Three-loop Four-gluon Amplitudes in YM

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1710.10208 R. Boels & HL

1802.06761 R. Boels, Q. Jin & HL

1910.05889 Q. Jin & HL

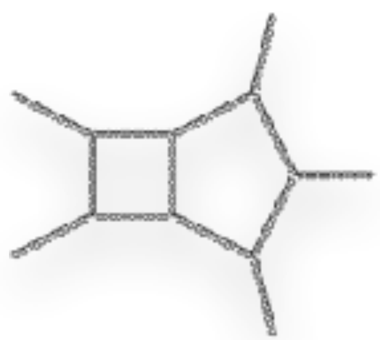
17/10/2019 @USTC, Hefei

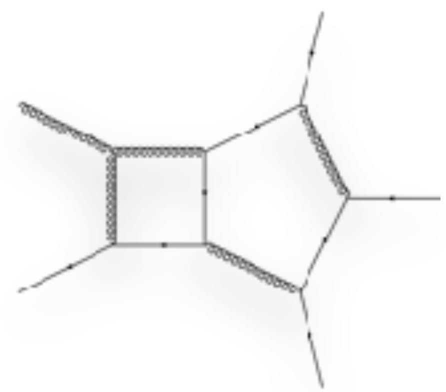
Theoretical Precision Calculations

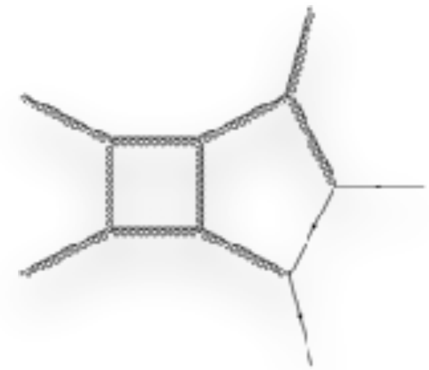
- LHC is undergoing a major update to HL-LHC in the 2020s
- Precision experimental measurements v.s. theoretical predictions
  Physics Signal = Precision Exp. - Precision Theo.
- Maybe significant anomalies and insights to physics effects from higher energy scales
- Focus on perturbative computations of amplitudes in massless QCD, expand w.r.t. the coupling constant

Interesting and Useful Objects

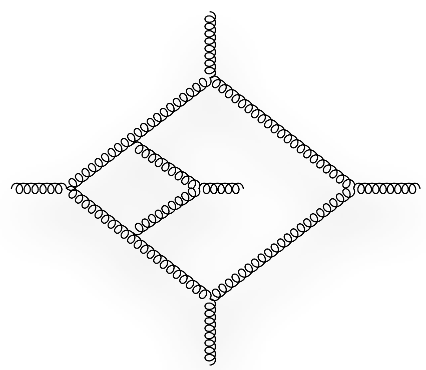
High multiplicity multi-loop amplitudes (CP even), eg.

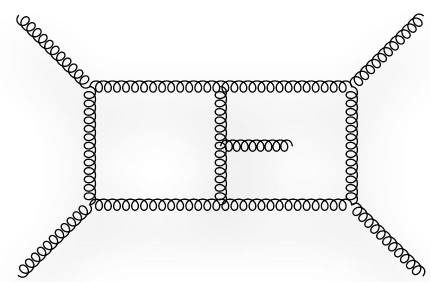




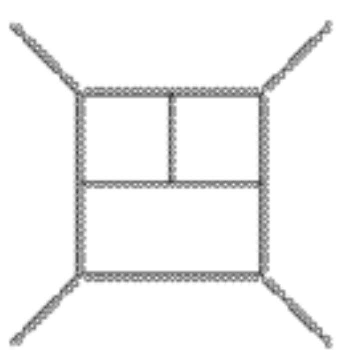


[Badger et al., 13’, 15’, 16’, 17’, 18’;
Abreu et al., 17’*2, 18’ *2, 19’*2 ;
Gehrmann et al., 15’, 18’;
Dunbar et al. 16;
Papadopoulos et al. 15’;
Boels et al., 18’;
Chawdhry et al., 18’]





