Minimalism in Modified Gravity

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Based on collaborations with
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Why modified gravity?

- Can we address **mysteries in the universe**?
  Dark energy, dark matter, inflation, big-bang singularity, cosmic magnetic field, etc.
- Help constructing a **theory of quantum gravity**?
  Superstring, Horava-Lifshitz, etc.
- Do we really **understand GR**?
  One of the best ways to understand something may be to break (modify) it and then to reconstruct it.
- …

of d.o.f. in general relativity

- 10 metric components → 20-dim phase space @ each point
- Einstein-Hilbert action does not contain time derivatives of N & N^i → $\pi_N = 0$ & $\pi_i = 0$

ADM decomposition

- Lapse N, shift N^i, 3d metric h_{ij}

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$$

- Einstein-Hilbert action

$$I = \frac{M_{\rm Pl}^2}{2} \int d^4x \sqrt{-g}\, {}^{(4)}R$$
$$= \frac{M_{\rm Pl}^2}{2} \int dt d^3\vec{x} N \sqrt{h} \left[K^{ij}K_{ij} - K^2 + {}^{(3)}R\right]$$

- Extrinsic curvature

$$K_{ij} = \frac{1}{2N}(\partial_t h_{ij} - D_i N_j - D_j N_i)$$

of d.o.f. in general relativity

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  All constraints are independent of N & N^i → π_N & π_i "commute with" all constraints → 1st-class

1^{st}-class vs 2^{nd}-class

- **2^{nd}-class constraint S**
  $\{ S , C_i \} \not\approx 0$ for $^{\exists} i$
  Reduces 1 phase space dimension
- **1^{st}-class constraint F**
  $\{ F , C_i \} \approx 0$ for $^{\forall} i$
  Reduces 2 phase space dimensions
  Generates a symmetry
  Equivalent to a pair of 2^{nd}-class constraints

$\{ C_i \mid i = 1,2,\ldots \}$: complete set of independent constraints

$A \approx B$ ⟺ $A = B$ when all constraints are imposed

(weak equality)

of d.o.f. in general relativity

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  All constraints are independent of N & N^i → π_N & π_i "commute with" all constraints → 1st-class
- 4 generators of 4d-diffeo: 1st-class constraints
- 20 – (4+4) x 2 = 4 → 4-dim physical phase space @ each point → 2 local physical d.o.f.

Minimal # of d.o.f. in modified gravity = 2

of d.o.f. in general relativity

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- Einstein-Hilbert action does not contain time derivatives of N & N^i → $\pi_N = 0$ & $\pi_i = 0$
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Minimal # of d.o.f. in modified gravity = 2

Can this be saturated?

Is general relativity unique?

- Lovelock theorem says **"yes"** if we assume:
(i) 4-dimensions; (ii) diffeo invariance; (iii) metric only; (iv) up to 2nd-order eom's of the form $E_{ab}=0$.
- Effective field theory (derivative expansion) says **"yes"** at low energy if we assume:
(i) 4-dimensions; (ii) diffeo invariance; (iii) metric only.
- However, cosmological backgrounds break 4d-diffeo while keeping 3d-diffeo.
- A metric theory with 3d-diffeo but with broken 4d-diffeo typically has 3 local physical d.o.f. (e.g. scalar-tensor theory, EFT of inflation/dark energy, Horava-Lifshitz gravity)

Example: simple scalar-tensor theory

- Covariant action

$$I = \frac{1}{2}\int d^4x\sqrt{-g}\left[\Omega^2(\phi)^{(4)}R + P(X,\phi)\right] \qquad X \equiv -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

- ADM decomposition

$$ds^2 = -N^2dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$$

$$g^{\mu\nu} = \begin{pmatrix} -\frac{1}{N^2} & \frac{N^i}{N^2} \\ \frac{N^j}{N^2} & h^{ij} - \frac{N^iN^j}{N^2} \end{pmatrix}$$

- Unitary gauge

$$\phi = t \quad \Rightarrow \quad X = \frac{1}{2}\frac{1}{N^2}$$

This is a good gauge iff derivative of ϕ is timelike.

- Action in unitary gauge

$$I = \int dt d^3\vec{x} N\sqrt{h}\left\{f_1(t)\left[K^{ij}K_{ij} - K^2 + {}^{(3)}R\right] + \frac{2}{N}\dot{f}_1(t)K + f_2(N,t)\right\}$$

$$\Omega^2(\phi) = f_1(t) \qquad P(X,\phi) = f_2(N,t)$$

Is general relativity unique?

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- A metric theory with 3d-diffeo but with broken 4d-diffeo typically has 3 local physical d.o.f. (e.g. scalar-tensor theory, EFT of inflation/dark energy, Horava-Lifshitz gravity)
- Is GR unique when we assume: (i) 4-dimensions; (ii) 3d-diffeo invariance; (iii) metric only; (iv) 2 local physical d.o.f. (= 2 polarizations of TT gravitational waves)?

A class of minimally modified gravity

Chushan Lin and SM, JCAP1710 (2017), 033

- 4d theories invariant under 3d-diffeo: $x^i \rightarrow x^i + \xi^i(t,\mathbf{x})$
- ADM decomposition

$$ds^2 = -N^2dt^2 + h_{ij}\,(dx^i+N^idt)\,(dx^j+N^jdt)$$

- Ansatz: actions linear in the lapse function N

$$S = \int dt d^3x \sqrt{h} N F\left(K_{ij}, R_{ij}, \nabla_i, h^{ij}, t\right)$$

$$K_{ij} \ = \ (\partial_t h_{ij} - \nabla_i N_j - \nabla_j N_j)/(2N)$$

- For simplicity, exclude mixed-derivative terms, i.e. those that contain spatial derivatives acted on K_{ij}
- Relation between K_{ij} and π^{ij} (momenta conjugate to h_{ij}) assumed to be invertible

$$\det\left(\frac{\partial^2 F}{\partial K_{ij}\partial K_{kl}}\right) \neq 0$$

- Seek theories with 2 local physical d.o.f.!

What we expect/need

- 10 metric components → 20-dim phase space @ each point
- $\pi_N = 0$ & $\pi_i = 0$: 1st-class constraints
- 3 generators of 3d spatial diffeo : 1st-class constraints
- If there is no other constraint then 20 – (4+3) x 2 = 6 → 6-dim physical phase space @ each point → 3 local physical d.o.f.
- We thus need a 1st-class constraint or a pair of 2nd-class constraints to find theories with 2 local physical d.o.f.

What we found

- The necessary and sufficient condition under which a theory in this class has 2 or less local physical degrees of freedom.
- Simple examples with 2 local physical degrees of freedom

An example of MMG: square-root gravity

- Action

$$S = \int d^4x\sqrt{h}N\left[\xi M(t)^4\sqrt{\left(1+\frac{c_1(t)}{M(t)^2}\mathcal{K}\right)\left(1+\frac{c_2(t)}{M(t)^2}R\right)}-\Lambda(t)\right]$$

$$\mathcal{K} = K_{ij}K^{ij} - K^2,\ K = K^i{}_i,\ \xi = \pm 1$$

- In the weak gravity limit,

$$S \simeq \int d^4x\sqrt{h}N\left[\xi M^4 - \Lambda + \frac{\xi}{2}M^2(c_1\mathcal{K}+c_2R)+\ldots\right]$$

GR with $M_p^2 = \xi c_1 M^2\,,\quad c_g^2 = \frac{c_2}{c_1}\,,\quad \Lambda_{\rm eff} = \frac{\Lambda - \xi M^4}{\xi c_1 M^2}$ is recovered.

- Flat FLRW with a canonical scalar $\qquad \xi = 1$

$$S = \int dx^3 \int dt a^3\left[M^4\sqrt{N^2 - \frac{6c_1}{M^2}\frac{\dot{a}^2}{a^2}} - N\Lambda + \frac{1}{2N}\dot{\phi}^2 - NV(\phi)\right]$$

$$1 - 6c_1\frac{H^2}{M^2} = \frac{M^8}{(\Lambda+\rho_m)^2}$$

$$H^2 \to \frac{1}{6c_1^2}M_p^2 \quad \text{as} \quad \rho_m \to \infty$$

H remains finite

What we found

- The necessary and sufficient condition under which a theory in this class has 2 or less local physical degrees of freedom.
- Simple examples with 2 local physical degrees of freedom

- However, it was not clear how to couple matter to gravity in a consistent way…

Matter coupling in scalar tensor theory

- Jordan (or matter) frame

$$I = \frac{1}{2}\int d^4x\sqrt{-g^{\rm J}}\,\left[\Omega^2(\phi)\,R[g^{\rm J}] + \cdots\right] + I_{\rm matter}[g^{\rm J}_{\mu\nu};{\rm matter}]$$

- Einstein-frame $\quad g^{\rm E}_{\mu\nu} = \Omega^2(\phi) g^{\rm J}_{\mu\nu} \quad$ K.Maeda (1989)

$$I = \frac{1}{2}\int d^4x\sqrt{-g^{\rm E}}\,\left[R[g^{\rm E}] + \cdots\right] + I_{\rm matter}[\Omega^{-2}(\phi) g^{\rm E}_{\mu\nu};{\rm matter}]$$

- Do we call this GR? No. This is a modified gravity because of **non-trivial matter coupling → <u>type-I</u>**
- There are more general scalar tensor theories where there is **no Einstein frame → <u>type-II</u>**

Type-I & type-II modified gravity

- **Type-I:**
  There exists an Einstein frame
  Can be recast as GR + extra d.o.f. **+ matter, which couple(s) non-trivially**, by change of variables
- **Type-II:**
  No Einstein frame
  Cannot be recast as GR + extra d.o.f. + matter by change of variables

Type-I minimally modified gravity (MMG)

Katsuki Aoki, Chunshan Lin and SM, arXiv:1804.03902, to appear in PRD

- **# of local physical d.o.f. = 2**
- There exists an Einstein frame
- Can be recast as GR **+ matter, which couple(s) non-trivially**, by change of variables
- **The most general change of variables = canonical tr.**
- Matter coupling just after canonical tr. → breaks diffeo → 1st-class constraint downgraded to 2nd-class → leads to extra d.o.f. in phase space → inconsistent
- Gauge-fixing after canonical tr. → splits 1st-class constraint into pair of 2nd-class constraints
- **Matter coupling after canonical tr. + gauge-fixing → a pair of 2nd-class constraints remain → consistent**

Simple example of type-I MMG

Katsuki Aoki, Chunshan Lin and SM, arXiv:1804.03902, to appear in PRD

- Start with the Hamiltonian of GR
  phase space: (N, N^i, Γ_{ij}) & (π_N, π_i, Π^{ij})
- **Simple canonical tr. $(\Gamma_{ij}, \Pi^{ij}) \rightarrow (\gamma_{ij}, \pi^{ij})$**

$$\Gamma_{ij} = -\frac{\delta F}{\delta \Pi^{ij}} \qquad \pi^{ij} = -\frac{\delta F}{\delta \gamma_{ij}} \qquad F = -\int d^3x \sqrt{\gamma} f(\tilde{\Pi}) \qquad \tilde{\Pi} := \Pi^{ij}\gamma_{ij}/\sqrt{\gamma}$$

- Gauge-fixing $\mathcal{G} \approx 0$

$$\boxed{\{\mathcal{G}, \mathscr{H}_0\} \not\approx 0} \qquad \{\mathcal{G}, \pi_N\} \approx 0 \qquad \{\mathcal{G}, \pi_i\} \approx 0 \qquad \{\mathcal{G}, \mathscr{H}_i\} \approx 0$$

- Lagrangian for $g^J_{\mu\nu} = (N, N^i, \gamma_{ij})$

$$\sqrt{-g^J}\mathcal{L} = \dot{\gamma}_{ij}\pi^{ij} - \mathcal{H}^{\rm GF}_{\rm tot}$$

$\mathcal{H}^{\rm GF}_{\rm tot}$: gauge-fixed total Hamiltonian density

- Adding matter

$$I_{\rm matter}[g^{\rm J}_{\mu\nu}; \text{matter}]$$

c.f. Carballo-Rubio, Di Filippo & Liberati (2018) argued that the square-root gravity should be of type-I but did not find a consistent matter coupling.

