CHY formalism and null string theory beyond tree level

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We generalize the CHY formalism to one-loop level, based on the framework of the null string theory. The null string, a tensionless string theory, produces the same results as the ones from the chiral ambitwistor string theory, with the latter believed to give a string interpretation of the CHY formalism. A key feature of our formalism is the interpretation of the modular parameters. We find that the S modular transformation invariance of the ordinary string theory does not survive in the case of the null string theory. Treating the integration over the modular parameters this way enable us to derive the n-gons scattering amplitude in field theory, thus proving the n-gons conjecture.

References

Based on the work with Chi Zhang and Yao-zhong Zhang, arXiv:1704.01290 to appear in JHEP。

References therein and further reading:

1)Henriette Elvang and Yu-tin Huang, "Scattering Amplitudes", arXiv:1308.1697 [hep-th]
269 pages, 134 figures. This is part of textbook to be published by Cambridge University Press

2)Johannes M. Henn and Jan C. Plefka, "Scattering Amplitudes in Gauge Theories", Lecture Notes in Physics, Volume 883 2014, Springer Berlin Heidelberg

Null string

$\alpha' \to 0$ limit in the string theory $\to$ field theory limit.
CHY formalism $\to$ Null string $\to \alpha' \to \infty$ limit in the string theory.
Vacuum state for the null string is changed $\bar{a}_{-n}||0>= 0$.
the null string theory can be seen as the ultra-relativistic limit of the string theory, $e^{2\pi n(i\sigma+\tau)} \sim e^{2\pi n(i\sigma)}(1+2\pi n\tau)$.
$X = X(\sigma) - i\tau\alpha' P(\sigma)$ Alternatively, we can consider the chiral ambitwistor string theory
$S = \int (P(x)\cdot\bar{\partial}X(x) + \lambda(x)P^2(x))d^2x$
or equivalently, after integrating out $P(x)$
$S = \int (\bar{\partial}X(x))^2 d^2x$
In this formalism, one can see that the 2d metric is degenerate, $\det(g) = 0$
X-P system is quantized just as the γ β bosonic ghost system.

CHY formalism

For N String vertices $\sim$
$\prod \int e^{k\cdot(X+(\bar{w}-w)P)} I(z) d^2z \sim \prod \int \delta(k\cdot P) I(z) dz$
the z-integral localizes to z_i which satisfies the scattering equation in CHY formalism.

$$\sum_{j\neq i} \frac{k_i \cdot k_j}{z_i - z_j} = 0 \ .$$

$k_i^2 = 0$. However, for off-shell loop momentum $p^2 \neq 0$, scattering equation needs modification.
$\det(g) = 0$, integration limits over modular parameters also needs more consideration.
At one-loop level, the off-shell scattering equation turns out to be

$$\frac{\ell \cdot k_i}{z_i} + \sum_{j\neq i} \frac{k_i \cdot k_j}{z_i - z_j} = 0 \ .$$

Here l is the loop momentum. The modular parameter is localized at $q \to 0$, the pinching limit.

Integration in modular space

The modular invariant integrand does not not exist in case of torus.
In the ultra-relativistic limit, the zero modes's contribution
$Z_0 = \int e^{i2\pi(\tau-\bar{\tau})\ell^2} d^d\ell \sim (\tau-\bar{\tau})^{-d/2}$, τ the modular parameter.
the non-zero modes coming from both left and right moving parts
$Z_{\rm osc} \sim (\prod_{n=1}^{\infty}(1-q^n))^{-2d}$, with $q = \exp(2\pi i\tau)$.
$Z(\tau,\bar{\tau}) = Z_0\, Z_{\rm osc} \sim (\tau-\bar{\tau})^{-d/2}\, (\prod_{n=1}^{\infty}(1-q^n))^{-2d}$
$Z_{\rm osc}$ is holomorphic in τ while Z_0 is not, $Z = Z_0 Z_{\rm osc}$ is invariant under the T but not the S modular transformation $\tau \to -1/\tau$.
Contribution from ghosts, but that would only change the chiral part at most.