[Badger et al., 15’,19’;
Abreu et al., 18’, 19’ ;
Chicherin et al., 18’*3, 19’*2]



[Vogt et al., 04’;
Gehrmann et al., 10’;
Almeid et al., 15’;
Henn et al., 13’, 16’;
Jin and HL, 19’]

Difficulties in High-loop & -multiplicity

- Eg. Pure Yang-Mills (main examples in this talk)

Spin info., momentum, internal group info. ↑   *Rational func. of scales* ↑

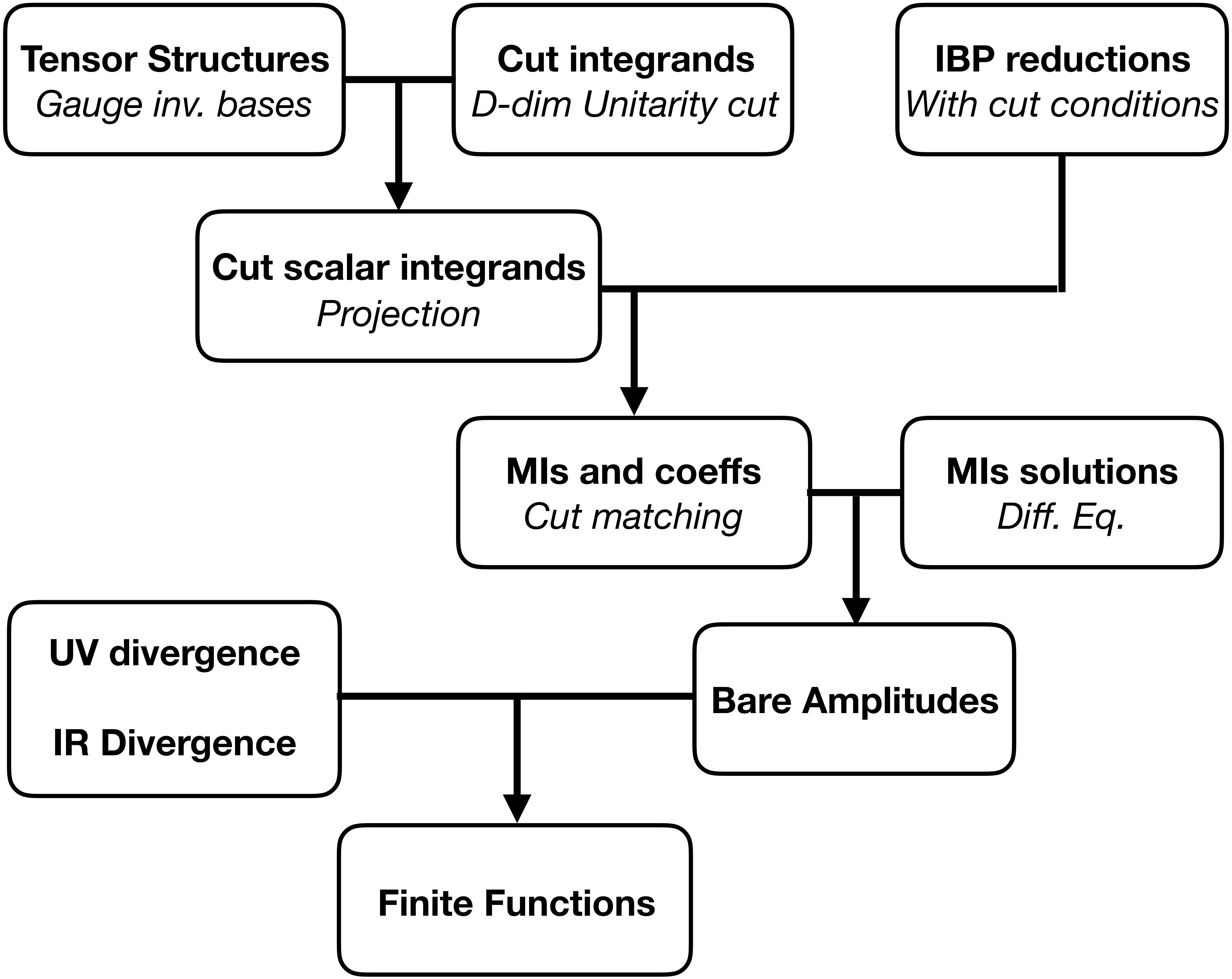
$$\mathcal{A}^L(\{\xi, p, a\}) = \sum \mathcal{C}(\{a_1, \cdots, c_N\}) f(\{p_i \cdot p_j\}) \mathcal{I}^L(\{\xi_1, p_1\}, \cdots, \{\xi_N, p_N\})$$