More general example of type-I MMG & phenomenology

Katsuki Aoki, Antonio De Felice, Chunshan Lin, SM and Michele Oliosi, arXiv: 1810.01047

- Original phase space: (M, N^i, Γ_{ij}) & (Π_M, π_i, Π^{ij})
- **Canonical tr. $(\mathcal{N}, \Gamma_{ij}, \Pi_{\mathcal{N}}, \Pi^{ij}) \rightarrow (N, \gamma_{ij}, \pi_N, \pi^{ij})$**

$$\mathcal{N} = -\frac{\delta F}{\delta \Pi_{\mathcal{N}}} \quad \Gamma_{ij} = -\frac{\delta F}{\delta \Pi^{ij}} \quad \pi_N = -\frac{\delta F}{\delta N} \quad \pi^{ij} = -\frac{\delta F}{\delta \gamma_{ij}}$$

$$F = -\int d^3x (M^2 \sqrt{\gamma} f(\tilde{\Pi}, \tilde{\mathcal{H}}) + N^i \Pi_i) \qquad \tilde{\Pi} = \frac{1}{M^2\sqrt{\gamma}} \Pi^{ij}\gamma_{ij}$$

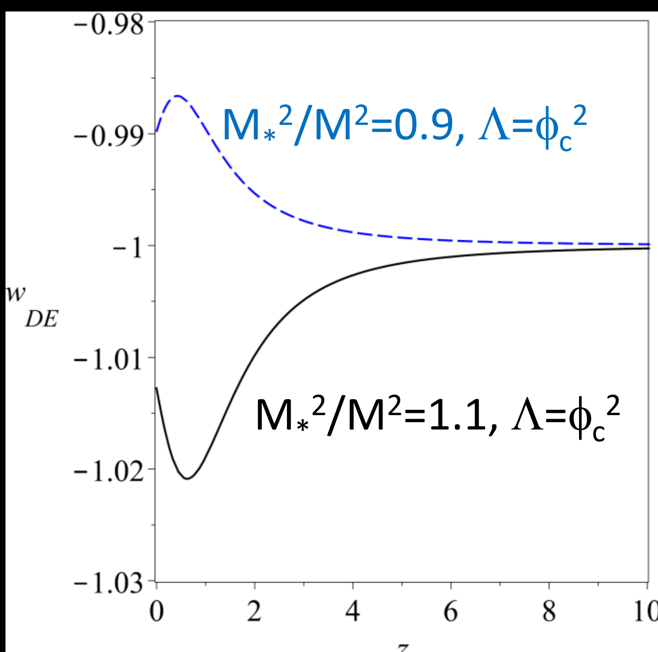
$$f(\phi, \psi) = f_0(\phi) + f_1(\phi)\psi + \mathcal{O}(\psi^2) \qquad \tilde{\mathcal{H}} = \frac{1}{M^2\sqrt{\gamma}} \Pi_{\mathcal{N}} N$$

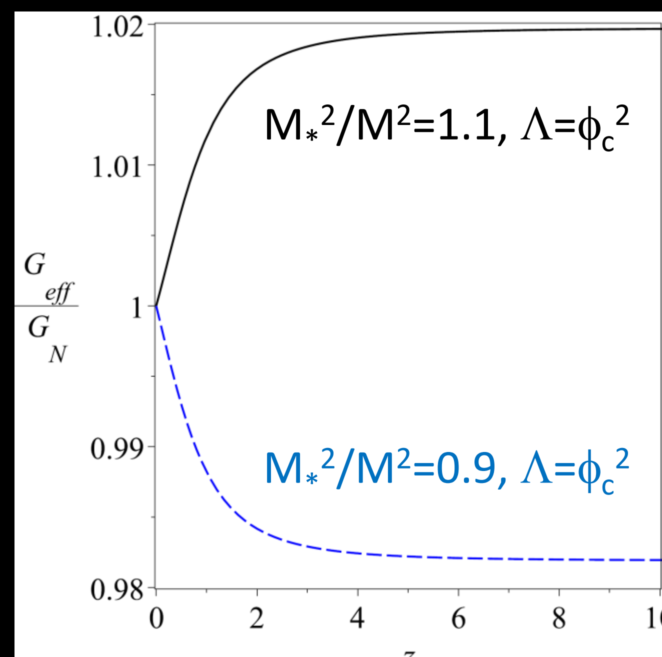
- Same sign for $\mathcal{N}$ & N, Γ_{ij} & γ_{ij} → $f_0 > 0$, $f_1 > 0$
- $c_T^2 = f_1^2/f_0'$ → $f_0' = f_1^2$
- $w_{DE} \neq -1$ in general (without dynamical DE)
- $G_{eff}/G = 1/f_0' \neq 1$ in general while $\Psi/\Phi = 1$

Example with $w_{DE} \neq -1$ & $G_{eff}/G \neq 1$

- $\Lambda \neq 0$ before canonical tr.
- $c_T^2 = f_1^2/f_0' \rightarrow f_0' = f_1^2$
- A choice of f_0

$$f_0' = \frac{(M_*/M_{\rm pl})^2 + (\phi/\phi_c)^2}{1 + (\phi/\phi_c)^2}$$





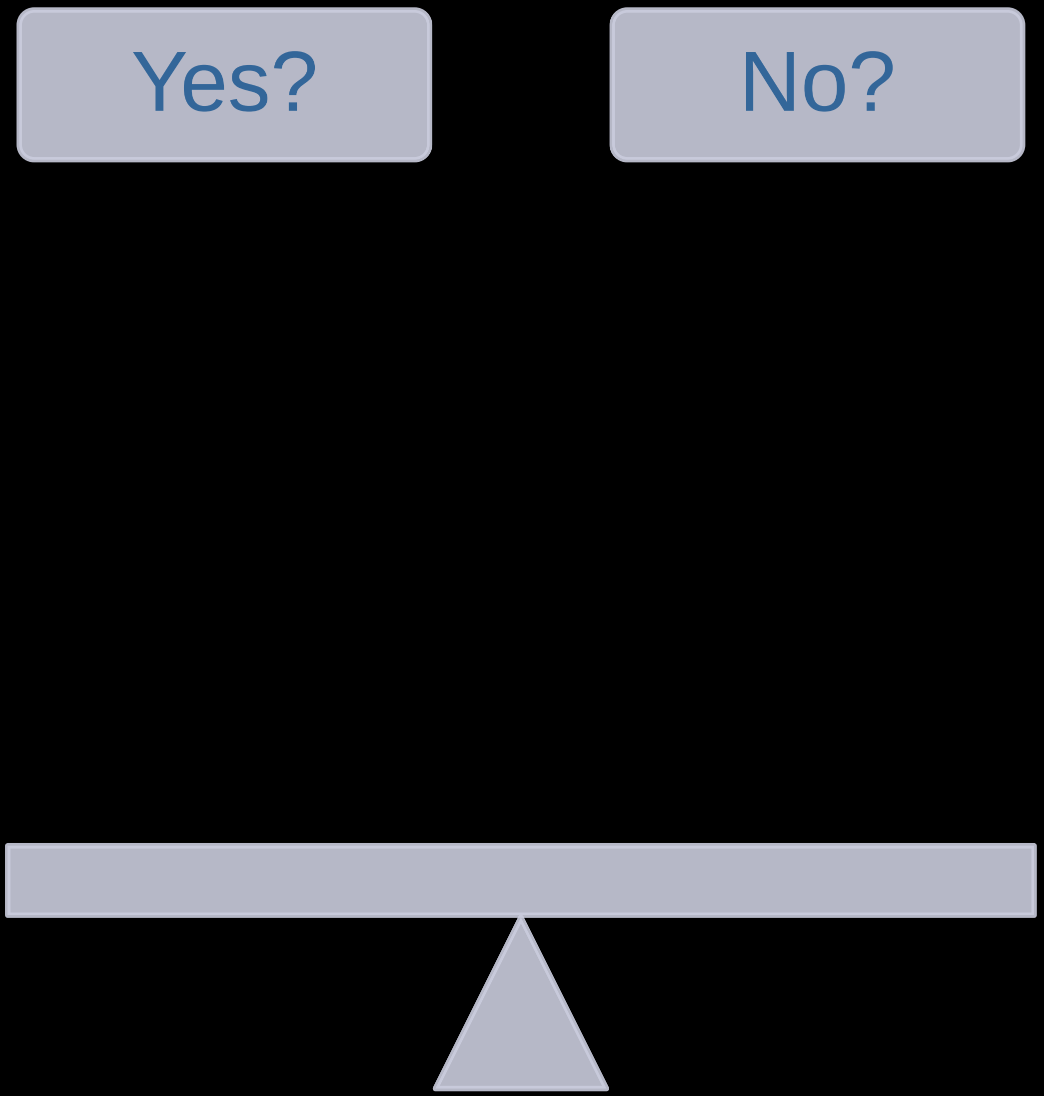
Type-II minimally modified gravity (MMG)

- **# of local physical d.o.f. = 2**
- **No Einstein frame**
- Cannot be recast as GR + matter by change of variables
- **Is there such a theory? Yes!**
- Example: Minimal theory of massive gravity [Antonio De Felice and SM, PLB752 (2016) 302; JCAP1604 (2016) 028; PRL118 (2017) 091104]
- Another example? : Ghost-free nonlocal gravity (if extended to nonlinear level?)

Massive gravity in a nutshell

Simple question: Can graviton have mass?

