

# Toward a Worksheet Theory of Entanglement Entropy

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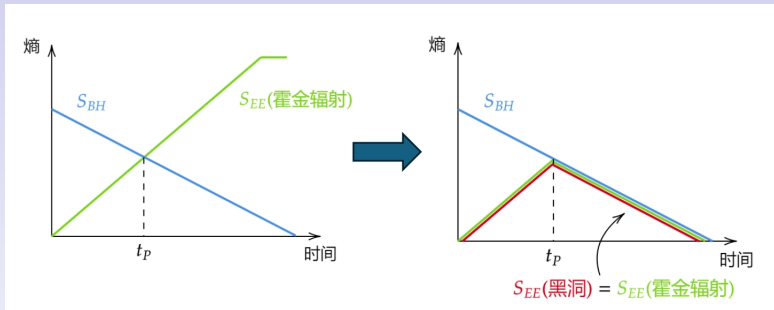
- 1 Motivation
- 2 The action
- 3 Point 1: Emergent gravity
- 4 Point 2: Predictions
- 5 Unified Framework for ER=EPR and Susskind-Uglum
- 6 Quantized RT surface and connections to LQG
- 7 Conclusion

Based on HW and Shuxuan Ying, "Toward a worldsheet theory of entanglement entropy," Phys. Rev. D **113**, no.6, 066006 (2026), [arXiv:2511.16586 [hep-th]].

# Motivation

# Motivation

Black hole information paradox resolution:



If unitarity is never broken, information is never lost.

Hawking's mistake is in using the Coarse-grained entropy:

- ① **Fine grained entropy:** Von Neuman entropy / entanglement entropy.
- ② **Coarse grained entropy:** Thermodynamic entropy (The horizon area computes this entropy).

The gravitational fine grained entropy formula gives the consistent entropy of black hole radiation with unitarity, namely  $S_{EE}(\text{黑洞}) = S_{EE}(\text{霍金辐射})$ .

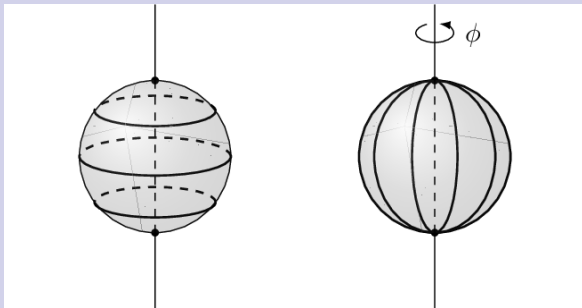
### Is the information puzzle solved?

- One aspect: computing the entropy, yes.
- Another aspect: Understanding what the state is, no.
- Replica wormholes compute the entropy, but do not tell us what the actual state is.
- As with black hole entropy, it is an accounting “oracle”. The explicit gravitational representation of the states is still mysterious. But the semiclassical solution is representing some aspects of the state.

However, this is challenging, since holographic methods—such as the computation of geodesic lengths—do not directly provide access to the underlying quantum state.

String theory is a possible resolution:

## Susskind and Uglum's conjecture (1994)



Inserting a conical singularity, corresponding to the tip of the cigar geometry.

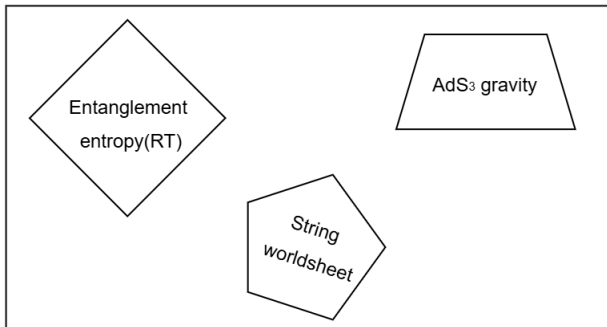
- **Closed string perspective:** Tseytlin's off-shell formalism can be used in this picture to calculate the tree-level BH entropy (A. Ahmadain and A. Wall, 2023).
- **Open string perspective:** Speculatively, von Neumann entropy (entanglement entropy) may be obtained by tracing out the open string inside the horizon.

If entanglement entropy is derived from string theory, the state can be read off directly. But how to prove it?

# Motivation

- If the Susskind–Uglum conjecture is correct, there should exist a correspondence between the **string worldsheet** and the **RT surface**.
- Since the **RT surface** arises from  $\text{AdS}_3$  in our setup, there must also exist a corresponding **AdS<sub>3</sub> gravity** description.

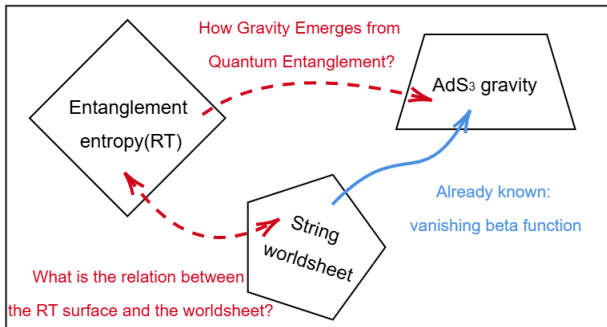
This leads to a nontrivial consistency requirement: the three components must fit together precisely.



# Motivation

- If the Susskind–Ugulum conjecture is correct, there should exist a correspondence between the **string worldsheet** and the **RT surface (entanglement entropy)**.
- Since the **RT surface** arises from  $\text{AdS}_3$  in our setup, there must also exist a corresponding **AdS<sub>3</sub> gravity** description.

This leads to a nontrivial consistency requirement: the three components must fit together precisely.



Previous studies suggest that this problem may appear intractable.

Rather than pursuing a bottom-up strategy that attempts to collect fragmented evidence from different theories,

we instead adopt a top-down approach:

**Construct a fundamental action for entanglement entropy of  $\text{CFT}_2$**

The central question then becomes: how can such an action be constructed?

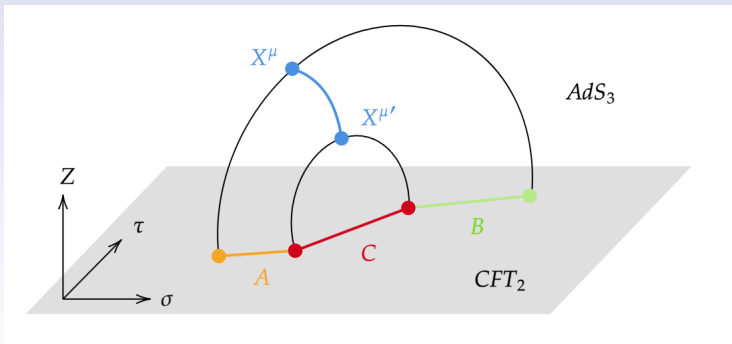
At present, our only input is entanglement entropy itself.

# The action

# The action

Let us first recall the entanglement wedge cross section (EWCS) in  $AdS_3$ , which measures the entanglement between subsystems  $A$  and  $B$ :

$$S_{EE}(A : B) = \frac{L(X, X')}{4G_N} = \frac{c}{6} \cosh^{-1} \left( 1 + \frac{\Delta Z^2 + \Delta\sigma^2 - \Delta\tau^2}{2ZZ'} \right). \quad (1)$$



# The action

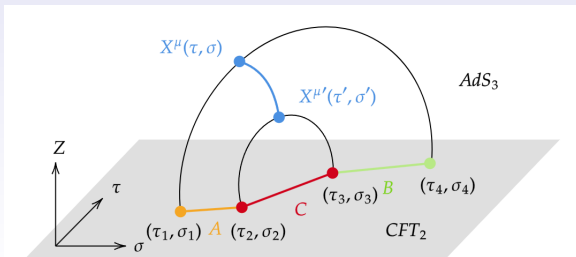
To better understand the geodesic length  $L(X, X')$ , we observe that the bulk radial coordinates  $Z$  and  $Z'$  can be expressed in terms of CFT coordinates (**Q. Wen and H. Zhong, 2022; X. Jiang, P. Wang, HW and H. Yang, 2025**):

$$Z = \frac{|\xi_{14}| \sqrt{|\xi_{12}| |\xi_{13}| |\xi_{34}| |\xi_{24}|}}{|\xi_{12}| |\xi_{13}| + |\xi_{34}| |\xi_{24}|}, \quad Z' = \frac{|\xi_{23}| \sqrt{|\xi_{12}| |\xi_{13}| |\xi_{34}| |\xi_{24}|}}{|\xi_{13}| |\xi_{34}| + |\xi_{12}| |\xi_{24}|}, \quad (2)$$

where  $\xi_{ij} \equiv \xi_i - \xi_j$ ,  $\xi_i = \sigma_i + i\tau_i$ . This suggests that bulk points can be parametrized by boundary data as