modular parameters and Schwinger parameters

Without S modular invariance, the integration in modular space is over the half infinite cylinder $\tau_1 \in [-1/2, 1/2]$, $\tau_2 \in (0, \infty)$.
Integrating over τ_1 projects out the non-zero modes, τ_2 the Schwinger parameter.
Integrate first either the loop momentum ℓ or the Schwinger parameter τ_2
Correspondence between higher loop Feynman diagrams and correlators in null string theory on Riemann surfaces.
Assume three point interactions. n # of external momenta, m # of propagators , g # of loop momenta.

$$
\begin{aligned}
m &= 3(g-1)+n, && \text{for} \quad g > 1, \\
m &= g+n-1, && \text{for} \quad g = 1, \\
m &= g+n-3, && \text{for} \quad g = 0 \ .
\end{aligned}
$$

m the dimension of the modular space of genus g Riemann surface with n punctures. Each propagator with a Schwinger parameter, which becomes the imaginary part of the modular parameter.

Green's function on torus

Green's function is the inverse of the operator D, $D = \bar{\partial}^2$
$\bar{\partial}^2 G = 2\pi\delta^2(z)$. On the infinite cylinder, its solution is given by
$G(z) = (\bar{z} - z)\pi\cot(\pi z)$
On torus we have to take into account the background charge,
$\bar{\partial}^2 G = 2\pi\delta^2(z) - \frac{\pi}{\tau_2}$
Its solution
$G(z) = (\bar{\tau} - \tau)\partial_\tau \ln\theta_1(z|\tau) + (\bar{z} - z)\partial_z \ln\theta_1(z|\tau) + \frac{\pi \mathrm{i}}{\bar{\tau}-\tau}(\bar{z} - z)^2$
where$\theta_1(z|\tau) = -\mathrm{i}\sum_{n=-\infty}^{\infty}(-1)^n q^{(n-1/2)^2/2} e^{2\mathrm{i}\pi z(n-1/2)}$ The two periods of the Green function $z \to z+1, \qquad z \to z+\tau$

Path integral evaluation

$Z[J] = \int [\mathrm{d}Y] \exp\left(-\int \mathrm{d}^2\sigma\, Y(\sigma) D Y(\sigma) + \mathrm{i} J(\sigma)\cdot Y(\sigma)\right)$
D is $\bar{\partial}^2$ in complex coordinates, The path integral is Gaussian
$Z[J] =$
$\delta^d(J^0)\left(\det{}' \frac{D}{2\pi}\right)^{-d/2} \exp\left(-\frac{1}{2}\int \mathrm{d}^2\sigma \int \mathrm{d}^2\sigma'\, J(\sigma)\cdot J(\sigma') G(\sigma,\sigma')\right)$
d the spacetime dimension, $\delta^d(J^0)$ the zero modes integration, primed determinant excludes the zero eigenvalues, $G(\sigma,\sigma')$ the Green's function of the operator D.
In our case,
$Z[J] =$
$C\delta^d(J^0)\left(\det{}' \frac{D}{2\pi}\right)^{-d/2} \exp\left(-\frac{1}{2}\sum_{i\neq j} k_i\cdot k_j G(\sigma_i,\sigma_j)\right) \mathcal{I}(\tau,\{z_i\})$
The other parts are holomorphic in z_i coordinates, which are evaluated according to the solutions of the scattering equation. However, instead of integrating out $\bar{z}_i$'s to obtain the scattering equation, we choose to deal with such terms in a different way at the one-loop level.

