{\alpha}\hat{\beta}} &= 2\mathbb{Y}^{[\hat{\alpha}}\partial_\mu\mathbb{Y}^{\hat{\beta}]}, \qquad (\mho_{\mu\nu})^{\hat{\alpha}\hat{\beta}} = 2\partial_\mu\mathbb{Y}^{[\hat{\alpha}}\partial_\nu\mathbb{Y}^{\hat{\beta}]} = -(\mho_{\nu\mu})^{\hat{\alpha}\hat{\beta}}, \\ (\mho_\Box)^{\hat{\alpha}\hat{\beta}} &= \frac{2}{d}\mathbb{Y}^{[\hat{\alpha}}\Box\mathbb{Y}^{\hat{\beta}]}, \qquad (\mho_{\Box\mu})^{\hat{\alpha}\hat{\beta}} = \frac{2}{d}\Box\mathbb{Y}^{[\hat{\alpha}}\partial_\mu\mathbb{Y}^{\hat{\beta}]}.\end{aligned}$$

The commutators are

$$[\mho_\mu,\mho_\nu]=0\,,\qquad [\mho_{\mu_1\mu_2},\mho_\nu] = -2\eta_{\nu[\mu_1}\mho_{\mu_2]}\,,\qquad [\mho_\Box,\mho_\mu] = -\mho_\mu\,,\qquad [\mho_\Box,\mho_{\Box\mu}] = \mho_{\Box\mu}\,,$$
$$[\mho_{\mu_1\mu_2},\mho_{\nu_1\nu_2}] = 2(\eta_{\nu_1[\mu_2}\mho_{\mu_1]\nu_2} - \eta_{\nu_2[\mu_2}\mho_{\mu_1]\nu_1})\,,\qquad [\mho_{\mu_1\mu_2},\mho_\Box]=0\,,\qquad [\mho_{\Box\mu},\mho_{\Box\nu}]=0\,,$$
$$[\mho_{\mu_1\mu_2},\mho_{\Box\nu}] = -2\eta_{\nu[\mu_1}\mho_{\Box\mu_2]}\,,\qquad [\mho_{\Box\mu},\mho_\nu] = \mho_{\mu\nu} + \eta_{\mu\nu}\mho_\Box\,.$$

- The previous $\mathbb{J}^\mu$ is in the Cartan sub-algebra. In general

$$\mathbb{J}^\mu = T^{\mu\nu}\mho_\nu + \tfrac{1}{2}S^{\mu\nu_1\nu_2}\mho_{\nu_1\nu_2} + U^\mu\mho_\Box + V^{\mu\nu}\mho_{\Box\nu}\,.$$

The duality between A and $\mathbb{J}$

- What is the explicit relation between A and $\mathbb{J}$?
- The asymptotic AdS B.C is

$$F \sim O(z^{d-2})$$

  We can simply choose

$$A = \sum z^n A^{(n)}\,, \qquad (n \geq d-2).$$

- In general, the boundary duality relation is given by the double dual formalism

$$(\mathbb{J}^\mu)_{\hat{\alpha}\hat{\beta}} = \lambda \epsilon^{\mu\mu_1\cdots\mu_d} \epsilon_{\hat{\alpha}\hat{\beta}\hat{\alpha}_1\cdots\hat{\alpha}_d} (A^{(d-2)}_{\mu_1})^{\hat{\alpha}_1\hat{\alpha}_2} \partial_{\mu_2} \mathbb{Y}^{\hat{\alpha}_3} \cdots \partial_{\mu_d} \mathbb{Y}^{\hat{\alpha}_d}\,.$$