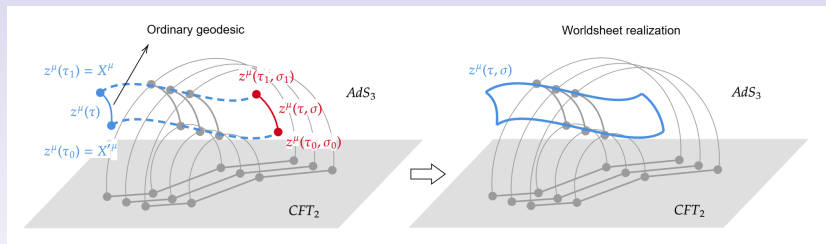
$$X^\mu(\tau, \sigma) \quad \text{and} \quad X^{\mu'}(\tau', \sigma'), \quad \mu = 0, 1, 2. \quad (3)$$

A similar structure also appears in conformal blocks involving internal twist operators in  $\text{CFT}_2$ .



# The action

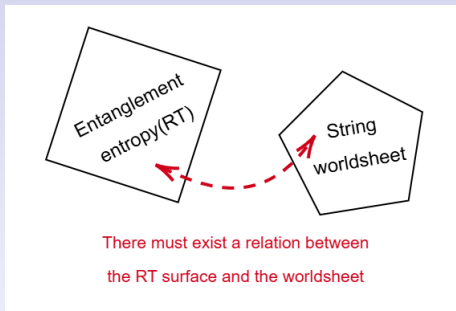
Now consider deforming the entangling regions  $A$  and  $B$ —through translation, rotation, stretching, or shrinking. The corresponding bulk geodesic is then continuously deformed.



- This is not a string worldsheet.
- Rather, it is a two-parameter family of geodesics. The union (envelope) of this family defines an emergent two-dimensional surface in the bulk.

## The action

Based on this observation, let us return to our central puzzle:



We are thus led to a bold step:

**Can we construct an action for this emergent surface—analogue to the Polyakov action—that captures the dynamics of entanglement entropy under deformations of the entangling regions?**

Let us recall how to construct the action for a string worldsheet:

- 1 Integrate the minimal area:  $\int dA$ .
- 2 Introduce appropriate prefactors (e.g.,  $\ell^2$ ) to ensure the action is dimensionless.
- 3 Parametrize the worldsheet by suitable coordinates, e.g.,  $(\tau, \sigma)$ .

This suggests that, to construct an analogous action, we require the following ingredients:

Area term,                      target space coordinates  $X^\mu$

These quantities must be constructed purely from entanglement entropy.

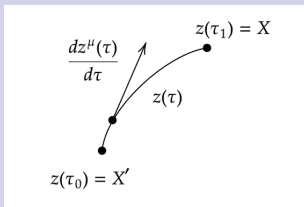
## The action

Luckily, these quantities can be constructed using Synge's world function:

Consider a geodesic  $z^\mu(\tau)$  connecting two spacetime points  $X'$  and  $X$ . The affine parameter  $\tau$  runs from  $\tau_0$  to  $\tau_1$ :

$$z(\tau_0) \equiv X', \quad z(\tau_1) \equiv X. \quad (4)$$

The tangent vector along the geodesic is  $dz^\mu(\tau)/d\tau$ :



The Synge world function  $\Omega(X, X')$  is defined as

$$\Omega(X, X') = \frac{1}{2} (\tau_1 - \tau_0) \int_{\tau_0}^{\tau_1} g_{\mu\nu}(z) \frac{dz^\mu}{d\tau} \frac{dz^\nu}{d\tau} d\tau. \quad (5)$$

It can equivalently be expressed in terms of the geodesic length:

$$\Omega(X, X') = \frac{1}{2} L(X, X')^2 = \frac{1}{2} \left( 4G_N^{(3)} S_{EE} \right)^2. \quad (6)$$

## The action

Based on Sygne's world function, we obtain the following building blocks:

$$\begin{array}{l} \text{Vector} \\ \Omega_\mu := \partial\Omega/\partial X^\mu \\ \Omega_{\mu'} := \partial\Omega/\partial X^{\mu'} \end{array}$$

$$\begin{array}{l} \text{Tensor} \\ \Omega_{\mu\nu} := \nabla_\nu \Omega_\mu \\ \Omega_{\mu\nu'} := \partial_{\nu'} \Omega_\mu \end{array}$$

$$\text{Area} \quad 2\Omega = \Omega_{\mu\nu} \Omega^\mu \Omega^\nu$$

We are now in a position to mimic the string worldsheet action and propose a fundamental action for entanglement entropy:

$$S_E = \frac{1}{\ell^2} \int d^2\sigma' \Omega_{\mu'\nu'}(X, X') \partial_{\alpha'} \Omega^{\mu'}(X, X') \partial^{\alpha'} \Omega^{\nu'}(X, X') \quad (7)$$

where  $\partial_{\alpha'} = (\partial_{\tau'}, \partial_{\sigma'})$  acts only on  $X'(\tau', \sigma')$ .

$$S_E = \frac{1}{\ell^2} \int d^2\sigma' \Omega_{\mu'\nu'}(X, X') \partial_{\alpha'} \Omega^{\mu'}(X, X') \partial^{\alpha'} \Omega^{\nu'}(X, X') \quad (8)$$

- 1 In this formulation, entanglement entropy plays the role of a fundamental field.
- 2 The points  $X$  and  $X'$  are interchangeable, allowing a symmetric description of physics at either spacetime point.

A crucial question remains: how do we verify that this action is correct?

We propose two essential consistency checks:

- 1 It must reproduce known results:
  - ◇ How does gravity emerge from this action?
  - ◇ How can we recover the entanglement entropy (or equivalently, the geodesic length) from it?
- 2 It should lead to new, testable predictions.

# Point 1: Emergent gravity

$$S_E = \frac{1}{\ell^2} \int d^2\sigma' \Omega_{\mu'\nu'}(X, X') \partial_{\alpha'} \Omega^{\mu'}(X, X') \partial^{\alpha'} \Omega^{\nu'}(X, X') \quad (9)$$

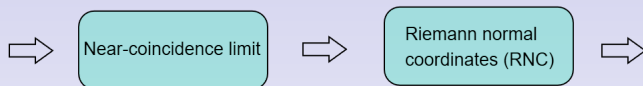
At first sight, this action is difficult to interpret physically, as it is manifestly nonlocal. To extract its physical content, we must perform a **localization** procedure.

However, the standard localization method—the **coincidence limit**  $X' \rightarrow X$  (denoted by  $[\dots]$ )—removes essential information:

$$[\Omega_{\mu\nu}] = g_{\mu\nu}(X), \quad [\Omega_\mu] = [\Omega_{\mu'}] = 0. \quad (10)$$

The key idea is therefore to work instead with the **near-coincidence limit**.

We implement localization in two steps:



**Step 1: Near-coincidence expansion**

$$\Omega_{\mu'\nu'}(X, X') = g_{\mu'\nu'}(X') - \frac{1}{3} R_{\mu'\lambda'\nu'\kappa'}(X') \Omega^{\lambda'}(X, X') \Omega^{\kappa'}(X, X') + \dots \quad (11)$$

**Step 2: Riemann normal coordinates (RNC)  $\mathbb{X}$**

$$\ell \mathbb{X}^a = -e_{\mu'}^a(X') \Omega^{\mu'}(X, X'), \quad (12)$$

where  $e_{\mu'}^a$  is the vielbein and  $\mathbb{X}^a$  is dimensionless.

## Point 1: Emergent gravity

After performing the expansion, the action becomes

$$S_E = \int d^2\sigma \left( g_{ab} \partial_\alpha \mathbb{X}^a \partial^\alpha \mathbb{X}^b - \frac{\ell^2}{3} R_{acbd} \mathbb{X}^c \mathbb{X}^d \partial_\alpha \mathbb{X}^a \partial^\alpha \mathbb{X}^b + \dots \right), \quad (13)$$

where  $R_{acbd} := R_{\alpha'\gamma'\beta'\delta'} e_a^{\alpha'} e_c^{\gamma'} e_b^{\beta'} e_d^{\delta'}$ .

