Calabi-Yau Varities: Enumerative Geometry, Arithmetic Geometry and Physics

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Aspects of Calabi-Yau manifolds (historically)

Differential geometry question

$\exists!$ g on Kähler manifolds with $R_{i\bar{\jmath}}(g) = 0$?

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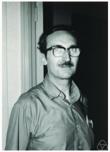

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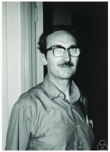

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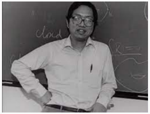

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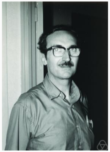

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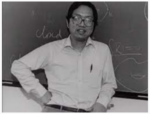

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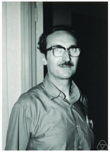

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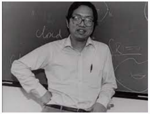

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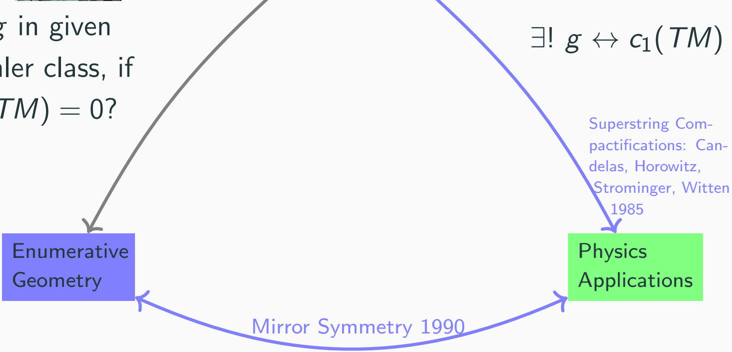

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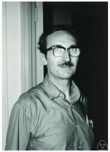

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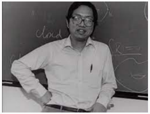

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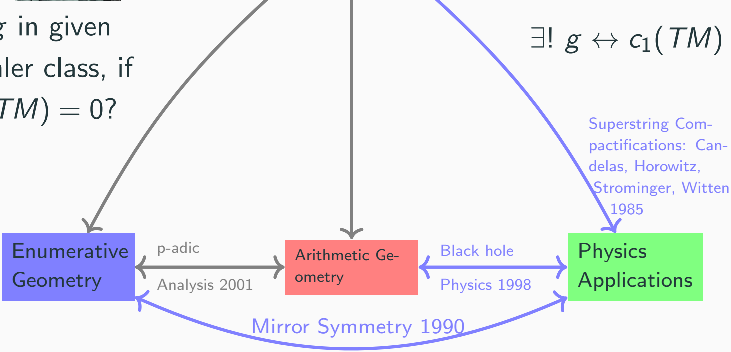

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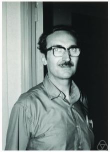

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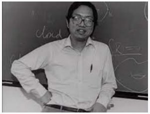

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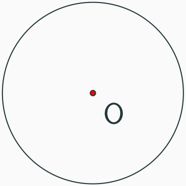
non-perturbative QFT, Int. Sys., QM, Feynman integrals,. . .

Special role of CY 3-folds in enumerative geometry

Enumerative geometry enumerates geometric objects. Euclid: How many tangents has a circle passing through further points? E.g. P:



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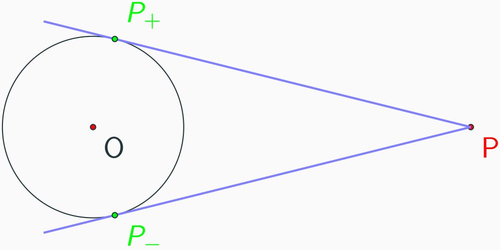

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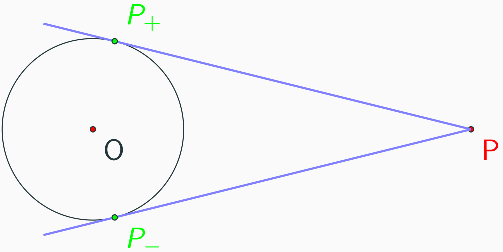
$\{x, y, r \in \mathbb{R} | x^2 + y^2 = r^2\}$



# Points	0 Points	1 Point	$>$ 1 Points
Tangents	∞	2	0
$\dim_{\mathbb{R}}(\mathcal{M})$	1	0	< 0

$\mathcal{M}$: moduli space of solutions

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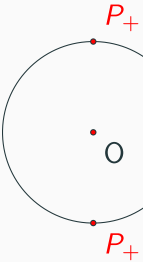

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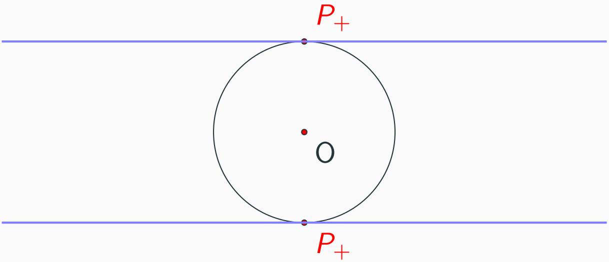

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- To put P where ever, it might be desirable to go to an algebraically closed field.

A beautiful critical enumerative question

Lines on a cubic surface:

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The Riemann-Roch theorem yields the (virtual) dimension of the moduli space of bi-holomorphic embeddings $\mathcal{C}_\beta$ of genus g curves Σ_g into a Kähler manifold M

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- Note the genus one contributions is critical for CY n-folds.

Formal definition of a Calabi-Yau manifold

Definition: A Calabi -Yau n-fold (M, ω, Ω) is a Kähler manifold, with (1,1)-Kähler form ω, of complex dimension n with the following additional (equivalent) properties

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Theorem (C.T.C Wall): The topological type of a Calabi-Yau 3-fold is fixed by their Hodge numbers, their triple intersection $D_i \cap D_j \cap D_k \in \mathbb{N}$ and $[c_2] \wedge D_k,\ D_k \in H_4(M, \mathbb{Z})$.

Construction of Calabi-Yau n-folds

Let M be a degree $\mathcal{N} = dH$ embedding of M into $H \subset \mathbb{P}^{n+1}$.
Then the splitting of the exact sequence

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$$\frac{(1+H)^{n+2}}{1+dH} = 1 + \underbrace{[(n+2)-d]H}_{c_1(TM)} + \underbrace{[(1-d)^2 + \frac{1}{2}n(n+3-2d)]H^2}_{c_2(TM)} + \ldots$$

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- A quintic $d=5$ in $\mathbb{P}^4$ is the simplest CY 3-fold, $\chi = -200$.

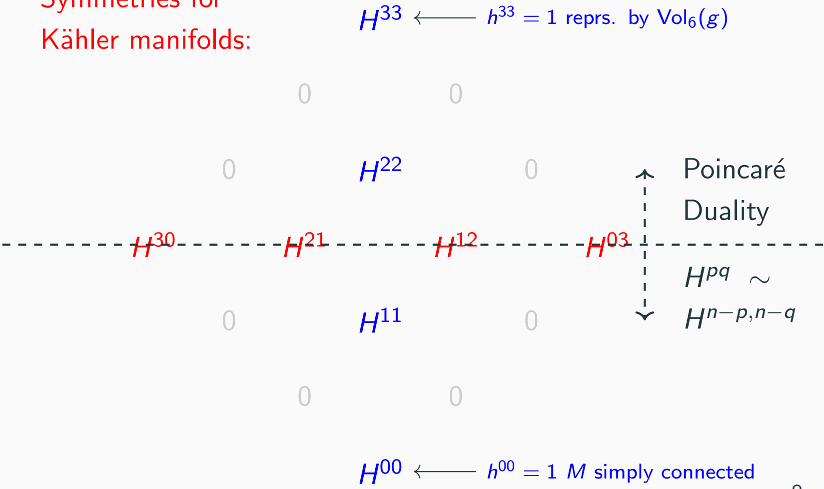
Cohomology and deformations of 3-folds

The cohomology groups of a CY-3fold:

$$\begin{array}{ccccccc}
 & & & H^{33} & & & \\
 & & 0 & & 0 & & \\
 & 0 & & H^{22} & & 0 & \\
H^{30} & & H^{21} & & H^{12} & & H^{03} \\
 & 0 & & H^{11} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & H^{00} & & &
\end{array}$$

Cohomology and deformations of 3-folds

Symmetries for Kähler manifolds:



H^{33} ⟵ $h^{33} = 1$ reprs. by $\mathrm{Vol}_6(g)$

0 0

0 H^{22} 0

H^{30} H^{21} H^{12} H^{03}

0 H^{11} 0

0 0

H^{00} ⟵ $h^{00} = 1$ M simply connected

Poincaré Duality

$H^{pq} \sim H^{n-p,n-q}$

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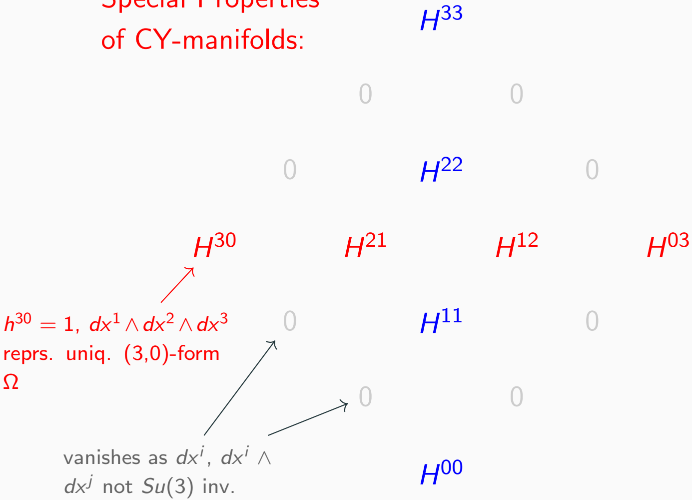
Dolbault Symmetry: $\overline{H^{p,q}} = H^{q,p}$

Cohomology and deformations of 3-folds

Special Properties
of CY-manifolds:



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\end{array}$$

$h^{30} = 1$, $dx^1 \wedge dx^2 \wedge dx^3$ reprs. uniq. (3,0)-form Ω

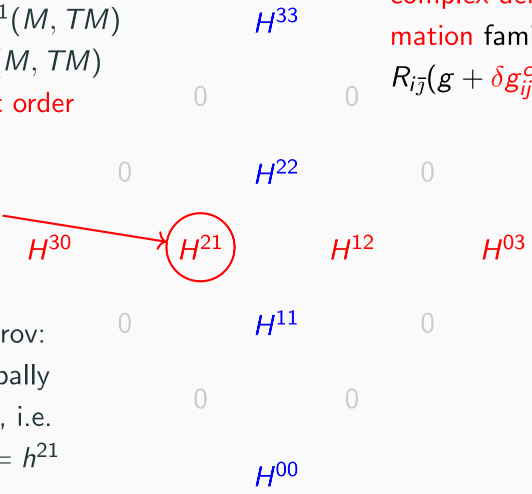
vanishes as dx^i, $dx^i \wedge dx^j$ not $Su(3)$ inv.