May lead to acceleration without dark energy

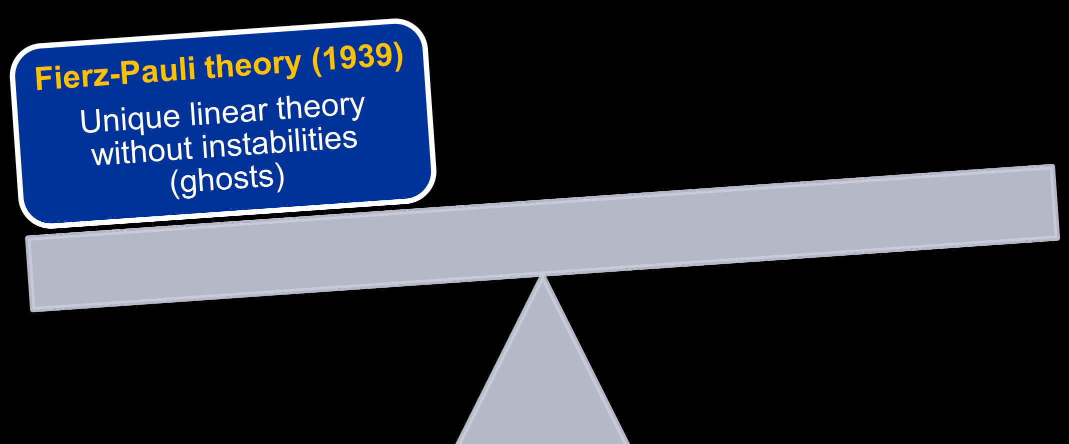


Massive gravity in a nutshell

Simple question: Can graviton have mass?

May lead to acceleration without dark energy

Yes? | No?



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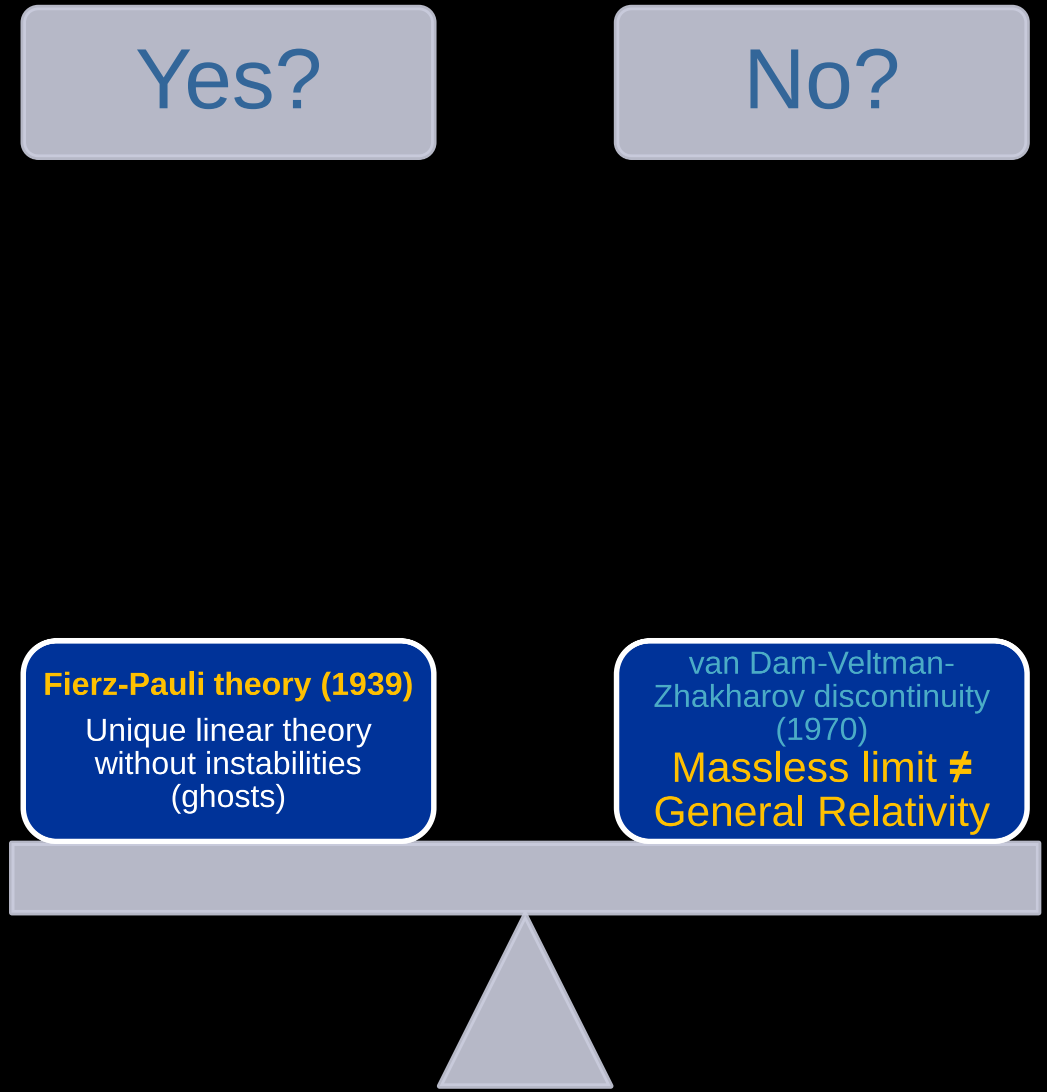
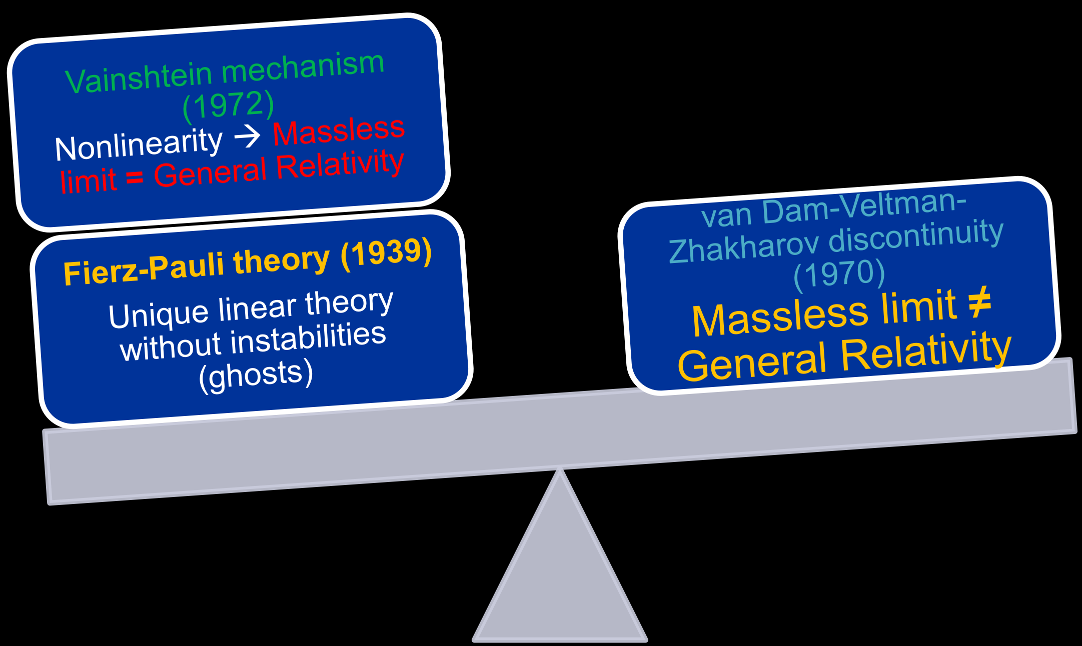


Massive gravity in a nutshell

Simple question: Can graviton have mass?

May lead to acceleration without dark energy

Yes? No?



Massive gravity in a nutshell

Simple question: Can graviton have mass?

May lead to acceleration without dark energy

Yes?

No?

Vainshtein mechanism (1972)
Nonlinearity → Massless limit = General Relativity

Fierz-Pauli theory (1939)
Unique linear theory without instabilities (ghosts)

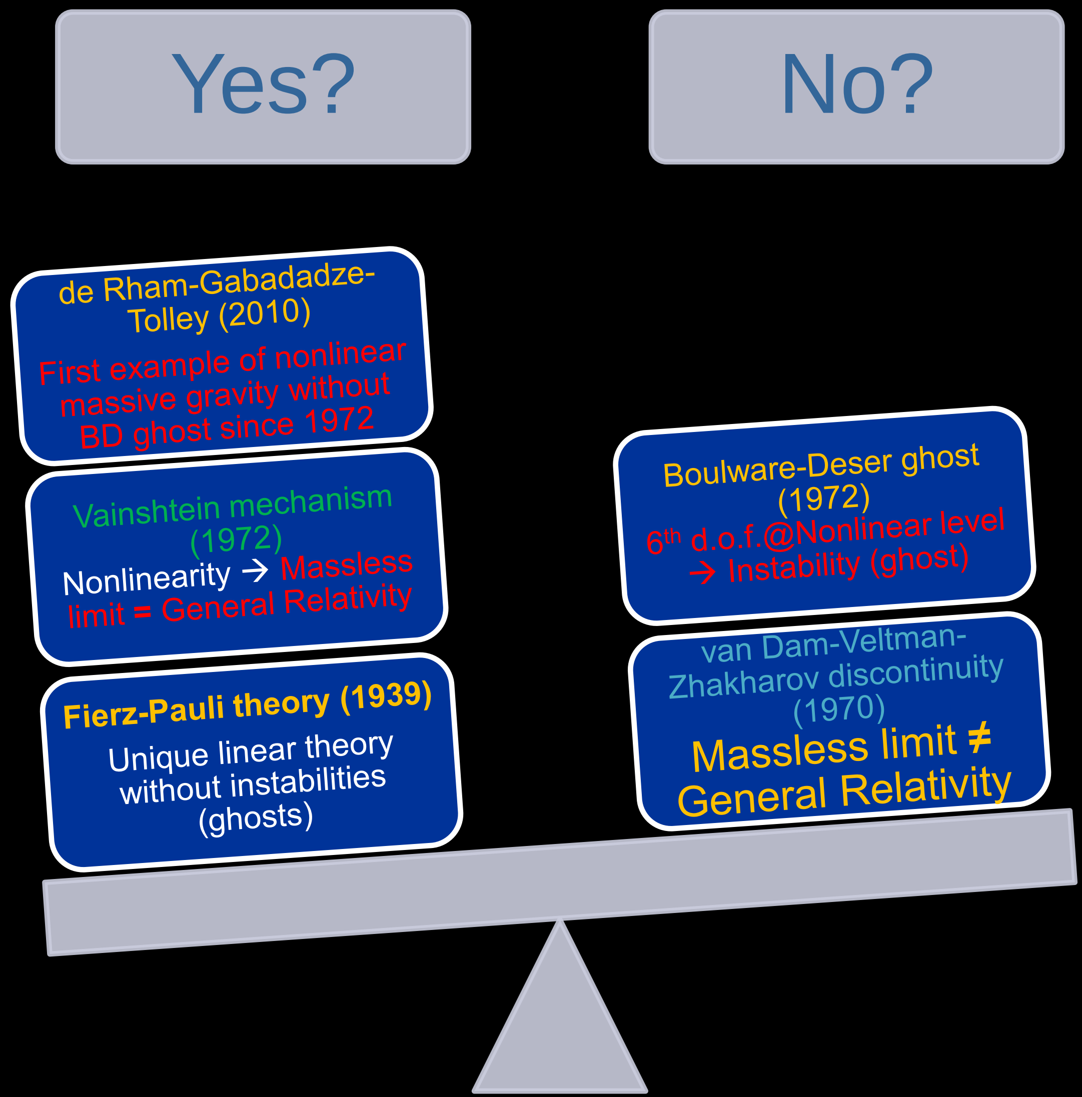
Boulware-Deser ghost (1972)
6^{th} d.o.f.@Nonlinear level → Instability (ghost)

van Dam-Veltman-Zhakharov discontinuity (1970)
Massless limit ≠ General Relativity

Massive gravity in a nutshell

Simple question: Can graviton have mass?