Partition function

$Z(\tau) = \sqrt{A} \prod_n \left(\frac{2\pi}{\lambda_n}\right)^{1/2}$

A the area of the torus and λ_n nonzero eigenvalues

The eigen equation is$\bar{\partial}^2 f = \lambda_n f$

$f = \exp\left(2\pi \mathrm{i} z \frac{(m - n\bar{\tau})}{\tau - \bar{\tau}}\right) \exp\left(2\pi \mathrm{i} \bar{z} \frac{-m + n\tau}{\tau - \bar{\tau}}\right)$ and

eigenvalues$\lambda_n = \left(2\pi \mathrm{i} \frac{m - n\tau}{\tau - \bar{\tau}}\right)^2$

Then the partition function becomes$Z(\tau) = \sqrt{A} \prod_{m,n} \frac{\sqrt{2}\tau_2}{\sqrt{\pi}} \frac{1}{m + n\tau}$

We use the ζ-function regularization technique and obtain

$Z(\tau) = \sqrt{A} \exp G'(0) = \frac{\mathrm{i}}{\sqrt{8\pi\tau_2}} \prod_{n>0} (1 - q^n)^{-2}$

Integrating out τ_1, we obtain the zero point one-loop amplitude for the null string $\quad A_{\text{one-loop}} = \int_0^\infty \mathrm{d}\tau_2 \, (\frac{\mathrm{i}}{\sqrt{8\pi\tau_2}})^d$

Comparing with that of massless particles.

$A(\text{vac bubble}) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \int_0^\infty \mathrm{d}l \exp(-k^2 l) = \mathrm{i} \int_0^\infty \mathrm{d}l \, (2\pi l)^{-d/2}$

Schwinger form of scattering amplitude

we find that Green function $G(z)$ can be written as

$$G(z) = (\bar{\tau} - \tau)\partial_\tau \ln \theta_1(z|\tau) + (\bar{z} - z)\partial_z \ln \theta_1(z|\tau) + \frac{\pi \mathrm{i}}{\bar{\tau} - \tau}(\bar{z} - z)^2$$
$$= \frac{\pi \tau_2}{2} + (\bar{z} - z)\pi \cot(\pi z) - \frac{\pi}{2\tau_2}(\bar{z} - z)^2 + O(q).$$

Integrating out τ_1 first, we get the partial amplitude

$$A(k_1, \cdots, k_n) = \int_0^\infty \mathrm{d}\tau_2 (\tau_2)^{-d/2} \prod_{\mu=2}^{n} \int \mathrm{d}z_\mu \mathrm{d}\bar{z}_\mu \exp(-\sum_{i<j}^{n} k_i \cdot k_j \tilde{G}_{ij})$$

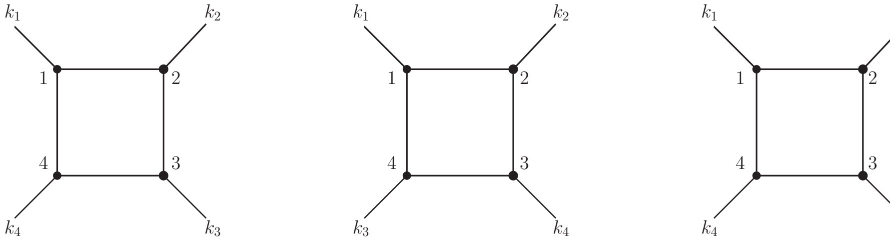
where

$$\tilde{G}_{ij} = (\bar{z}_{ij} - z_{ij})\pi \cot(\pi z_{ij}) - \frac{\pi}{2\tau_2}(\bar{z}_{ij} - z_{ij})^2 \ ,$$

$z_{ij} \equiv z_i - z_j$ and $n = 4$ for the present example.

Schwinger form of scattering amplitude

Set $z_1 = 0$ by the two conformal Killing vectors on the torus.
Then there are three possible ways in dividing the integration region up to topological equivalence:
(i) $\text{Im}\, z_2 < \text{Im}\, z_3 < \text{Im}\, z_4 < \tau_2$,
(ii)$\text{Im}\, z_2 < \text{Im}\, z_4 < \text{Im}\, z_3 < \tau_2$,
(iii)$\text{Im}\, z_4 < \text{Im}\, z_2 < \text{Im}\, z_3 < \tau_2$.
the case (i) corresponds to figure (a), the case (ii) to figure (b), and the case (iii) to figure (c).
In what follows, we will focus on the case (i).