- 1 This is precisely a **nonlinear sigma model** in RNC.
- 2 Locally, the dynamics of geodesic fluctuations matches that of a string worldsheet theory.
- 3 Gravity emerges from the **vanishing of the beta function**.

However, two key issues remain:

- 1 Since  $\Omega_\mu$  lives in  $\text{AdS}_3$ , the induced worldsheet fields  $\mathbb{X}^a$  propagate in  $\text{AdS}_3$ .  
**How does the full  $\text{AdS}_3$  gravitational dynamics emerge from this action?**
- 2 How is the geodesic itself encoded in the worldsheet theory? In particular, the  $\text{CFT}_2$  entanglement entropy must be recovered from this action.

## Emergent AdS<sub>3</sub> gravity

The crucial step is to include an antisymmetric field from the outset. This leads to the modified action:

$$S_E = \frac{1}{\ell^2} \int d^2\sigma' \sqrt{\gamma} \left( \gamma^{\alpha'\beta'} \Omega_{\mu'\nu'}(X, X') + i\epsilon^{\alpha'\beta'} \mathcal{B}_{\mu'\nu'}(X, X') + \ell^2 \sqrt{\gamma} \mathcal{R}\phi(X, X') \right) \partial_{\alpha'} \Omega^{\mu'}(X, X') \partial_{\beta'} \Omega^{\nu'}(X, X'), \quad (14)$$

where  $\mathcal{B}_{\mu'\nu'}(X, X')$  admits the near-coincidence expansion:

$$\begin{aligned} \mathcal{B}_{\mu'\nu'}(X') &= B_{\mu'\nu'}(X') + \nabla_{\lambda'} B_{\mu'\nu'} \Omega^{\lambda'}(X, X') \\ &\quad + \frac{1}{2} \left( \nabla_{\alpha'} \nabla_{\beta'} B_{\mu'\nu'}(X') - \frac{1}{3} R^{\lambda'}_{\alpha'\mu'\beta'} B_{\lambda'\nu'} \right. \\ &\quad \left. - \frac{1}{3} R^{\lambda'}_{\alpha'\nu'\beta'} B_{\lambda'\mu'} \right) \Omega^{\alpha'}(X, X') \Omega^{\beta'}(X, X') + \dots, \end{aligned} \quad (15)$$

- ① We only know  $\Omega$  denotes the square of entanglement entropy.
- ② But, what does  $\mathcal{B}_{\mu'\nu'}(X, X')$  stands for?

## Emergent AdS<sub>3</sub> gravity

Employing RNC and using the relation  $B_{ab} := B_{\alpha'\beta'} e_a^{\alpha'} e_b^{\beta'}$ , the vanishing of the  $\beta$ -functions for  $g_{ab}$ ,  $B_{ab}$  and  $\phi$  leads to the following gravitational field equations:

$$\begin{aligned} R_{ab} + 2\nabla_a \nabla_b \phi - \frac{1}{4} H_{acd} H_b{}^{cd} &= 0, \\ \nabla^a \left( e^{-2\phi} H_{abc} \right) &= 0, \\ 4\nabla^2 \phi - 4(\nabla\phi)^2 + R - \frac{1}{12} H^2 + \frac{4}{k} &= 0. \end{aligned} \tag{16}$$

This system admits a solution fully consistent with our entanglement entropy setup (G. T. Horowitz and D. L. Welch, 1993):

- 1  $g_{ab}$ : the AdS<sub>3</sub> metric,
- 2  $H_{abc} = \frac{2}{\ell_{\text{AdS}}} \epsilon_{abc}$ ,
- 3 constant dilaton  $\phi$ ,
- 4 cosmological constant fixed by non-critical string theory:  $k = \ell_{\text{AdS}}^2$ .

## Point 1: Emergent gravity

Finally, we must show how the entanglement entropy (or equivalently, the RT surface) is recovered from this action.

### Extract AdS<sub>3</sub> geodesic

We derive the EOM by varying  $\Omega_\mu$ . Imposing the coincidence limit:

$$[\Omega_{\mu'\nu'}] = g_{\mu\nu}, \quad [\Omega_{\mu'}] = 0, \quad [\partial_{\alpha'}\Omega^{\mu'}] = \partial_\alpha X^\mu, \quad [\mathcal{B}_{\mu'\nu'}] = b_{\mu\nu}, \quad (17)$$

we obtain

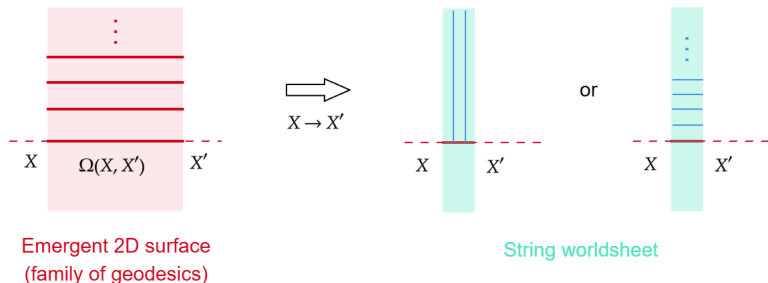
$$\begin{aligned} \text{EOM :} \quad & \partial_\alpha \partial^\alpha X^\lambda + \Gamma_{\mu\nu}^\lambda \partial_\alpha X^\mu \partial^\alpha X^\nu = 0, \\ \text{B.C. :} \quad & (g + b)_{\mu\nu} n_\alpha \partial^\alpha X^\mu \delta X^\nu \Big|_{\partial\Sigma} = 0. \end{aligned} \quad (18)$$

Decomposing  $\mu$  into tangent directions  $a$  (along the RT surface) and normal directions  $i$ , the boundary equation reduces to

$$\ddot{X}^a + \Gamma_{bc}^a \dot{X}^b \dot{X}^c = 0. \quad (19)$$

where  $\Gamma_{bc}^a \rightarrow \Gamma_{bc}^a - \frac{1}{2} H_{bc}^a$ . This is precisely the geodesic equation governing endpoints on the RT surface in AdS<sub>3</sub>.

## Extract $\text{AdS}_3$ geodesic

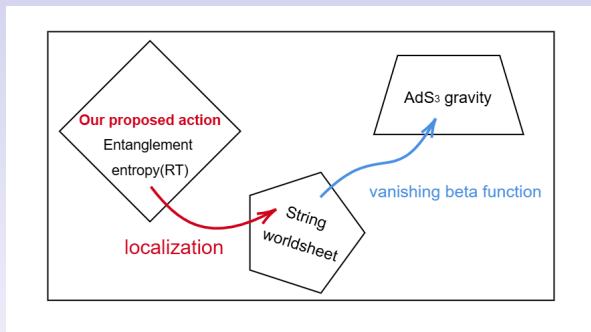


- 1 The geodesic sweeps out a two-dimensional surface.
- 2 In the limit  $X \rightarrow X'$ , this surface acquires the structure of a string worldsheet.
- 3 With appropriate boundary conditions, the string endpoints obey the geodesic equation encoding the RT surface.

This suggests that the RT surface can be interpreted as an effective D-brane (an “entangling brane” (W. Donnelly, G. Wong, 2017)), with open strings attached to it.

## Point 1: Emergent gravity

We can now resolve the original puzzle:



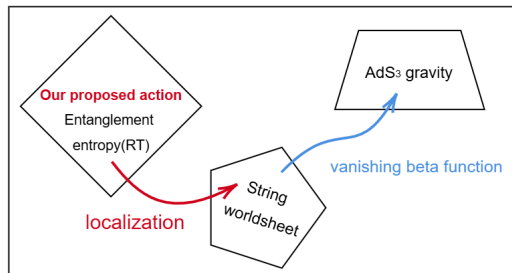
- 1 The action is constructed entirely from entanglement entropy:  
 $\Omega(X, X') \sim S_{EE}(A : B)^2$ .
- 2 Its EOM and boundary conditions reproduce the geodesic equation in AdS<sub>3</sub>.
- 3 In the near-coincidence limit, the action yields AdS<sub>3</sub> gravity.