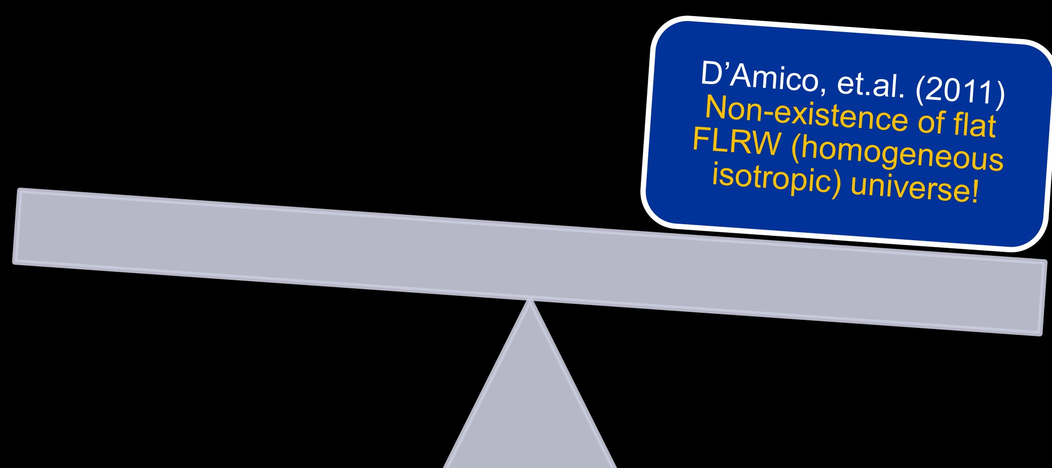
May lead to acceleration without dark energy



Cosmological solutions in nonlinear massive gravity

Good?

Bad?



Cosmological solutions in nonlinear massive gravity

Good?

Bad?

Open universes with self-acceleration
GLM (2011a)

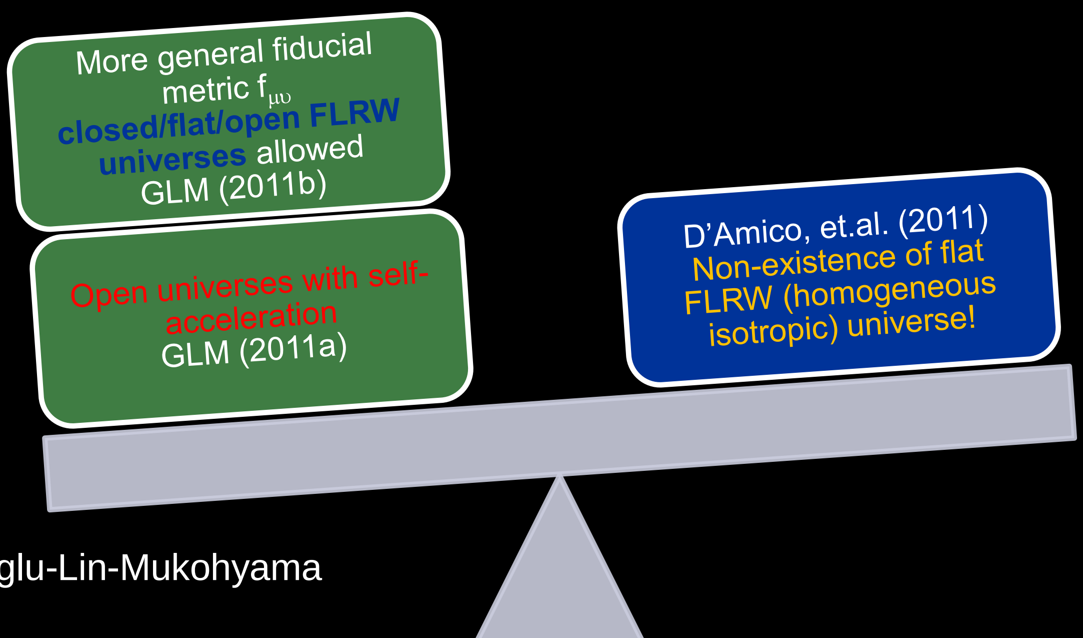
D’Amico, et.al. (2011)
Non-existence of flat FLRW (homogeneous isotropic) universe!

GLM = Gumrukcuoglu-Lin-Mukohyama

Cosmological solutions in nonlinear massive gravity

Good?

Bad?



GLM = Gumrukcuoglu-Lin-Mukohyama

Cosmological solutions in nonlinear massive gravity

Good?

Bad?

More general fiducial metric $f_{\mu\upsilon}$
closed/flat/open FLRW universes allowed
GLM (2011b)

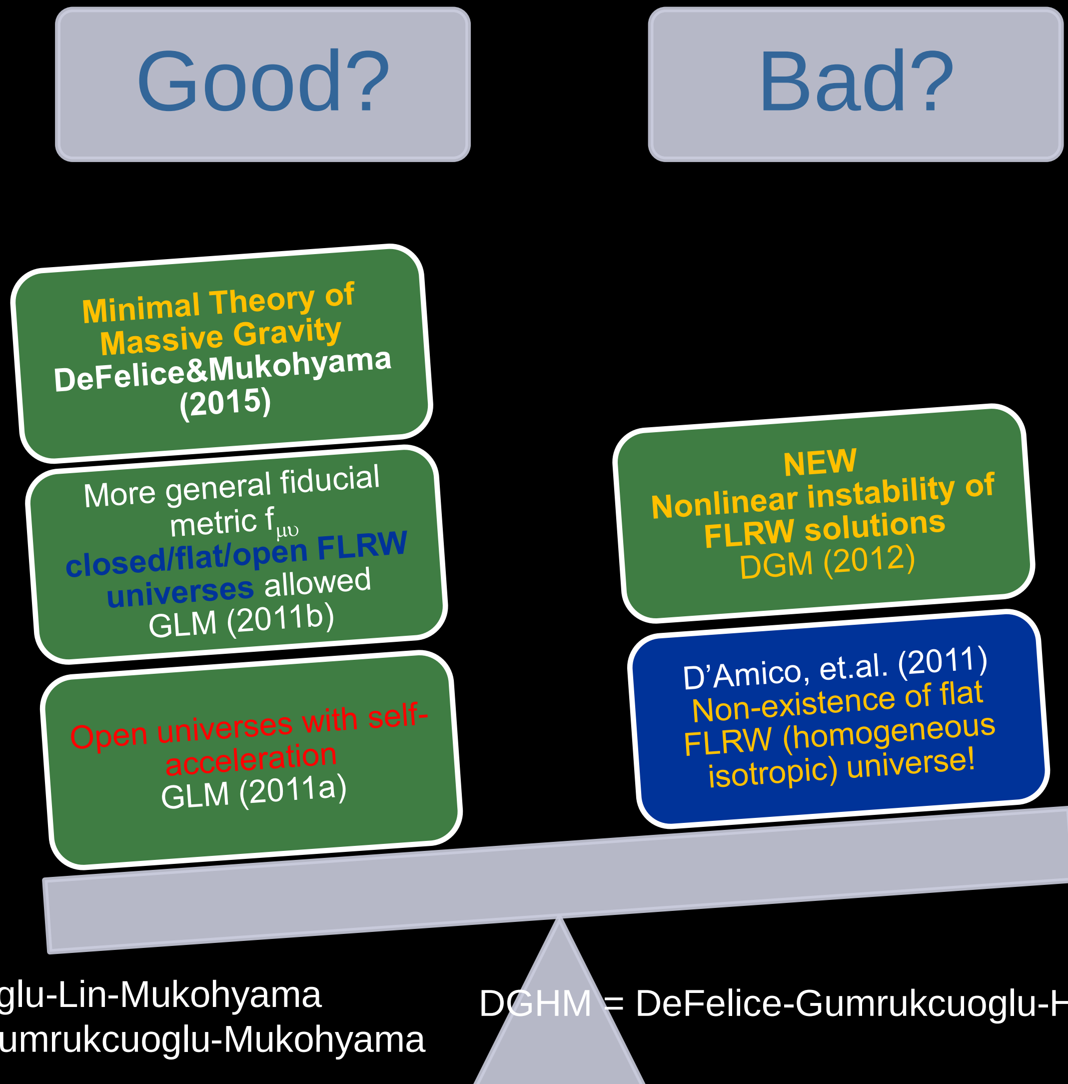
Open universes with self-acceleration
GLM (2011a)

NEW
Nonlinear instability of FLRW solutions
DGM (2012)

D'Amico, et.al. (2011)
Non-existence of flat FLRW (homogeneous isotropic) universe!

GLM = Gumrukcuoglu-Lin-Mukohyama
DGM = DeFelice-Gumrukcuoglu-Mukohyama

Cosmological solutions in nonlinear massive gravity



GLM = Gumrukcuoglu-Lin-Mukohyama
DGM = DeFelice-Gumrukcuoglu-Mukohyama

DGHM = DeFelice-Gumrukcuoglu-Heisenberg-Mukohyama

Minimal theory of massive gravity (MTMG)

De Felice & Mukohyama, PLB752 (2016) 302; JCAP1604 (2016) 028

- 2 physical dof only = massive gravitational waves
- exactly same FLRW background as in dRGT
- no BD ghost, no Higuchi ghost, no nonlinear ghost
- positivity bound does not apply

Three steps to the Minimal Theory

1. Fix local Lorentz to realize ADM vielbein in dRGT
2. Switch to Hamiltonian
3. Add 2 additional constraints

(It is easy to go back to Lagrangian after 3.)