Schwinger form of scattering amplitude

In terms of Schwinger parameters s_i $(i = 1, 2, 3, 4)$ the amplitude for figure (a)

$$\int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{1}{p^2(p+k_1)^2(p+k_1+k_2)^2(p-k_4)^2}$$

$$= \int \frac{\mathrm{d}^d p}{(2\pi)^d} \int_0^\infty \mathrm{d}s_1 \cdots \int_0^\infty \mathrm{d}s_4$$

$$\exp\left[-s_4 p^2 - s_1(p+k_1)^2 - s_2(p+k_1+k_2)^2 - s_3(p-k_4)^2\right]$$

$$= \prod_{i=1}^4 \int_0^1 \mathrm{d}\alpha_i\, \delta\left(\sum_{i=1}^4 \alpha_i - 1\right) \int_0^\infty \mathrm{d}\tau_2 (\tau_2)^{3-d/2} \exp\left[\tau_2(\ell^2 - 2\alpha_2 k_1 \cdot k_2)\right]$$

$$\ell = k_1(\alpha_1+\alpha_2+\alpha_3) + k_2(\alpha_2+\alpha_3) + k_3\alpha_3 \ ,$$

$$\tau_2 = s_1 + s_2 + s_3 + s_4 \ ,$$

$$\alpha_i = \frac{s_i}{\tau_2} \ , \qquad i = 1, 2, 3, 4 \ ,$$

$$k_i^2 = 0 \ , \qquad i = 1, 2, 3, 4 \ .$$

Schwinger form of scattering amplitude

For null string, after integrating out τ_1,

$$A = \int_0^\infty \mathrm{d}\tau_2 (\tau_2)^{-d/2} \prod_{\mu=2}^{4} \int \frac{\mathrm{d}\phi_i \mathrm{d}\left|\sigma_i\right|}{4\pi^2 \left|\sigma_i\right|} \exp\left(- \sum_{i<j} k_i \cdot k_j \tilde{G}_{ij} \right) ,$$

$$\tilde{G}_{ij} = \frac{1}{2} \ln \left| \frac{\sigma_i}{\sigma_j} \right|^2 \frac{\sigma_i + \sigma_j}{\sigma_i - \sigma_j} - \frac{\pi}{2\tau_2} \left(\frac{1}{2\pi \mathrm{i}} \ln \left| \frac{\sigma_i}{\sigma_j} \right|^2 \right)^2 .$$

Integrating out ϕ_i's we derive the reduced effective Green functions

$$\tilde{G}_{ij}^{\mathrm{eff}} = \frac{1}{2} \ln \left| \frac{\sigma_i}{\sigma_j} \right|^2 - \frac{\pi}{2\tau_2} \left(\frac{1}{2\pi \mathrm{i}} \ln \left| \frac{\sigma_i}{\sigma_j} \right|^2 \right)^2 .$$

Schwinger form of scattering amplitude

Let $|\sigma_i| = e^{-2\pi y_i}$, then

$$A = \int_0^\infty \mathrm{d}\tau_2 (\tau_2)^{-d/2} \prod_{\mu=2}^{4} \int_0^{\tau_2} \mathrm{d}y_\mu \prod_i \theta(y_{i+1,i})$$
$$\times \exp\left(-\sum_{i<j} k_i \cdot k_j \left[y_{ij} + \frac{1}{\tau_2}(y_{ij})^2 \right] 2\pi \right)$$

$\theta(s)$ has value $+1$ for $s > 0$ and 0 for $s < 0$ and $y_{ij} \equiv y_{i,j} \equiv y_i - y_j$.
After some rescaling it is ready for a comparison,

$$A(k_1, \cdots, k_n) = C \int_0^\infty \mathrm{d}\tau_2 (\tau_2)^{-d/2+n-1} \prod_{\mu=2}^{n} \int_0^1 \mathrm{d}y_\mu$$
$$\times \exp\left(-\sum_{i<j} k_i \cdot k_j \left[y_{ij} + (y_{ij})^2 \right] \tau_2 \right) \prod_i \theta(y_{i+1,i}) \, ,$$

with $n=4$, the two formulas are the same after identifying $\alpha_i = y_{i+1} - y_i, \quad i = 1, \cdots, n, \quad$ letting $\quad y_{n+1} = 1$,
For general n, proving the n-gons conjecture.