The duality between A and $\mathbb{J}$

- In terms of the components,

$$A_\mu^{(d-2)} = \mathcal{T}_\mu{}^\rho \mathbb{T}_\rho + \tfrac{1}{2}\mathcal{S}_\mu{}^{\rho_1\rho_2}\mathbb{T}_{\rho_1\rho_2} + \mathcal{U}_\mu \mathbb{T}_\Box + \mathcal{V}_\mu{}^\rho \mathbb{T}_{\Box\rho}\,,$$

we have

$$T^{\mu\nu} = \lambda_d(\mathcal{T}^{\nu\mu} - \mathcal{T}^\rho{}_\rho \eta^{\mu\nu})\,, \qquad S^{\mu\nu_1\nu_2} = 2\lambda_d \eta^{\mu[\nu_1}\mathcal{U}^{\nu_2]}\,,$$
$$U^\mu = -\lambda_d \mathcal{S}_\nu{}^{\mu\nu}\,, \qquad V^{\mu\nu} = -\lambda_d(\mathcal{V}^{\nu\mu} - \mathcal{V}^\rho{}_\rho \eta^{\mu\nu})\,.$$

- For $d > 2$, conservation equation $\partial_\mu \mathbb{J}^\mu = 0$ is equivalent to

$$(\mathrm{d}A^{(d-2)})^{[\hat\alpha_1\hat\alpha_2} \wedge \mathrm{d}\mathbb{Y}^{\hat\alpha_3} \wedge \cdots \wedge \mathrm{d}\mathbb{Y}^{\hat\alpha_d]} = 0$$

which is the leading z^0 order of the bulk EOM

$$F^{[\hat\alpha_1\hat\alpha_2} \wedge \mathrm{D}Y^{\hat\alpha_3} \wedge \cdots \wedge \mathrm{D}Y^{\hat\alpha_{D-1}]} = 0\,.$$

- For $d = 2$, the above is still valid if $A^{(0)}$ is valued in the cartan sub-algebra.

Boundary $SO(2,d)$ gauge field

- The bulk configuration is described by $\{Y, A\}$. It allows $A \to UAU^{-1} - \mathrm{d}UU^{-1}$. If U contains terms lower than z^{d-2} , we will go beyond the choice $A \sim O(z^{d-2})$.
- Especially, one can turning on the boundary $SO(2,d)$ gauge field $\mathbb{A} = A^{(0)}$.
- From the CFT point of view, it means localizing the original rigid conformal symmetry.
- The initial choice of $\mathbb{Y}$ can be viewed as the boundary embedding gauge.
- Fixing in the boundary embedding gauge, any boundary diff$\times$weyl transformation can be mapped to a gauge transformation up to the orthogonal $ISO(1,d-1)$. The original conformal symmetry is mapped to the rigid part.

Boundary $SO(2,d)$ gauge field

- Turning on $\mathbb{A}$, the boundary metric $\mathbb{g}_{\mu\nu} = \mathbb{D}_\mu \mathbb{Y}^{\hat{\alpha}} \mathbb{D}_\nu \mathbb{Y}_{\hat{\alpha}}$ is not always flat.
- Since it is pure gauge, we can turn it to be zero by $SO(2,d)$ transformation. Then in general $Y^{\hat{\alpha}}$ is non-longer in the embedding gauge. We can take the coordinate system

$$x^\mu = \mathbb{Y}^{\hat{\mu}} / \mathbb{Y}^{\hat{+}} , \qquad \mathbb{Y}^{\hat{+}} = \mathbb{Y}^{\hat{d}} + \mathbb{Y}^{\hat{\bullet}} .$$

Thus the corresponding metric is

$$\mathbb{D}_\mu \mathbb{Y}^{\hat{\alpha}} \mathbb{D}_\nu \mathbb{Y}_{\hat{\alpha}} = (\mathbb{Y}^{\hat{+}})^2 \eta_{\mu\nu} .$$

It means that the general boundary metric allowed by the $F \sim O(z^{d-2})$ boundary condition is conformal flat.

