Closed-form Schur indices and free fields

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based on 1903.03623, 2104.12180, work in progress

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October 28, 2021

Introduction

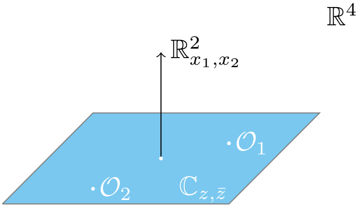
Associated VOA

[Beem, Lemos, Liendo, Peelaers, Rastelli, Rees]

- 4d $\mathcal{N} = 2$ SCFT $\mathcal{T}$ on $\mathbb{R}^4$
- Superconformal algebra generators

$$P_\mu, K_\mu, D, M, R^I{}_J, Q^I_\alpha, \tilde{Q}_{I\dot{\alpha}}, S^\alpha_I, \tilde{S}^{I\dot{\alpha}} \quad (1)$$

- Pick $\mathbb{R}^2_{x_3,x_4} \equiv \mathbb{C}_{z,\bar{z}}$



Associated VOA

- **Schur operators**: special type of operators on $\mathbb{C}_{z,\bar{z}}$
  (a) Cohomology classes $\mathcal{O}(z)$ of

$$\mathcal{Q}_1 \equiv Q^1_- + \tilde{Q}^{\dagger}_{2\dot{-}}, \qquad \mathcal{Q}_2 \equiv (Q^1_-)^{\dagger} - \tilde{Q}_{2\dot{-}} \qquad (2)$$

  (b) Schur conditions (from cohomology requirement):

$$E - 2R - j_1 - j_2 = 0, \qquad r + j_1 - j_2 = 0 \qquad (3)$$

  (c) $\mathcal{O}(z)$ depend on z only
  (d) $\mathcal{O}(z)\mathcal{O}'(0)$ OPE coefficients depend on z only
- 4d/2d correspondence: Schur ops. span a **vertex operator algebra $\mathbb{V}[\mathcal{T}]$**

Associated VOA

- Encode important info about the 4d SCFTs [Lemos, Liendo][Bonetti, Rastelli][Cordorva, Gaiotto, Shao] [Song, Xie, Yan] ...
- Simplest examples:
  - free hypermultiplet: $\beta\gamma$ (symplectic boson) system
  - free vector multiplet: small bc ghost
  - $SU(2)$ with 4 flavors: $\widehat{\mathfrak{so}}(8)_{-2}$
  - $\mathcal{N}=4$ $SU(2)$ SYM: 2d small $\mathcal{N}=4$ SCFA
  - T_3: $(\widehat{\mathfrak{e}}_6)_{-3}$
  - ...
- 4d $\mathcal{R}$-symmetry current $\to$ stress tensor T, $c_{\text{2d}} = -12c_{\text{4d}}$
- 4d $\mathfrak{f}$ flavor-symmetry moment map $\to$ $\widehat{\mathfrak{f}}$-current, $k_{\text{2d}} = -\frac{1}{2}k_{\text{4d}}$

Associated VOA

- Large body of literature on the subject
- identification of associated VOAs, VOA structure and modular differential equations, bounds, indices ...

  Class-$\mathcal{S}$ and T_N: [Beem, Peelaers, Rastelli, van Rees][Lemos, Peelaers][Kiyoshige, Nishinaka] ...

  Argyres-Douglas: [Song, Xie, Yan] [Xie, Yan, Yau] [Dedushenko, Wang] [Buican, Nishinaka] [Kozcaz, Shakirov, Yan][Creutzig] ...

  MDE, defects: [Cordova, Gaiotto, Shao][Nishinaka, Sasa, Zhu][Beem, Rastelli][YP, Wang, Zheng] ...

  Free field realization: [Adamovic][Beem, Meneghelli, Rastelli][Bonetti, Meneghelli, Rastelli] ...

- Schur ops counted by the **Schur index** [Gadde, et.al.],

$$\mathcal{I}[\mathcal{T}] \equiv \mathrm{str}_{\mathbb{V}[\mathcal{T}]}\, q^{E-R+\frac{c_{4\mathrm{d}}}{2}}\mathbf{b}^{\mathbf{f}} = \underbrace{\mathrm{str}_{\mathbb{V}}\, q^{L_0-\frac{c_{2\mathrm{d}}}{24}}\mathbf{b}^{\mathbf{f}}}_{\text{vacuum character of } \mathbb{V}[\mathcal{T}]} \tag{4}$$

  where $q \equiv e^{2\pi i\tau}$, $\mathbf{b}, \mathbf{f}$ are flavor fugacities and Cartans

- **Schur limit** of the full $\mathcal{N} = 2$ SCFI [Kinney, et.al.]

$$\mathcal{I}(p, q, t) \equiv \mathrm{str}\, e^{-\beta\tilde{\delta}_{1\dot{-}}}\, p^{\frac{\delta_{1+}}{2}}\, q^{\frac{E-2j_2-2R-r}{2}}\, t^{R+r}\mathbf{b}^{\mathbf{f}} \tag{5}$$

$$\xrightarrow{t\to q} \mathcal{I}(q) = \mathrm{str}\, e^{-\beta\tilde{\delta}_{1\dot{-}}}\, p^{\delta_{1+}}\, q^{\frac{E-2j_2+r}{2}}\mathbf{b}^{\mathbf{f}} \tag{6}$$

$\Rightarrow$ independence of p, contrib. only from **Schur operators**

Computing Schur indices (focus on Lagrangian theories):

- Direct counting Schur opereators or identifying the VOA [Gadde, Rastelli, Razamat, Yan]: a series expansion
- From **2d q-Yang-Mills** partition functions [Gadde, Rastelli, Razamat, Yan]: an infinite sum over representations
- From **localization** on $S^3 \times S^1$, or zero-coupling limit (independence of g_{YM}) [Gadde, et.al.][YP, Peelaers][Dedushenko, Fluder][Jeong]: a contour integral; also compute Schur correlators on $S^3 \times S^1$

  Multivariate contour integral formula

$$\mathcal{I} = \oint_{|a|=1} \left[\frac{da}{2\pi i a}\right] \mathcal{Z}(a) \tag{7}$$

Goal

- Task: compute the Schur indices **analytically** in closed-form
- Different from previous results on closed-forms [Bourdier, Drukker, Felix]
- S-duality, modular propreties, additional solutions to (flavored) modular differential equations [Gaberdiel, Keller][Krauel, Mason][Beem, Rastelli]

Free field realization and characters

Novel free field realization

Free field realization (special cases): VOAs $\mathbb{V}^{G}_{\mathcal{N}=4}$ associated to 4d $\mathcal{N}=4$ G-SYM realized with $bc\beta\gamma$ **systems** [Bonetti, Meneghelli, Rastelli]