**Conclusion:** *Does gravity emerge from quantum entanglement?*

## Point 1: Emergent gravity

**Conclusion:** *Does gravity emerge from quantum entanglement?*

**No, string theory emerges from quantum entanglement, and gravity emerges from string theory!**



## Point 1: Emergent gravity

One may still wonder whether this result is merely coincidental, or whether alternative constructions could also lead to emergent gravity.

We now turn to nontrivial predictions of this framework, which provide strong evidence for its validity:

- 1 A correspondence between RT surfaces and string worldsheets.
- 2 A proof of the Susskind–Uglum conjecture.

## Point 2: Predictions

(a) Kalb-Ramond charge/bit threads duality

## Point 2(a): Kalb-Ramond charge/bit threads duality

Let us recall the localized form of our proposed action:

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \left( \sqrt{\gamma} \gamma^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu} + \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu} + \alpha' \sqrt{\gamma} \mathcal{R}\phi \right)$$

spacetime metric

anti-symmetric  
Kalb-Ramond field

scalar dilaton



RT surface



???



coupling

- 1 In the worldsheet framework, the spacetime metric  $g_{ab}$  and the antisymmetric field  $B_{ab}$  enter on equal footing.
- 2 The entanglement entropy (equivalently, the geodesic length) is computed from  $g_{ab}$ , while  $B_{ab}$  does not explicitly appear.

If our proposed action is correct, there must exist an additional—previously unrecognized—mechanism in which  $B_{ab}$  also contributes to the computation of entanglement entropy.

## Point 2(a): Kalb-Ramond charge/bit threads duality

To identify this mechanism, recall the physical role of the Kalb–Ramond field  $B_{ab}$ :

- ① It generalizes the electromagnetic coupling of a point particle to a **string-like coupling**  $\int B$  on the worldsheet.
- ② It couples to oriented strings and encodes an antisymmetric flux (field strength  $H = dB$ ).

Electromagnetic field

action

$$S = -m \int ds + q \int A_\mu dx^\mu - \frac{1}{4\kappa_0} \int d^D x F_{\mu\nu} F^{\mu\nu}$$

current

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = j^\mu$$

$j^0$  : electric charge density

Kalb-Ramond field

$$S = S_{str} - \frac{1}{2} \int d^2\sigma \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu} - \frac{1}{6\kappa^2} \int d^D x H_{\mu\nu\rho} H^{\mu\nu\rho}$$

$$\frac{\partial H^{\mu\nu\rho}}{\partial x^\rho} = j^{\mu\nu}$$

$\vec{j}^0 \equiv j^{0\nu}$  : string charge density

## Point 2(a): Kalb-Ramond charge/bit threads duality

Now consider the Kalb–Ramond charge density:

### Kalb-Ramond charge density

$$\frac{\partial H^{\mu\nu\rho}}{\partial x^\rho} = \kappa^2 j^{\mu\nu}, \quad (20)$$

- 1 The current  $j^{\mu\nu}$  is associated with string-like sources.
- 2 The string charge density is a divergenceless vector  $\nabla \cdot \vec{j}^0 = 0$ .
- 3 The boundary of the worldsheet (a D-brane) is naturally identified with the RT surface.

Therefore, treating  $g_{ab}$  and  $B_{ab}$  on equal footing suggests that this conserved flux should also encode entanglement entropy.

Remarkably, in the context of entanglement entropy, such a divergenceless quantity already exists: **bit threads**.

## Point 2(a): Kalb-Ramond charge/bit threads duality

### Bit threads (M. Freedman and M. Headrick, 2017)

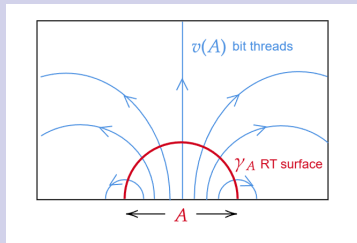
Consider the flux through a boundary region  $A$ :

$$\int_A v \equiv \int_A \sqrt{h} n_\mu v^\mu = \int_{\gamma_A} v \leq C \text{ area}(\gamma_A), \quad (21)$$

where  $h$  is the induced metric on  $A$ , and  $n_\mu$  is the unit normal vector. The entanglement entropy is given by

$$S_{\text{vN}} = \frac{\text{area}(\gamma_A)}{4G_N} = \max_v \int_A v, \quad C = 1/4G_N^{(3)}. \quad (22)$$

- 1  $v^\mu$  is divergenceless  $\nabla \cdot v = 0$  and bounded  $|v| \leq 1/4G_N^{(3)}$ .
- 2 The flow is orthogonal to the RT surface.

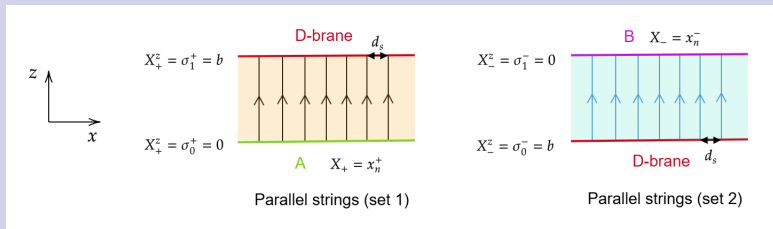


*Is there a precise correspondence between  
Kalb–Ramond charge density and bit threads?*

## Step 1: Two sets of parallel open strings

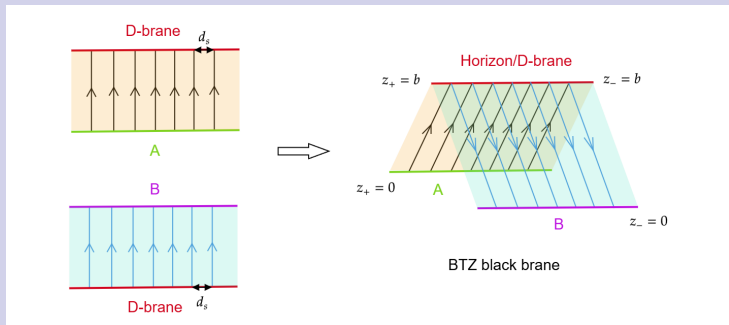
We consider two sets of stationary, parallel open strings stretched along the  $z_+$ - and  $z_-$ -directions in two copies of three-dimensional flat spacetime:

$$(t_{\pm}, z_{\pm}, x_{\pm}).$$



- 1 The worldsheet embeddings are  $X_{\pm}^{\mu}(\tau^{\pm}, \sigma^{\pm})$ ,  $\mu = t, z, x$ .
- 2 We choose the static gauge:  $X_{\pm}^t = \tau^{\pm}$ ,  $X_{\pm}^z = \sigma^{\pm}$ ,  $X_{\pm}^x = x_n^{\pm}$ .
- 3 Dirichlet boundary conditions are imposed along the endpoints:  $\delta X_{\pm}^x|_{\partial\Sigma} = 0$ .
- 4 The separation between adjacent strings is denoted by  $d_s$ .

## Step 2: Mapping to a time slice of the BTZ black brane

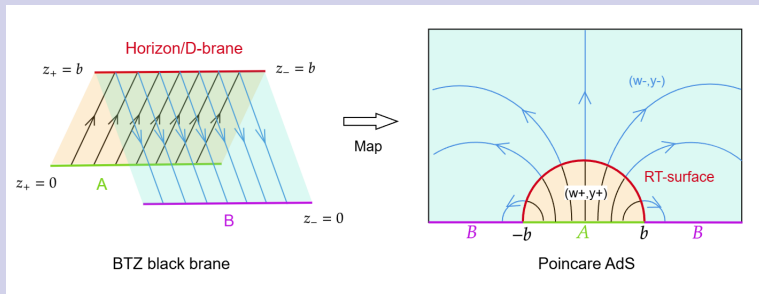


The resulting metric takes the form

$$dS_{\pm}^2 = \frac{l_{\text{AdS}}^2}{z_{\pm}^2} \left( - \left( 1 - (z_{\pm}/b)^2 \right) dt^2 + \frac{dx_{\pm}^2}{1 - (z_{\pm}/b)^2} + dz_{\pm}^2 \right), \quad (23)$$

- 1  $t_+ = t_- = t$ ,  $x_{\pm} \in \mathbb{R}$ ,  $z_{\pm} \in (0, b]$ .
- 2 The planar horizon, located at  $z_{\pm} = b$ , is identified with the RT surface.
- 3 The open strings on the two sides can be smoothly connected across this surface.