Boundary $SO(2,d)$ gauge field

- We can establish the corresponding $so(2,d)$ basis as following:

$$(\hat{\mathfrak{v}}_\mu)^{\hat\alpha\hat\beta} = 2\mathbb{Y}^{[\hat\alpha}\hat{\mathbb{D}}_\mu\mathbb{Y}^{\hat\beta]}\,, \qquad (\hat{\mathfrak{v}}_{\mu\nu})^{\hat\alpha\hat\beta} = 2\partial_\mu\mathbb{Y}^{[\hat\alpha}\hat{\mathbb{D}}_\nu\mathbb{Y}^{\hat\beta]} = -(\hat{\mathfrak{v}}_{\nu\mu})^{\hat\alpha\hat\beta}\,,$$
$$(\hat{\mathfrak{v}}_\Box)^{\hat\alpha\hat\beta} = \frac{2}{d}\mathbb{Y}^{[\hat\alpha}\hat\Box\mathbb{Y}^{\hat\beta]}\,, \qquad (\hat{\mathfrak{v}}_{\Box\mu})^{\hat\alpha\hat\beta} = \frac{2}{d}\hat\Box\mathbb{Y}^{[\hat\alpha}\partial_\mu\mathbb{Y}^{\hat\beta]}\,.$$

where $\hat{\mathbb{D}}$ is the covariant derivative for diff+gauge, and

$$\hat\Box\mathbb{Y}^{\hat\alpha} = \left(\hat{\mathbb{D}}^2 + \frac{\hat{\mathbb{D}}^2\mathbb{Y}^{\hat\beta}\hat{\mathbb{D}}^2\mathbb{Y}_{\hat\beta}}{2d}\right)\mathbb{Y}^{\hat\alpha} = \left(\hat{\mathbb{D}}^2 + \frac{\mathbb{R}}{2(d-1)}\right)\mathbb{Y}^{\hat\alpha}\,.$$

- The commutators are now compatible with the metric $\mathbb{g}_{\mu\nu}$

$$[\hat{\mathfrak{v}}_\mu,\hat{\mathfrak{v}}_\nu] = 0\,, \quad [\hat{\mathfrak{v}}_{\mu_1\mu_2},\hat{\mathfrak{v}}_\nu] = -2\mathbb{g}_{\nu[\mu_1}\hat{\mathfrak{v}}_{\mu_2]}\,, \quad [\hat{\mathfrak{v}}_\Box,\hat{\mathfrak{v}}_\mu] = -\hat{\mathfrak{v}}_\mu\,, \quad [\hat{\mathfrak{v}}_\Box,\hat{\mathfrak{v}}_{\Box\mu}] = \hat{\mathfrak{v}}_{\Box\mu}\,,$$
$$[\hat{\mathfrak{v}}_{\mu_1\mu_2},\hat{\mathfrak{v}}_{\nu_1\nu_2}] = 2(\mathbb{g}_{\nu_1[\mu_2}\hat{\mathfrak{v}}_{\mu_1]\nu_2} - \mathbb{g}_{\nu_2[\mu_2}\hat{\mathfrak{v}}_{\mu_1]\nu_1})\,, \quad [\hat{\mathfrak{v}}_{\mu_1\mu_2},\hat{\mathfrak{v}}_\Box] = 0\,, \quad [\hat{\mathfrak{v}}_{\Box\mu},\hat{\mathfrak{v}}_{\Box\nu}] = 0\,,$$
$$[\hat{\mathfrak{v}}_{\mu_1\mu_2},\hat{\mathfrak{v}}_{\Box\nu}] = -2\mathbb{g}_{\nu[\mu_1}\hat{\mathfrak{v}}_{\Box\mu_2]}\,, \quad [\hat{\mathfrak{v}}_{\Box\mu},\hat{\mathfrak{v}}_\nu] = \hat{\mathfrak{v}}_{\mu\nu} + \mathbb{g}_{\mu\nu}\hat{\mathfrak{v}}_\Box\,.$$