- $\text{rank } G(\leq \text{dim } G)$ copies of $bc\beta\gamma$ systems
- Weights h and $\mathfrak{u}(1)$ charges m ($i = 1, \ldots, \text{rank } G$)

	h	m
(b_i, c_i)	$(\frac{d_i+1}{2}, \frac{1-d_i}{2})$	$(\frac{d_i-1}{2}, \frac{1-d_i}{2})$
(β_i, γ_i)	$(\frac{d_i}{2}, 1-\frac{d_i}{2})$	$(\frac{d_i}{2}, -\frac{d_i}{2})$

d_i: degs of fund. invariants/Casimirs

- $\mathbb{V}^{G}_{\mathcal{N}=4} \leq \mathbb{V}^{G}_{bc\beta\gamma}$

Novel free field realization

- Example: $G = SU(2)$, $d_1 = 2$ [Bonetti, Meneghelli, Rastelli][Adamovic]

	h	m
(b,c)	$(\frac{3}{2},-\frac{1}{2})$	$(\frac{1}{2},-\frac{1}{2})$
(β,γ)	$(1,0)$	$(1,-1)$

- VOA = 2d $\mathcal{N}=4$ small SCFA $\mathbb{V}^{SU(2)}_{\mathcal{N}=4} \leq \mathbb{V}_{bc\beta\gamma}$

$$J^+ = \beta, \quad G^+ = b, \quad T = -\frac{3}{2}b\partial c - \frac{1}{2}\partial bc - \beta\partial\gamma \ , \ldots$$

- As a subalgebra, $\mathbb{V}^{SU(2)}_{\mathcal{N}=4} = \ker S$

$$S = \oint dz (b e^{-\frac{1}{2}(\chi+\phi)})(z) \ , \tag{8}$$

with $\beta = e^{\chi+\phi}$, $\gamma = \partial\chi e^{-\chi-\phi}$.

Residues as free field characters

- $\mathcal{N} = 4$ SYMs with simple gauge groups G, the Schur indices

$$\mathcal{I} \sim \oint \prod_{i}^{r} \frac{da_i}{2\pi i a_i} \underbrace{\frac{\eta(\tau)^{3r}}{\vartheta_4(\mathfrak{b})^r} \prod_{\alpha} \frac{\vartheta_1(\alpha(\mathfrak{a}))}{\vartheta_4(\alpha(\mathfrak{a}) + \mathfrak{b})}}_{\mathcal{Z}(a)} \tag{9}$$

- Simply-laced gauge group: all poles of of $\mathcal{Z}(a)$ share identical residues as analytic functions (up to numerical factors)
  Others: finitely many different residues as analytic functions
- Consider only the simplest poles of $\mathcal{Z}(a)$

$$e^{2\pi i \alpha_i(\mathfrak{a})} = b q^{\frac{1}{2}} \ , i = 1, \dots, r \ . \tag{10}$$

- The residue:

$$\operatorname*{Res}_{e^{2\pi i \alpha_i(\mathfrak{a})} \to b q^{\frac{1}{2}}} \left(\prod_i \frac{1}{a_i} \right) \mathcal{Z}(a) \tag{11}$$

- Massive cancellation between numerator and denominator
  - $\prod_\alpha \to \prod_{H \geq 0}$: $H(\alpha) \equiv \sum_{i=1}^r m_i$, $\alpha = \sum_{i=1}^r m_i \alpha_i$
  - Almost complete cancellation between H and $H+1$
  - **Incomplete cancellation** when $\#(H+1) < \#(H)$:

The residue

$$= q^{\frac{\dim \mathfrak{g}}{8}} \prod_{\substack{H \geq 0 \\ \#(H+1) < \#(H)}} \frac{(b^H q^{\frac{1}{2} + \frac{H+1}{2}}; q)(b^{-H} q^{\frac{1}{2} - \frac{H+1}{2}}; q)}{(b^{H+1} q^{\frac{H+1}{2}}; q)(b^{-H-1} q^{1 - \frac{H+1}{2}}; q)} \tag{12}$$

Residues as free field characters

- $\#(H+1) < \#(H) \iff H+1 = d_i$:
  d_i are the **degrees of the fund. invariants** of $\mathfrak{g}$
  [Kostant][Collingwood, McGovern]:

- Residue [Peelaers][YP, Wang, Zheng]

$$\begin{aligned}\text{Res} &= q^{\frac{\dim \mathfrak{g}}{8}} \prod_{i=1}^{r} \frac{(b^{d_i-1} q^{\frac{d_i+1}{2}}; q)(b^{-d_i+1} q^{\frac{1-d_i}{2}}; q)}{(b^{d_i} q^{\frac{d_i}{2}}; q)(b^{-d_i} q^{1-\frac{d_i}{2}}; q)} \\ &= \text{ch}(\mathbb{V}^G_{bc\beta\gamma}) \\ &= \text{str}_{\mathbb{V}^G_{bc\beta\gamma}}\, q^{L_0 - \frac{c_{\mathsf{2d}}}{24}} b^f \,.\end{aligned}$$

Some immediate implications

- $\mathbb{V}^G_{\mathcal{N}=4} \leq \mathbb{V}^G_{bc\beta\gamma} \Rightarrow \mathrm{Res}$ must automatically satisfy all the flavored modular differential equations from the special nulls

  [Beem, Rastelli][Beem, Peelaers]

- Consider the projection $P : \mathbb{V}^G_{bc\beta\gamma} \to \mathbb{V}^G_{\mathcal{N}=4}$, we conjecture existence of $\mathcal{P}$:

$$\mathcal{I} = \ \mathrm{str}_{\mathbb{V}^G_{\mathcal{N}=4}} \ q^{L_0 - \frac{c_{\text{2d}}}{24}} b^f = \mathrm{str}_{\mathbb{V}^G_{bc\beta\gamma}} \ P q^{L_0 - \frac{c_{\text{2d}}}{24}} b^f$$

$$\equiv \mathcal{P} \underbrace{\mathrm{str}_{\mathbb{V}^G_{bc\beta\gamma}} \ q^{L_0 - \frac{c_{\text{2d}}}{24}} b^f}_{\mathrm{Res} \ \text{ of } \mathcal{Z}(a)}$$

$\Rightarrow$ Question: $\mathcal{N} = 4$ Schur indices completely determined by the residues of the one-loop $\mathcal{Z}(\mathfrak{a})$?