$$A(k_1,\cdots k_n) = \int \frac{\mathrm{d}^d p}{(2\pi)^d} \prod_{i=1}^{n} \int_0^1 \mathrm{d}\alpha_i \, \delta\left(\sum_{i=1}^{n} \alpha_i - 1\right) \int_0^\infty \mathrm{d}\alpha \, \alpha^{n-1}$$
$$\exp\left(-\alpha\left(\sum_{i}^{n-1} \alpha_i (p+q_i)^2 + \left(1 - \sum_{i}^{n-1} \alpha_i\right) p^2\right)\right) \quad ,$$

where $q_i = \sum_{j=1}^{i} k_j$. After the integration of p,

$$A(k_1,\cdots k_n) = \prod_{\mu=2}^{n} \int_0^1 \mathrm{d}y_\mu \prod_i \theta(y_{i+1,i}) \int_0^\infty \mathrm{d}\alpha \, \alpha^{n-1-d/2}$$
$$\exp\left(\alpha\left[(\sum_{i=1}^{n} k_i y_i)^2 + \sum_{i<j} 2 y_j k_i \cdot k_j\right]\right) \quad .$$

Operastor formalism

We begin with the tensionless string. On cylinder coordinates, the action is
$S = \int d^2 w (\bar{\partial} Y)^2$
with e.o.m. $\bar{\partial}^2 Y = 0$
The general solution $Y = X(w) + \frac{(w-\bar{w})}{2} P(w)$
$z = e^{iw}$, $w = \sigma - \mathrm{i}\tau, \sigma \in [0, 2\pi]$, $\tau \in R$ are the coordinates on the cylinder. $X(w) = \sum_{n \in Z} x_n z^{-n}$, $P(w) = \sum_{n \in Z} p_n z^{-n}$
$< 0|P^\mu(w) \sum_i k_i \cdot X(w_i')|0 >= -\frac{1}{2} \sum_i k_i^\mu \cot(\frac{w - w_i'}{2})$
Constraints,

$$T(z) = \mathrm{i} : \partial_z X(z) P(z) + T_{KM} :$$
$$M(z) = (P(z))^2$$

$M(z)$ is a primary field with conformal weight 2

String vertex

$$V_k(z,\bar{w}) :=: e^{ik\cdot Y(z,\bar{w})}\epsilon \cdot \partial_{\bar{w}} Y(z,\bar{w}) : J(z)$$

$$=: \partial_{\bar{z}} \frac{1}{k\cdot P(z)} e^{ik\cdot X(z)}\epsilon \cdot P(z) : J(z),$$

$$V_k := \int V_k(z,\bar{z}) dz d\bar{z}$$

Including worldsheet fermions,

$$V_k(w,\bar{w}) :=: e^{ik\cdot Y(w,\bar{w})}(\epsilon \cdot P(w) + k\cdot \psi(w)\epsilon \cdot \psi(w)) : J(w)$$

$$V_k := \int V_k(w,\bar{w}) dw d\bar{w}$$

Graviton scattering

$$\begin{aligned}
V_k(w,\bar{w}) &:=: e^{ik\cdot Y(w,\bar{w})}(\epsilon_1\cdot P(w)+k\cdot\psi_1(w)\epsilon_1\cdot\psi_1(w)) \\
&\times(\epsilon_2\cdot P(w)+k\cdot\psi_2(w)\epsilon_2\cdot\psi_2(w)) \\
&=: \partial_{\bar{z}}\frac{1}{k\cdot P(z)}e^{ik\cdot X(z)}(\epsilon_1\cdot P(w)+k\cdot\psi_1(w)\epsilon_1\cdot\psi_1(w)) \\
&\times(\epsilon_2\cdot P(w)+k\cdot\psi_2(w)\epsilon_2\cdot\psi_2(w)) \\
V_k &:= \int V_k(w,\bar{w})dwd\bar{w}
\end{aligned}$$