- The $\mathbb{J}^\mu$ should also be written in terms of the new basis

$$\mathbb{J}^\mu = T^{\mu\nu}\hat{\mathfrak{v}}_\nu + \tfrac{1}{2}S^{\mu\nu_1\nu_2}\hat{\mathfrak{v}}_{\nu_1\nu_2} + U^\mu\hat{\mathfrak{v}}_\Box + V^{\mu\nu}\hat{\mathfrak{v}}_{\Box\nu}\,.$$

Schwarzian derivative from $SO(2,2)$

- As an explicit example, let us consider the $d=2$ non-rigid conformal transformation.
- In terms of the complex coordinates $\{x^\mu\} = \{w, \bar{w}\}$, the embedding gauge is

$$\mathbb{Y}^{\hat{w}}(x) = w\,, \quad \mathbb{Y}^{\hat{\bar{w}}}(x) = \bar{w}\,, \quad \mathbb{Y}^{\hat{+}}(x) = 1\,, \quad \mathbb{Y}^{\hat{-}}(x) = -w\bar{w}\,.$$

- After a general conformal transformation $\tilde{w} = f(w)$, the original background primary transforms as

$$\tilde{\mathbb{Y}}^{\hat{\alpha}}(\tilde{x}) = (f'\bar{f}')^{\frac{1}{2}}\mathbb{Y}^{\hat{\alpha}}(x)$$

In the new coordinate system, it is non-longer in the standard form of $\mathbb{Y}^{\hat{\alpha}}(\tilde{x})$ the embedding gauge.

Schwarzian derivative from $SO(2,2)$

- We can find a $SO(2,2)$ transformation takes $\mathbb{Y}^{\hat{\alpha}}(\tilde{x})$ to $\tilde{\mathbb{Y}}^{\hat{\alpha}}(\tilde{x})$. It is given by

$$(\Lambda^{(1)})^{\hat{\alpha}}{}_{\hat{\beta}}(w,\bar{w}) = (\Lambda^{(L)})^{\hat{\alpha}}{}_{\hat{\gamma}}(w)(\Lambda^{(R)})^{\hat{\gamma}}{}_{\hat{\beta}}(\bar{w}) = (\Lambda^{(R)})^{\hat{\gamma}}{}_{\hat{\beta}}(\bar{w})(\Lambda^{(L)})^{\hat{\alpha}}{}_{\hat{\gamma}}(w)$$

$$= \begin{pmatrix} (\psi_2 - w\psi_2')\bar{\psi}_1' & (\psi_1 - w\psi_1')\bar{\psi}_2' & -(\psi_1 - w\psi_1')\bar{\psi}_1' & (\psi_2 - w\psi_2')\bar{\psi}_2' \\ \psi_2'(\bar{\psi}_1 - \bar{w}\bar{\psi}_1') & \psi_1'(\bar{\psi}_2 - \bar{w}\bar{\psi}_2') & -\psi_1'(\bar{\psi}_1 - \bar{w}\bar{\psi}_1') & \psi_2'(\bar{\psi}_2 - \bar{w}\bar{\psi}_2') \\ -\psi_2'\bar{\psi}_1' & -\psi_1'\bar{\psi}_2' & \psi_1'\bar{\psi}_1' & -\psi_2'\bar{\psi}_2' \\ (\psi_2 - w\psi_2')(\bar{\psi}_1 - \bar{w}\bar{\psi}_1') & (\psi_1 - w\psi_1')(\bar{\psi}_2 - \bar{w}\bar{\psi}_2') & -(\psi_1 - w\psi_1')(\bar{\psi}_1 - \bar{w}\bar{\psi}_1') & (\psi_2 - w\psi_2')(\bar{\psi}_2 - \bar{w}\bar{\psi}_2') \end{pmatrix}$$

where

$$\psi_2 = (f')^{-\frac{1}{2}}\,, \quad \psi_1 = \psi_2 f = (f')^{-\frac{1}{2}} f\,.$$