Closed-form Schur indices

Ellipticity

- Some convention: normal v.s. fraktur font

$$z = e^{2\pi i \mathfrak{z}}, \qquad y = e^{2\pi i \mathfrak{y}}, \qquad a = e^{2\pi i \mathfrak{a}}, \qquad b = e^{2\pi i \mathfrak{b}}$$

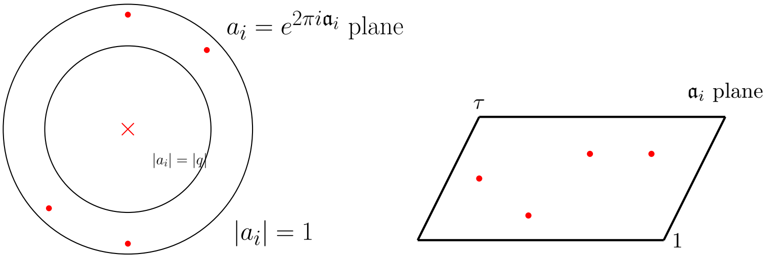
except

$$q = e^{2\pi i \tau} \; .$$

Ellipticity

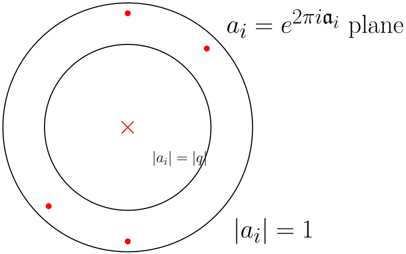
- One-loop $\mathcal{Z}(a_i \equiv e^{2\pi i \mathfrak{a}_i})$ is elliptic w.r.t. each $\mathfrak{a}_i$ [Razamat]

$$\mathcal{Z}(\mathfrak{a}_i, \ldots) = \mathcal{Z}(\mathfrak{a}_i + 1, \ldots) = \mathcal{Z}(\mathfrak{a}_i + \tau, \ldots) \ , \forall i \ .$$



- Ellipticity: highly constraining
- Task: compute multivariate contour integrals of $\mathcal{Z}$ by collecting residues

Ellipticity



Problem:

- non-isolated singularity at the origin: no residue
- All residues outside cancel: no naive "Higgs branch localization"

Elliptic functions

- Crucial family of (almost) elliptic functions:

$\sigma(\mathfrak{a}\vert\tau)$	$\zeta(\mathfrak{a}\vert\tau)$	$\wp(\mathfrak{a}\vert\tau)$	$\frac{\partial^n}{\partial\mathfrak{a}^n}\wp(\mathfrak{a}\vert\tau)$
almost elliptic	almost elliptic	elliptic	elliptic
$\sim \log \mathfrak{a}$	$\sim \mathfrak{a}^{-1}$	$\sim \mathfrak{a}^{-2}$	$\sim \mathfrak{a}^{-n-2}$

  e.g.,

$$\zeta(\mathfrak{z}) \equiv \frac{1}{\mathfrak{z}} + \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}}' \left[\frac{1}{\mathfrak{z} - m - n\tau} + \frac{1}{m + n\tau} + \frac{\mathfrak{z}}{(m + n\tau)^2}\right]$$

  From left to right: take derivative

- **Build any elliptic function**

Elliptic functions

- Example: elliptic $f(\mathfrak{a})$ with only simple poles $\mathfrak{a}_i$ in the **fund. parallelogram**

$$f(\mathfrak{a}) = \underbrace{C_f(\tau)}_{\mathfrak{a}\text{-const.}} + \overbrace{\frac{1}{2\pi i}\sum_i \underbrace{\left[\operatorname*{Res}_{a\to a_i}\frac{1}{a}f(a)\right]}_{R_i}\underbrace{\zeta(\mathfrak{a}-\mathfrak{a}_i)}_{\text{unit residue at } a_i}}^{\text{same pole/res struc. as } f} . \qquad (13)$$

Note: under $\mathfrak{a} \to \mathfrak{a} + \tau$, all ζ's shift by identical constant $\Rightarrow$ the RHS is invariant due to $\sum_i \mathrm{res}_i = 0$.

- Elliptic functions with higher order poles: include $\partial_{\mathfrak{a}}^n \wp(\mathfrak{a} - \mathfrak{a}_i)$

Elliptic functions

- Translation to Jacobi-theta

$$\zeta(\mathfrak{z}) = \frac{\vartheta_1'(\mathfrak{z})}{\vartheta_1(\mathfrak{z})} - 4\pi^2 \mathfrak{z} E_2(\tau) \ . \qquad (14)$$

- Can be Fourier expanded, $0 < \lambda < 1$

$$\zeta(\mathfrak{z}) = \ - 4\pi^2 \mathfrak{z} E_2(\tau) - \pi i + \pi \sum_n{}' \frac{q^{-\frac{n}{2}}}{\sin n\pi\tau} e^{2\pi i n \mathfrak{z}}, \quad \mathfrak{z} \in \mathbb{R}$$

$$\zeta(\mathfrak{z}) = \ - 4\pi^2 \mathfrak{z} E_2(\tau) + \pi i + \pi \sum_n{}' \frac{q^{+\frac{n}{2}}}{\sin n\pi\tau} e^{2\pi i n \mathfrak{z}}, \quad \mathfrak{z} \in \mathbb{R} - \lambda\tau$$

Eisenstein Series

- Twisted Eisenstein series $E_k\begin{bmatrix}\phi\\ \theta\end{bmatrix}$: quasi-Jacobi/modular forms
- Relatively simple shift properties,

$$E_k\begin{bmatrix}\pm 1\\ zq^{\frac{n}{2}}\end{bmatrix} = \sum_{\ell=0}^{k}\left(\frac{n}{2}\right)^{\ell}\frac{1}{\ell!}E_{k-\ell}\begin{bmatrix}(-1)^n(\pm 1)\\ z\end{bmatrix}\ . \tag{15}$$

- Constant terms ($\mathcal{S}_{2n} \equiv \left[\frac{y}{2\sinh\frac{y}{2}}\right]_{2n}$)

$$E_{2n+1}\begin{bmatrix}\pm 1\\ z\end{bmatrix} \sim 0, \qquad E_1\begin{bmatrix}+1\\ z\end{bmatrix} \sim -\frac{1}{2}, \tag{16}$$

$$E_{2n}\begin{bmatrix}+1\\ z\end{bmatrix} \sim -\frac{B_{2n}}{(2n)!}, \qquad E_{2n}\begin{bmatrix}-1\\ z\end{bmatrix} \sim -\mathcal{S}_{2n} \tag{17}$$

- Translation to Jacobi theta functions, e.g.

$$E_k \begin{bmatrix} \pm 1 \\ z \end{bmatrix} = - \underbrace{\left[e^{-\frac{y}{2\pi i}\mathcal{D}_{\mathfrak{z}} - P_2(y)} \right]_k}_{\text{coeff of } y^k \text{ in } y\text{-Taylor}} \vartheta_{1/4}(\mathfrak{z}) \tag{18}$$

where

$$\mathcal{D}_{\mathfrak{z}}^n \vartheta_i(\mathfrak{z}) \equiv \frac{\vartheta_i^{(n)}(\mathfrak{z})}{\vartheta_i(\mathfrak{z})} \; . \tag{19}$$