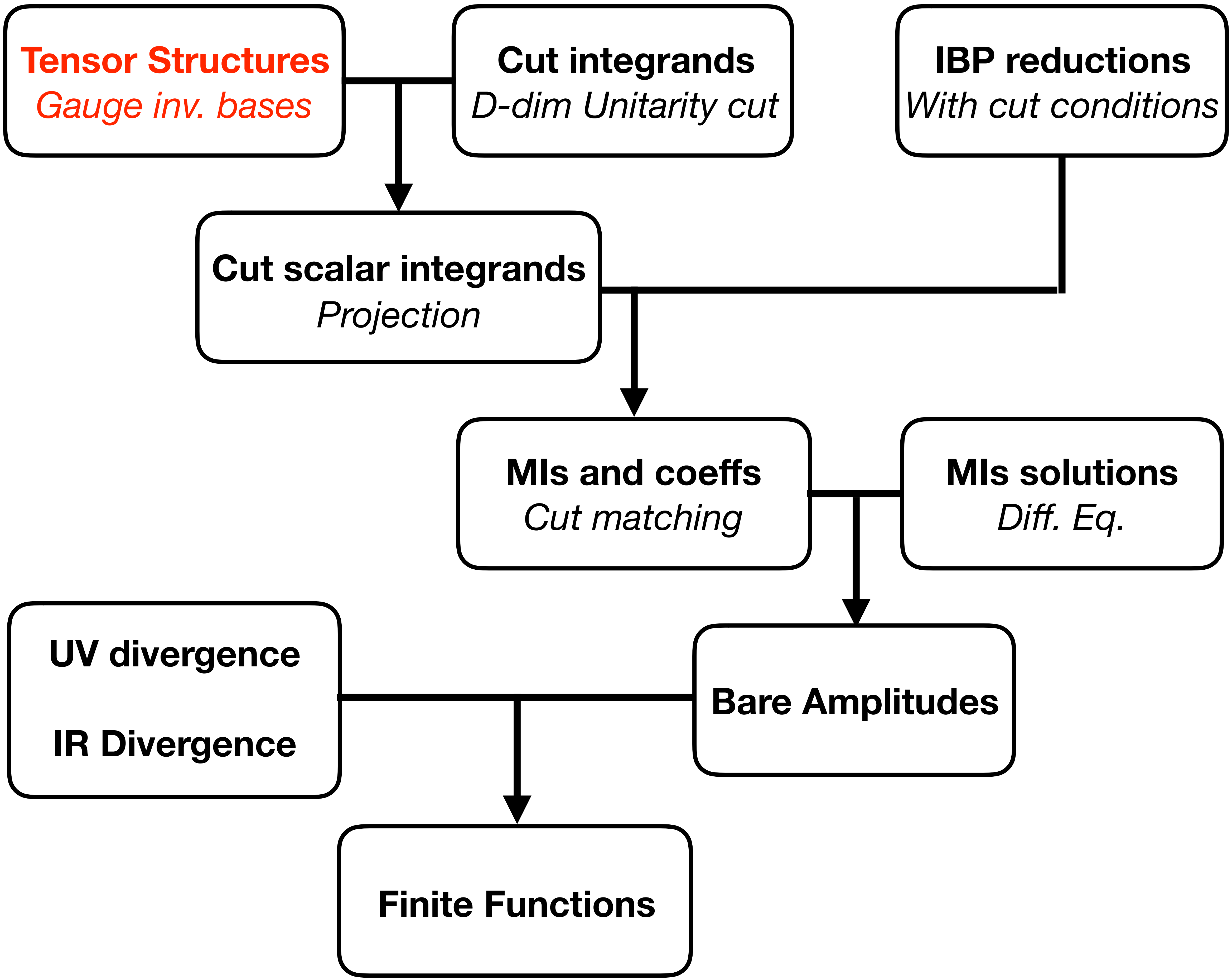
↓ *color structures*   ↓ *Integration of tensor integrands*