Its left and right movers are respectively

$$(\Lambda^{(L)})^{\hat{\alpha}}{}_{\hat{\beta}}(w) \; = \; (\Lambda^{(1)})^{\hat{\alpha}}{}_{\hat{\beta}}|_{\bar{f}=\bar{w}}\,, \qquad (\Lambda^{(R)})^{\hat{\alpha}}{}_{\hat{\beta}}(\bar{w}) = (\Lambda^{(1)})^{\hat{\alpha}}{}_{\hat{\beta}}|_{f=w}\,.$$

Schwarzian derivative from $SO(2,2)$

- The vacuum $|\tilde{\Omega}\rangle$ defined in $\{\tilde{x}\}$ frame is decided by

$$\langle\tilde{\Omega}|\tilde{T}_{\tilde{w}\tilde{w}}(\tilde{w})|\tilde{\Omega}\rangle = 0\,.$$

  After the double dual, the corresponding gauge field is also vanishing $\mathbb{A}(\tilde{x}) = 0$.
- Now going back to the $\{x\}$ frame, the corresponding gauge field becomes non-zero due to the gauge transformation

$$\mathbb{A}(x) \;=\; (\Lambda^{(1)})^{\hat{\alpha}}{}_{\hat{\gamma}}\mathrm{d}(\Lambda^{(1)})^{\hat{\beta}\hat{\gamma}} = -\mathbb{S}\mathbb{t}_{\bar{w}}\mathrm{d}w - \bar{\mathbb{S}}\mathbb{t}_{w}\mathrm{d}\bar{w}\,.$$

  where $\mathbb{S}$ is the Schwarzian derivative

$$\mathbb{S} = \{f, w\}_S = \frac{2f'''f' - 3(f'')^2}{2(f')^2}\,.$$

- The $\mathbb{A}(x)$ is valued in the cartan sub-algebra. Thus we have $\hat{\mathbb{t}}_\mu = \mathbb{t}_\mu$.

Schwarzian derivative from $SO(2,2)$

- The double dual gives the corresponding $SO(2,d)$ current

$$\mathbb{J}_\mu = T_\mu{}^\rho \hat{\mathbb{t}}_\rho = -\lambda_2(\mathbb{S}\hat{\mathbb{t}}_{\bar{w}}\mathrm{d}w + \bar{\mathbb{S}}\hat{\mathbb{t}}_w\mathrm{d}\bar{w})\,.$$

- Thus measured by the E-M tensor defined in $\{x\}$ frame

$$\langle\Omega|T_{ww}(w)|\Omega\rangle = 0$$

the state $|\tilde{\Omega}\rangle$ gives rise to

$$\langle\tilde{\Omega}|T_{ww}|\tilde{\Omega}\rangle = -\tfrac{1}{2}\lambda_2\mathbb{S}\,.$$

- Comparing with the conformal transformation rule

$$\tilde{T}_{\tilde{w}\tilde{w}}(\tilde{w}) = (f')^{-2}\left(T_{ww}(w) - \frac{c}{12}\mathbb{S}\right)\,,$$

we can fix $\lambda_2 = -\frac{c}{6}$.

Schwarzian derivative from $SO(2,2)$

- By acting the same $SO(2,2)$ transformation $(\Lambda^{(1)})^{\hat{\alpha}}{}_{\hat{\beta}}(w,\bar{w})$ on the bulk field $Y^{\hat{\alpha}}$, it will induce the bulk coordinate transformation

$$\tilde{w} = f - \frac{2\ell^2 z^2 (f')^2 \bar{f}''}{4f'\bar{f}' + \ell^2 z^2 f'' \bar{f}''}\,, \quad \tilde{\bar{w}} = \bar{f} - \frac{2\ell^2 z^2 f'' (\bar{f}')^2}{4f'\bar{f}' + \ell^2 z^2 f'' \bar{f}''}\,, \quad \tilde{z} = \frac{4z (f')^{\frac{3}{2}} (\bar{f}')^{\frac{3}{2}}}{4f'\bar{f}' + \ell^2 z^2 f'' \bar{f}''}\,.$$