$\Rightarrow$ Modular properties under, e.g.

$$\mathfrak{z} \to \frac{\mathfrak{z}}{\tau}, \qquad \tau \to -\frac{1}{\tau} \; . \tag{20}$$

Eisenstein Series

- Can be Fourier expanded

$$E_{2n}\begin{bmatrix}+1\\ z\end{bmatrix} = \sum_{m=0}^{n} c_{2n}(2m) \sum_{\ell}{}' \frac{1}{\sin^{2m}\ell\pi\tau} e^{2\pi i\ell\mathfrak{z}}$$

$$E_{2n+1}\begin{bmatrix}-1\\ z\end{bmatrix} = \sum_{m=0}^{n} c_{2n+1}(2m+1) \sum_{\ell}{}' \frac{1}{\sin^{2m+1}\ell\pi\tau} e^{2\pi i\ell\mathfrak{z}}$$

- Difference equations of E provide recursion relations for c's

$$2ic_{2n+1}(2m+1) = \sum_{\ell=0}^{n-m} \frac{1}{2^{2\ell}(2\ell+1)!} c_{2n-2\ell}(2m) , \qquad m \in \mathbb{N} ,$$

$$2ic_{2n+2}(2m+2) = \sum_{\ell=0}^{n-m} \frac{1}{2^{2\ell}(2\ell+1)!} c_{2n+1-2\ell}(2m+1) , \qquad m \in \mathbb{N} ,$$

Eisenstein Series

- Conversely, Fourier series $\sum_\ell' \frac{1}{\sin^k \ell\pi\tau} e^{2\pi i\ell\mathfrak{z}} \sim$ combinations of twisted Eisenstein series.

- Integrating an elliptic function $f(\mathfrak{a})$

$$\oint f(a)\frac{da}{2\pi i a} = C_f + \frac{1}{2\pi i}\sum_i R_i \oint \zeta(\mathfrak{a}-\mathfrak{a}_i)\frac{da}{2\pi i a}\,. \qquad (21)$$

(a) The ζ integral is doable (ζ is a total derivative/Fourer)

(b) C_f can be replaced by f and R_i

- Final result (a_0 is an arbitrary reference value): sum over poles in the fundamental parallelogram

$$\oint_{|a|=1} f(a)\frac{da}{2\pi i a} = f(a_0) \;+ \sum_{\text{real/img. } \mathfrak{a}_i} R_i E_1\begin{bmatrix} -1 \\ \frac{a_i}{a_0} q^{\pm\frac{1}{2}} \end{bmatrix}\,,$$

real/imaginary poles: $\operatorname{Im}\mathfrak{a}_i = 0$ **or** $\operatorname{Im}\mathfrak{a}_i > 0$.

- $\mathcal{T}:\mathcal{N}=4$ $SU(2)$ SYM, $\mathbb{V}[\mathcal{T}]=$ 2d small $\mathcal{N}=4$ SCFA
- The Schur index (two imaginary poles, common residue)

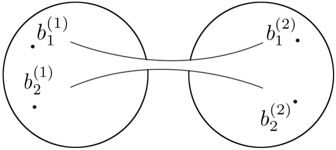
$$\mathcal{I}_{\mathcal{N}=4\ SU(2)}(b)=\frac{1}{2}\oint\frac{da}{2\pi i a}\frac{\vartheta_1(\pm 2\mathfrak{a})\eta(\tau)^3}{\vartheta_4(\pm 2\mathfrak{a}+\mathfrak{b})\vartheta_4(\mathfrak{b})}$$
$$=E_1\begin{bmatrix}-1\\ b\end{bmatrix}\underbrace{\frac{i\vartheta_4(\mathfrak{b})}{\vartheta_1(2\mathfrak{b})}}_{\mathrm{ch}^{A_1}_{\mathbb{V}_{bc\beta\gamma}}}=\frac{1}{2\pi}\frac{\vartheta_4'(\mathfrak{b})}{\vartheta_1(2\mathfrak{b})}\ . \qquad (22)$$

- Genus-one with one puncture

$$\mathcal{I}_{1,1}(b)=\frac{1}{2\pi}\frac{\vartheta_4'(\mathfrak{b})}{\vartheta_1(2\mathfrak{b})}\frac{\eta(\tau)}{\vartheta_4(\mathfrak{b})}\ . \qquad (23)$$

Example: $SU(2)$ SQCD

- $SU(2)$ gauge theory with four fundamental hypers: the associated VOA is $\widehat{\mathfrak{so}}(8)_{-2}$



- Schur index $\mathcal{I}_{0,4}(b)$

$$= -\frac{1}{2}\oint \frac{da}{2\pi i a}\vartheta_1(\pm 2\mathfrak{a})^2 \prod_{j=1}^{4} \frac{\eta(\tau)}{\vartheta_4(\pm\mathfrak{a}+\mathfrak{m}_j)} \qquad (24)$$

Example: $SU(2)$ SQCD

- 8 imaginary poles, 4 different residues: $R_j \equiv \underset{a \to m_j q^{\frac{1}{2}}}{\mathrm{Res}}$ integrand

$$\mathcal{I}_{0,4} = \sum_{j=1}^{4} E_1 \begin{bmatrix} -1 \\ m_j \end{bmatrix} \frac{i\vartheta_1(2\mathfrak{m}_j)}{\eta(\tau)} \prod_{\ell \neq j} \frac{\eta(\tau)}{\vartheta_1(\mathfrak{m}_j + \mathfrak{m}_\ell)} \frac{\eta(\tau)}{\vartheta_1(\mathfrak{m}_j - \mathfrak{m}_\ell)} \ .$$

- m's recombines to fugacities of the four punctures

$$m_1 = b_1^{(1)} b_2^{(1)}, \quad m_2 = \frac{b_1^{(1)}}{b_2^{(1)}}, \quad m_3 = b_1^{(2)} b_2^{(2)}, \quad m_4 = \frac{b_1^{(2)}}{b_2^{(2)}} \ .$$

  Manifest permutation invariance among $b_a^{(i)}$ is lost

- We will derive alternative more elegant expression

General integral formula

- Higher ranks:

$$\oint \cdots \frac{da_2}{2\pi i a_2} \frac{da_1}{2\pi i a_1} \underbrace{\mathcal{Z}(a_1, \ldots, a_n)}_{\text{individually elliptic}} \tag{25}$$

- Problem: ellipticity is lost as function of $\mathfrak{a}_{2,\ldots,n}$

$$\oint \cdots \frac{da_2}{2\pi i a_2} \underbrace{\oint \frac{da_1}{2\pi i a_1} \mathcal{Z}(a_1, \ldots, a_n)}_{\text{non-elliptic in } \mathfrak{a}_{2,3,\ldots}} \tag{26}$$