## Step 3: Mapping to a time slice of Poincare AdS<sub>3</sub>



We perform the coordinate transformation (S. Caggioli et al., 2024):

$$y_{\pm} = \frac{b \sinh(x_{\pm}/b)}{\cosh(x_{\pm}/b) \pm \sqrt{1 - (z_{\pm}/b)^2}}, \quad w_{\pm} = \frac{z_{\pm}}{\cosh(x_{\pm}/b) \pm \sqrt{1 - (z_{\pm}/b)^2}}. \quad (24)$$

This maps the BTZ black brane geometry to the Poincare AdS<sub>3</sub> metric:

$$dS_{\pm}^2 = \frac{l_{\text{AdS}}^2}{w_{\pm}^2} (dy_{\pm}^2 + dw_{\pm}^2). \quad (25)$$

How can this construction be realized in our framework?

Carrying out this program is highly nontrivial:

- 1 The  $\text{AdS}_3$  background must be a solution of our EOM.
- 2 The multiple parallel open strings should propagate in this background without significantly backreacting on the geometry.
- 3 One must derive the corresponding conserved vector (Kalb–Ramond charge density) in this setup.

Consistency requires that all these ingredients emerge naturally from the proposed action.

## The resolution: string solitons

(M. J. Duff, R. R. Khuri and J. X. Lu, 1995)

**We consider macroscopic string configurations carrying Kalb–Ramond charge, propagating in  $\text{AdS}_3$  and interacting with the background fields.**

## Point 2(a): Kalb-Ramond charge/bit threads duality

We now incorporate macroscopic string sources into the localized action:

$$S = \frac{1}{2\kappa^2} \int d^3x \sqrt{-g} e^{-2\phi} \left( R + 4(\nabla\phi)^2 - \frac{1}{12} H^2 + \frac{4}{k} \right) + S_\sigma, \quad (26)$$

where the string source is described by the sigma model

$$S_\sigma = -\frac{1}{4\pi\alpha'} \int d^2\sigma \left( \sqrt{\gamma} \gamma^{mn} \partial_m X^a \partial_n X^b g_{ab} + \epsilon^{mn} \partial_m X^a \partial_n X^b B_{ab} \right), \quad (27)$$

The resulting EOM are

$$\begin{aligned} R^{ab} + 2\nabla^a \nabla^b \phi - \frac{1}{4} H^{acd} H^b{}_{cd} &= \kappa^2 T^{ab}, \\ \nabla_a \left( e^{-2\phi} H^{abc} \right) &= \frac{\kappa^2}{\pi\alpha'} j^{bc} \\ 4\nabla^2 \phi - 4(\nabla\phi)^2 + R - \frac{1}{12} H^2 + \frac{4}{k} &= 0, \end{aligned} \quad (28)$$

with source terms

$$\begin{aligned} T^{ab} &= -\frac{e^{2\phi}}{2\pi\alpha' \sqrt{-g}} \int d^2\sigma \sqrt{\gamma} \gamma^{mn} \partial_m X^a \partial_n X^b \delta^{(3)}(x - X(\tau, \sigma)), \\ j^{ab} &= \frac{1}{\sqrt{-g}} \int d^2\sigma \left( \frac{\partial X^a}{\partial \tau} \frac{\partial X^b}{\partial \sigma} - \frac{\partial X^b}{\partial \tau} \frac{\partial X^a}{\partial \sigma} \right) \delta^{(3)}(x - X(\tau, \sigma)). \end{aligned} \quad (29)$$

## Point 2(a): Kalb-Ramond charge/bit threads duality

To realize an  $\text{AdS}_3$  background, we work in the regime of weak bulk gravity  $\kappa^2 = 8\pi G_N^{(3)} \rightarrow 0$ ,  $c \rightarrow \infty$ . The  $D$ -dimensional Newton constant is

$$16\pi G_N^{(D)} = \left(2\pi\sqrt{\alpha'}\right)^{D-2} g_s^2 (2\pi)^{-1}, \quad (30)$$

which implies

$$\frac{\kappa^2}{\pi\alpha'} = \frac{g_s^2}{2\pi\sqrt{\alpha'}}. \quad (31)$$

To suppress backreaction from  $\mathcal{N}$  string sources, we require

$$g_s^2 = e^{2\phi_0} \ll \frac{\sqrt{\alpha'}}{l_{\text{AdS}}} \cdot \frac{1}{\mathcal{N}}, \quad (32)$$

A convenient choice is

$$g_s^2 = e^{2\phi_0} = \frac{\sqrt{\alpha'}}{l_{\text{AdS}}} \cdot \frac{\epsilon}{\mathcal{N}}, \quad \epsilon \ll 1. \quad (33)$$

This explains why the dilaton must be included in the action, and is similar the scaling in the D1–D5 system:  $g_s^2 = \frac{\sqrt{\alpha'}}{l_{\text{AdS}}} \cdot \frac{1}{Q_1}$ .

## Point 2(a): Kalb-Ramond charge/bit threads duality

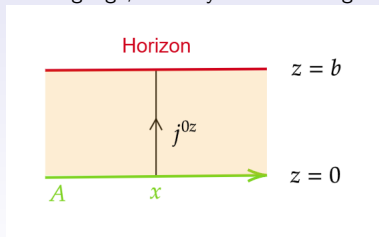
For simplicity, we consider a single side of the brane. In the probe limit  $\kappa^2/(\pi\alpha') \rightarrow 0$ , the background is

$$dS^2 = \frac{l_{\text{AdS}}^2}{z^2} \left( - \left( 1 - \left( \frac{z}{b} \right)^2 \right) dt^2 + \frac{dz^2}{1 - \left( \frac{z}{b} \right)^2} + dx^2 \right), \quad (34)$$

We introduce a static string stretched along the  $z$ -direction:

$$X^t(\tau, \sigma) = \tau, \quad X^z(\tau, \sigma) = \sigma, \quad X^x(\tau, \sigma) = 0. \quad (35)$$

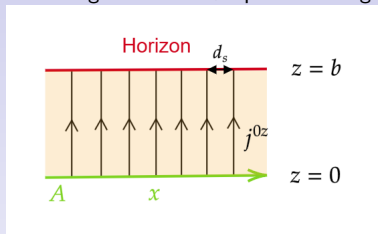
In this gauge, the only non-vanishing current component is



$$j^{0z} = \frac{z^3}{l_{\text{AdS}}^3} \delta(x), \quad 0 \leq z \leq b. \quad (36)$$

## Point 2(a): Kalb-Ramond charge/bit threads duality

We now generalize to  $\mathcal{N}$  parallel strings uniformly distributed along the  $x$ -direction:



$$j^{0z} = \frac{z^3}{l_{\text{AdS}}^3} \sum_{n \in \mathbb{Z}} \delta(x - nd_s). \quad (37)$$

- 1 The sum  $\sum_n \delta(x - nd_s)$  defines a Dirac comb.
- 2 Strings are located at  $x = \dots, -2d_s, -d_s, 0, d_s, 2d_s, \dots$

For a boundary region  $A = [-b, b]$ , we distribute  $\mathcal{N}$  strings uniformly over length  $2b$ , giving

$$d_s = \frac{2b}{\mathcal{N}}. \quad (38)$$

With this choice, the charge density  $j^{0z}$  becomes

$$j^{0z} = \frac{1}{d_s} d_s \frac{z^3}{l_{\text{AdS}}^3} \sum_{n \in \mathbb{Z}} \delta(x - nd_s) = \frac{1}{2b} \frac{z^3}{l_{\text{AdS}}^3} d_s \sum_{n \in \mathbb{Z}} \delta(x - nd_s), \quad (39)$$

## Point 2(a): Kalb-Ramond charge/bit threads duality

We now take the continuum limit  $\mathcal{N} \rightarrow \infty$  (hence  $d_s \rightarrow 0$ ). The discrete Dirac comb converges weakly to a constant:

$$j^{0z} = \frac{1}{2b} \frac{z^3}{l_{\text{AdS}}^3} \lim_{d_s \rightarrow 0} d_s \sum_n \delta(x - nd_s) = \frac{1}{2b} \frac{z^3}{l_{\text{AdS}}^3}. \quad (40)$$

This procedure is analogous to that used in string cosmology, where a single string (delta-function source) is replaced by a continuous string gas.