$$\mathcal{I}^L(\{\xi_1, p_1\}, \cdots, \{\xi_N, p_N\}) = \int d^D l_1 \cdots d^D l_L \frac{\mathcal{N}(\xi\cdot\xi, \xi\cdot p, \xi\cdot l, p\cdot l, l^2)}{D_1 D_2 \cdots D_M}$$

$$D_i = (\theta(l_1) l_1 + \cdots + \theta(l_L) l_L + p_m + \cdots + p_n)^2, \qquad \theta(l_i) = 0 \text{ or } 1$$

- Higher-multiplicity and higher-loop:
  - Independent scales (only high multiplicity)
  - Color structures
  - Higher numerator power
  - Integrand reductions
  - Evaluate master integrals
  - Integrand topologies
  - Kinematic tensor structures
  - Integral reductions
  - etc.


Tensor Structures
Gauge inv. bases
Cut integrands
D-dim Unitarity cut
IBP reductions
With cut conditions
Cut scalar integrands
Projection
MIs and coeffs
Cut matching
MIs solutions
Diff. Eq.
UV divergence
IR Divergence
Bare Amplitudes
Finite Functions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Physical Properties of Amplitudes

A color stripped scattering amplitude

$$\mathcal{A}_n(\{p_i,\lambda_i\}) = \overline{v}_{\dot{a}_1}(p_{f_1})u_{a_1}(p_{f_{M+1}})\cdots\overline{v}_{\dot{a}_M}(p_{f_M})u_{a_M}(p_{f_{2M}})\,\xi_1^{\mu_1}\cdots\xi_N^{\mu_N}\,\hat{A}(\{\eta_{\mu\nu},\gamma_\rho,p_{k,\mu}\})$$

- A Lorentz scalar and multilinear of spin variables, satisfying
  - Momentum conservation $\sum_i p_i^\mu = 0$
  - Dirac eq.
    $$(\not{p}-m)u(p) = \bar{u}(p)(\not{p}-m) = (\not{k}+m)v(k) = \bar{v}(k)(\not{k}+m) = 0$$
  - Transversality of polarization vector $p_i^\mu \xi_{i,\mu} = 0$
  - On-shell gauge invariance $\mathcal{A}_n(\xi_i \to p_i) = 0$
  - Branch cuts and poles

Amplitude Decomposition

Color stripped amplitude decomposes into

$$\mathcal{A}_n = \sum \alpha_i(\{p,l\}) B_i(\{p,\lambda\})$$

- A linear combination of a group of kinematic basis B_i

- Kinematic basis: external particle informations, multilinear of spins satisfying all physical properties except unitarity, but with locality

$$B_i(\{p,\lambda\}) = \overline{v}_{\dot{a}_1}(p_{f_1}) u_{a_1}(p_{f_{M+1}}) \cdots \overline{v}_{\dot{a}_M}(p_{f_M}) u_{a_M}(p_{f_{2M}})\, \xi_1^{\mu_1} \cdots \xi_N^{\mu_N} f_B(\{\eta_{\mu\nu}, \gamma_\rho, p_k\})$$

- Coefficients of kinematic basis:

$$\alpha_i(\{p,l\}) = \sum f_\alpha(\{p_j \cdot p_k\}) \int (d^D l)^L\, I(\{l\cdot l, p\cdot l\})$$

[Glover et al. , 03’, 04’, 12’; Boels et al., 16’; Arkarni-Hamed et al., 16’; Bern et al., 17’]

Kinematic Basis Construction

Brute-force construction by solving physical constraints $\mathcal{A}_n = \sum \alpha_i B_i$

[R. Boels & R. Medina, 16'; R. Boels & HL, 17']