Lorentz-violation due to graviton loops is suppressed by m^2/M_{Pl}^2 and thus consistent with all constraints for $m = O(H_0)$

Cosmology of MTMG I

- Constraint $\mathcal{C}_0 \approx 0$ $\qquad X \doteq \tilde{a}/a$

$$(c_3 + 2c_2 X + c_1 X^2)(\dot{X} + NHX - MH) = 0$$

- Self-accelerating branch

$$X = X_\pm \doteq \frac{-c_2 \pm \sqrt{c_2^2 - c_1 c_3}}{c_1} \qquad \lambda = 0$$

$$3M_{\rm P}^2 H^2 = \frac{m^2 M_{\rm P}^2}{2}\left(c_4 + 3c_3 X + 3c_2 X^2 + c_1 X^3\right) + \rho$$

$\Lambda_{\rm eff}$ from graviton mass term (even with c_4=0)
Scalar/vector parts are the same as ΛCDM
Time-dependent mass for gravity waves

Cosmology of MTMG II

- Constraint $\mathcal{C}_0 \approx 0$ $X \doteq \tilde{a}/a$

$$(c_3 + 2c_2 X + c_1 X^2)(\dot{X} + NHX - MH) = 0$$

- "Normal" branch

$$H = XH_f \qquad \lambda = \frac{4(H_f X - H)\,N}{m^2\,(c_1 X^2 + 2c_2\,X + c_3)\,M}$$

$$3M_{\rm P}^2 H^2 = \frac{m^2 M_{\rm P}^2}{2}\left(c_4 + 3c_3 X + 3c_2 X^2 + c_1 X^3\right) + \rho$$

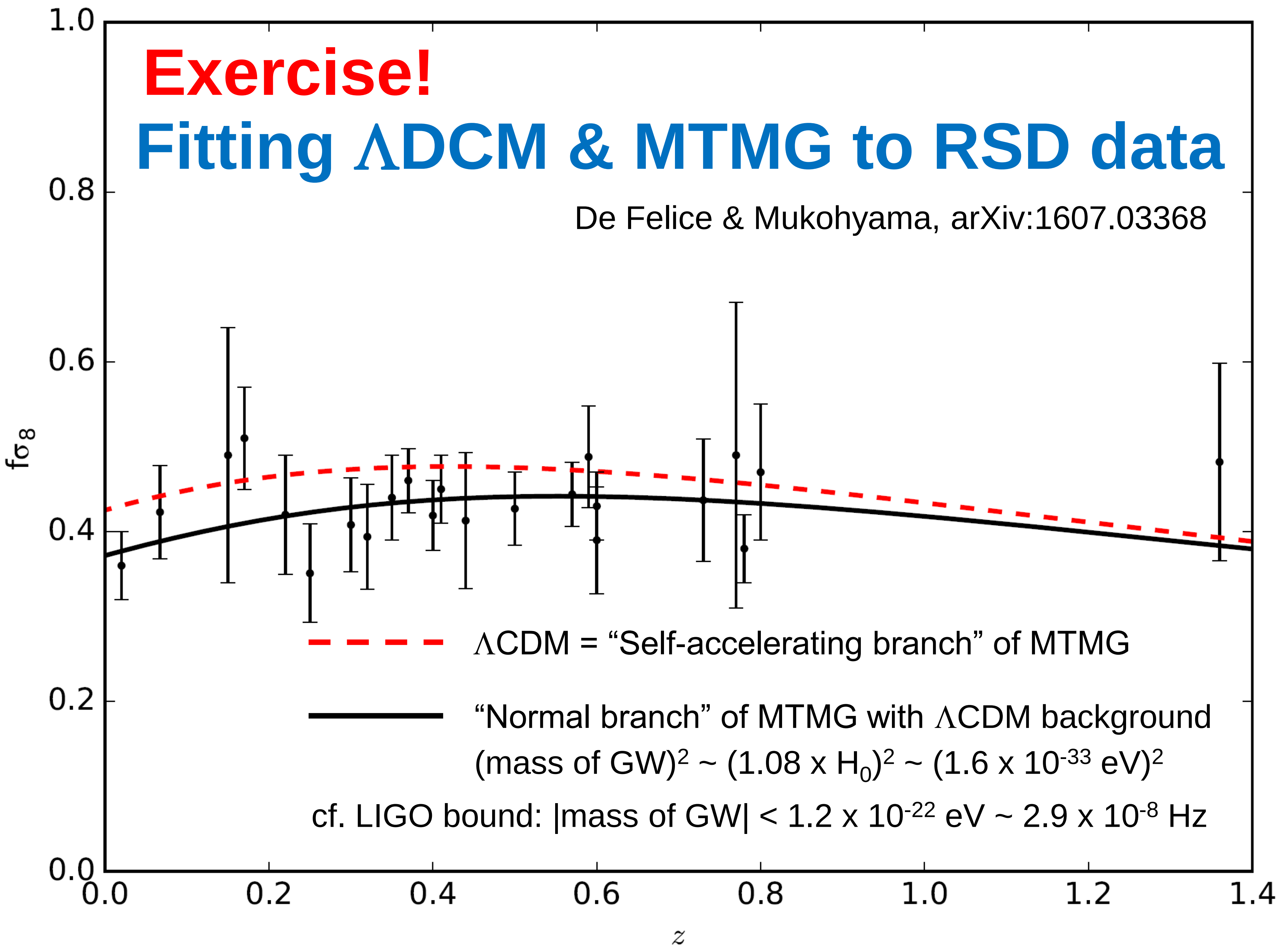
Dark component **without extra dof**

Scalar part recovers GR in UV (L≪m^{-1}) but

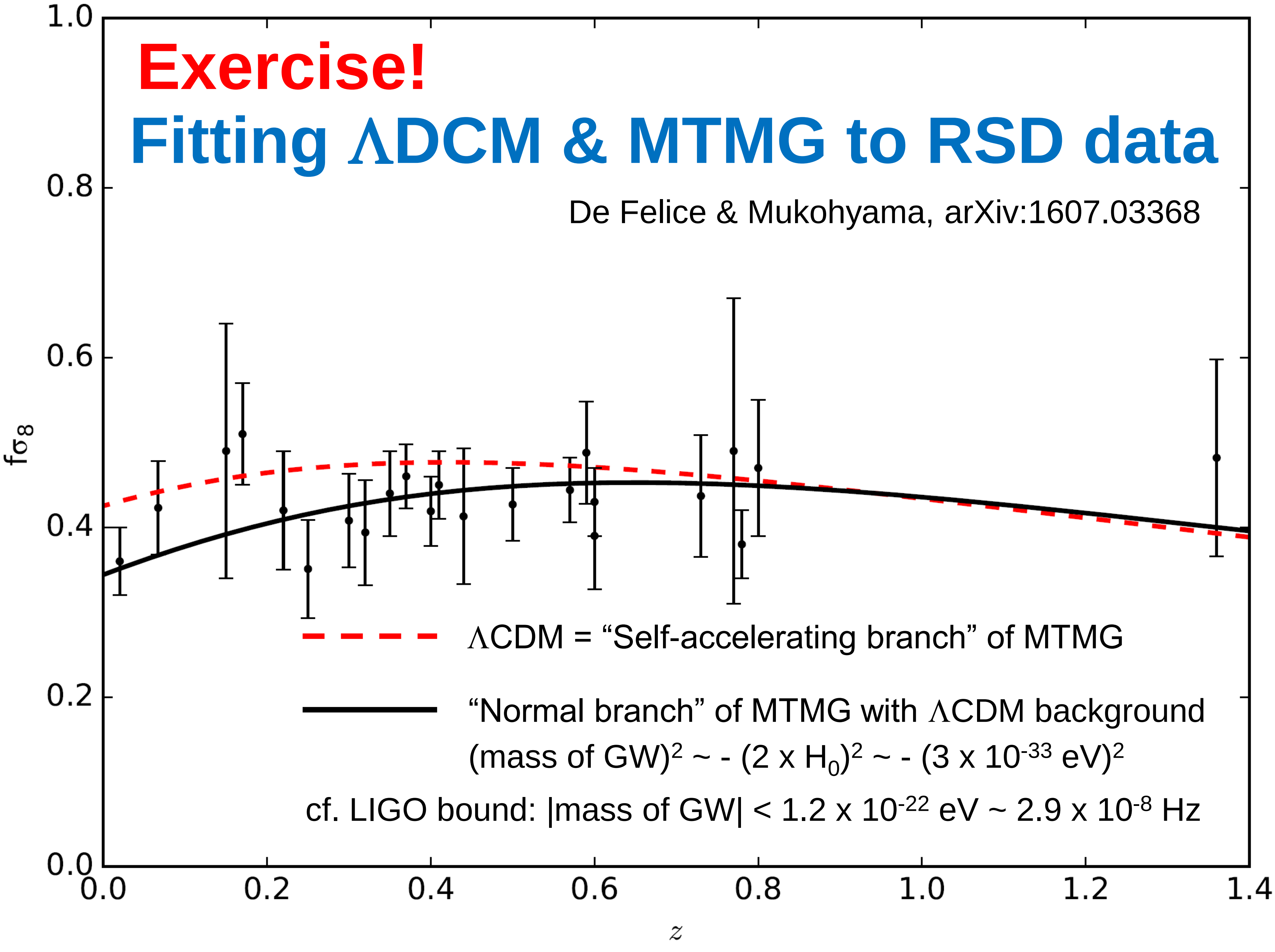
deviates from GR in IR (L≫m^{-1})

Vector part is the same as GR

Non-zero mass for gravity waves



Exercise!
Fitting ΛDCM & MTMG to RSD data
De Felice & Mukohyama, arXiv:1607.03368
ΛCDM = "Self-accelerating branch" of MTMG
"Normal branch" of MTMG with ΛCDM background
(mass of GW)^2 ~ (1.08 x H_0)^2 ~ (1.6 x 10^{-33} eV)^2
cf. LIGO bound: |mass of GW| < 1.2 x 10^{-22} eV ~ 2.9 x 10^{-8} Hz
$f\sigma_8$
z
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4


Exercise!
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ΛCDM = "Self-accelerating branch" of MTMG
"Normal branch" of MTMG with ΛCDM background
(mass of GW)^2 ~ - (2 x H_0)^2 ~ - (3 x 10^{-33} eV)^2
cf. LIGO bound: |mass of GW| < 1.2 x 10^{-22} eV ~ 2.9 x 10^{-8} Hz
$f\sigma_8$
z
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4

BH and Stars in MTMG

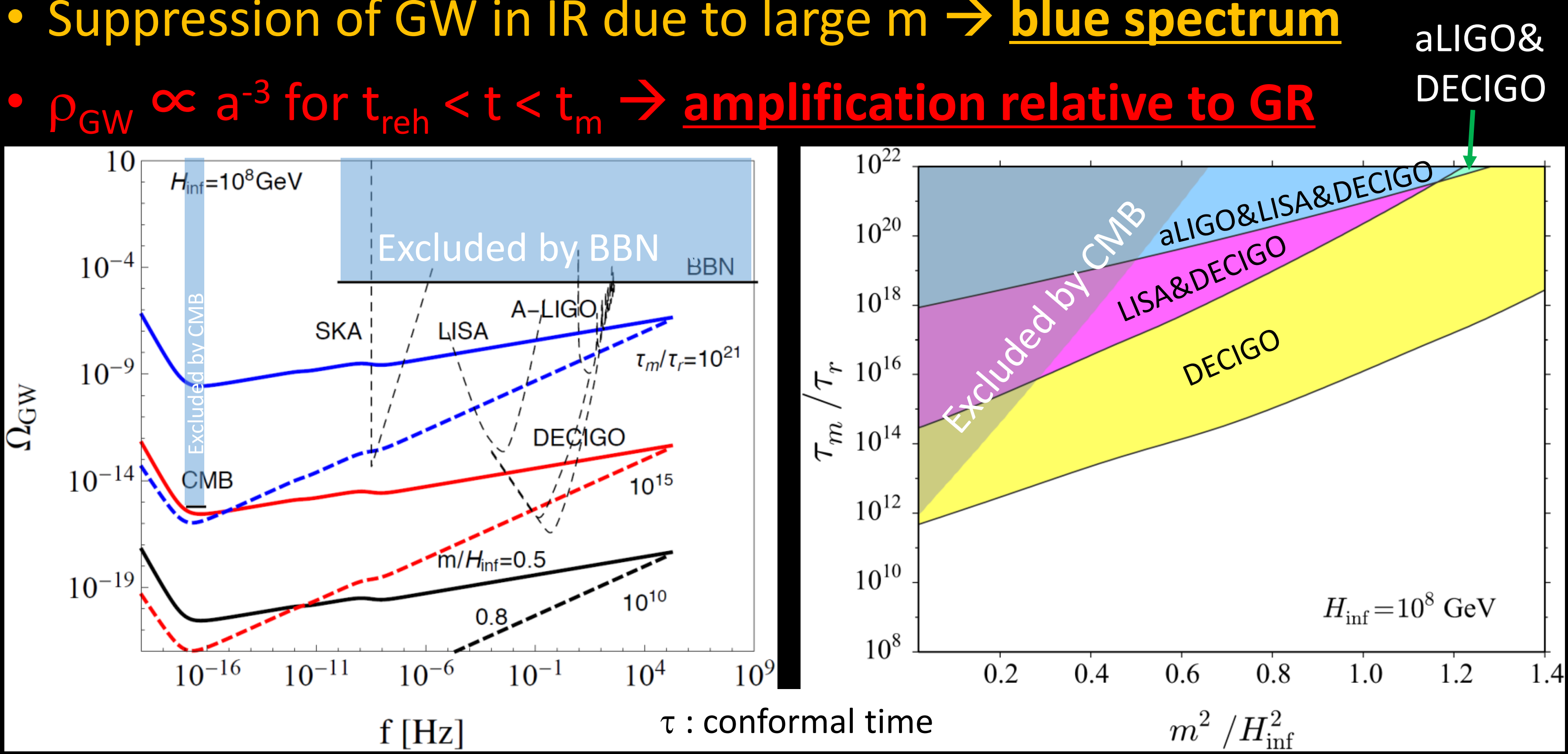
De Felice, Larrouturou, Mukohyama, Oliosi,
arXiv: 1808.01403