Cohomology and deformations of 3-folds

$H^{2,1}(M) \underset{\lrcorner\Omega}{\sim} H^1(M, TM)$

Kodaira: $H^1(M, TM)$ describes first order complex structure deformations

Tian & Todorov: They are globally unobstructed, i.e. $\dim_{\mathbb{C}}(\mathcal{M}_{cs}) = h^{21}$



H^{33}

0 0

0 H^{22} 0

H^{30} $\boxed{H^{21}}$ H^{12} H^{03}

0 H^{11} 0

0 0

H^{00}

CY 3-folds as complex deformation families $R_{i\bar{\jmath}}(g + \delta g_{ij}^{cs}) = 0$:

Cohomology and deformations of 3-folds

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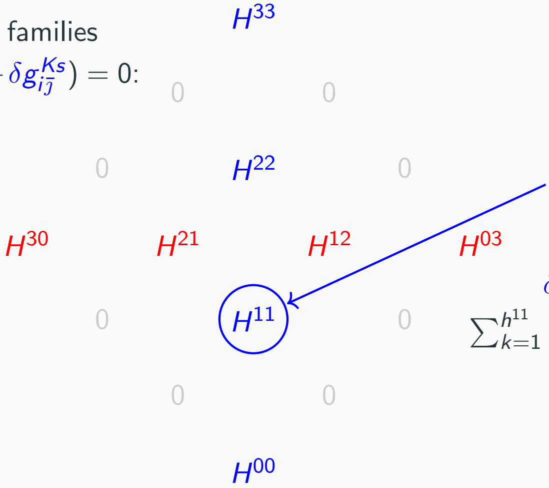

H^{33}

$0 \quad 0$

$0 \quad H^{22} \quad 0$

$H^{30} \quad H^{21} \quad H^{12} \quad H^{03}$

$0 \quad H^{11} \quad 0$

$0 \quad 0$

H^{00}

Kähler structure deformations

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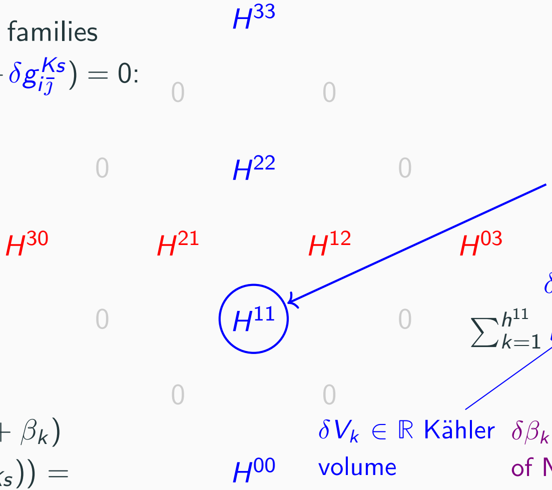

$$\begin{array}{ccccccc} & & & H^{33} & & & \\ & & 0 & & 0 & & \\ & 0 & & H^{22} & & 0 & \\ H^{30} & & H^{21} & & H^{12} & & H^{03} \\ & 0 & & \boxed{H^{11}} & & 0 & \\ & & 0 & & 0 & & \\ & & & H^{00} & & & \end{array}$$

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$\delta V_k \in \mathbb{R}$ Kähler volume

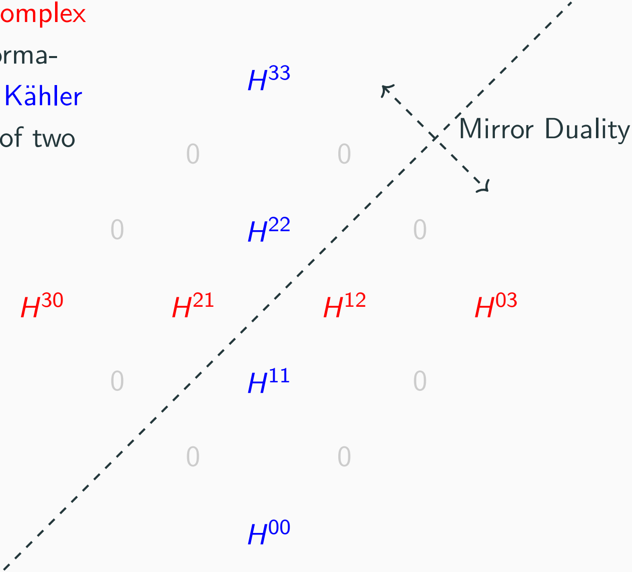
↖

$\delta\beta_k \in \mathbb{R}$ value of N-S b-field (stringy)

$t_k = 2\pi i(iV_k + \beta_k)$ i.e. $\dim_{\mathbb{C}}(\mathcal{M}_{cKs}) = h^{11}$

Cohomology and deformations of 3-folds

Mirror Symmetry exchanges the complex structure deformations and the Kähler deformations of two CY (M, W)



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\end{array}$$

Mirror Duality

Construction of mirror pairs

Quintic in $\mathbb{P}^4$: $[p_5 = \sum_{i=0}^4 x_i^5 - z \prod_{i=0}^4 x_i = 0] = [5H] \subset \mathbb{P}^4$

Generalisation Batyrev: $(\Delta, \hat{\Delta})$ a pair of reflexive pair of lattice polyhedra, $\mathbb{P}_\Delta$ the associated toric space and $[H_i]$ its divisors.

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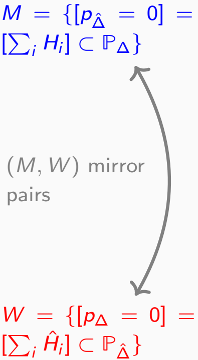

$M = \{[p_{\hat{\Delta}} = 0] = [\sum_i H_i] \subset \mathbb{P}_\Delta\}$

(M, W) mirror pairs

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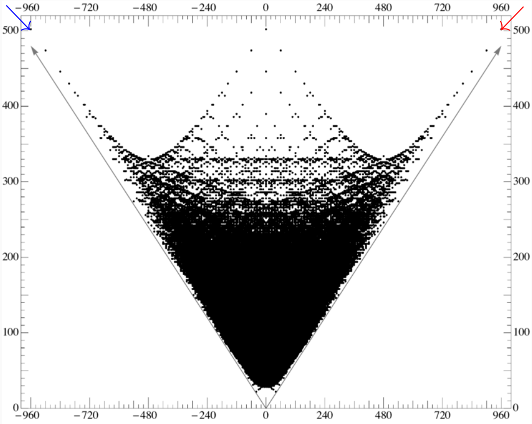

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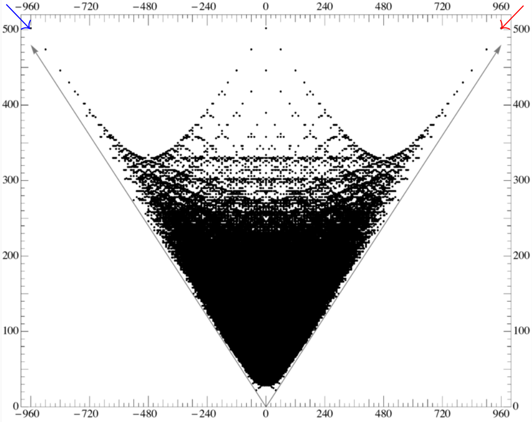
Geom.	$\#(\Delta_k, \hat{\Delta}_k)$
2 *pts*	1
ell crv	16
K3	4319[1]
CY 3-flds	473800776[1]
CY 4-flds	$\mathcal{O}(10^{22})$?
⋮	⋮

[1] Kreuzer and Skarke '02, $k = 3, 4$



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The special role of CY 3-folds in String Theory

Super string theory is defined by the $X : \Sigma_g \to \mathcal{C}_\beta \subset$ space-time, weighted by an action S that is a super symmetric extension of the area of $\mathcal{C}_\beta$.

The first quantised theory is defined by a variational integral. E.g for the partition function

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Dilaton

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Background fields

CY-metric

Neveu-Schwarz b-field

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Superstring string theory is Weyl invariant in ten dimensions:

$$\int \mathcal{D}h \to \sum_{g=0}^{\infty} \int_{\mathcal{M}_{\Sigma_g}} \mu_{3g-3},$$

Functional integral → discrete sum over finite dim. int. in 10d.