BRST charge

i)bosonic ambitwistor string

$$
\begin{aligned}
c_{total} &= c_D + c_{KM} + c_{gh} = 0 \\
c_{KM} &= 52 - 2D
\end{aligned}
$$

In particular, for $D = 4$, we have $c_{KM} = 44$. If $c_{total} = 0$ is satisfied, we then have

$$
\begin{aligned}
Q &= \oint \left((T(z) + \frac{1}{2} T_{gh}(z))c(z) + M(z)\tilde{c}(z) \right) dz \\
Q^2 &= 0 \\
\{Q, b(z)\} &= T_{total}(z) \\
\{Q, \tilde{b}(z)\} &= M_{total}(z)
\end{aligned}
$$

ii)heterotic ambitwistor string

$$
\begin{aligned}
G_{total}(z) &= G(z) + G_{gh}(z) \\
G_{gh}(z) &= \gamma(z)\tilde{b}(z) \\
T_{gh}(z) &= 2\partial c(z)b(z) + c(z)\partial b(z) + 2\partial\tilde{c}(z)\tilde{b}(z) + \tilde{c}(z)\partial\tilde{b}(z) \\
&\quad - \frac{1}{2}\partial\beta(z)\gamma(z) - \frac{3}{2}\beta(z)\partial\gamma(z) \\
c_{gh} &= -52 + 11 = -41
\end{aligned}
$$

$$
c_{total} = c_D + c_{KM} - 41 = 5D/2 + c_{KM} - 41 = 0
$$

$$
Q = \oint ((T(z) + \frac{1}{2}T_{gh}(z))c(z) + M(z)\tilde{c}(z) + (G(z) + \frac{1}{2}G_{gh}(z))\gamma(z))dz,
$$

iii)fermionic ambitwistor string

$$T_{gh}(z) = 2\partial c(z)b(z) + c(z)\partial b(z) + 2\partial\tilde{c}(z)\tilde{b}(z) + \tilde{c}(z)\partial\tilde{b}(z)$$
$$- \frac{1}{2}\partial\beta(z)\bar{\gamma}(z) - \frac{3}{2}\beta(z)\partial\bar{\gamma}(z) - \frac{1}{2}\partial\bar{\beta}(z)\gamma(z) - \frac{3}{2}\bar{\beta}(z)\partial\gamma(z)$$
$$c_{gh} = -52 + 22 = -30$$
$$G_{gh}(z) = \gamma(z)\tilde{b}(z)$$
$$\bar{G}_{gh}(z) = \bar{\gamma}(z)\tilde{b}(z),$$

$c_{total} = 3D + c_{KM} - 30 = 0$ leads to $c_{KM} = 3(10 - D)$
For $D = 10$, we have $c_{KM} = 0$, we have pure 10d gravity theory.
For $D = 4$, we have $c_{KM} = 18$, we then have a 4d theory contains massless graviton, gauge boson and scaler.

$$Q = \oint ((T(z) + \frac{1}{2}T_{gh}(z))c(z) + M(z)\tilde{c}(z)$$
$$+ (\bar{G}(z) + \frac{1}{2}\bar{G}_{gh}(z))\gamma(z) + (G(z) + \frac{1}{2}G_{gh}(z))\bar{\gamma}(z))dz,$$

One loop level

$$A_{1-loop}(1;2;\cdots;m) := \mathrm{tr}\prod_{i=1}^{m}\exp\{ik_i\cdot(X(w_i)+2i(\bar{w}_i-w_i)P(w_i))\}$$
$$\times\exp\{i(\bar{\tau}-\tau)M_0\}\exp\{i\tau L_0\}$$

Conclusion and perspective

1) In more general cases one has to calculate Feynman diagrams other than the ones for n-gons and the factor $\mathcal{I}(\tau, \{z_i\})$ can no longer be taken as a constant.
2) It would be interesting to generalize the null string formulation to higher genus Riemann surfaces which gives rise to multi-loop scattering amplitudes in point particle field theory.
3) BMS symmetry, an asymptotic symmetry, should emerge from null string formalism.

Thanks