- The corresponding metric is

$$\begin{aligned} \langle\tilde{\Omega}|ds^2|\tilde{\Omega}\rangle \;&=\; \mathrm{D}Y^{\hat{\alpha}}(x,z)\mathrm{D}Y_{\hat{\alpha}}(x,z) = \mathrm{d}Y^{\hat{\alpha}}(\tilde{x},\tilde{z})\mathrm{d}Y_{\hat{\alpha}}(\tilde{x},\tilde{z}) = \frac{1}{\tilde{z}^2}\left[\hat{\alpha}\mathrm{d}\tilde{z}^2 + \mathrm{d}\tilde{w}\mathrm{d}\tilde{\bar{w}}\right] \\ &=\; \frac{1}{z^2}\left[\ell^2\mathrm{d}z^2 + (\mathrm{d}w - \tfrac{1}{2}\ell^2 z^2 \bar{\$}\,\mathrm{d}\bar{w})(\mathrm{d}\bar{w} - \tfrac{1}{2}\ell^2 z^2 \$\,\mathrm{d}w)\right]. \end{aligned}$$

This is the most general bulk solution for Brown-Heneaux boundary condition.

General $SO(2,2)$ transformation

- A general $SO(2,2)$ transformation is $\Lambda = \Lambda^{(0)}\Lambda^{(W)}\Lambda^{(L)}\Lambda^{(R)}$.
- The $\Lambda^{(W)}$ corresponds to the Weyl transformation. The $\Lambda^{(L)}$ and $\Lambda^{(R)}$ are the generalized left and right mover.

$$\Lambda^{(W)}(w,\bar{w}) = \begin{pmatrix} 1 & 0 & (e^{\sigma}-1)w & 0 \\ 0 & 1 & (e^{\sigma}-1)\bar{w} & 0 \\ 0 & 0 & e^{-\sigma} & 0 \\ (e^{-\sigma}-1)\bar{w} & (e^{-\sigma}-1)w & -(e^{\sigma}-e^{-\sigma})^2 w\bar{w} & e^{\sigma} \end{pmatrix},$$

$$\Lambda^{(L)}(w,\bar{w}) = \begin{pmatrix} \psi_2 - w\partial_w\psi_2 & 0 & -(\psi_1 - w\partial_w\psi_1) & 0 \\ 0 & \partial_w\psi_1 & 0 & \partial_w\psi_2 \\ -\partial_w\psi_2 & 0 & \partial_w\psi_1 & 0 \\ 0 & \psi_1 - w\partial_w\psi_1 & 0 & \psi_2 - w\partial_w\psi_2 \end{pmatrix},$$

$$\Lambda^{(R)}(w,\bar{w}) = \begin{pmatrix} \partial_{\bar{w}}\bar{\psi}_1 & 0 & 0 & \partial_{\bar{w}}\bar{\psi}_2 \\ 0 & \bar{\psi}_2 - \bar{w}\partial_{\bar{w}}\bar{\psi}_2 & -(\bar{\psi}_1 - \bar{w}\partial_{\bar{w}}\bar{\psi}_1) & 0 \\ 0 & -\partial_{\bar{w}}\bar{\psi}_2 & \partial_{\bar{w}}\bar{\psi}_1 & 0 \\ \bar{\psi}_1 - \bar{w}\partial_{\bar{w}}\bar{\psi}_1 & 0 & 0 & \bar{\psi}_2 - \bar{w}\partial_{\bar{w}}\bar{\psi}_2 \end{pmatrix}$$

General $SO(2,2)$ transformation

- ψ_1 and ψ_2 are decide by the general coordinate transformation $w \to \tilde{w} = f(w,\bar{w})$ instead of the holomorphic one.