- Any solution of GR that can be rendered spatially flat by a coordinate change is also a solution of the self-accelerating branch of MTMG, with or without matter.
- Schwarzschild sol → **BH, star exterior**
- Spherical GR sol with matter → **gravitational collapse, star interior**
- **No strong coupling**
- **No singularities** except for those in GR

Blue-tilted & amplified primordial GW from MTMG

Fujita, Kuroyanagi, Mizuno, Mukohyama,
arxiv: 1808.02381

- Simple extension: $c_i \rightarrow c_i(\phi)$ with $\phi = \phi(t)$
- m large until t_m ($t_{reh} < t_m < t_{BBN}$) but small after t_m
  cf. no Higuchi bound in MTMG
- Suppression of GW in IR due to large m → **blue spectrum**
- $\rho_{GW} \propto a^{-3}$ for $t_{reh} < t < t_m$ → **amplification relative to GR**



τ : conformal time

Summary

- Minimal # of d.o.f. in modified gravity = 2
  can be saturated → minimally modified gravity (MMG)
- Type-I MMG: $^{\exists}$Einstein frame
  Type-II MMG: no Einstein frame
- Example of type-I MMG
  GR + canonical tr. + gauge-fixing + adding matter
  Rich phenomenology: w_{DE}, G_{eff}, etc.
- Example of type-II MMG
  Minimal theory of massive gravity (MTMG)
  Cosmology: self-accelerating branch & normal branch
  BHs and stars: no strong coupling, no new singularity
  Stochastic GWs: blue-tilted & largely amplified

Backup slides

Setup

- 4d theories invariant under 3d-diffeo: $x^i \rightarrow x^i + \xi^i(t,\mathbf{x})$
- ADM decomposition

$$ds^2 = -N^2 dt^2 + h_{ij} (dx^i + N^i dt)(dx^j + N^j dt)$$

- Ansatz: actions linear in the lapse function N

$$S = \int dt d^3x \sqrt{h} N F\left(K_{ij}, R_{ij}, \nabla_i, h^{ij}, t\right)$$

$$K_{ij} = (\partial_t h_{ij} - \nabla_i N_j - \nabla_j N_j)/(2N)$$

- For simplicity, exclude mixed-derivative terms, i.e. those that contain spatial derivatives acted on K_{ij}
- Relation between K_{ij} and π^{ij} (momenta conjugate to h_{ij}) assumed to be invertible $\det\left(\frac{\partial^2 F}{\partial K_{ij} \partial K_{kl}}\right) \neq 0$
- Seek theories with 2 local physical d.o.f.!

Equivalent action and Hamiltonian

- Equivalent action

$$S=\int dtd^3x\sqrt{h}N\left[F\left(Q_{ij},R_{ij},\nabla_i,h^{ij},t\right)+\upsilon^{ij}\left(Q_{ij}-K_{ij}\right)\right]$$

- Conjugate momenta

$$\pi^{ij}=\frac{\partial\mathcal{L}}{\partial\dot{h}_{ij}}=-\frac{1}{2}\sqrt{h}\upsilon^{ij},\qquad \pi_N=\frac{\partial\mathcal{L}}{\partial\dot{N}}=0,\qquad \pi_i=\frac{\partial\mathcal{L}}{\partial\dot{N}^i}=0\,,$$

$$P^{ij}=\frac{\partial\mathcal{L}}{\partial\dot{Q}_{ij}}=0,\qquad U_{ij}=\frac{\partial\mathcal{L}}{\partial\dot{\upsilon}^{ij}}=0\,.$$

- 22 primary constraints

$$\pi_N\approx 0,\qquad \pi_i\approx 0,\qquad P^{ij}\approx 0\,,\qquad U_{ij}\approx 0,\qquad \Psi^{ij}\approx 0\,.$$

- Hamiltonian

$$\Psi^{ij}\equiv\pi^{ij}+\frac{1}{2}\sqrt{h}\upsilon^{ij}$$

$$H=\int d^3x\left[N\mathcal{C}+N^i\mathcal{H}_i+\lambda_N\pi_N+\lambda^i\pi_i+\chi_{ij}P^{ij}+\varphi^{ij}U_{ij}+\lambda_{ij}\Psi^{ij}\right]$$

$$\mathcal{C}\equiv-\sqrt{h}\left[F\left(Q_{ij},R_{ij},\nabla_i,h^{ij},t\right)+\upsilon^{ij}Q_{ij}\right]\,,$$

$$\mathcal{H}_i\equiv\sqrt{h}\nabla_j\upsilon^j{}_i\,,$$

Primary and secondary constraints

- 10 secondary constraints

$$0 \approx \frac{d\pi_N}{dt} = \{\pi_N, H\} = -\mathcal{C}\,,$$
$$0 \approx \frac{d\pi_i}{dt} = \{\pi_i, H\} = -\mathcal{H}_i\,,$$
$$0 \approx \frac{dP^{ij}}{dt} = \{P^{ij}, H\} = N\Phi^{ij}\,, \qquad \Phi^{ij} \equiv \sqrt{h}\left(\frac{\partial F}{\partial Q_{ij}} + v^{ij}\right)$$

- Total Hamiltonian

$$H_{\rm tot} = \int d^3x \left[\lambda_c\mathcal{C} + \tilde{N}^i\mathcal{H}_i + \lambda_N\pi_N + \lambda^i\pi_i + \chi_{ij}P^{ij} + \varphi^{ij}U_{ij} + \lambda_{ij}\Psi^{ij} + \phi_{ij}\Phi^{ij}\right]$$

- 7 constraints are 1st-class

$$\pi_N \approx 0 \qquad \pi_i \approx 0 \qquad \mathcal{H}_i^E \approx 0$$

$$\mathcal{H}_i^E = -2\sqrt{h}\nabla_j\left(\frac{\pi^j{}_i}{\sqrt{h}}\right) + P^{jk}\nabla_i Q_{jk} - 2\sqrt{h}\nabla_k\left(\frac{P^{jk}}{\sqrt{h}}Q_{ij}\right) + U_{jk}\nabla_i v^{jk} + 2\sqrt{h}\nabla_k\left(v^{jk}\frac{U_{ij}}{\sqrt{h}}\right) + \pi_N\partial_i N$$

- 25 remaining constraints

$$\mathcal{C} \approx 0\,, \qquad P^{ij} \approx 0\,, \qquad U_{ij} \approx 0\,, \qquad \Phi^{ij} \approx 0\,, \qquad \Psi^{ij} \approx 0\,.$$

If they are 2nd-class and there is no tertiary constraint then 44 – 7 x 2 – 25 = 5 → 5-dim phase space @ each point → inconsistent theory

Functional determinant

- If Det $M_{ab}(x,y) \not\approx 0$, where

$$M_{ab}(x,y) \equiv \{\phi_a(x), \phi_b(y)\} \approx \begin{pmatrix} 0 & 0_6^T & u_1^T & 0_6^T & \hat{u}_2^T \\ 0_6 & 0_{6,6} & 0_{6,6} & A_1 & 0_{6,6} \\ -u_1 & 0_{6,6} & 0_{6,6} & a\mathbf{1}_{6,6} & b\mathbf{1}_{6,6} \\ 0_6 & -A_1^T & -a\mathbf{1}_{6,6} & 0_{6,6} & \hat{A}_2 \\ -\hat{u}_2 & 0_{6,6} & -b\mathbf{1}_{6,6} & -\hat{A}_2 & A_3 \end{pmatrix},$$

$$\phi_a \equiv (\mathcal{C}, P^{ij}, U_{ij}, \Phi^{ij}, \Psi^{ij})$$

then all ϕ_a are 2nd-class and there is no tertiary constraint.

- Det $M_{ab}(x,y) \approx 0$ if and only if $\exists\, \mathbf{v} = (\alpha, v_1, v_2, v_3, v_4)^T \neq 0$ s.t.

$$\int d^3y \left[u_1^T v_2(y) + \hat{u}_2^T v_4(y)\right] \approx 0\,, \qquad \int d^3y \left[-u_1\alpha(y) + av_3(y) + bv_4(y)\right] \approx 0\,,$$

$$\int d^3y \left[\hat{A}_1 v_3(y)\right] \approx 0\,, \quad \int d^3y \left[-A_1^T v_1(y) - av_2(y) + \hat{A}_2 v_4(y)\right] \approx 0\,,$$

$$\int d^3y \left[-\hat{u}_2\alpha(y) - bv_2(y) - \hat{A}_2 v_3(y) + A_3 v_4(y)\right] \approx 0\,, \qquad \text{for } {}^\forall x\,.$$

This is equivalent to the existence of a function $\alpha(x)$ that does not vanish everywhere and that satisfies

$$\int d^3x \left[\frac{\delta\bar{\mathcal{C}}[\beta]}{\delta h_{ij}(x)} Q_{ij}(x)\alpha(x) - \frac{\delta\bar{\mathcal{C}}[\alpha]}{\delta h_{ij}(x)} Q_{ij}(x)\beta(x)\right] \approx 0\,, \quad \text{for } {}^\forall\beta(x)$$

$$\bar{\mathcal{O}}[\lambda] \equiv \int d^3x \lambda \mathcal{O}$$

Necessary and sufficient condition

- Suppose $^{\exists}\alpha$ s.t.

$$\int d^3x \left[\frac{\delta\bar{\mathcal{C}}[\beta]}{\delta h_{ij}(x)}Q_{ij}(x)\alpha(x) - \frac{\delta\bar{\mathcal{C}}[\alpha]}{\delta h_{ij}(x)}Q_{ij}(x)\beta(x)\right] \approx 0\,, \quad \text{for } {}^{\forall}\beta(x)$$

If $\int d^3x \sum_a v^a(x)\frac{\partial\phi_a(x)}{\partial t}$ is non-vanishing then it will be a tertiary constraint.