The resulting current vector is

$$j^{0\mu} = (j^{0t}, j^{0z}, j^{0x}) = \frac{1}{2b} \frac{z^3}{l_{\text{AdS}}^3} (0, 1, 0), \quad \mu = t, z, x, \quad (41)$$

which satisfies the 3D divergenceless condition  $\partial_\mu (\sqrt{-g} j^{0\mu}) = 0$ . Restricting to the spatial slice  $(z, x)$ , we obtain the projected divergenceless condition:

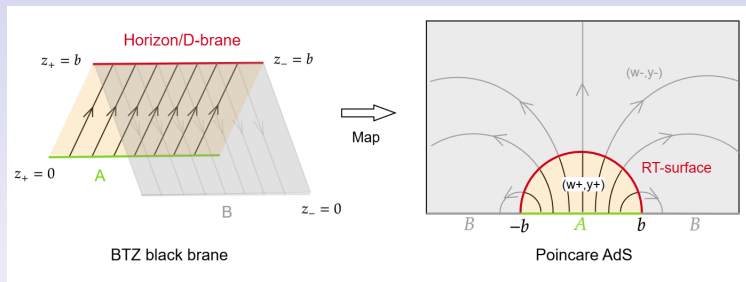
$$\partial_i \left( \sqrt{h^{(2)}} j^{0i} \right) = 0, \quad i = z, x, \quad (42)$$

The induced current on this slice is

$$j^{0i} = \sqrt{-g_{tt}} j^{0i} = \frac{1}{2b} \frac{z^2}{l_{\text{AdS}}^2} \sqrt{1 - \left(\frac{z}{b}\right)^2} (1, 0). \quad (43)$$

## Point 2(a): Kalb-Ramond charge/bit threads duality

This describes a Kalb–Ramond charge density vector flowing from the boundary ( $z = 0$ ) to the RT surface ( $z = b$ ).



We now map this configuration to Poincare AdS<sub>3</sub> using

$$y_{\pm} = \frac{b \sinh(x_{\pm}/b)}{\cosh(x_{\pm}/b) \pm \sqrt{1 - (z_{\pm}/b)^2}}, \quad w_{\pm} = \frac{z_{\pm}}{\cosh(x_{\pm}/b) \pm \sqrt{1 - (z_{\pm}/b)^2}}, \quad (44)$$

## Point 2(a): Kalb–Ramond charge / bit threads duality

Under this mapping, we define

$$v^i \equiv \frac{\ell_{\text{AdS}}}{2G_N^{(3)}} j^{0i}, \quad (45)$$

which yields

$$v^i = \frac{1}{4G_N^{(3)}} \frac{1}{\ell_{\text{AdS}}} \left( \frac{2bw}{\sqrt{(b^2 - y^2 - w^2)^2 + 4b^2w^2}} \right)^2 \left( \frac{b^2 - y^2 + w^2}{2b}, \frac{yw}{b} \right). \quad (46)$$

The normalization ensures the bound

$$|v| \leq \frac{1}{4G_N^{(3)}},$$

so that  $v$  saturates the max-flow/min-cut constraint.

**This precisely reproduces the bit thread configuration.**

Therefore,

Kalb–Ramond charge density  $\longleftrightarrow$  bit threads

## Point 2(a): Kalb-Ramond charge/bit threads duality

A natural question is whether parallel strings interact or annihilate.

This is determined by the string EOM:

$$\nabla_m (\gamma^{mn} \nabla_n X^a) = -\Gamma_{bc}^a \partial_m X^b \partial_n X^c \gamma^{mn} + \frac{1}{2} H_{bc}^a \partial_m X^b \partial_n X^c \epsilon^{mn}. \quad (47)$$

For two nearby strings, treating one as a source and the other as a test. We choose the static gauge for the test string:

$$X^0 = \tau, \quad X^z = \sigma. \quad (48)$$

We now begin by the configuration of the test string at a given time,

$$\partial_\tau X^x = \partial_\sigma X^x = 0. \quad (49)$$

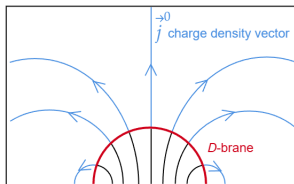
Then we have

$$\frac{d^2}{d\tau^2} X^x = 0. \quad (50)$$

- 1 The test string does not encounter a force in the transverse direction.

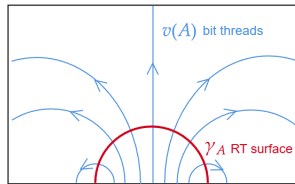
# Point 2(a): Kalb-Ramond charge/bit threads duality

## Step 3: Kalb-Ramond charge/bit threads duality



Worksheet picture

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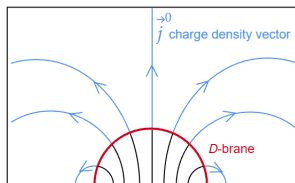
EE picture

Worksheet	Entanglement
<i>D</i> -brane/ <i>E</i> -brane	RT surface
Charge density vector $\vec{j}^0$	Bit threads $v$

$g_{ab}$	RT surface	$S_{EE} = \frac{\text{area}(\gamma_A)}{4G_N}$
$B_{ab}$	Bit threads $v$	$S_{EE} = \max_v \int_A v$

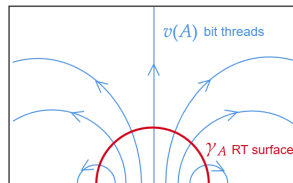
## Point 2(a): Kalb-Ramond charge/bit threads duality

### Step 3: Kalb-Ramond charge/bit threads duality



Worldsheet picture

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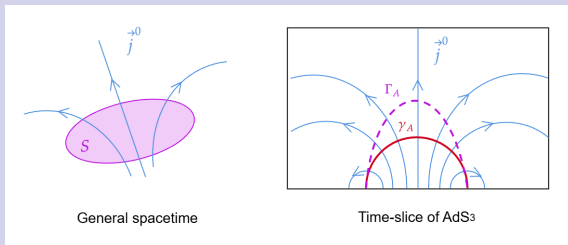
EE picture

- 1 Red geodesics: D-brane/RT surface.
- 2 Blue fluxes: Kalb-Ramond charge density/bit threads.
- 3 All of them determine bulk geometry and dynamics in  $AdS_3$ .

## Point 2: Predictions

(b) A realization of the Susskind–Uglum conjecture in  $\text{AdS}_3$

## Entanglement entropy from open string charge



We first introduce the **string number**  $\mathcal{N}$ :

$$\mathcal{N} \equiv \int_S \vec{j}^0 \cdot d\vec{a}. \quad (51)$$

where  $S$  is a surface pierced by strings, and the integral counts the number of strings crossing  $S$ .

On a constant-time slice of AdS<sub>3</sub>, this reduces to an integral over a curve  $\Gamma_A$ . Using the charge/threads duality, we obtain

$$\mathcal{N} = \int \vec{j}^0 \cdot dS_{\Gamma_A} = \frac{2G_N^{(3)}}{l_{\text{AdS}}} \int \vec{v} \cdot dS_{\Gamma_A} = \frac{1}{2} \frac{L\gamma_A}{l_{\text{AdS}}}, \quad (52)$$

## Entanglement entropy from open string charge

We then obtain

$$S_{\text{vN}} = \frac{L\gamma_A}{4G_N^{(3)}} = \frac{l_{\text{AdS}}}{4G_N^{(3)}} \cdot 2\mathcal{N} = \frac{c}{3} \ln\left(\frac{2b}{\epsilon}\right). \quad (53)$$

**Interpretation:**

- Entanglement entropy is proportional to the **number of open strings** piercing the entangling surface.
- Each open string carries one unit of Kalb–Ramond charge and contributes a single unit of information.
- The entangling surface (RT surface) plays the role of a **brane** storing these degrees of freedom holographically.

# Point 2(b): Susskind–Ugulum conjecture in $\text{AdS}_3$

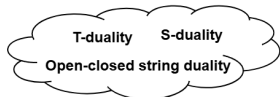
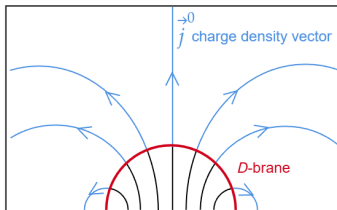
Entanglement entropy  $\iff$  string worldsheet physics



String theory exhibits rich dualities

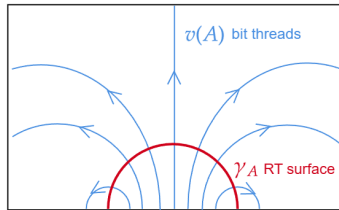


What is the entanglement counterpart of open–closed string duality?