$\mathfrak{a}_1$-integral contains Eisenstein series in $\mathfrak{a}_{2,3,\ldots}$

- New task: compute

$$\mathcal{I}_k^{\pm}[\,f\,] \equiv \oint \frac{da}{2\pi i a} E_k \begin{bmatrix} \pm 1 \\ ab \end{bmatrix} \underbrace{f(a)}_{\text{elliptic}} \,. \qquad (27)$$

- Tool: Fourier series

$$E_k \begin{bmatrix} \pm 1 \\ a \end{bmatrix} \leftrightarrow \sum_{\ell=0}^{k} C_k(\ell) \sum_n{}' \frac{1}{\sin^\ell n\pi\tau} e^{2\pi i n \mathfrak{a}} \qquad (28)$$

General integral formula

- **General formula for $\mathcal{I}_k^{\pm}[f]$** as finite sum of residues $\times$ Eisenstein

$$\begin{aligned}\mathcal{I}_k^{-}[\,f\,] &= \oint_{|a|=1} \frac{da}{2\pi i a} f(a) E_k \begin{bmatrix} -1 \\ ab \end{bmatrix} \\ &= -\,\mathcal{S}_{2k} \left(f(a_0) + \sum_{\text{poles } a_i} R_i E_1 \begin{bmatrix} -1 \\ a_i b q^{\pm\frac{1}{2}} \end{bmatrix} \right) \\ &\quad - \sum_{\text{poles } a_i} R_i \sum_{\ell=0}^{k-1} \mathcal{S}_{2\ell} E_{k-\ell+1} \begin{bmatrix} 1 \\ a_i b q^{\pm\frac{1}{2}} \end{bmatrix} ,\end{aligned} \tag{29}$$

where $\frac{1}{2}\frac{y}{\sinh\frac{y}{2}} = \sum_{\ell\geq 0} \mathcal{S}_{2\ell}\, y^{2\ell}$.

Higher-rank computable examples

- Compact formula for all A_1-theories of class-$\mathcal{S}$
- $SU(N)$ with $2N$ flavors (computable, compact formula not available yet)
- $\mathcal{N}=4$ $G=SU(3), SU(4), SO(4), SO(5)$ SYM
- $\mathcal{N}=4$ $SU(N)$ unflavored indices (conjectural compact formula)
- Schematic structure:

$$\mathcal{I} = \sum_{\text{poles}} (\text{res}) E_* \begin{bmatrix} \pm 1 \\ \text{pole info} \end{bmatrix} \dots E_* \begin{bmatrix} \pm 1 \\ \text{pole info} \end{bmatrix} . \qquad (30)$$

A quick review: 4d $\mathcal{N}=2$ SCFT of class-$\mathcal{S}$ [Gaiotto]

- Starting point: 6d (0,2) SCFT of type $\mathfrak{g} \in$ ADE
- Put on $\mathbb{R}^{3,1} \times \Sigma_{g,n}$: genus g, n punctures (co-dim two defects in 6d, labeled by "some discrete data")
- Compactify $\Sigma_{g,n} \Rightarrow$ 4d $\mathcal{N}=2$ SCFT $\mathcal{T}_{g,n}[\mathfrak{g}, \text{discrete data}]$
- The discrete data at each puncture: implies a flavor symmetry subgroup in 4d
- Complex structure moduli of $\Sigma_{g,n}$: gauge couplings
- Pants-decompositions of one $\Sigma_{g,n}$: different gauge theory descriptions, S-duality

A quick review: 4d $\mathcal{N}=2$ SCFT of class-$\mathcal{S}$

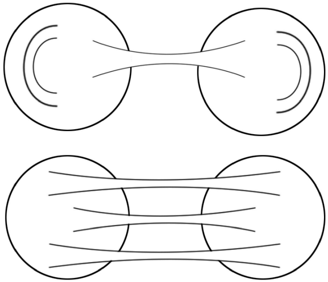
- Simplest examples: $\mathfrak{g}=\mathfrak{su}(2)$
- "discrete data": trivial
- Punctured Riemann surface $\Sigma_{g,n} \leftrightarrow \mathcal{T}_{g,n}$
- Examples

(1) $\mathcal{T}_{g=0,n=3}$: "trinion theory" = 4 free hypers
(2) $\mathcal{T}_{g=0,n=4}$: $SU(2)$ theory with 4 fundamental hypers
(3) $\mathcal{T}_{g=0,n=5}$: $SU(2)\times SU(2)$ theory with hypers in $(\mathbf{2},\mathbf{1})$, $(\mathbf{2},\mathbf{2})$, $(\mathbf{1},\mathbf{2})$
(3) $\mathcal{T}_{g=1,n=1}$: $\mathcal{N}=2$ $SU(2)$ theory + one free hyper

Examples: A_1 theories of class-$\mathcal{S}$

$\mathcal{I}_{2,0}(b)$

- Two S-duality frames [Gadde, et.al.][Kiyoshige, Nishinaka]



- No puncture: hidden $U(1)$-flavor symmetry (invisible in the class-$\mathcal{S}$ picture) with fugacity b [Kiyoshige, Nishinaka]

$\mathcal{I}_{2,0}(b)$

- First frame $\mathcal{I}_{2,0}(\mathfrak{b})$

$$\begin{aligned}&\frac{1}{2}\oint \frac{da}{2\pi i a}\vartheta_1(2\mathfrak{a})\vartheta_1(-2\mathfrak{a})\mathcal{I}_{1,1}(\mathfrak{a},\mathfrak{b})\mathcal{I}_{1,1}(\mathfrak{a},-\mathfrak{b})\\ =&\frac{i\vartheta_1(\mathfrak{b})^2}{\eta(\tau)\vartheta_1(2\mathfrak{b})}\left(E_3\begin{bmatrix}+1\\b\end{bmatrix}+E_1\begin{bmatrix}+1\\b\end{bmatrix}E_2\begin{bmatrix}+1\\b\end{bmatrix}-E_2(\tau)E_1\begin{bmatrix}+1\\b\end{bmatrix}\right.\\ &\qquad\left.+E_2(\tau)E_1\begin{bmatrix}-1\\b\end{bmatrix}+\frac{1}{12}E_1\begin{bmatrix}-1\\b\end{bmatrix}\right)\\ &+\frac{\eta(\tau)^2}{2}\left(E_2+\frac{1}{12}\right)\frac{\vartheta_4(0)^2}{\vartheta_4(\mathfrak{b})^2}\ .\end{aligned}$$