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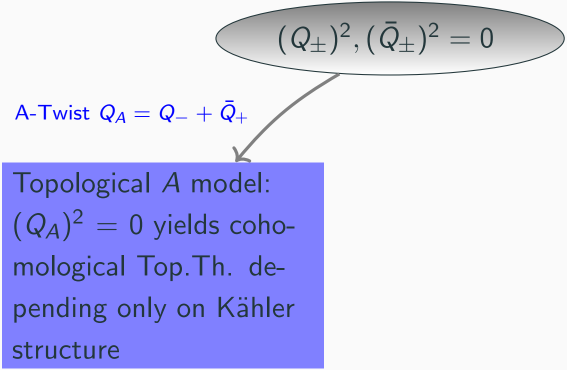

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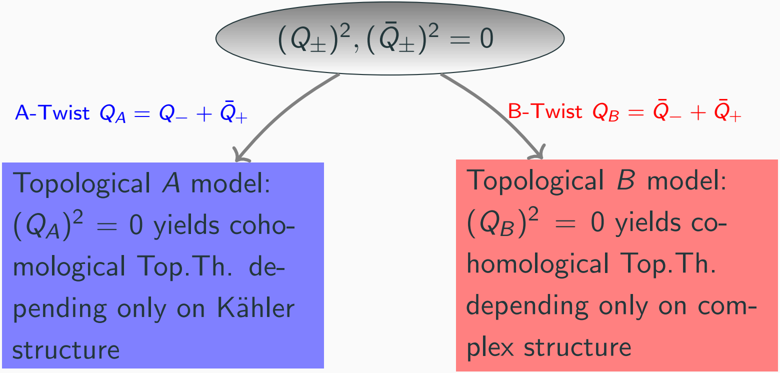

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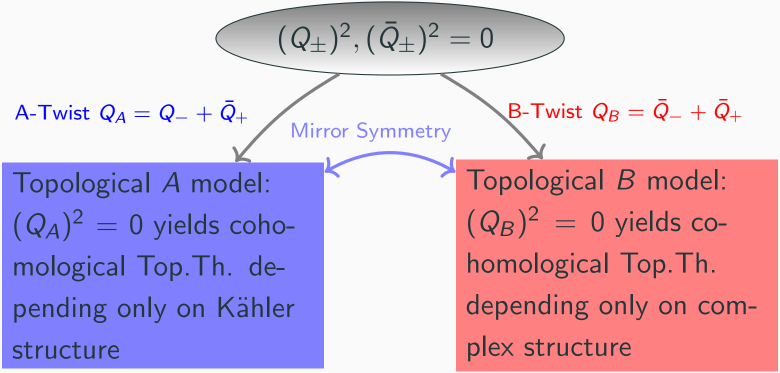

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In the A model terms depending on the complex structure are Q_A exact and the variational integral localizes to the bi-holmorphic maps depending only on the Kähler structure

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$Q = e^{2\pi i \int_{\mathcal{C}_\beta} i\omega + b} = e^{\boldsymbol{t}\cdot\beta}$ and also in full string theory these maps are stationary "points" of the action! World-Sheet instantons

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$n_g^\beta \in \mathbb{Z}$ the BPS indices or Pandharipande Thomas invariants

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By descend rel. to marginal ops. parametrising compl. structure def.

1. Can be calculated by periods

2. $t(z)$ Mirror map can be specified only for $-\mathrm{Re}(t_k) \sim V_k \to \infty$

Periods on Calabi-Yau n-folds

Add 1 : Periods are integrals

$$\Pi_{ij}(\underline{z}) = \int_{\lambda_i} \Lambda^j(\underline{z})$$

that define a pairing between between homology and cohomology (n odd) well defined by the theorem of Stokes:

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$\underline{\lambda}$ is topol. and so is $\underline{\Lambda}$ via $\int_{A^I} \alpha_J \;=\; \int_{B_J} \beta^I \;=\; \delta^I_J$. A basis moving with the comp. str. in $\underline{\Lambda}$ are the meromorphic forms $\Omega(z), \partial_z\Omega(z), \ldots$.

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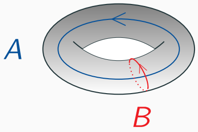

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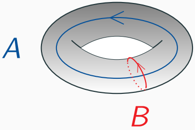

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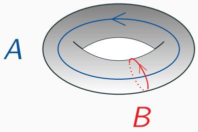

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$$E_1(z) = \oint_A \Omega,\ E_2(z) = \oint_B \Omega$$ Elliptic integrals.

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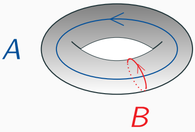
Calabi-Yau 1-fold



$$p_3 = wy^2 - 4x^3 - g_2(z)xw^2 - g_3(z)w^3 = 0 \subset \mathbb{P}^2$$

$$\Omega(z) = \oint \tfrac{2dx\wedge dy}{p_3} = \tfrac{dx}{y},\ \partial_z\Omega(z) \sim \tfrac{xdx}{y}$$

$$E_1(z) = \oint_A \Omega,\ E_2(z) = \oint_B \Omega \text{ Elliptic integrals.}$$

Well studied in part because they solve Keplers problem

Periods on Calabi-Yau n-folds

Add 1 : Periods are integrals

$$\Pi_{ij}(\underline{z}) = \int_{\lambda_i} \Lambda^j(\underline{z})$$

that define a pairing between between homology and cohomology (n odd) well defined by the theorem of Stokes:

$$\Pi : H_n(M_n, \mathbb{Z}) \times H^n(M_n, \mathbb{C}) \to \mathbb{C}\,.$$

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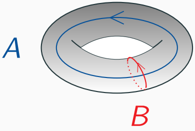

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Fullfill linear diff eq. of 2cd order. Picard(1891)-Fuchs(1881) eq.

Period geometry on CY n-fold

The main constraints which govern the period geometry of CY-folds are the Riemann bilinear relations

$$e^{-K} = i^{n^2} \int_{M_n} \Omega \wedge \bar{\Omega} > 0 \tag{1}$$

defining the real positive exponential of the Kähler potential $K(z)$ for the Weil-Peterssen metric $G_{i\bar{\jmath}} = \partial_{z_i} \bar{\partial}_{\bar{z}_{\bar{\jmath}}} K(z)$ on $\mathcal{M}_{cs}(M_n)$.

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$$\int_{M_n} \Omega \wedge \underline{\partial}^k_{I_k} \Omega = \begin{cases} 0 & \text{if } k < n \\ C_{I_n}(z) & \text{if } k = n \ . \end{cases} \tag{2}$$

Here $\underline{\partial}^k_{I_k} \Omega = \partial_{z_{I_1}} \ldots \partial_{z_{I_k}} \Omega \in F^{n-k} := \bigoplus_{p=0}^{k} H^{n-p,p}(W)$ are arbitrary combinations of derivatives w.r.t. to the z_i, $i = 1, \ldots, r$.

Period geometry on CY n-fold

The $C_{I_n}(z)$ are rational functions labelled by I_n up to permutations. The differential ideals $\mathcal{L}\vec{\Pi} = 0$ also determine the $C_{I_n}(z)$ up to a multiplicative normalisation

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Remark 1:W.r.t the Hodge decomposition the pairings (1) and (2) have the property that if $\alpha_{m,n} \in H^{m,n}(M_n)$ and $\beta_{p,q} \in H^{r,s}(M_n)$ then $\int_W \alpha_{m,n} \wedge \beta_{p,q} = 0$ unless $m + p = n + q = 3$.

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Remark 2: In terms of a basis of periods compatible with Σ they can be written as

$$\int_{M_n}\Omega\wedge\bar{\Omega}=\vec{\Pi}^\dagger\Sigma\vec{\Pi}, \qquad \int_{M_n}\Omega\wedge\underline{\partial}^k_{I_k}\Omega=-\vec{\Pi}^T\Sigma\underline{\partial}^k_{I_k}\vec{\Pi}\ ,$$

Periods on 3-folds

Consider the mirror quintic W

$$\hat{p}_5 = \sum_{i=0}^{4} x_k^5 - 5z^{-\frac{1}{5}} \prod_{k=0}^{4} z_i = 0 \subset \hat{\mathbb{P}}^4$$

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Hodge diamond of elliptic curve

Hodge diamond of W

$$\begin{array}{ccc} & 1 & \\ 1 & & 1 \\ & 1 & \end{array} \longrightarrow \begin{array}{ccccccc} &&&1&&& \\ &&0&&0&& \\ &0&&101&&0& \\ 1&&1&&1&&1 \\ &0&&101&&0& \\ &&0&&0&& \\ &&&1&&& \end{array}$$

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The period vector $\Pi(z) = \left(\int_{A^0} \Omega, \int_{A^1} \Omega(z), \int_{B^0} \Omega(z), \int_{B^1} \Omega(z)\right)^T$ fullfils a 4th order Picard-Fuchs diff. eq. ($\theta = zd/dz$)

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$$\boxed{[\theta^4 - 5z(5\theta+1)(5\theta+2)(5\theta+3)(5\theta+4)]\Pi(z) = 0}$$

Periods on 3-folds

Local → global: How to find the periods over cycles in $H_3(W,\mathbb{Z})$? Find the basis in which mondromies $\Pi \mapsto M_* \Pi$ around the singular points $*$ are in $\mathrm{Sp}(4,\mathbb{Z})$

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$$\mathcal{P}\left\{\begin{array}{cccc} 0 & 5^{-5} & \infty & * \\ 0 & 0 & \frac{1}{5} & \\ 0 & 1 & \frac{2}{5} & z \\ 0 & 2 & \frac{3}{5} & \\ 0 & 1 & \frac{4}{5} & \end{array}\right\}$$

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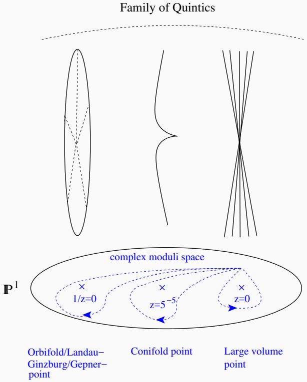

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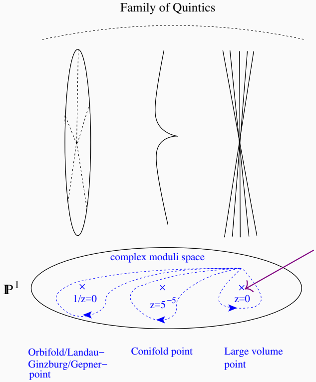

Identifies also the expansion point for the mirror map as point of maximal unipotent monodromy

Periods on 3-folds

Special geometry Bryant and Griffiths '83 implies that the periods can be expressed by a prepotential $\mathcal{F}$

triple logarithmic solution

double logarithmic solution

analytic solution

logarithmic solution

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$$\begin{pmatrix} \int_{B_0}\Omega \\ \int_{B_1}\Omega \\ \int_{A_0}\Omega \\ \int_{A_1}\Omega \end{pmatrix} = \begin{pmatrix} F_0 \\ F_1 \\ X^0 \\ X^1 \end{pmatrix} = X^0 \begin{pmatrix} 2\mathcal{F}_0 - t\partial_t\mathcal{F}_0 \\ \partial_t\mathcal{F}_0 \\ 1 \\ t \end{pmatrix} \begin{array}{l} \longleftarrow \text{triple logarithmic solution} \\ \longleftarrow \text{double logarithmic solution} \\ \longleftarrow \text{analytic solution} \\ \longleftarrow \text{logarithmic solution} \end{array}$$

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and Candelas et al '91 identified near the MUM point $z = 0$

$$\boxed{\mathcal{F}(z) \equiv \mathcal{F}_0(t(z)), \qquad t = \frac{X^1}{X^0} = \log(z) + \mathcal{O}(z)}$$