$$\begin{aligned} \psi_2 &= (\partial_w f)^{-\frac{1}{2}} , \quad \psi_1 = \psi_2 f = (\partial_w f)^{-\frac{1}{2}} f , \\ \bar{\psi}_2 &= (\partial_{\bar{w}} \bar{f})^{-\frac{1}{2}} , \quad \bar{\psi}_1 = \bar{\psi}_2 \bar{f} = (\partial_{\bar{w}} \bar{f})^{-\frac{1}{2}} \bar{f} . \end{aligned}$$

- $\Lambda^{(0)}$ is the $ISO(1,1)$ which leaves $\mathbb{Y}$ intact.

$$\Lambda^{(0)}(w,\bar{w}) = \begin{pmatrix} 1 & 0 & 0 & w \\ 0 & 1 & 0 & \bar{w} \\ 0 & 0 & 0 & 1 \\ -\bar{w} & -w & 1 & -w\bar{w} \end{pmatrix} \begin{pmatrix} e^{\phi} & 0 & f_1 & 0 \\ 0 & e^{-\phi} & f_2 & 0 \\ 0 & 0 & 1 & 0 \\ -e^{\phi} f_2 & -e^{-\phi} f_1 & -f_1 f_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -w & 0 \\ 0 & 1 & -\bar{w} & 0 \\ \bar{w} & w & -w\bar{w} & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

- The resulting metric is

$$\begin{aligned} \mathrm{d}s^2 &= \mathbb{D}\mathbb{Y}^{\hat{\alpha}} \mathbb{D}\mathbb{Y}_{\hat{\alpha}} = e^{-2\Omega} \left[\left(\partial_w f \partial_{\bar{w}} \bar{f} + \partial_{\bar{w}} f \partial_w \bar{f} \right) \mathrm{d}w \mathrm{d}\bar{w} + \partial_w f \partial_w \bar{f} \mathrm{d}w^2 + \partial_{\bar{w}} f \partial_{\bar{w}} \bar{f} \mathrm{d}\bar{w}^2 \right] \\ &= e^{-2\Omega} \mathrm{d}f \mathrm{d}\bar{f} = e^{-2\sigma} \left(\partial_w f \partial_{\bar{w}} \bar{f} \right)^{-1} \mathrm{d}f \mathrm{d}\bar{f} \end{aligned}$$

Weyl anomaly

- For simplicity, we take f to be holomorphic again, and

$$f_1 \;=\; -e^{\phi+\sigma}\partial_{\bar w}\sigma\,, \qquad\qquad f_2 = -e^{-\phi+\sigma}\partial_w\sigma\,.$$

- Then after the double dual, we get

$$\begin{aligned} J^{\bar w} =& -\frac{2c}{3}e^{4\sigma}\partial_w\partial_{\bar w}\sigma\,\hat\tau_w + \frac{2c}{3}e^{4\sigma}\left[\partial_w^2\sigma + (\partial_w\sigma)^2 + \frac{2\partial_w f\,\partial_w^3 f - 3(\partial_w^2 f)^2}{4(\partial_w f)^2}\right]\hat\tau_{\bar w} \\ &+\frac{c}{3}e^{2\sigma}\left(\partial_w\phi + \partial_w\sigma\right)\hat\tau_\Box - \frac{c}{3}e^{2\sigma}\left(1-e^{-\phi+\sigma}\right)\hat\tau_{\Box w}\,, \\ J^w =& \frac{2c}{3}e^{4\sigma}\left[\partial_{\bar w}^2\sigma + (\partial_{\bar w}\sigma)^2 + \frac{3(\partial_{\bar w}^2\bar f)^2 - 2\partial_{\bar w}\bar f\,\partial_{\bar w}^3\bar f}{4(\partial_{\bar w}\bar f)^2}\right]\hat\tau_w - \frac{2c}{3}e^{4\sigma}\partial_{\bar w}\partial_w\sigma\,\hat\tau_{\bar w} \\ &-\frac{c}{3}e^{2\sigma}\left(\partial_{\bar w}\phi - \partial_{\bar w}\sigma\right)\hat\tau_\Box + \frac{c}{3}e^{2\sigma}\left(1-e^{\phi+\sigma}\right)\hat\tau_{\Box\bar w}\,. \end{aligned}$$