- Application: up to 6-pt tree and 4-pt 2-loop pure-YM amplitudes

- Shortcomings: complicated for (>=) 5-pt, ie. $P_{ij} = \sum_{\text{spin}} B_i B_j$

Projector

$$\sum_s u(p)\bar{u}(p) = \not{p} + m, \qquad \sum_s v(k)\bar{v}(k) = \not{k} - m$$

$$\sum_{\text{helicities}} \xi_\mu \xi_\nu = \eta_{\mu\nu} - \left(\frac{p_\mu q_\nu + p_\nu q_\mu}{q \cdot p}\right) \qquad \sum_{\text{helicities}} \xi \cdot \xi = d - 2$$

eg. 5 gluons {142, 142}
2 fermions + 3 gluons {144, 144}
6 gluon {2364, 2364} full matrix, impossible to inverse

- This construction way is kind of arbitrary, linear combinations of bases are still on-shell gauge invariant kinematic bases

Kinematic Basis Construction for Pure-YM

[R. Boels, Q. Jin and HL,18']

"Canonical" kinematic basis construction

- A-type building block: $A_i(j,k) = (p_k \cdot p_i)\, p_j \cdot \xi_i - (p_j \cdot p_i)\, p_k \cdot \xi_i$

$$\{A_i(j) = A_i(i+j, i+j+1) | j \in \{1, \ldots, n-3\}\}$$

  - Solutions for 1 gluon (n-1) scalar scattering [R. Boels and HL,17']
  - For n-gluon scattering, n copies A form a basis

- C-type building block: $C_{i,j} = (\xi_i \cdot \xi_j)(p_i \cdot p_j) - (p_i \cdot \xi_j)(p_j \cdot \xi_i)$

  - One solution for 2-gluon (n-2)-scalar (Another from 2-copies of A-type building blocks) [R. Boels and HL,17']
  - Proportional to two contracted linearized field strength tensor

$$F_{\mu\nu}(\xi_1) F^{\mu\nu}(\xi_2)$$

A &C-type building blocks: on-shell gauge invariant

Kinematic Basis Construction for Pure-YM

[R. Boels, Q. Jin and HL,18']

"Canonical" kinematic basis construction

- D-type building block: $D_{i,j} = C_{i,j} - \sum_{k,l=1}^{n-3} X_{ij}(k,l) A_i(k) A_j(l)$

Require orthogonality $\sum_{h_i} A_i(k) D_{i,j} = 0 = \sum_{h_j} A_j(k) D_{i,j}, \ \forall k$

Fix the constructions with $P_i^A(k,l) = \sum_{h_i} A_i(k) A_i(l)$

$$A^i(k) \equiv \sum_l (P_i^A)^{-1}(k,l) A_i(l) \qquad A^i(k) A_i(l) \equiv \sum_{\text{helicities},i} A^i(k) A_i(l) = \delta(k,l)$$

$$D_{i,j} = C_{i,j} - \sum_{k,l=1}^{n-3} A_i(k) A_j(l) \left(A^m(k) A^n(l) C_{m,n} \right)$$

$$\sum_{\text{helicities}} D_{i,j} D_{i,j} = (p_i \cdot p_j)^2 (d - n + 1) \qquad \sum_{\text{helicities},i} D_{i,j} D_{i,k} = \frac{(p_i \cdot p_j)(p_i \cdot p_k)}{(p_j \cdot p_k)} D_{j,k}$$

Four gluon kinematic basis

[Used in Q. Jin and HL, 19']

"Canonical" kinematic basis: 10 in total

- Expressed in terms of A and C:

$$
\begin{aligned}
B_1 &= A_1A_2A_3A_4, & B_2 &= C_{13}C_{24}, \\
B_3 &= C_{13}A_2A_4, & B_4 &= C_{24}A_1A_3, \\
B_5 &= C_{12}C_{34}, & B_6 &= C_{23}C_{14}, \\
B_7 &= C_{12}A_3A_4, & B_8 &= C_{23}A_4A_1, \\
B_9 &= C_{34}A_1A_2, & B_{10} &= C_{41}A_2A_3,
\end{aligned}
$$

$$A_i(j,k) = (p_k \cdot p_i)\, p_j \cdot \xi_i - (p_j \cdot p_i)\, p_k \cdot \xi_i$$

$$\{A_i(j) = A_i(i+j, i+j+1) | j = 1\}$$

$$C_{i,j} = (\xi_i \cdot \xi_j)(p_i \cdot p_j) - (p_i \cdot \xi_j)(p_j \cdot \xi_i)$$