If it vanishes then $\int d^3x \sum_a v^a(x)\phi_a(x)$ is 1st-class.

- We need a 1st-class constraint or a tertiary constraint @ each point. We thus require

$$\int d^3x \left[\frac{\delta\bar{\mathcal{C}}[\beta]}{\delta h_{ij}(x)}Q_{ij}(x)\alpha(x) - \frac{\delta\bar{\mathcal{C}}[\alpha]}{\delta h_{ij}(x)}Q_{ij}(x)\beta(x)\right] \approx 0\,, \quad \text{for } {}^{\forall}\alpha(x),\ {}^{\forall}\beta(x)\,.$$

- Under this condition, we define $\mathcal{C}^E(x)$ by

$$\int d^3x \sum_a v^a(x)\phi_a(x) = \bar{\mathcal{C}}^E[\alpha] + \text{boundary terms}$$

- If $\partial\mathcal{C}^E/\partial t \approx 0$ then $\mathcal{C}^E$ is 1st-class. Otherwise, $\partial\mathcal{C}^E/\partial t \approx 0$ should be imposed as a tertiary constraint.

Further modification: lapse-independent term

- If F satisfies the consistency condition then the theory

$$S=\int dtd^3x\sqrt{h}\left[NF+\boxed{G\left(R_{ij},\nabla_i,h^{ij},t\right)}\right]$$

  also satisfies the consistency condition. [G does not contribute to the primary and secondary constraints and the Poisson brackets among them.]
- Generically, $0\approx\partial\mathcal{C}^E/\partial t+\{\mathcal{C}^E,H_{\rm tot}\}$ leads to a tertiary constraint. [G contributes to the Hamiltonian.]
- An example: GR + higher spatial curvature terms

For F = F(Q_{ij}, R_{ij}, h^{ij}, t)

- The consistency condition:

$$\int \sqrt{h}\,(\alpha\nabla^i\beta - \beta\nabla^i\alpha)\left[-\frac{\partial F}{\partial R_{kl}}\nabla^j\left(Q_{il}h_{jk} - \frac{1}{2}Q_{kl}h_{ij} - \frac{1}{2}Qh_{jk}h_{il}\right)\right.$$
$$\left.+\nabla^j\left(\frac{\partial F}{\partial R_{kl}}\right)\cdot\left(Q_{jl}h_{ik} - \frac{1}{2}Q_{kl}h_{ij} - \frac{1}{2}Qh_{ik}h_{jl}\right)\right] \approx 0\,, \quad \text{for } {}^\forall\alpha(x)\,,\ {}^\forall\beta(x)\,.$$

- A sufficient condition:

$$\boxed{\begin{aligned}&-\frac{\partial F}{\partial R_{kl}}\nabla^j\left(Q_{il}h_{jk} - \frac{1}{2}Q_{kl}h_{ij} - \frac{1}{2}Qh_{jk}h_{il}\right)\\ &\quad+\nabla^j\left(\frac{\partial F}{\partial R_{kl}}\right)\cdot\left(Q_{jl}h_{ik} - \frac{1}{2}Q_{kl}h_{ij} - \frac{1}{2}Qh_{ik}h_{jl}\right) \approx 0\,.\end{aligned}}$$

- F = F(Q_{ij}, h^{ij}, t) (independent of R_{ij}) is too simple: gravitational waves do not propagate, static masses do not gravitate, …

Example 1: general relativity

- Ansatz

$$F = f_1(\mathcal{Q}) + f_2(R)\,, \quad f_1'(\mathcal{Q}) \neq 0\,,\ f_2'(R) \neq 0$$

$$\mathcal{Q} \equiv Q_{ij}Q^{ij} - Q^2 \qquad Q = Q^i{}_i$$

- Consistency condition

$$-\nabla^i\left(Q_{ij} - Qh_{ij}\right) + \left(Q_{ij} - Qh_{ij}\right)\nabla^i \ln f_2' \approx 0$$

- Momentum constraint

$$\nabla^i\left(Q_{ij} - Qh_{ij}\right) + \left(Q_{ij} - Qh_{ij}\right)\nabla^i \ln f_1' \approx 0$$

- Consistency condition rewritten

$$\partial_i \ln\left(f_1' f_2'\right) \approx 0\,, \qquad \text{i.e.} \qquad f_1'(\mathcal{Q})f_2'(R) \approx \text{constant in space.}$$

- Solution

$$F = c_1(t)\mathcal{Q} + c_2(t)R - \Lambda(t)\,, \quad c_1(t) \neq 0\,,\ c_2(t) \neq 0\,.$$

Details of GR and its relatives

- If $c_1 c_2 = \text{constant}\,, \quad c_1 \Lambda = \text{constant}\,.$ then $\mathcal{C}^E$ is 1st-class and the theory is GR up to redefinition of N, provided that c_1 and c_2 are positive.
- If $c_1 c_2 = \text{constant}\,, \quad c_1 \Lambda \neq \text{constant}$ then $\partial \mathcal{C}^E / \partial t = \partial \mathcal{C} / \partial t \approx \Lambda \partial_t \ln(c_1 \Lambda)$ is non-vanishing and thus the theory is inconsistent.
- If $c_1 c_2 \neq \text{constant}$ then $0 \approx \partial \mathcal{C}^E / \partial t = \partial \mathcal{C} / \partial t$ gives a tertiary constraint. Both $\mathcal{C}^E \approx 0$ and $\partial \mathcal{C}^E / \partial t \approx 0$ are 2nd-class.

Example 2: a square root gravity

- Ansatz

$$F = f_1(\mathcal{Q})\, f_2(R) - \Lambda(t)\,, \quad f_1'(\mathcal{Q}) \neq 0\,,\ f_2'(R) \neq 0\,.$$

- Consistency condition

$$-\nabla^i \left(Q_{ij} - Q h_{ij}\right) + \nabla^i \ln\left(f_1 f_2'\right) \cdot \left(Q_{ij} - Q h_{ij}\right) \approx 0$$

- Momentum constraint

$$\nabla^i \left(Q_{ij} - Q h_{ij}\right) + \nabla^i \ln\left(f_1' f_2\right) \cdot \left(Q_{ij} - Q h_{ij}\right) \approx 0$$

- Consistency condition rewritten

$$\partial_i \ln\left(f_1 f_1' f_2 f_2'\right) \approx 0, \qquad \text{i.e.}$$

$$f_1(\mathcal{Q}) f_1'(\mathcal{Q}) f_2(R) f_2'(R) \approx \text{constant in space.}$$

- Solution

$$f_1^2 = A(t)\mathcal{Q} + B(t)\,, \quad f_2^2 = C(t)R + D(t)\,,$$

$$A(t) \neq 0\,,\ C(t) \neq 0\,.$$

Details of square root gravity

- If AC and (Λ^2-BD)A/B are constant then $\mathcal{C}^E$ is 1st-class.
- If AC is constant and if (Λ^2-BD)A/B is not constant then $\partial\mathcal{C}^E/\partial t = \partial\mathcal{C}/\partial t$ is non-vanishing and thus the theory is inconsistent.
- If AC is not constant then $0 \approx \partial\mathcal{C}^E/\partial t = \partial\mathcal{C}/\partial t$ gives a tertiary constraint. Both $\mathcal{C}^E \approx 0$ and $\partial\mathcal{C}^E/\partial t \approx 0$ are 2nd-class.

- If B = Λ = 0 and if A (≠0), C (≠0), D are constant then the theory is equivalent to the shape dynamics description of GR [Baierlein-Sharp-Wheeler 1962].

Phenomenology of square root gravity

- For BD>0,

$$S=\int d^4x\sqrt{h}N\left[\xi M(t)^4\sqrt{\left(1+\frac{c_1(t)}{M(t)^2}\mathcal{K}\right)\left(1+\frac{c_2(t)}{M(t)^2}R\right)}-\Lambda(t)\right]$$

$$\mathcal{K}=K_{ij}K^{ij}-K^2,\ K=K^i{}_i,\ \xi=\pm 1$$

- In the weak gravity limit,

$$S\simeq\int d^4x\sqrt{h}N\left[\xi M^4-\Lambda+\frac{\xi}{2}M^2(c_1\mathcal{K}+c_2R)+\ldots\right]$$

GR with $M_p^2=\xi c_1 M^2\,,\quad c_g^2=\frac{c_2}{c_1}\,,\quad \Lambda_{\rm eff}=\frac{\Lambda-\xi M^4}{\xi c_1 M^2}$ is recovered.

- Flat FLRW with a canonical scalar $\quad \xi=1$

$$S=\int dx^3\int dt a^3\left[M^4\sqrt{N^2-\frac{6c_1}{M^2}\frac{\dot{a}^2}{a^2}}-N\Lambda+\frac{1}{2N}\dot{\phi}^2-NV(\phi)\right]$$

$$1-6c_1\frac{H^2}{M^2}=\frac{M^8}{(\Lambda+\rho_m)^2}$$

$$H^2\rightarrow\frac{1}{6c_1^2}M_p^2\quad\text{as}\quad\rho_m\rightarrow\infty$$

Example 3: an exponential gravity

- Ansatz

$$F = f_1(Q) + \exp\left[c_1 R + f_2(Q)\right]$$

- Hamiltonian constraint

$$0 \approx F - \frac{\partial F}{\partial Q_{ab}} Q^{ab} = (f_1 - 2f_1' Q) + e^{c_1 R + f_2}\,(1 - 2f_2' Q)$$

- Momentum constraint

$$0 \approx \left(2f_1' + e^{c_1 R + f_2} \cdot 2f_2'\right) \nabla_i \left(Q^{ij} - Qh^{ij}\right) + \left(Q^{ij} - Qh^{ij}\right) \nabla_i \left(2f_1' + e^{c_1 R + f_2} \cdot 2f_2'\right)$$

- Consistency condition

$$\nabla_i \left[\frac{f_1 - 2f_1' Q}{1 - 2f_2' Q}\left(f_1' - \frac{f_1 - 2f_1' Q}{1 - 2f_2' Q} f_2'\right)\right] \approx 0$$

- Special solution

$$f_1 = c_4(t) Q + \Lambda(t) \qquad f_2 = c_3(t) Q + \ln c_2(t)$$

$$c_4 = 2\Lambda c_3 \qquad c_2 \Lambda < 0$$

Details of exponential gravity

- If $c_1 c_3 \Lambda^2$ and $\ln(-\Lambda/c_2)/c_1$ are constant then $\mathcal{C}^E$ is 1st-class.
- If $c_1 c_3 \Lambda^2$ is constant and if $\ln(-\Lambda/c_2)/c_1$ is not constant then $\partial\mathcal{C}^E/\partial t = \partial\mathcal{C}/\partial t$ is non-vanishing and thus the theory is inconsistent.
- If $c_1 c_3 \Lambda^2$ is not constant then $0 \approx \partial\mathcal{C}^E/\partial t = \partial\mathcal{C}/\partial t$ gives a tertiary constraint. Both $\mathcal{C}^E \approx 0$ and $\partial\mathcal{C}^E/\partial t \approx 0$ are 2nd-class.