$\mathcal{I}_{2,0}(b)$

- Second frame $\mathcal{I}'_{2,0}(\mathfrak{b})$

$$\frac{1}{8}\oint\prod_{i=1}^{3}\left[\frac{da_i}{2\pi i a_i}\vartheta_1(2\mathfrak{a}_i)\vartheta_1(-2\mathfrak{a}_i)\right]\prod_{\pm\pm\pm}\frac{\eta(\tau)}{\vartheta_4(\pm\mathfrak{a}_1\pm\mathfrak{a}_2\pm\mathfrak{a}_3+\mathfrak{b})}$$
$$=\frac{i\vartheta_1(2\mathfrak{b})^2}{\eta(\tau)\vartheta_1(4\mathfrak{b})}\left(E_3\begin{bmatrix}+1\\b^2\end{bmatrix}+E_1\begin{bmatrix}+1\\b^2\end{bmatrix}E_2\begin{bmatrix}+1\\b^2\end{bmatrix}+\frac{1}{12}E_1\begin{bmatrix}+1\\b^2\end{bmatrix}\right).$$

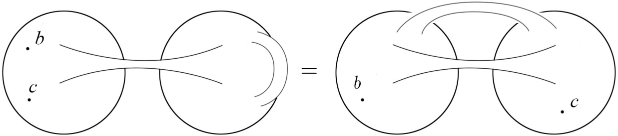
- S-duality $\mathcal{I}_{2,0}(2\mathfrak{b})=\mathcal{I}'_{2,0}(\mathfrak{b})$ thanks to “duplication formula”

$$E_1\begin{bmatrix}+1\\z\end{bmatrix}-E_1\begin{bmatrix}-1\\z\end{bmatrix}=\frac{\eta(\tau)^3}{2i}\frac{\vartheta_1(2\mathfrak{z})\vartheta_4(0)^2}{\vartheta_1(\mathfrak{z})^2\vartheta_4(\mathfrak{z})^2}.$$

Examples: A_1 theories of class-$\mathcal{S}$

$\mathcal{I}_{1,2}(b,c)$ [Gadde, et.al.]

- Two duality frames



$$\frac{\eta(\tau)^2}{\vartheta_1(2\mathfrak{b})\vartheta_1(2\mathfrak{c})}\left(\pm E_2\begin{bmatrix}1\\ bc^{\pm}\end{bmatrix}\right) = \frac{\eta(\tau)^2}{\vartheta_1(2\mathfrak{b})\vartheta_1(2\mathfrak{c})}\left(E_2\begin{bmatrix}1\\ \pm\sqrt{bc}\end{bmatrix} + \ldots\right)$$

- S-duality: due to identities

$$\sum_{\pm} E_k\begin{bmatrix}\phi\\ \pm z\end{bmatrix}(\tau) = 2E_k\begin{bmatrix}\phi\\ z^2\end{bmatrix}(2\tau), \;\ldots \tag{31}$$

Examples: A_1 theories of class-$\mathcal{S}$

$$\mathcal{I}_{1,2}(b) = -\frac{1}{2}\frac{\eta(\tau)^2}{\prod_{i=1}^{2}\vartheta_1(2\mathfrak{b}_i)}\sum_{\alpha_1,\alpha_2=\pm1}\alpha_1\alpha_2 E_2\begin{bmatrix}+1\\ \prod_{i=1}^{2} b_i^{\alpha_i}\end{bmatrix}\ .$$

- Gauging in trinion $\mathcal{T}_{0,3}$ to increase puncture numbers n
- Gluing two punctures to increase genus g
- Observation 1 (for adding $\mathcal{T}_{0,3}$)

$$\underset{a=b^\beta c^\gamma q^{\frac{1}{2}}}{\operatorname{Res}}\ \frac{1}{a}\frac{\eta(\tau)^n}{\vartheta_1(2\mathfrak{a})\prod_{i=1}^{n-1}\vartheta_1(2\mathfrak{b}_i)}\mathcal{I}_{\mathsf{VM}}(\mathfrak{a})\mathcal{I}_{0,3}(\mathfrak{a},\mathfrak{b}_n,\mathfrak{b}_{n+1})$$

$$= \beta\gamma\frac{-i\eta(\tau)^{n+1}}{2\prod_{i=1}^{n+1}\vartheta_1(2\mathfrak{b}_i)}$$

- Observation 2 (for increasing g)

$$\frac{\eta(\tau)^n}{\vartheta_1(2\mathfrak{a})\vartheta_1(-2\mathfrak{a})\prod_{i=1}^{n-2}\vartheta_1(2\mathfrak{b}_i)}\frac{1}{2}\vartheta_1(2\mathfrak{a})^2 \qquad (32)$$

independent of $\mathfrak{a}$

- Observation 3 (partial proof in [Gadde, Rasstelli, Razamat, Yan])

$$\mathcal{I}_{0,3}(b) = \frac{1}{2i}\frac{\eta(\tau)}{\prod_{i=1}^{3}\vartheta_1(2\mathfrak{b}_i)}\sum_{\alpha_i=\pm}\Big(\prod_{i=1}^{3}\alpha_i\Big)E_1\left[\frac{-1}{\prod_{i=1}^{3}b_i^{\alpha_i}}\right] .$$

Examples: A_1 theories of class-$\mathcal{S}$

- Finaly result for all A_1 indices at any (g, n)

$$\mathcal{I}_{g,n}(b) = \frac{i^n}{2} \frac{\eta(\tau)^{n+2g-2}}{\prod_{i=1}^{n} \vartheta_1(2\mathfrak{b}_i)} \sum_{\alpha=\pm} \Big(\prod_{i=1}^{n} \alpha_i\Big) \sum_{k=1}^{n+2g-2} \lambda_k^{(n+2g-2)} E_k \left[\frac{(-1)^n}{\prod_{i=1}^{n} b_i^{\alpha_i}} \right]$$

$$\mathcal{I}_{g,n=0} = \frac{1}{2} \eta(\tau)^{2g-2} \sum_{k=1}^{g-1} \lambda_{2k}^{(2g-2)} \left(E_{2k} + \frac{B_{2k}}{(2k)!} \right)$$

- λ's are rational numbers: recursion relations

$$\lambda_0^{(\text{even})} = \lambda_{\text{even}}^{(\text{odd})} = \lambda_{\text{odd}}^{(\text{even})} = 0, \qquad \lambda_2^{(2)} = 1 \ ,$$

$$\lambda_{2m+1}^{(2k+1)} = \sum_{\ell=m}^{k} \lambda_{2\ell}^{(2k)} \mathcal{S}_{2(\ell-m)}, \quad \lambda_{2m+2}^{(2k+2)} = \sum_{\ell=m}^{k} \lambda_{2\ell+1}^{(2k+1)} \mathcal{S}_{2(\ell-m)},$$

$$\lambda_1^{(2k+1)} = \sum_{\ell=1}^{k} \lambda_{2\ell}^{2k} \left(\frac{B_{2\ell}}{(2\ell)!} - \mathcal{S}_{2\ell} \right) \quad .$$