String worldsheet

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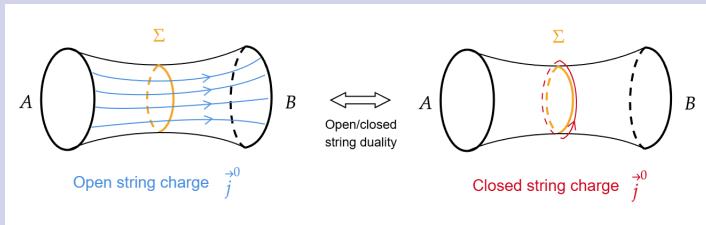


Entanglement entropy

## Bekenstein-Hawking entropy from closed string charge

To explore open–closed string duality in this framework, we compactify the spatial direction.

Under open–closed duality, open string charge maps to closed string winding charge:



The total closed string charge is

$$\vec{Q} = \int d^d x \sqrt{h^{(d)}} \vec{j}^0. \quad (54)$$

## Bekenstein-Hawking entropy from closed string charge

Substituting the background solution, we obtain

$$\vec{Q} = \int_0^{2\pi} d\sigma \partial_\sigma \vec{X}(t_0, \sigma). \quad (55)$$

- For contractible closed strings:  $\vec{Q} = 0$ .
- For winding strings around the compact direction:

$$\vec{Q} = \vec{X}(2\pi) - \vec{X}(0) \neq 0. \quad (56)$$

Parameterizing  $X = r\sigma$ , we find

$$\vec{Q} = (0, Q), \quad Q = 2\pi r. \quad (57)$$

### Bekenstein-Hawking entropy from closed string charge

The Bekenstein–Hawking entropy becomes

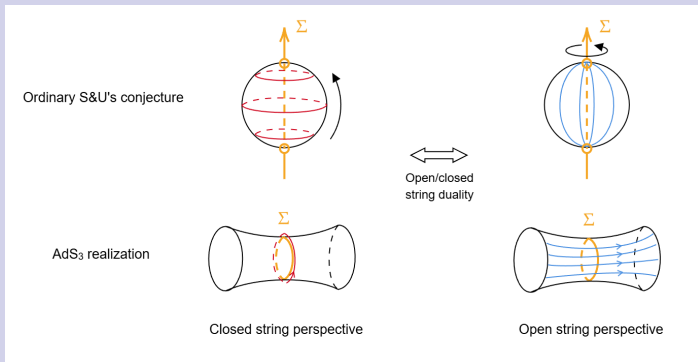
$$S_{BH} = \frac{Q}{4G_N^{(3)}} = \frac{2\pi r}{4G_N^{(3)}}, \quad (58)$$

This reproduces the entropy of the BTZ black hole.

#### Implications:

- Black hole entropy is determined by **closed string winding charge**.
- This result is consistent with the previous observation, calculated in Horowitz-Polchinski theory, that the string charge is proportional to the entropy (R. Brustein, Y. Zigdon, 2022).
- This provides a direct geometric interpretation of microscopic degrees of freedom.
- The structure is consistent with the D1–D5 system, where entropy arises from three kinds of charges.

Susskind–Uglum conjecture in  $\text{AdS}_3$



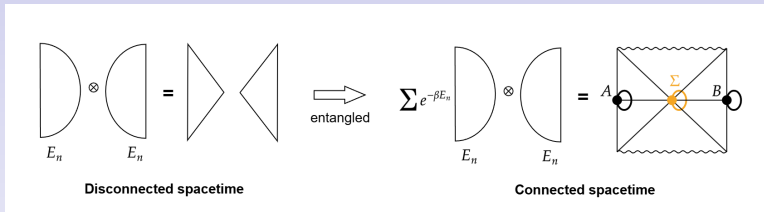
- ① Open String Charge = Entanglement Entropy:
  - ◇ The number of open strings crossing the boundary gives the entanglement entropy.
- ② Closed String Charge = Bekenstein–Hawking Entropy:
  - ◇ The closed string winds around the black hole tells us the black hole entropy.

**Entanglement Entropy and Black Hole Entropy are just two sides of the same coin.**

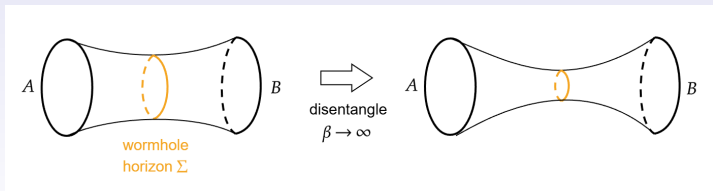
# Unified Framework for ER=EPR and Susskind-Uglum

# Unified Framework for ER=EPR and Susskind-Uglum

Let us first recall the ER=EPR thought experiment.

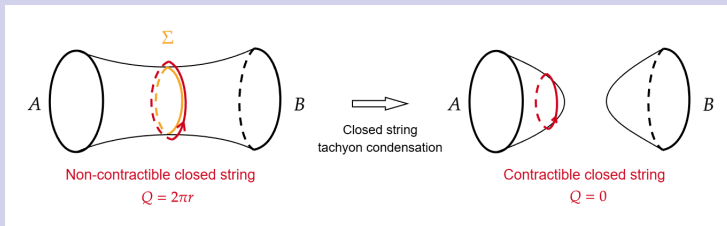


- 1 Once the two subsystems are entangled, they can be described by the TFD state.
- 2 The TFD state may be viewed as a quantum superposition of disconnected geometries, which effectively yields a wormhole.



- 1 In the zero-temperature limit  $\beta \rightarrow \infty$ , the entangled TFD state reduces to the unentangled product state, and the wormhole disappears.

## ER=EPR in our framework



1 The string cylinder is stretched, leading to a reduction of the horizon size.



2 Once the horizon shrinks below the string length scale, the winding closed string becomes tachyonic.



3 Tachyon condensation then drives the closed string charge  $Q \rightarrow 0$ , signaling that the spacetime splits into two disconnected components.

This process is similar to the open-closed string duality (Susskind-Uglum conjecture):

D-brane separation is large:

- 1 The propagation of closed strings dominates at long distances: the massive modes decouple, leaving only the massless sector.
- 2 The open strings are long, so their massive excitations significantly contribute, leading to strong gauge coupling.

D-brane separation is close:

- 1 The supergravity approximation breaks down, while the massless sector of open strings dominates.

D-brane separation increases  $\implies$  the open string description becomes closed string description  $\implies$  Causally disconnected as the closed-string propagation time tends to infinity.

It therefore unifies the ER=EPR, open-closed string duality and Susskind-Uglum in same underlying principle: the transition between dual descriptions (geometric vs. entanglement) driven by string charges.

# Quantized RT surface and connections to LQG

*In previous analyses, we considered  $\mathcal{N}$  parallel open strings, which reproduce the standard entanglement entropy of  $CFT_2$ .*

*If this framework is correct, what is the result for a single string?*

To address this, let us recall the charge density vector for a single string source:

$$j^{0a} = (j^{0t}, j^{0z}, j^{0x}) = \frac{z^3}{l_{AdS}^3} (0, \delta(x), 0), \quad 0 \leq z \leq b. \quad (59)$$

Repeating the same analysis as before, we obtain

$$(j^{0w}, j^{0y}) = \left( \frac{w^2}{l_{AdS}^2} \delta(y), 0 \right), \quad (60)$$

defined on the Poincare patch of a time slice of  $AdS_3$ :

$$dS^2 = \frac{l_{AdS}^2}{w^2} (dy^2 + dw^2). \quad (61)$$

Using the Kalb–Ramond charge / bit-thread correspondence, we have

$$v = \frac{l_{AdS}}{2G_N^{(3)}} (j^{0w}, j^{0y}). \quad (62)$$

We now compute the entanglement entropy using the bit-thread formalism.