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Hosono et. al '93 generalised to multiparameter CY and related the classical terms to the CTC Wall data $\kappa = D^3$, $\sigma = (\kappa \bmod 2)/2$ in

$$\mathcal{F} = -\frac{\kappa}{6}t^3 + \frac{\sigma}{2}t^2 + \frac{c_2 \cdot D}{24}t + \frac{\chi(M)}{2}\frac{\zeta(3)}{(2\pi i)^3} - \frac{1}{(2\pi i)^3}\sum_{\substack{\beta\in H_2(M,\mathbb{Z})\\ \beta\neq 0}} n_0^\beta \mathrm{Li}_3(Q^\beta).$$

Mirror symmetry predictions for the quintic

g	d=1	d=2	d=3	d=4	d=5	. . .
0	2875	609250	317206375	242467530000	229305888887625	
1	0	0	609250	3721431625	12129909700200	
2	0	0	0	534750	75478987900	
3	0	0	0	8625	-15663750	
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5	0	0	0	0	1100	
6	0	0	0	0	10	
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Alexandrov, Feyzbakash, Pioline, Schimannek & A.K. 2d elliptic genus $r = 1$, DT: $g \leq 69$, $r = 2$, DT: $g \leq 80$ involving mock modularity.

http://www.th.physik.uni-bonn.de/Groups/Klemm/data.php

An elliptic fibration over the Hirzebruch surface F_1

Rational curves in elliptically fibered CY 3-fold Klemm,Mayr,Vafa '96

d_f	$d_e = 0$	1	2	3	4	5
0		480	480	480	480	480
1	1	252	5130	54760	419895	2587788
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In geometric engineering E_8 corresponds to a real flavour goup.

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E_1	$E_2 = 0$	1	2	3	4	5	6	$\Sigma = 252$
0	1							1
1	1	27	27	1				56
2			27	84	27			138
3				1	27	27	1	56
4							1	1

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In geometric engineering E_8 corresponds to a real flavour goup. By blowing up the geometry we can break it to $U(1) \times E_7$, $U(1)^2 \times E_6$.

Refined BPS states

Conceptional inputs from physics to enumerative geometry: The reorganisation of $\mathcal{F}(Q, g_s)$ in BPS indices comes from calculating a BPS saturated Schwinger-loop amplitudes that contribute to the $R_+^2 F_+^{2g-2}$ term in the effective supergravity action Gopakumar & Vafa '02

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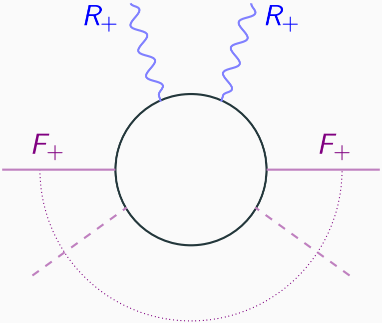

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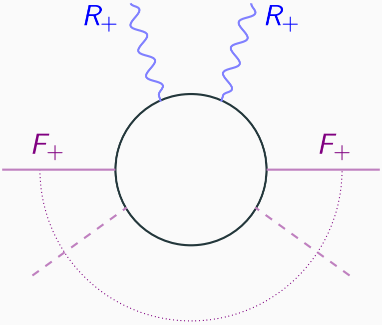

2g − 2: F_+ Graviphoton insertions

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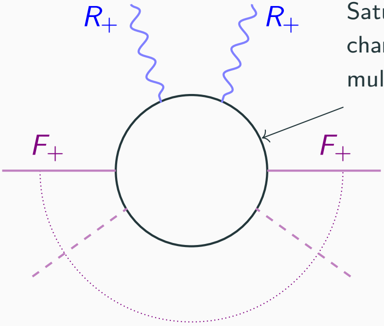

Saturated by BPS states of charge β and spin j_L, j_R and multiplicity $N^{\beta}_{j_R, J_L}$

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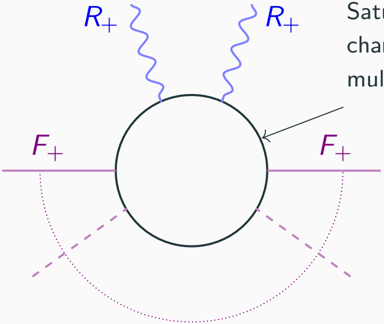

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$$\sum_{j_R \in \frac{1}{2}\mathbb{Z}_{\geq 0}} (-1)^{2j_r}(2j_R+1) N^{\beta}_{j_R j_L} [j_L] = \sum_{g \in \mathbb{Z}_{g \geq 0}} I_g^L n_g^{\beta} \ .$$

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$I_g^L = \left(2[\mathbf{0}] + \left[\tfrac{\mathbf{1}}{\mathbf{2}}\right]\right)^{\otimes g}$

Spin statistic sign

Repr. of left spin

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actual BPS number
$N^\beta_{j_R, j_L} \in \mathbb{N}$

BPS index $n_g^\beta \in \mathbb{Z}$

Refined BPS states

Conceptional inputs from physics to enumerative geometry: The reorganisation of $\mathcal{F}(Q, g_s)$ in BPS indices comes from calculating a BPS saturated Schwinger-loop amplitudes that contribute to the $R_+^2 F_+^{2g-2}$ term in the effective supergravity action Gopakumar & Vafa '02

$I^L_g = \left(2[\mathbf{0}] + \left[\tfrac{\mathbf{1}}{\mathbf{2}}\right]\right)^{\otimes g}$

Spin statistic sign

Repr. of left spin

$$\sum_{j_R \in \frac{1}{2}\mathbb{Z}_{\geq 0}} (-1)^{2j_r}(2j_R+1) N^{\beta}_{j_R j_L}\,[j_L] = \sum_{g \in \mathbb{Z}_{g\geq 0}} I^L_g\, n^{\beta}_g \ .$$

actual BPS number
$N^{\beta}_{j_R, j_L} \in \mathbb{N}$

BPS index $n^{\beta}_g \in \mathbb{Z}$

One should and can Huang& Klemm '13 in the local CY-3-folds calculate the refined BPS states $N^{\beta}_{j_R j_L}$. These detect dimensions of representations of group actions, e.g. M_{24} on K3.

BPS asymptotics

The classical entropy of 5d spinning black holes with angular moment is one quarter of the horizon area

$$S_0 = 2\pi\sqrt{\mathcal{Q}^3 - m^2}, \qquad \mathcal{Q}^{3/2} = \frac{1}{6}\kappa t^3 \qquad Q = \frac{1}{2}\kappa t^2.$$

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If the microscopic entropy is from the BPS states n_r^Q, then the number of states is

$$\Omega(Q, m) = \sum_r \binom{2r+2}{m+r+1} n_Q^r,$$

and the asymptotic of $A(Q = d) = \log(\Omega(Q, 0)) \sim \frac{4\pi}{3\sqrt{2\kappa}}$.

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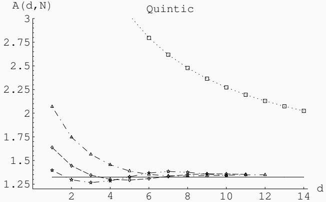

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$$Z(X_p, T) = \exp\left(\sum_{n=1}^{\infty} \#X_p(\mathbb{F}_{p^n})\frac{T^n}{n}\right)$$

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the local zeta function of X_p. $Z(X_p, T)$ turns out to be rational

$$Z(X_p, T) = \prod_{r=0}^{2d} P_r(X_p, T)^{(-1)^{r+1}}$$

where $P_r(X_p, T)$ is a polynomial of degree $b_r(X)$ with integral coefficients and with all roots of absolute value $p^{-r/2}$.

Calabi-Yau modularity

For elliptic curves $b_1 = 2$ and

$$P_1(\mathcal{E}/\mathbb{F}_p, T) = (1 - a_p T + pT^2)\ .$$

The modularity theorem implies that

$$f_2 = \sum_n a_n q^n$$

is a weight 2 Hecke eigenform in the space of cusp forms $S_2(\Gamma_0(N))$ for some conductor N.

For one parameter CY 3-fold families one has

$$P_3(M_z/\mathbb{F}_p, T) = 1 + \alpha_p T + \beta_p p T^2 + \alpha_p p^3 T^2 + p^6 T^4 \qquad (3)$$

for integers α_p and β_p. There is a conjecture that the α_p and β_p are Hecke eigenvalues of Siegel para modular Hecke form. Checking this conjecture is rather difficult but examples have been found Golyshev, Van Straten 2021.

Calabi-Yau modularity

With Böhnisch, Scheidegger and Zagier CMP '24 we are studying special fibers where the Galois action on the middle cohomology is reducible, which signals a factorization of P_3

$$P_3(M_{z_*}/\mathbb{F}_p, T) = (1 - a_p T + p^3 T^2)(1 - b_p(pT) + p(pT)^2)$$

where a_p and b_p are the Hecke eigenvalues of f_4 and g_2 cusp forms. This a rank two attractor point, i.e. $H^3(W_{z_*}, \mathbb{Q}) = \Lambda \oplus \Lambda_\perp$ where

$$\Lambda \subseteq H^{3,0}(M_{z_*}) \oplus H^{0,3}(M_{z_*}) \quad \text{and} \quad \Lambda_\perp \subseteq H^{2,1}(M_{z_*}) \oplus H^{1,2}(M_{z_*})\,.$$

At z^* one can construct a stable $N = 2$ flux vacua/super symmetric black holes.

New Physics Applications of Calabi Period motives

The Period Motive of Calabi-Yau manifolds that we know in string theory especially the B-model approach to mirror symmetry and the B-type topological string has new applications to

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- **III.** Post Minkowskian (PM) Worldline Quantum Field Theory approximation to General Relativity predict the gravitational wave forms in black hole scattering/mergers detected by LIGO,....