- Manifest permutation invariance among all flavor fugacities b_i
- Unflavoring limit of $\mathcal{I}_{0,4}(b)$:

$$\mathcal{I}_{0,4}(b \to 1) = 3\frac{q\partial_q E_4(\tau)}{\eta(\tau)^{10}}$$

directly recovers Arakawa and Kawasetsu's unflavored character of $\widehat{\mathfrak{so}}(8)_{-2}$

Examples: $\mathcal{N} = 4$ theories

- Flavored index with lower ranks ($SU(2,3,4)$, $SO(4)$, $SO(5)$) can be easily computed, e.g.

$$\mathcal{I}_{\mathcal{N}=4\ SU(3)} = -\frac{1}{8}\underbrace{\frac{\vartheta_4(\mathfrak{b})}{\vartheta_4(3\mathfrak{b})}}_{\mathrm{ch}_{\mathbb{V}^{A_2}_{bc\beta\gamma}}}\left(-\frac{1}{3} + 4E_1\begin{bmatrix}-1\\ b\end{bmatrix}^2 - 4E_2\begin{bmatrix}+1\\ b^2\end{bmatrix}\right) \ ,$$

- Compact formula for general $SU(N)$ flavor indices out of reach at the moment

- Conjectural unflavored indices for $SU(N)$,

$$\mathcal{I}_{\mathcal{N}=4\ SU(2N+1)} = (-1)^N \tilde{\lambda}_2^{(2N+3)} + (-1)^N \sum_{k=1}^{N} \frac{\tilde{\lambda}_{2k+2}^{(2N+3)}(2)}{2k} \widetilde{\mathbb{E}}_{2k} \ ,$$

$$\mathcal{I}_{\mathcal{N}=4\ SU(2N)} = (-1)^N \sum_{k=1}^{N} \frac{(-1)^k \tilde{\lambda}_{2k+1}^{(2N+2)}(2)}{(2k)!} \left(\frac{1}{2\pi}\right)^{2k-1} \frac{\vartheta_4^{(2k)}(0)}{\vartheta_1'(0)} \ .$$

where

$$\widetilde{\mathbb{E}}_{2k} = \sum_{\substack{\{n_p\} \\ \sum_{p\geq 1} 2pn_p = 2k}} \prod_{p\geq 1} \frac{1}{n_p!} \left(-\frac{1}{2p} E_{2p}\right)^{n_p} \tag{33}$$

Examples: $\mathcal{N} = 4$ theories

- The $\tilde{\lambda}$'s are related to those in $\mathcal{I}_{g,n}$

$$\tilde{\lambda}_{\ell}^{(n)}(K) \equiv \sum_{\ell'=\max(\ell,1)}^{n} \left(\frac{K}{2}\right)^{\ell'-\ell} \frac{1}{(\ell'-\ell)!} \lambda_{\ell'}^{(n)} . \qquad (34)$$

- More precisely, $\tilde{\lambda}$ appears in residues of $\mathcal{I}_{g,n}$

- e.g., E_6, E_7 SCFT,
  With the Spiridonov-Warnaar inversion [Spiridonov, Warnaar] [Razamat] [Agarwal, Maruyoshi, Song]
  $\mathcal{I}_{SU(3)\text{SQCD}} \to \mathcal{I}_{E_6}$, $\mathcal{I}_{SU(4)\text{SQCD}} \to \mathcal{I}_{R_{0,4}} \to \mathcal{I}_{E_7}$

- Conformal gauging multiple $D_p(G)$ theories: $\widehat{\Gamma}[G]$ [Kang, Lawrie, Song]
  $\widehat{E_6}[SO(4)]$, $\widehat{D}_4[SU(3)]$, $\widehat{E}_7[SU(3)]$, $\widehat{E}_6[SU(4)]$, ...:
  Schur indices given basically by $\mathcal{N}=4$ indices

$$\mathcal{I}_{\widehat{\Gamma}[G]} = q^{\#}\mathcal{I}_{\mathcal{N}=4\ G}(b = q^{\frac{\alpha_\Gamma}{2}-1}, q^{\alpha_\Gamma}) \ . \tag{35}$$

For example,

$$\begin{aligned} \mathcal{I}_{\widehat{D_4}[SU(3)]} &= q\mathcal{I}_{\mathcal{N}=4\ SU(3)}(b=1,q^2) \\ &= \frac{1}{24}+\frac{1}{2}E_2(2\tau)\ . \end{aligned} \tag{36}$$

and (defining $\widehat{\vartheta}_i(z) \equiv \vartheta_i(z,4\tau)$)

$$\begin{aligned} \mathcal{I}_{\widehat{E_7}[SU(3)]} &= q^{-1}\mathcal{I}_{\mathcal{N}=4\ SU(3)}(b=q,q^4) \\ &= \frac{1}{12\pi}\frac{\widehat{\vartheta}_4(\tau)}{\widehat{\vartheta}_1(\tau)}\Bigg[-\frac{\widehat{\vartheta}'_4(0)}{\widehat{\vartheta}_4(0)}-\frac{\widehat{\vartheta}'_4(\tau)}{\widehat{\vartheta}_4(\tau)}-\frac{i}{\pi}\frac{\widehat{\vartheta}'_4(0)}{\widehat{\vartheta}_4(0)}\frac{\widehat{\vartheta}'_4(\tau)}{\widehat{\vartheta}_4(\tau)} \\ &\quad -\frac{i}{\pi}\frac{\widehat{\vartheta}'_4(\tau)^2}{\widehat{\vartheta}_4(\tau)^2}-\frac{i}{2\pi}\frac{\widehat{\vartheta}''_4(0)}{\widehat{\vartheta}_4(0)}+\frac{i}{2\pi}\frac{\widehat{\vartheta}''_4(\tau)}{\widehat{\vartheta}_4(\tau)}\Bigg]\ . \end{aligned} \tag{37}$$

Modular properties: $\mathcal{N}=4$ $SU(2)$ theory

- S-transform

$$\mathcal{I} = \frac{1}{2\pi}\frac{\vartheta_4'(\mathfrak{b})}{\vartheta_1(2\mathfrak{b})} \xrightarrow{STS} \mathcal{I}_{\mathsf{log}} \equiv \frac{1}{2\pi i}(\log q - 2\pi i)\mathcal{I} + (\log b)\,\mathrm{ch}_{\mathbb{V}_{bc\beta\gamma}}$$

- $\mathrm{ch}_M = \mathrm{ch}_{\mathbb{V}_{bc\beta\gamma}} - \mathcal{I}$: character of the only non-vacuum irreducible module M from category-$\mathcal{O}$ (Adamovic)
- Three solutions to all the flavored modular differential equations [Beem, Rastelli][Beem, Peelaers]

$$\mathcal{I}, \qquad \mathrm{ch}_{\mathbb{V}_{bc\beta\gamma}}, \qquad \mathcal{I}_{\log}\ .$$

$\mathcal{I}, \mathcal{I}_{\log}$ have smooth unflavoring limit.