- The corresponding E-M tensor is

$$\begin{aligned} &T_{ww} = \frac{c}{6}\left[\partial_w^2\sigma + (\partial_w\sigma)^2 + \frac{1}{2}\mathbb{S}\right], \qquad T_{\bar w\bar w} = \frac{c}{6}\left[\partial_{\bar w}^2\sigma + (\partial_{\bar w}\sigma)^2 + \frac{1}{2}\bar{\mathbb{S}}\right], \\ &T_{w\bar w} = T_{\bar w w} = -\frac{c}{6}\partial_w\partial_{\bar w}\sigma\,. \end{aligned}$$

- It gives rise to the correct Weyl anomaly

$$T^\mu{}_\mu = -\frac{2c}{3}e^{2\sigma}\partial_w\partial_{\bar w}\sigma = -\frac{c}{12}R\,.$$

Weyl anomaly

- For $d > 2$, let us look at the components of the conservation equation directly

$$\hat{\mathbb{D}}_\mu \mathbb{J}^\mu = \left(\nabla_\mu T^{\mu\rho} - \frac{1}{d}\mathbb{f}_{\mu\nu} S^{\mu\nu\rho} - \frac{1}{d}\mathbb{f}_\mu{}^\rho U^\mu\right)\hat{\mathfrak{t}}_\rho + \left(\frac{1}{2}\nabla_\mu S^{\mu\nu\rho} + T^{[\nu\rho]} - \frac{1}{d}\mathbb{f}_\mu{}^{[\nu} V^{|\mu|\rho]}\right)\hat{\mathfrak{t}}_{\nu\rho}$$
$$+ \left(\nabla_\mu U^\mu + T_\mu{}^\mu + \frac{1}{d}\mathbb{f}_{\mu\nu} V^{\mu\nu}\right)\hat{\mathfrak{t}}_\Box + \left(\nabla_\mu V^{\mu\nu} + S_\mu{}^{\mu\nu} - U^\nu\right)\hat{\mathfrak{t}}_{\Box\nu}\,,$$

where $\mathbb{f}_{\mu\nu} = \frac{d}{d-2}\left(\mathbb{R}_{\mu\nu} - \frac{1}{2(d-1)}\mathbb{R}\,\mathbb{g}_{\mu\nu}\right)$.

- If $S^{\mu\nu\rho} = 0$ and $U^\mu = 0$, we get

$$\nabla_\mu T^{\mu\rho} = 0\,, \quad T^{[\nu\rho]} = \frac{1}{d}\mathbb{f}_\mu{}^{[\nu} V^{|\mu|\rho]}\,, \quad T_\mu{}^\mu = -\frac{1}{d}\mathbb{f}_{\mu\nu} V^{\mu\nu}\,, \quad \nabla_\mu V^{\mu\nu} = 0\,.$$

- In the embedding gauge, to get the non-flat metric, we must have $V^{\mu\nu} \neq 0$.
- For $d = 2k$, $V^\mu{}_\nu = \delta^{\mu\mu_2\cdots\mu_k}_{\nu\nu_2\cdots\nu_k}\mathbb{f}_{\mu_2}{}^{\nu_2}\cdots\mathbb{f}_{\mu_k}{}^{\nu_k}$ satisfies the above constraints, and $\mathbb{f}_{\mu\nu} V^{\mu\nu}$ is proportion to the Euler class.

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  - The bulk Einstein equation=The conservation law of $\mathbb{J}$ on any given hyper-plane.