Based on work with

Kilian Bönisch, Claude Duhr, Fabian Fischbach, Florian Loebbert, Christoph Nega, Jan Plefka, Franzika Porkert, Reza Safari, Benjamin Sauer, Lorenzo Tancredi

[1]=arXiv:1912.06201v2, [2]=arXiv:2008.10574v1,
[3]=arXiv:2108.05310, in JHEP
[4]=arXiv:2209.05291 in PRL and [5]=arXiv: 2212.09550 in JHEP,
[6]= arXiv:2310.08625 acc. JHEP, [7]= arXiv:2402.19034,
[8]= arXiv:2401.07899 sub. PRL

Introduction perturbative QFT

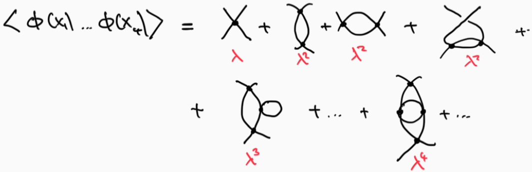
$$Z[J] = \int \mathcal{D}\phi \exp\left[\frac{i}{\hbar}\int \mathrm{d}^D x(\mathcal{L} + J\phi)\right] \ .$$

E.g. with $\mathcal{L} = \int \mathrm{d}^D x \left[\frac{1}{2}(\partial_u \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4\right]$.

All physical correlators are of the form

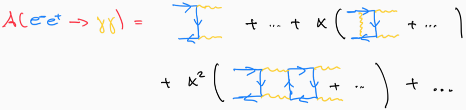
$$\langle\phi(x_1)..\phi(x_n)\rangle = Z[J]^{-1}\left(\frac{\delta}{\delta J(x_1)}\right)..\left(\frac{\delta}{\delta J(x_n)}\right) Z[J]\bigg|_{J=0}$$

In interacting theories $\lambda \neq 0$ this is expanded asymptotically in Feynman graphs

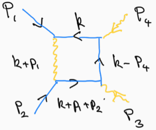


Introduction perturbative QFT

Realistic theories: Probability for e^- e^+ to annihilate to two photons $P(e^-e^+ \to \gamma\gamma) \sim |\mathcal{A}(e^-e^+ \to \gamma\gamma)|^2$, $\alpha \sim \frac{1}{137}$



Scalar part e.g. for e.g. the box integral I: Propagators $\frac{1}{q^2 - m^2 + i \cdot 0}$



$\sum_{i=1}^{4} p_i = 0$ momentum conservation

$$\sim \int \frac{d^D k}{(k^2 - m^2)\,(k+p_1)^2\,((k+p_1+p_2)^2 - m^2)((k-p_4)^2 - m^2)}$$

$D = D_0 - 2\epsilon$, $I = \sum_{k=-n}^{\infty} I_k \epsilon^n$ with I_k functions of masses and Lorentz invariant products of the external momenta that we need to know!

Feyman graphs and relative Calabi-Yau periods

In the Feynman representation the contribution of an l-loop graph yields an integral with a rational integrand defined by the graph polynomials $\mathcal{U}(\underline{x})$ and $\mathcal{F}(\underline{x}, \underline{p}, \underline{m})$, $\underline{p}$ independent momenta, $\underline{m}$ masses

$$I_{\sigma_{n-1}}(\underline{k}, \underline{m}) = \int_{\sigma_{n-1}} \prod_i x_i^{\nu_i - 1} \frac{\mathcal{U}^{\omega - \frac{D}{2}}}{\mathcal{F}^{\omega}} \mu_{n-1}$$

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$\omega = \sum_{i=1}^{n} \nu_i - lD/2$, l # of loops

n # of edges, ν_i their multiplicity D space time dim

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$\sigma_{n-1} = \{[x_1 : \ldots : x_n] \in \mathbb{P}^{n-1} | x_i \in \mathbb{R}_{\geq 0} \,\forall\, 1 \leq i \leq n\}$ an open domain.

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Generally one needs dimensional regularisation and evaluates in $D = 4 - 2\epsilon$ dimensions and gets a Laurent expansion $I = \frac{I_{-1}}{\epsilon} + I_0 + \ldots$. The Laurent coefficients are also to expected to be (twisted) periods Bogner, Weinzierl.

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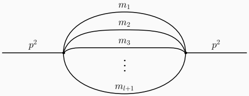

This graph leads in $t = \frac{p^2}{\mu^2}$, $\xi_i = \frac{m_i}{\mu}$ to the period integral

$$I_{\sigma_l} = \int_{\sigma_l} \frac{\mu_l}{P_l(t,\xi_i;x)} = \int_{\sigma_l} \frac{\mu_l}{\left(t - \left(\sum_{i=1}^{l+1} \xi_i^2 x_i\right)\left(\sum_{i=1}^{l+1} x_i^{-1}\right)\right) \prod_{i=1}^{l+1} x_i}$$

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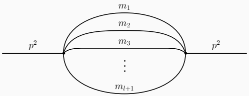

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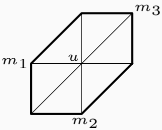
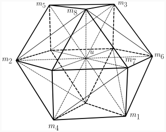
The Newton polytope of the numerator is reflexive. For example for $l = 2, 3$ they look like





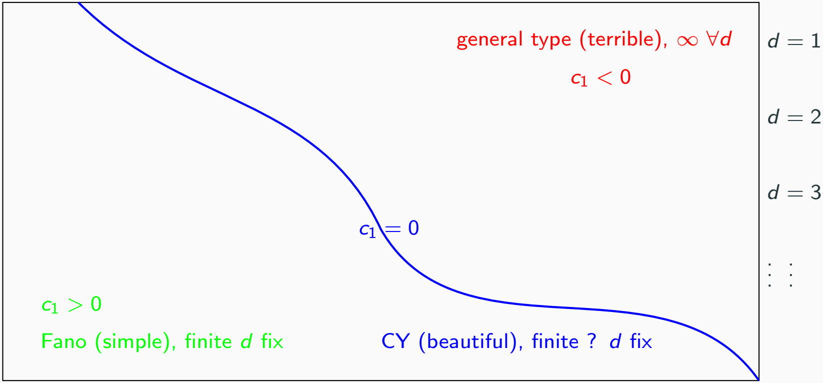
Feyman graphs and relative Calabi-Yau periods

Feynman integrals ⇔ Periods of algebraic varities

Planar Feynman graph	Max. Cut Integrals	Period - Geometry
1-loop	rational functions	Pts in Fano 1-fold
2-loop	elliptic functions	families of elliptic curve
3-loop	fullfil 3 ord. hom diff eqs.	families of K3
4-loop	fullfil 4 ord. hom diff eqs.	families of CY-3-fold
$\vdots$	$\vdots$	$\vdots$

For the full Feynman integral the rational functions are replaced by rational polylogarithms ✓ and the elliptic functions by elliptic polylogarithms (✓) . I. Gel'fand, S. Bloch, P. Vanhove, M.Kerr, C. Duran, S. Weinzierl, F. Brown, O. Schnetz, J. Bourjaily, A. Mc Leod, M. Hippel, M. Wilhelm, J. Broedel, L Trancredi, S. Müller-Stach, Klemm, Nega, Safari, '19, +Böhnisch, Fischbach '20, + Duhr '21 . . . + 248 cits. in the latter

Kodaira map of algebraic varieties



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$l = 0 \quad l = 1 \quad l = 2 \quad l = 3 \quad \cdots$

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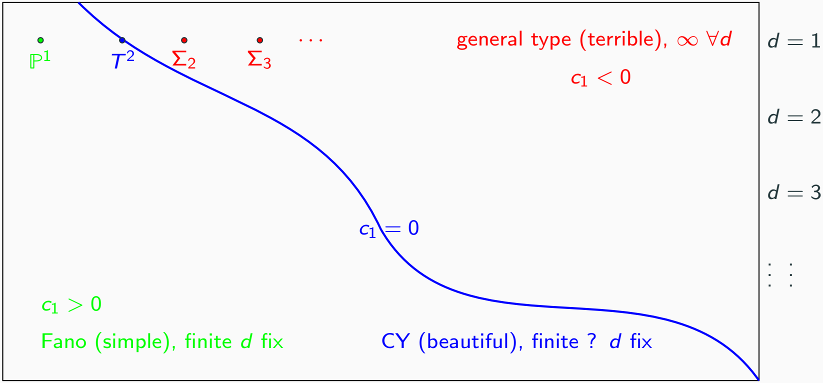

$\mathbb{P}^1$ T^2 Σ_2 Σ_3 $\cdots$

general type (terrible), $\infty \; \forall d$

$c_1 < 0$

$c_1 = 0$

$c_1 > 0$

Fano (simple), finite d fix

CY (beautiful), finite ? d fix

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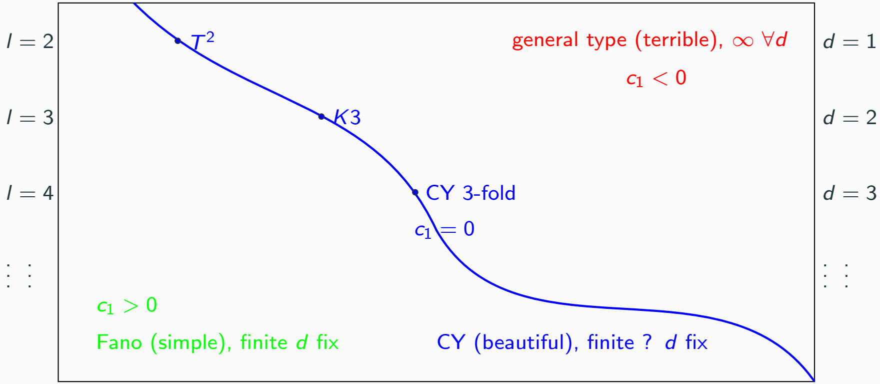

$l = 2$ T^2 general type (terrible), $\infty\ \forall d$ $d = 1$

$c_1 < 0$

$l = 3$ $K3$ $d = 2$

$l = 4$ CY 3-fold $d = 3$

$c_1 = 0$

$\vdots\ \vdots$ $\vdots\ \vdots$

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Fano (simple), finite d fix

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Dictionary Feynman graphs/amplitudes and geometry

Perturbative QFT	Geometry X	Differential eq.	Arithmetic Geometry
maximal cut Feynman integral	Period integral $\underline{\Pi}$ (ϵ-deformed)	Homogeneous Gauss Manin $(d-A(z))\underline{\Pi}=0$	Motive defined by l-adic coh $H^k_{et}(\overline{X},\mathbb{Q}_l)$
	$\circlearrowleft$ Monodromy group $\in \Gamma(\mathbb{Z})$; irreducible ?		$\circlearrowleft$ Galois group $\mathrm{Gal}(\overline{K}/K)$ irreducible ?
actual Feynman integral	Chain integral (ϵ-deformed)	Inhomogeneous Gauss Manin connection $(d-A(z))\underline{\Pi}=B(z)$	Extended motive

Gauss Manin connection and sub sectors

One way to get the Gauss-Manin connection and the inhomogeneous term is to use the integration by by parts relations IBP relation between so called master integrals. Consider l-loop Feynman integrals in general dimensions $D \in \mathbb{R}_+$ of the form

$$I_{\underline{\nu}}(\underline{x}, D) := \int \prod_{r=1}^{l} \frac{\mathrm{d}^D k_r}{i\pi^{\frac{D}{2}}} \prod_{j=1}^{p} \frac{1}{D_j^{\nu_j}} \tag{4}$$

$D_j = q_j^2 - m_j^2 + i \cdot 0$ for $j = 1, \ldots, p$ are the propagators, q_j is the j^{th} momenta through D_j, $m_j^2 \in \mathbb{R}_+$ are masses, $i \cdot 0$ indicates the choice of contour/branchcut in $\mathbb{C}$. Subject to momentum conservation the q_j are linear in the external momenta $p_1, \ldots, p_E$, $\sum_{i=j}^{E} p_j = 0$ and the loop momenta k_r. We defined $\epsilon := \frac{D_0 - D}{2}$.