Modular properties: $SU(2)$ with four flavors

- The index

$$\mathcal{I}_{0,4} = \frac{1}{2}\frac{\eta(\tau)^2}{\prod_{i=1}^4 \vartheta_1(2\mathfrak{b}_i)} \sum_{\alpha_i=\pm}\left(\prod_{i=1}^4 \alpha_i\right) E_2\begin{bmatrix} +1 \\ \prod_{i=1}^4 b_i^{\alpha_i}\end{bmatrix} .$$

- Under S-transformation,

$$e^{-\frac{4\pi i}{\tau}\sum_i \mathfrak{b}_i^2}\mathcal{I}_{0,4}\left(-\frac{1}{\tau}\right)$$
$$= \frac{\log q}{2\pi}\mathcal{I}_{0,4}(\tau)$$
$$+ \frac{\eta(\tau)^2}{4\pi\prod_{i=1}^4 \vartheta_1(2\mathfrak{b}_i)} \sum_{\alpha_i}\left(\prod_i \alpha_i\right)\log(\prod_i b_i^{\alpha_i}) E_1\begin{bmatrix} 1 \\ \prod_i b_i^{\alpha_i}\end{bmatrix} .$$

Modular properties: $SU(2)$ with four flavors

- Reorganized into

$$e^{-\frac{4\pi i}{\tau}\sum_i \mathfrak{b}_i^2}\mathcal{I}_{0,4}\left(-\frac{1}{\tau}\right) = \frac{\log q}{2\pi}\mathcal{I}_{0,4}(\tau) + \frac{1}{\pi}\sum_{i=1}^{4}(\log m_j)R_j$$

$$e^{+\frac{4\pi i}{\tau}\sum_i \mathfrak{b}_i^2}R_j\left(-\frac{1}{\tau}\right) = iR_j(\tau)$$

- R_j: characters of four non-vacuum modules of $\widehat{\mathfrak{so}}(8)_{-2}$ [Arakawa]; highest weight

$$\lambda = w(\omega_1 + \omega_2 + \omega_3) - \rho, \quad w = 1, s_{1,3,4}\ . \tag{38}$$

conformal weight $h = -1$.

Modular properties: $SU(2)$ with four flavors

- Consistency check: R_j satisfy all the required flavored modular differential equations [Peelaers]

Surface Defect from Higgsing

- Focus on A_1 theories
- poles $b_i = q^{\frac{k}{2}}$ of $\mathcal{I}_{g,n+1}$
- $k=1$: recovers

$$\operatorname*{Res}_{b\to q^{\frac{1}{2}}} \frac{2\eta(\tau)^2}{b}\mathcal{I}_{g,n+1}(b) = \mathcal{I}_{g,n} \ . \tag{39}$$

- $k>1$, residue (using shift properties of E_i's)

$$\sim \frac{\eta(\tau)^{n+2g-2}}{\prod_{i=1}^{n}\vartheta_1(2\mathfrak{b}_i)} \sum_{\alpha_i}\left(\prod_{i=1}^{n}\alpha_i\right)\sum_{\ell=1}^{n+1+2g-2}\tilde{\lambda}_{\ell}^{n+1+2g-2}(k)E_\ell\begin{bmatrix}(-1)^{n+k+1}\\ b_1^{\alpha_1}\dots b_n^{\alpha_n}\end{bmatrix}$$

$\sim$ difference operator on $\mathcal{I}_{g,n}$ [Gaiotto, Rastelli, Razamat][Alday, et.al.][Bullimore, et.al.]

Surface Defect from Higgsing

- $\tilde{\lambda}$'s are rational numbers

$$\tilde{\lambda}_{\ell}^{(n)}(K) \equiv \sum_{\ell'=\max(\ell,1)}^{n} \lambda_{\ell'}^{(n)} \left(\frac{K}{2}\right)^{\ell'-\ell} \frac{1}{(\ell'-\ell)!} \,. \qquad (40)$$

already appeared in $\mathcal{N}=4$ unflavored indices ($K=2$)

- Example ($k=$ even)

$$\operatorname*{Res}_{b\to q^{\frac{k}{2}}} \frac{\eta(\tau)^2}{b}\mathcal{I}_{g,5}(b) = \frac{k}{2}\frac{\eta(\tau)^2}{\prod_{i=1}^{4}\vartheta_1(2\mathfrak{b}_i)} \sum_{\alpha_i}\left(\prod_{i=1}^{4}\alpha_i\right) E_2\left[\frac{-1}{\prod_{i=1}^{4} b_i^{\alpha_i}}\right] \,.$$

$$= \frac{kq^{-\frac{1}{2}}}{2}\sum_{\pm} b_4^{\mp 2}\mathcal{I}_{0,4}(b_4 q^{\pm\frac{1}{2}}) \qquad (41)$$

Outlook

- Identify the projection $P : \mathbb{V}^G_{bc\beta\gamma} \to \mathbb{V}^G_{\mathcal{N}=4}$ (generalize to $\mathcal{N} = 3$?), clarify the VOA interpretation of the computation method
- BRST-reduction in two steps? ($\dim G$ copies of $\beta\gamma bc$ $\xrightarrow{\text{“Higgsing”}}$ $\text{rank}\, G$ copies of $\beta\gamma bc \xrightarrow{\ker} \mathcal{N} = 4$ VOA)
- The residues for other $\mathcal{N} = 2$ Lagrangian theories: new free field realization? Are they module characters of the associated VOA? Modular properties?
- For $\mathcal{N} = 4$ theories with non-ADE gauge group: physical meaning of the other residues? Additional free field realization?

Outlook

- Closed-form for correlators with local/non-local ops
  E.g., coupling to $\mathbb{C}P^1$ model inserts $a^2 + a^{-2}$ factor, Wilson loop inserts polynomials of a's: recompute all the Fourier integrals
- Additional integral formula for more general Schur indices
  E.g., Gauging A_N-theories reqiures integrals of the form

$$\oint \frac{dz}{2\pi i z} f(z) E_{k_1}\begin{bmatrix} \pm 1 \\ z a_1 \end{bmatrix} E_{k_2}\begin{bmatrix} \pm 1 \\ z a_2 \end{bmatrix} \cdots \tag{42}$$

- Elliptic genera computation? (relation between JK and unit circle)
- Macdonald/Hall-Littlewood index: work with non-elliptic functions

Thank you!