Phenomenology of exponential gravity

- In the weak gravity limit,

$$F = 2\Lambda c_3 \mathcal{K} + \Lambda + c_2 \exp\left[c_1 R + c_3 \mathcal{K}\right]$$
$$= \Lambda + c_2 + (2\Lambda c_3 + c_2 c_3)\,\mathcal{K} + c_1 c_2 R + \frac{1}{2} c_2 (c_1 R + c_3 \mathcal{K})^2 + \cdots$$

  GR is recovered if Λ+c_2 is constant and if ($2\Lambda c_3$+$c_2 c_3$), $c_1 c_2$ are constant and positive.

- …

Step 1. Fix local Lorentz to realize ADM vielbein in dRGT

$$\|e^{\mathcal{A}}{}_{\mu}\| = \begin{pmatrix} N & \vec{0}^T \\ e^I{}_i N^i & e^I{}_j \end{pmatrix} \qquad \|E^{\mathcal{A}}{}_{\mu}\| \doteq \begin{pmatrix} M & \vec{0}^T \\ E^I{}_i M^i & E^I{}_j \end{pmatrix}$$

physical fiducial

$$S_{\rm pre} = \frac{M_{\rm P}^2}{2}\int d^4x\sqrt{-g}\,\mathcal{R}[g_{\mu\nu}]$$
$$+\frac{M_{\rm P}^2}{2}m^2\int d^4x\left[\frac{c_0}{24}\epsilon_{\mathcal{ABCD}}\epsilon^{\alpha\beta\gamma\delta}E^{\mathcal{A}}{}_{\alpha}E^{\mathcal{B}}{}_{\beta}E^{\mathcal{C}}{}_{\gamma}E^{\mathcal{D}}{}_{\delta}\right.$$
$$+\frac{c_1}{6}\epsilon_{\mathcal{ABCD}}\epsilon^{\alpha\beta\gamma\delta}E^{\mathcal{A}}{}_{\alpha}E^{\mathcal{B}}{}_{\beta}E^{\mathcal{C}}{}_{\gamma}e^{\mathcal{D}}{}_{\delta}$$
$$+\frac{c_2}{4}\epsilon_{\mathcal{ABCD}}\epsilon^{\alpha\beta\gamma\delta}E^{\mathcal{A}}{}_{\alpha}E^{\mathcal{B}}{}_{\beta}e^{\mathcal{C}}{}_{\gamma}e^{\mathcal{D}}{}_{\delta}$$
$$+\frac{c_3}{6}\epsilon_{\mathcal{ABCD}}\epsilon^{\alpha\beta\gamma\delta}E^{\mathcal{A}}{}_{\alpha}e^{\mathcal{B}}{}_{\beta}e^{\mathcal{C}}{}_{\gamma}e^{\mathcal{D}}{}_{\delta}$$

dRGT potential

Step2. Switch to Hamiltonian

$$H_{\rm pre} = \int d^3x [-N\mathcal{R}_0 - N^i\mathcal{R}_i$$

linear in lapse and shift
→ 4 primary constraints

$$+ m^2 M\mathcal{H}_1 + \tilde{\lambda}^\alpha \tilde{\mathcal{C}}_\alpha$$

2 secondary constraints (α=1,2)

$$+ \alpha_{MN}\mathcal{P}^{[MN]} + \beta_{MN}Y^{[MN]}]$$

6 (= 3 primary + 3 secondary) constraints associated with symmetry of spatial vielbein

9 x 2 – 4 – 2 – 6 = 6 → 3 d.o.f.

c.f. consistent with the analysis by Comelli, Nesti and Pilo 2014

Step2. Switch to Hamiltonian

$$H_{\rm pre} = \int d^3x [-N\mathcal{R}_0 - N^i\mathcal{R}_i$$

linear in lapse and shift
→ 4 primary constraints

$$+ m^2 M \mathcal{H}_1 + \tilde{\lambda}^\alpha \tilde{\mathcal{C}}_\alpha$$

2 secondary
constraints (α=1,2)

$$+ \alpha_{MN} \mathcal{P}^{[MN]} + \beta_{MN} \mathcal{Y}^{[MN]}]$$

Precursor theory with 3 d.o.f.

6 (= 3 primary + 3 secondary) constraints
associated with symmetry of spatial vielbein

9 x 2 – 4 – 2 – 6 = 6 → 3 d.o.f.

c.f. consistent with the analysis by Comelli, Nesti and Pilo 2014

Step3. Add 2 additional constraints

$$H = \int d^3x[-N\mathcal{R}_0 - N^i\mathcal{R}_i$$

$$+ m^2 M\mathcal{H}_1 + \lambda\mathcal{C}_0 + \lambda^i\mathcal{C}_i$$

4 constraints instead of 2

$$+ \alpha_{MN}\mathcal{P}^{[MN]} + \beta_{MN}Y^{[MN]}]$$

$$\mathcal{C}_0 \doteq \{\mathcal{R}_0, H_1\} + \frac{\partial \mathcal{R}_0}{\partial t} \qquad \mathcal{C}_l \doteq \{\mathcal{R}_l, H_1\}$$

➡ Only 2 among (C_0, C_i) are new

6 (from precursor theory) – 2 (additional constraints) = 4 ➔ 2 d.o.f.

Action for MTMG

$$S = S_{\rm pre} + \frac{M_{\rm P}^2}{2}\int d^4x N\sqrt{\gamma}\left(\frac{m^2}{4}\frac{M}{N}\lambda\right)^2\left(\gamma_{ik}\gamma_{jl} - \frac{1}{2}\gamma_{ij}\gamma_{kl}\right)\Theta^{kl}\Theta^{ij}$$
$$- \frac{M_{\rm P}^2}{2}\int d^4x(\lambda\bar{\mathcal{C}}_0 + \lambda^i\mathcal{C}_i) + S_{\rm mat}\,,$$

$$S_{\rm pre} = S_{\rm GR} + S_{\rm dRGT}$$ in ADM vielbein

$$\bar{\mathcal{C}}_0 = \frac{1}{2}m^2M\sqrt{\gamma}\left(\gamma_{ik}\gamma_{jl} - \frac{1}{2}\gamma_{ij}\gamma_{kl}\right)\Theta^{kl}(K^{ij} - K\gamma^{ij})$$
$$-m^2M\sqrt{\gamma}\left(\frac{\sqrt{\tilde{\gamma}}}{\sqrt{\gamma}}[c_1\tilde{\zeta} + c_2(\mathcal{K}\tilde{\zeta} - \mathcal{K}^m{}_n\tilde{\zeta}^n{}_m)] + c_3\mathfrak{K}^m{}_n\tilde{\zeta}^n{}_m\right)$$

$$\mathcal{C}_i = m^2\sqrt{\gamma}\mathcal{D}_n\left[M\left(\frac{\sqrt{\tilde{\gamma}}}{\sqrt{\gamma}}[c_1\mathcal{K}^n{}_i + c_2(\mathcal{K}\mathcal{K}^n{}_i - \mathcal{K}^n{}_l\mathcal{K}^l{}_i)] + c_3\delta^n{}_i\right)\right]$$

$$\Theta^{ij} = \left[\frac{\sqrt{\tilde{\gamma}}}{\sqrt{\gamma}}\{c_1(\gamma^{il}\mathcal{K}^j{}_l + \gamma^{jl}\mathcal{K}^i{}_l) + c_2[\mathcal{K}(\gamma^{il}\mathcal{K}^j{}_l + \gamma^{jl}\mathcal{K}^i{}_l) - 2\tilde{\gamma}^{ij}]\} + 2c_3\gamma^{ij}\right]$$

$$S_{\rm dRGT} = \frac{M_{\rm P}^2}{2} \sum_i \mathcal{S}_i,$$

in ADM vielbein

$$\mathcal{S}_1 = -m^2 c_1\, a_f^3 \left(N + M\mathcal{K}\right),$$

$$\mathcal{S}_2 = -\frac{1}{2} m^2 c_2 a_f^3 \left[2N\mathcal{K} + M\mathcal{K}^2 - M\mathcal{K}^i{}_j\mathcal{K}^j{}_i\right],$$

$$\mathcal{S}_3 = -m^2 c_3 \sqrt{\gamma} \left[M + N\,\mathfrak{K}\right],$$

$$\mathcal{S}_4 = -m^2 c_4 \sqrt{\gamma}\,.$$

$$\mathcal{K}^k{}_n \equiv \left(\sqrt{\tilde{\gamma}^{-1}\gamma}\right)^k{}_n \qquad \mathfrak{K}^k{}_n \equiv \left(\sqrt{\gamma^{-1}\tilde{\gamma}}\right)^k{}_n$$

Action for MTMG

$$S = S_{\rm pre} + \frac{M_{\rm P}^2}{2}\int d^4x N\sqrt{\gamma}\left(\frac{m^2}{4}\frac{M}{N}\lambda\right)^2\left(\gamma_{ik}\gamma_{jl} - \frac{1}{2}\gamma_{ij}\gamma_{kl}\right)\Theta^{kl}\Theta^{ij}$$

$$- \frac{M_{\rm P}^2}{2}\int d^4x(\lambda\bar{\mathcal{C}}_0 + \lambda^i\mathcal{C}_i) + S_{\rm mat}\,,$$

$S_{\rm pre} = S_{\rm GR} + S_{\rm dRGT}$ in ADM vielbein

Kinetic term is different from dRGT!

$$\bar{\mathcal{C}}_0 = \frac{1}{2}m^2 M\sqrt{\gamma}\left(\gamma_{ik}\gamma_{jl} - \frac{1}{2}\gamma_{ij}\gamma_{kl}\right)\Theta^{kl}(K^{ij} - K\gamma^{ij})$$

$$-m^2 M\sqrt{\gamma}\left(\frac{\sqrt{\tilde{\gamma}}}{\sqrt{\gamma}}[c_1\tilde{\zeta} + c_2(\mathcal{K}\tilde{\zeta} - \mathcal{K}^m{}_n\tilde{\zeta}^n{}_m)] + c_3\mathfrak{K}^m{}_n\tilde{\zeta}^n{}_m\right)$$

$$\mathcal{C}_i = m^2\sqrt{\gamma}\mathcal{D}_n\left[M\left(\frac{\sqrt{\tilde{\gamma}}}{\sqrt{\gamma}}[c_1\mathcal{K}^n{}_i + c_2(\mathcal{K}\mathcal{K}^n{}_i - \mathcal{K}^n{}_l\mathcal{K}^l{}_i)] + c_3\delta^n{}_i\right)\right]$$

$$\Theta^{ij} = \left[\frac{\sqrt{\tilde{\gamma}}}{\sqrt{\gamma}}\{c_1(\gamma^{il}\mathcal{K}^j{}_l + \gamma^{jl}\mathcal{K}^i{}_l) + c_2[\mathcal{K}(\gamma^{il}\mathcal{K}^j{}_l + \gamma^{jl}\mathcal{K}^i{}_l) - 2\tilde{\gamma}^{ij}]\} + 2c_3\gamma^{ij}\right]$$