Master Integrals and integration by parts relations

The Feynman integral depends besides D on dot products of p_i and the masses m_j^2, written compactly in a vector $\underline{w} = (w_1, \ldots, N) = (p_{i_1} \cdot p_{i_2}, m_j^2)$ and dimensional analysis of $I_{\underline{\nu}}$ shows that it depends only on the ratios of two parameters x_i, we chose

$$x_k := \frac{w_k}{w_N} \qquad \text{for } 1 \leq k < N$$

and label now the parameters of the integrals $I_{\underline{\nu}}$ by the dimensionless parameters $\underline{x}$.

Master Integrals and integration by parts relations

The propagator exponents and $D \in \mathbb{Z}$ span a lattice $(\underline{\nu}, D) \in \mathbb{Z}^{p+1}$. The $I_{\underline{\nu}}(\underline{x}, D)$ are called master integrals.

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The integration by parts (IBP) identities

$$\int \prod_{r=1}^{l} \frac{\mathrm{d}^D k_r}{i\pi^{\frac{D}{2}}} \frac{\partial}{\partial k_k^{\mu}} \left(q_l^{\mu} \prod_{j=1}^{p} \frac{1}{D_j^{\nu_j}} \right) = 0 \; .$$

relate the master integrals with different exponents $\underline{\nu}$.

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relate the master integrals with different exponents $\underline{\nu}$.

There is a finite region in the lattice that contains all non-vanishing master integrals. In a basis of master integrals one can express derivatives w.r.t. the z_k as a linear combination rational coefficients by the IBP relations.

Master Integrals and integration by parts relations

- The basis of master integrals (graph cohomology) corresponds to the basis of the cohomology $H^{l-1}(M_l, \mathbb{Z})$.

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Among the elements in the lattice $\mathbb{Z}^p$ and, in particular, for the master integrals one can define sectors and a semi-ordering on the latter by defining a map

$$\underline{\nu} \mapsto \underline{\vartheta}(\underline{\nu}) =: (\theta(\nu_j))_{1 \leq j \leq p} \ .$$

where θ is the Heaviside step function. The semi-ordering is then defined by $\underline{\vartheta}(\underline{\nu}) \leq \underline{\vartheta}(\underline{\tilde{\nu}})$, iff $\theta(\nu_j) \leq \theta(\tilde{\nu}_j)$, $\forall j$. This defines an inclusive order on subgraphs with less propagators and therefore simpler topology.

IBP relation summary:

The IBP relations characterise a suitable finite set of master integrals

$$I_{\underline{\nu}}(\underline{x}, D) := \int \prod_{r=1}^{l} \frac{\mathrm{d}^D k_r}{i\pi^{\frac{D}{2}}} \prod_{j=1}^{p} \frac{1}{D_j^{\nu_j}} \ ,$$

with $D_j = q_j^2 - m_j^2 + i \cdot 0$ for $j = 1, \ldots, p$ propagators and $(\underline{\nu}, D)$ in a finite region in $\mathbb{Z}^{p+1}$, by a first order Gauss Manin connection

$$\boxed{d\underline{I}(\underline{x}, \epsilon) = \mathbf{A}(\underline{x}, \epsilon)\underline{I}(\underline{x}, \epsilon)}$$

$\epsilon = (D_{cr} - D)/2.$

Master Integral Basis Change possibly to canonical form

$$\underline{I}(\underline{x},\epsilon) \to \underline{I}^{better}(\underline{z}(x);\epsilon) = R_0(\underline{z}(x);\epsilon)\underline{I}(\underline{z}(x);\epsilon)$$

$$\mathbf{A}(\underline{z};\epsilon)^{better} = [R_0(\underline{z};\epsilon)\mathbf{A} + dR_0(\underline{z};\epsilon)]R_0(\underline{z};\epsilon)^{-1}$$

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$$\mathbf{A}(\underline{z}; \epsilon)^{best} = [R_n(\underline{z}; \epsilon)\mathbf{A}^{n-1} + dR_n(\underline{z}; \epsilon)]R_n(\underline{z}; \epsilon)^{-1} = \epsilon A(\underline{z})$$

Master Integral Basis Change possibly to canonical form

$$\underline{I}(\underline{x},\epsilon) \to \underline{I}^{better}(\underline{z}(x);\epsilon) = R_0(\underline{z}(x);\epsilon)\underline{I}(\underline{z}(x);\epsilon)$$

$$\mathbf{A}(\underline{z};\epsilon)^{better} = [R_0(\underline{z};\epsilon)\mathbf{A} + dR_0(\underline{z};\epsilon)]R_0(\underline{z};\epsilon)^{-1}$$

$$\mathbf{A}(\underline{z};\epsilon)^{best} = [R_n(\underline{z};\epsilon)\mathbf{A}^{n-1} + dR_n(\underline{z};\epsilon)]R_n(\underline{z};\epsilon)^{-1} = \epsilon A(\underline{z})$$

$$\left[d_z - \epsilon \begin{pmatrix} & \mathbf{0} & & & \mathbf{0} & & & \mathbf{0} & \\ * & \dots & * & A^1_{11} & \cdots & A^1_{1r_1} & & & \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & & \mathbf{0} & \\ * & \dots & * & A^1_{r_1} & \cdots & A^1_{r_1 r_1} & & & \\ & \vdots & & & \ddots & & & & \\ * & \dots & * & & & & A^n_{11} & \cdots & A^n_{1r_1} \\ \vdots & \ddots & \vdots & & \mathbf{0} & & \vdots & \ddots & \vdots \\ * & \dots & * & & & & A^n_{r_2} & \cdots & A^n_{r_n r_n} \end{pmatrix} \right] \begin{pmatrix} I^{sub} \\ \Pi^1_1 \\ \vdots \\ \Pi^1_{r_1} \\ \vdots \\ \Pi_{n_1} \\ \vdots \\ \Pi_{n_r} \end{pmatrix}^{best} = 0$$

The blocks

Here $A_{ij}^{k}(z)$ are $d\log(\mathrm{alg}(z))$ and the $*$ are rational functions in z and we typically have a situation, where the l-loop block in this improved IBP first order flat connection above is described by period integrals in the sense of Kontsevich and Zagier fullfilling the Gauss-Manin flat connection of a geometry X, which is typically a (non-smooth) Calabi-Yau manifold.

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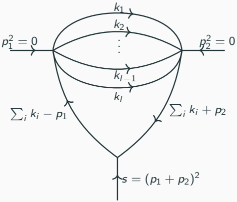
Example: From the (l+1)-loop ice-cone graph



it is clear that it contains l-loop banana graph as block(s).

Dictionary for the blocks

	$l = (n+1)$-**loop in block integrals in** D_{cr} **dimensions**	**Calabi-Yau (CY) geometry**
1	Maximal cut integrals in D_{cr} dimensions	$(n, 0)$-form periods of CY manifolds or CY motives
2	Dimensionless ratios $z_i = {m_i}^2/p^2$	Unobstructed compl. moduli of M_n, or equi'ly Kähler moduli of the mirror W_n
3	Integration-by-parts (IBP) reduction	Griffiths reduction method
4	Integrand-basis for maximal cuts of of master integrals in D_{cr}	Middle (hyper) cohomology $H^n(M_n)$ M_n
5	Complete set of differential operators annihilating a given maximal cut in D_{cr} dimensions	Homogeneous Picard-Fuchs differential ideal (PFI) / Gauss-Manin (GM) connection

Consequences of the Geometric representation

Advantages of the geometric representation of the Feynman graphs as periods of (complete intersection) Calabi-Yau manifolds

1.) The GKZ system in the yields immediately all period integrals $\underline{\Pi}$ and near the point of maximal unipotent monodromy $z_i = 0$ a canonical integral basis w.r.t. to the global monodromy $\mathcal{O}(\Sigma, \mathbb{Z})$. In particular one identifies the physical period and its analytic properties.

2.) Once the analytic continuation of $\underline{\Pi}$ to the other critical divisors in the discriminate locus is known they can be calculated to very high precision everywhere in the physical parameter space in extremely short time.

Consequences of the Geometric representation

3.) Griffith-transversality (2) implies

a.) The Inverse of the Wronskian is up rational factors linear in the periods $W^{-1} = \Sigma W^T Z^{-1}$

$$Z^{-1} = \frac{(2\pi i)^3}{C} \begin{pmatrix} 0 & \frac{C''}{C} - 2\frac{C'}{C} + \frac{c_2}{c_4} & -\frac{C'}{C} & 1 \\ 2\frac{C'}{C} - \frac{C''}{C} - \frac{c_2}{c_4} & 0 & -1 & 0 \\ \frac{C'}{C} & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

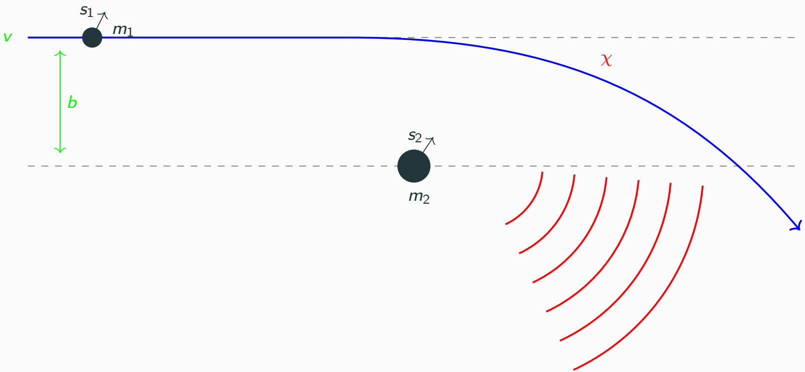
b.) The Gauss-Manin connection can be brought into a canonical form

$$\partial_{t_*^i} \begin{pmatrix} \mathcal{V}_0 \\ \mathcal{V}_j \\ \mathcal{V}^j \\ \mathcal{V}^0 \end{pmatrix} = \begin{pmatrix} 0 & \delta_{ik} & 0 & 0 \\ 0 & 0 & C_{ijk} & 0 \\ 0 & 0 & 0 & \delta_i^j \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{V}_0 \\ \mathcal{V}_k \\ \mathcal{V}^k \\ \mathcal{V}^0 \end{pmatrix} .$$

4.) a.) Implies that that in the "variation of constant" procedure the inhomogeneous solution is an iterated integral of the periods $\partial_n^k \Pi$ modulo rational functions. b.) implies that the higher terms in ϵ can be similar written as iterated integrals.