On a constant- $w$  slice, the induced metric is  $h = \frac{l_{\text{AdS}}^2}{w^2}$ , and the unit normal vector is  $n_\mu = (l_{\text{AdS}}/w, 0)$ . Thus,

$$\sqrt{h}n_\mu v^\mu = \frac{l_{\text{AdS}}}{2G_N^{(3)}}\delta(y) \quad (63)$$

Integrating over the boundary interval:

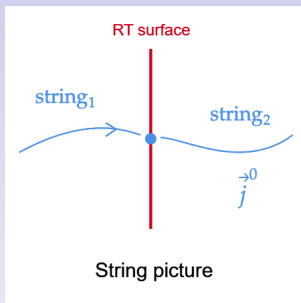
$$S_{EE} = \int_A \sqrt{h}n_\mu v^\mu = \int_{\epsilon-b}^{b-\epsilon} \sqrt{h}n_\mu v^\mu dy = \frac{l_{\text{AdS}}}{4G_N^{(3)}} \cdot 2, \quad (64)$$

This exactly matches the general result

$$S_{EE} = \frac{l_{\text{AdS}}}{4G_N^{(3)}} \cdot 2\mathcal{N}. \quad (65)$$

for  $\mathcal{N} = 1$ .

# Quantized RT surface and connections to LQG



$$S_{EE} = \frac{l_{\text{AdS}}}{4G_N^{(3)}} \cdot 2\mathcal{N}. \quad (66)$$

- 1 The open string is oriented and is intersected by the RT surface.
- 2 The pure state  $|\psi\rangle$  of the two segments ( $\text{string}_1$ ,  $\text{string}_2$ ) admits a decomposition:

$$|\psi\rangle \in \mathcal{H} \subseteq \mathcal{H}_{\text{string}_1} \otimes \mathcal{H}_{\text{string}_2}.$$

- 3 The RT surface length (entanglement entropy) is proportional to the number  $\mathcal{N}$  of open strings.

These properties closely parallel entanglement entropy in LQG.

## Entanglement entropy in LQG

Let us recall the simplest Wilson line state  $|\gamma, j, a, b\rangle$  in LQG, whose wave function is

$$\psi_{\gamma, j, a, b}[h(A)] = \langle h(A) | \gamma, j, a, b \rangle = \sqrt{2j+1} D_{ab}^{(j)}(h_\gamma(A)), \quad (67)$$

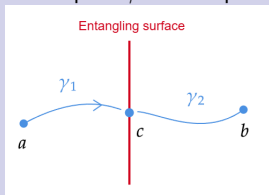
This defines a map from the  $\mathfrak{su}(2)$  connection  $A$  along a path  $\gamma$  to  $\mathbb{C}$ . Here

$$h_\gamma(A) = \mathcal{P} \exp \left( \int_\gamma A \right)$$

is the holonomy along  $\gamma$ , and  $D_{ab}^{(j)}(g)$  are Wigner matrices.

We now perform the Schmidt decomposition of this Wilson line state.

Suppose the path  $\gamma$  is decomposed as  $\gamma = \gamma_1 \circ \gamma_2$ . The Hilbert space then factorizes as



$$\mathcal{H}_\gamma \subseteq \mathcal{H}_{\gamma_1} \otimes \mathcal{H}_{\gamma_2}. \quad (68)$$

## Entanglement entropy in LQG

The Wilson line state admits the decomposition

$$\begin{aligned} \psi_{\gamma_1 \circ \gamma_2, j, a, b} [h(A)] &= \frac{1}{\sqrt{2j+1}} \sum_{c=1}^{2j+1} \psi_{\gamma_1, j, a, c} [h_{\gamma_1}(A)] \psi_{\gamma_2, j, c, b} [h_{\gamma_2}(A)] \\ &= \frac{1}{\sqrt{2j+1}} \sum_{c=1}^{2j+1} \langle h | \gamma_1, j, a, c \rangle \langle h | \gamma_2, j, c, b \rangle. \end{aligned} \quad (69)$$

Equivalently, in Hilbert space language,

$$|\gamma, j, a, b\rangle = \frac{1}{\sqrt{2j+1}} \sum_{c=1}^{2j+1} |\gamma_1, j, a, c\rangle \otimes |\gamma_2, j, c, b\rangle \quad (70)$$

This is precisely a Schmidt decomposition, with coefficients

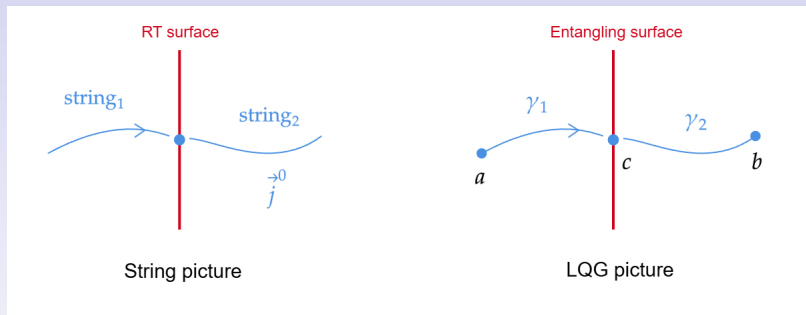
$$\lambda_i = \frac{1}{\sqrt{2j+1}}, \quad i = 1, 2, \dots, 2j+1. \quad (71)$$

The entanglement entropy is therefore (W. Donnelly, 2008)

$$S_A = S_B = - \sum_i \lambda_i^2 \log \lambda_i^2 = \log(2j+1). \quad (72)$$

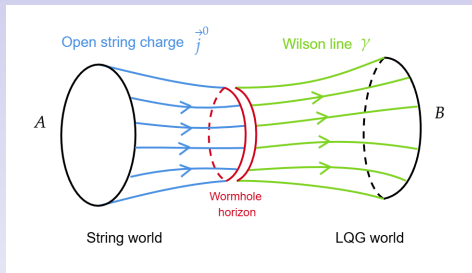
# Quantized RT surface and connections to LQG

Therefore, we obtain the following correspondence between the string and LQG perspectives:



Oriented open string cut by RT surface	Oriented Wilson line cut by entangling surface
$ \psi\rangle \in \mathcal{H} \subseteq \mathcal{H}_{\text{string}_1} \otimes \mathcal{H}_{\text{string}_2}$	$ \psi\rangle \in \mathcal{H}_\gamma = \mathcal{H}_{\gamma_1} \otimes \mathcal{H}_{\gamma_2}$ .
EE counts the number of strings	EE counts =s on the number of Wilson lines
String charge $\frac{l_{\text{AdS}}}{4G_N^{(3)}} \cdot 2 \int_{\gamma_A} \overrightarrow{j^0}  \psi\rangle = \frac{l_{\text{AdS}}}{4G_N^{(3)}} \cdot 2\mathcal{N}  \psi\rangle$	Noether charge $\widehat{\int_{\gamma_A} Q}  \psi\rangle = \log(2j+1)  \psi\rangle$

# Quantized RT surface and connections to LQG



- 1 Consider the Schmidt decomposition of both the open string charge density and the LQG Wilson line.
- 2 Split each system into two parts and recombine them into a mixed configuration:

$$|\psi\rangle \in \mathcal{H} \subseteq \mathcal{H}_{\text{string}_1} \otimes \mathcal{H}_{\gamma_2}.$$

- 3 Both descriptions exhibit the same quantized wormhole horizon. The endpoints of open strings can attach to Wilson lines at the same locations on the horizon, suggesting that the two descriptions coincide at the wormhole throat.

**Conjecture:** At the level of entanglement entropy, string theory and LQG may provide equivalent descriptions.

# Conclusion

Key steps:

- We use the entanglement entropy of  $\text{CFT}_2$  (i.e., the geodesic length in  $\text{AdS}_3$ ) to construct a fundamental action.
- Upon localization, the action reduces to the string worldsheet action, including  $g_{\mu\nu}$ ,  $B_{\mu\nu}$ , and  $\phi$ .
- $\text{AdS}_3$  gravity emerges naturally from this action.
- The equal footing of  $g_{\mu\nu}$  and  $B_{\mu\nu}$  implies that entanglement entropy can also be computed from  $B_{\mu\nu}$ . In particular, bit threads are exactly reproduced from the Kalb–Ramond charge.
- Entanglement entropy is computed from open string charge, while Bekenstein–Hawking entropy arises from closed string charge via open–closed duality. This provides a proof of the Susskind–Uglum conjecture.

*Thank you!*

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