Worldline Quantum Field Theory approach to General Relativity

Scattering of two black holes (BH) as starter to the description of BH mergers as the main sources for gravitational waves detected at LIGO, . . .



Worldline Quantum Field Theory approach to General Relativity

The action for the scattering process

$$S = -\sum_{i=1}^{2} m_i \int \mathrm{d}\tau \left[\frac{1}{2} g_{\mu\nu} \dot{x}_i^{\mu} \dot{x}_i^{\nu} \right] + S_{\mathrm{EH}}$$

is expanded in Post Minkowskian (PM) approximation in the Worldline Quantum Field Theory (WQFT) approach around the non-interacting background configurations

$$x_i^{\mu} = b_i^{\mu} + v_i^{\mu} \tau + z_i^{\mu}(\tau) \,, \quad g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{32\pi G}\, h_{\mu\nu}(x) \ .$$

Worldline Quantum Field Theory approach to General Relativity

The goal is to calculate from the initial data: the impact parameter $b^\mu = b_1^\mu - b_2^\mu$ and the incoming velocities v_1, v_2 the physical quantity of interest, which is the radiation induces change in the momentum say $\Delta p_1^\mu = m_1 \int \mathrm{d}\tau \ddot{x}(\tau)$ of the first particle.

In the PM approximation the latter can be expanded in the gravitational coupling G

$$\Delta p_1^\mu = \sum_{n=1}^{\infty} G^n \Delta p^{(n)\,\mu}(x) \ .$$

At each order the contributions $\Delta p^{(n)\,\mu}(x)$ are calculated in the WQFT approach in the Swinger-Keldysh in-in formalism in terms of a Feynman graph expansion with retarded propagators. Here $x = \gamma - \sqrt{\gamma^2 - 1}$ with γ the Lorentz factor of the relative velocities is the only parameter.

Worldline Quantum Field Theory approach to General Relativity

In the 4PM approximation the Feynman integral in the 1SF sector

(K3.0)

involve bilinear of elliptic function which are periods of the $K3$

$$Y^2 = X(X-1)(X-x)Z(Z-1)(Z-1/x)\,.$$

In the 5PM approximation we find in [8] that in the 5PM approximation the following graphs in the 1SF sector

(CY3)

Worldline Quantum Field Theory approach to General Relativity

The corresponding smooth CY three-fold one-parameter complex family $x = (2\psi)^{-8}$, can be defined as resolution of four symmetric quadrics

$$x_j^2 + y_j^2 - 2\psi x_{j+1} y_{j+1} = 0,\ j \in \mathbb{Z}/4\mathbb{Z}$$

in the homogeneous coordinates x_i, y_j, $j = 0, \ldots, 3$ of $\mathbb{P}^7$. The periods of the above $K3$ and CY threefold determine all special functions that are necessary to solve for $\Delta p^{(5)\,\mu}(x)$ in the 1SF sector.

In the 5PM 2SF further different CY and K3 appear.

N=4 Super-Yang-Mills and integrablity

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Integrable Deformations: Marginal β deformations Leigh, Strassler (95) Maldacena Luni (05). Here most relevant the supersymmetry breaking γ_i, $i = 1, 2, 3$ deformations in the double scaling limit $g \to 0$, $\gamma_3 \to i\infty$ with $\xi^2 = g^2 N_c e^{-i\gamma_3}$ fixed Gürdoğan, Kazakov (16), with Caetano (18) and the bi-scalar model χFT Kazakov, Olivucci (18) leading to holographic dual pairs of integrable fishnet and fishchain theories in D dimensions.

Orginal Fishnet Lagrangians

These bi-"scalar" fishnet theories in D dimensions have a Lagrangian with quartic interaction $V = 4$

$$\mathcal{L}^{\omega D}_{\mathrm{quad}} = N_{\mathrm{c}} \mathrm{tr} \big[-X(-\partial_\mu \partial^\mu)^\omega \bar{X} - Z(-\partial_\mu \partial^\mu)^{\frac{D}{2}-\omega} \bar{Z} + \xi^2 X Z \bar{X} \bar{Z} \big] \ .$$

ω determines the propagator power in the Feynman graphs. E.g. $D = 4$, $\omega = 1$ and $D = 2$, $\omega = 1/2$ are conformal choices.

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Hexagonal Fishnets Lagrangian

A generalization with analogous symmetry properties are Fishnet theories with cubic interaction $V = 3$ Kazakov, Olivucci (23) and Lagrangian

$$\begin{aligned}\mathcal{L}^{D}_{\mathrm{cub}} = \quad & N_{\mathrm{c}}\mathrm{tr}\big[-X(-\partial_\mu\partial^\mu)^{\omega_1}\bar{X} - Y(-\partial_\mu\partial^\mu)^{\omega_2}\bar{Y} - Z(-\partial_\mu\partial^\mu)^{\omega_3}\bar{Z} \\ & +\xi_1^2\bar{X}YZ + \xi_2^2 X\bar{Y}\bar{Z}\big]\,,\end{aligned}$$

with $\sum_{i=1}^{V}\omega_i = D$ at vertex, e.g. $D = 2$ and $\omega_1 = \omega_2 = \omega_3 = 2/3$.

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Then the (planar) fishnet graphs can be cut by a closed oriented curve from the three regular tilings of the plane:

Regular tilings and Calabi-Yau motives

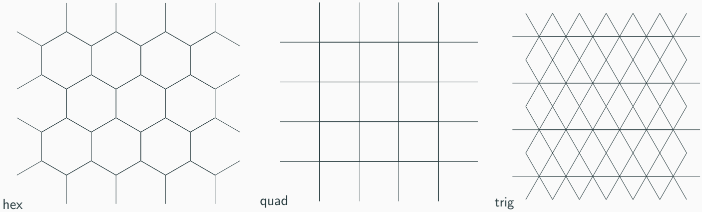

Figure 1: The three regular tilings of the plan with vertices of valence $\nu = 3, 4, 6$ respectively.

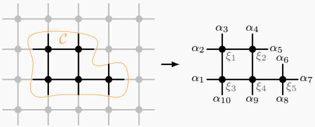

Figure 2: Ten-point five-loop fishnet integral cut out of a square tiling of the plane.

Regular tilings and Calabi-Yau motives

To obtain a graph G consider a convex closed oriented curve $\mathcal{C}$ that cuts edges of the tiling and does not pass to vertices. To each vertex inside the curve $\mathcal{C}$ we associate a $\mathbb{P}^1$ with homogeneous coordinates $[x_i : u_i]$, $i = 1, \ldots, l$ over which we want to integrate with the measure

$$\mathrm{d}\mu_i = u_i \mathrm{d}x_i - x_i \mathrm{d}u_i \ . \tag{5}$$

To the end point of each cut edge outside $\mathcal{C}$ we associate a parameter $a_j \in \mathbb{C}$, $j = 1, \ldots, r$. The graph is constructed by the l vertices with propagators

$$P^I_{ij} = \frac{1}{(x_i - x_j)^{w_{ij}}}, \qquad P^E_{ij} = \frac{1}{(x_i - a_j)^{w_{ij}}} \ . \tag{6}$$

To be conformal in D dimension the weights of propagators incident to each vertex V_i has to fullfill

$$\sum w_{ij} = D \tag{7}$$

Regular tilings and Calabi-Yau motives

We deal mainly with $D = 2$ and choose the propagator weights all equal $w_{ij} = w = 2/\nu(V)$, where $\nu(V)$ is the valence of the vertices, i.e. for the hexagonal tiling we have $w = \frac{2}{3}$, for the quartic tiling $w = \frac{1}{4}$ amd for the trigonal tiling $w = \frac{1}{3}$.

To the hexagonal and the quartic lattice we can associate an in general singular l-dimensional Calabi-Yau variety M_l as the $d = 3$ or $d = 2$ fold cover

$$W = \frac{y^d}{d} - P([\underline{x} : \underline{u}]; \underline{a}) = 0 \tag{8}$$

over the base $B = (\mathbb{P}^1)^l$ branched at

$$P([\underline{x} : \underline{w}]; \underline{a}) = \prod_{ij}(u_j x_i - x_j u_i) \prod_{ij}(x_i - a_j u_i) = 0 \ , \tag{9}$$

respectively. The orders of the covering automorpishm exchanging the sheets will play a crucial role in the following geometric analysis.

Regular tilings and Calabi-Yau motives

Note that (8) defines a Calabi-Yau manifold, because the canonical class of the base is with H_i the hyperplane class of the i'th $\mathbb{P}^1$ given by

$$K_B = 2 \bigoplus_{i=1} H_i, \tag{10}$$

and the Calabi-Yau condition ensuring $K_{M_l} = 0$

$$\frac{d}{d-1} K_B = [P([\underline{x} : \underline{u}]; \underline{a})] = \nu \bigoplus_{i=1} H_i \tag{11}$$

is true with $d = 3, 2$ as $\nu = 3, 4$ for graphs from the hexagonal and the quartic tiling, respectively.

Regular tilings and Calabi-Yau motives

Another way of stating this is that the periods over the unique holomorphic (ℓ,0)-form, given by the Griffiths residuum form Ω

$$\Pi_G = \int_C \Omega = \int_C \frac{1}{2\pi i} \oint_\gamma \frac{dy \prod_{i=1}^{l} d\mu_i}{W} = \int_C \frac{\prod_{i=1}^{l} d\mu_i}{\partial_y W} = \int_C \frac{\prod_{i=1}^{l} d\mu_i}{P^{\frac{d-1}{d}}} = \int_C \prod_{ij} P^I_{ij} \prod_{ij} P^E_{ij} \prod_{i=1}^{l} d\mu_i \,, \tag{12}$$

are well defined. The significance for the application is that these period integrals over cycles $C \in H_l(M_l, \mathbb{Z})$ are building blocks for the amplitudes.

$$I_G = \int_C \Omega = \int \sqrt{\left| \prod_{ij} P^I_{ij} \prod_{ij} P^E_{ij} \right|^2} \prod_{i=1}^{l} d\mu_i \wedge d\bar{\mu}_i \,, \tag{13}$$

Regular tilings and Calabi-Yau motives

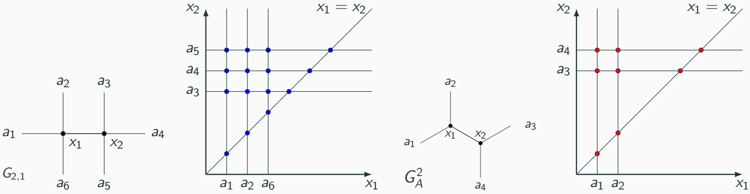

Figure 3: Singularities of the $K3$ denoted for the valence 4 graph $M_{G_{1,2}}$ and the valence 3 graph $M_{G_A^2}$. Note that 3 of the a_i can be set to $0, 1, \infty$ by a diagonal $PSL(2, \mathbb{C})$ acting on the projective plane in which the a_i lie

Regular tilings and Calabi-Yau motives

Claim 1: To each graph G we can associate a Calabi-Yau variety X whose periods determine I.

Regular tilings and Calabi-Yau motives

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Claim 2: Each l gives rise to a Calabi-Yau motive with integer symmetry (l even) or antisymmetric (l odd) intersection form Σ, a point of maximal unipotent monodromy and a period vector $\Pi(\underline{z}) = \int_{\Gamma_i} \Omega$ with $\Gamma_i \in H_l(W^{(m,n)}, \mathbb{Z})$. The Feynman amplitude is given near the Mum points by the quantum volume of the mirror

$$I = i^{l^2} \Pi^\dagger \Sigma \Pi = e^{-K(\underline{z},\bar{\underline{z}})} = Vol_q(M^{(m,n)})$$

and globally by analytic continuation of the periods. Here $M^{(m,n)}$ is the mirror of $W^{(m,n)}$.

Regular tilings and Calabi-Yau motives

Claim 3: There exist an integrable conformal fishnet theories (CFNT) developed first (Gürdogan, Kazakov 2015) as deformation of $N = 4$ $SU(N_c)$ SYM theory. Let X, Z be $SU(N_c)$ matrix fields then the Lagrangian is

$$\mathcal{L}_{FN} = N_c \mathrm{tr}\left(-\partial_\mu X \partial^m u \bar{X} - \partial_\mu Z \partial^m u \bar{Z} + \xi^2 X Z \bar{X} \bar{Z}\right)$$

Each $I_{m,n}$ integral is an amplitude in the CFNT, i.e. $I_{m,n}(\underline{z})$ has to be single valued i.e. a Bloch Wigner dilogarithm or in the $D = 2$ case e^{-K}.

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The factorisation of the amplitudes of the integrable system subject to the Yang-Baxter relations imply many non-trivial relations for he periods of the $W^{(m,n)}$. E.g. we the one parameter specialisation the periods of $W^{(n,m)}$ are $(m \times m)$ minors of the periods $W_l^{(1,m+m)}$ etc.

Regular tilings and Calabi-Yau motives

Claim 4: ($Y(SO(3,1)) = Y(Sl(2,\mathbb{R})) \oplus \overline{Y(Sl(2,\mathbb{R}))}$.) The holomorphic Yangian generated by the algebra

$$\begin{aligned} P_j^\mu &= -i\partial_{a_j}^\mu, & K_j^\mu &= -2ia_j^\mu(a_j^\nu\partial_{a_j,\nu} + \Delta_j) + ia_j^2\partial_{a_j}^\mu \\ L_j^{\mu\nu} &= i(a_j^\mu\partial_{a_j}^\nu - a_j^\nu\partial_{a_j}^\mu), & D_j &= -i(a_j^\mu\partial_{a_j,\mu}), \end{aligned}$$

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in differentials w.r.t. to the external position, generates together with the permutation symmetries of the latter a differential ideal that annihilates the $I(\underline{z})$ and is *equivalent* to the Picard-Fuchs differential ideal that describes the variation of the Hodge structure in the middle cohomology of X and annihilated the periods of Ω.

Regular tilings and Calabi-Yau motives

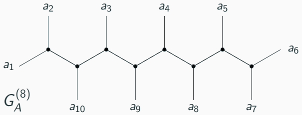


Figure 4: The $G_A^{(8)}$ graph. The A series starts from even dimensional Calabi-Yau spaces

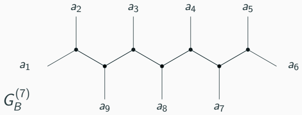


Figure 5: The $G_B^{(7)}$ graph. The B series starts from odd dimensional Calabi-Yau spaces

Regular tilings and Calabi-Yau motives

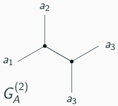
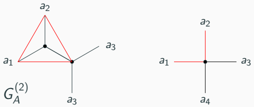


Figure 6: The $G_A^{(2)}$ graph and its transformation to a genus 2 Picard curve

$$y^3 = (x - a_1)(x - a_2)(x - a_3)^2(x - a_4)^2$$

Regular tilings and Calabi-Yau motives



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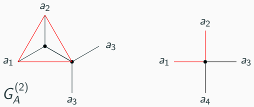

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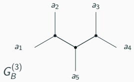


Figure 7: The $G_B^{(3)}$ graph and its transformation to a genus Picard curve

$$y^3 = (x - a_1)(x - a_2)(x - a_3)(x - a_4)(x - a_5)^2$$

Regular tilings and Calabi-Yau motives

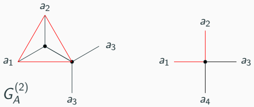


Figure 6: The $G_A^{(2)}$ graph and its transformation to a genus 2 Picard curve

$$y^3 = (x - a_1)(x - a_2)(x - a_3)^2(x - a_4)^2$$

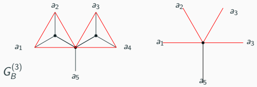
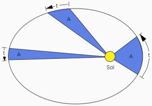
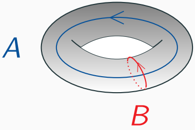


Figure 7: The $G_B^{(3)}$ graph and its transformation to a genus Picard curve

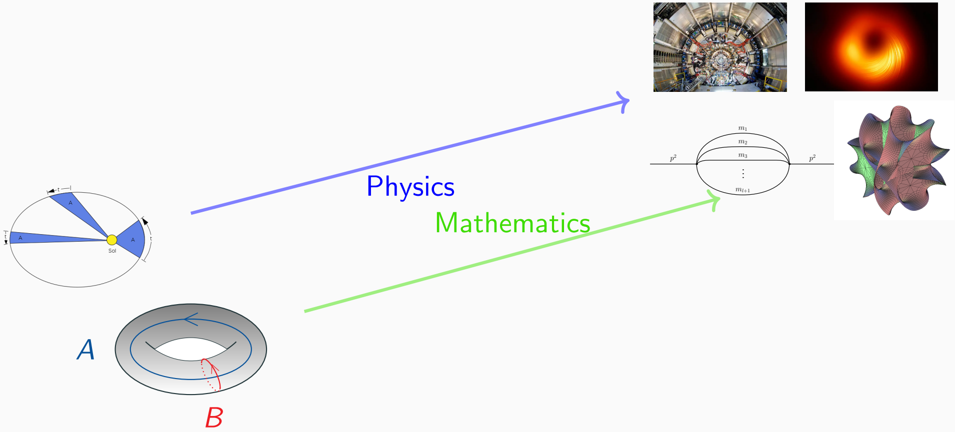
$$y^3 = (x - a_1)(x - a_2)(x - a_3)(x - a_4)(x - a_5)^2$$

Conclusion and Outlook





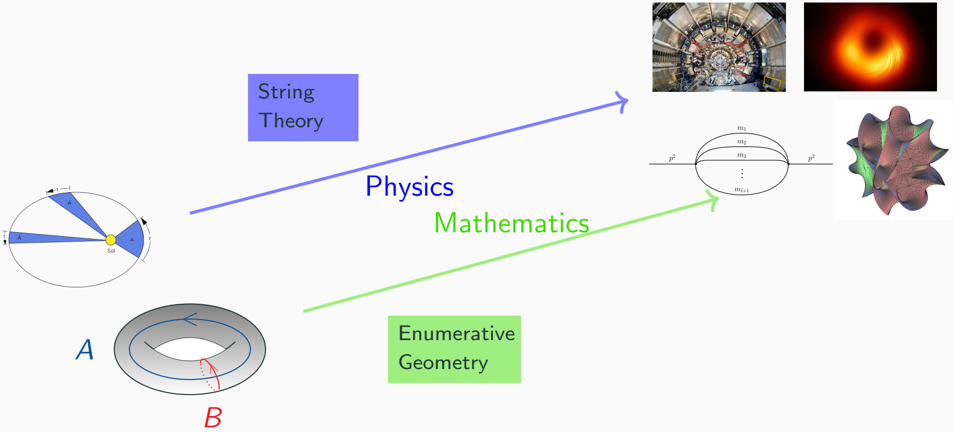
Conclusion and Outlook



Physics

Mathematics

Conclusion and Outlook



String Theory

Physics

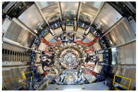
Mathematics

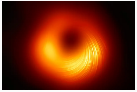
Enumerative Geometry

Conclusion and Outlook

String Theory

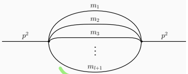
Black Hole Physics

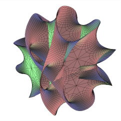


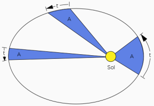


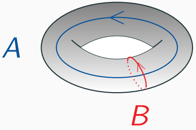
Physics

Mathematics









Enumerative Geometry

Arithmetic Geometry