- Under cyclic permutation:

$$p_i \to p_{i+1}$$

$$
\begin{aligned}
&B_1 \to B_1,\ B_2 \to B_2, \\
&B_3 \leftrightarrow B_4,\ B_5 \leftrightarrow B_6, \\
&B_7 \to B_8,\ B_8 \to B_9,\ B_9 \to B_{10},\ B_{10} \to B_7
\end{aligned}
$$

- Inner product with rank 10 in D-dim and rank 8 in 4-dim, eg.

$$
\begin{pmatrix}
\frac{1}{256}s^4t^4u^4 & \frac{1}{64}s^2t^2u^4 & \frac{1}{128}s^3t^3u^4 & \frac{1}{128}s^3t^3u^4 & \frac{1}{64}s^4t^2u^2 & \frac{1}{64}s^2t^4u^2 & -\frac{1}{128}s^4t^3u^3 & -\frac{1}{128}s^3t^4u^3 & -\frac{1}{128}s^4t^3u^3 & -\frac{1}{128}s^3t^4u^3 \\
\frac{1}{64}s^2t^2u^4 & \frac{1}{16}(-2+D)^2u^4 & \frac{1}{32}(-2+D)stu^4 & \frac{1}{32}(-2+D)stu^4 & \frac{1}{16}(-2+D)s^2u^2 & \frac{1}{16}(-2+D)t^2u^2 & -\frac{1}{32}s^2tu^3 & -\frac{1}{32}st^2u^3 & -\frac{1}{32}s^2tu^3 & -\frac{1}{32}st^2u^3 \\
\frac{1}{128}s^3t^3u^4 & \frac{1}{32}(-2+D)stu^4 & \frac{1}{64}(-2+D)s^2t^2u^4 & \frac{1}{64}s^2t^2u^4 & \frac{1}{32}s^3tu^2 & \frac{1}{32}st^3u^2 & -\frac{1}{64}s^3t^2u^3 & -\frac{1}{64}s^2t^3u^3 & -\frac{1}{64}s^3t^2u^3 & -\frac{1}{64}s^2t^3u^3 \\
\frac{1}{128}s^3t^3u^4 & \frac{1}{32}(-2+D)stu^4 & \frac{1}{64}s^2t^2u^4 & \frac{1}{64}(-2+D)s^2t^2u^4 & \frac{1}{32}s^3tu^2 & \frac{1}{32}st^3u^2 & -\frac{1}{64}s^3t^2u^3 & -\frac{1}{64}s^2t^3u^3 & -\frac{1}{64}s^3t^2u^3 & -\frac{1}{64}s^2t^3u^3 \\
\frac{1}{64}s^4t^2u^2 & \frac{1}{16}(-2+D)s^2u^2 & \frac{1}{32}s^3tu^2 & \frac{1}{32}s^3tu^2 & \frac{1}{16}(-2+D)^2s^4 & \frac{1}{16}(-2+D)s^2t^2 & -\frac{1}{32}(-2+D)s^4tu & -\frac{1}{32}s^3t^2u & -\frac{1}{32}(-2+D)s^4tu & -\frac{1}{32}s^3t^2u \\
\frac{1}{64}s^2t^4u^2 & \frac{1}{16}(-2+D)t^2u^2 & \frac{1}{32}st^3u^2 & \frac{1}{32}st^3u^2 & \frac{1}{16}(-2+D)s^2t^2 & \frac{1}{16}(-2+D)^2t^4 & -\frac{1}{32}s^2t^3u & -\frac{1}{32}(-2+D)st^4u & -\frac{1}{32}s^2t^3u & -\frac{1}{32}(-2+D)st^4u \\
-\frac{1}{128}s^4t^3u^3 & -\frac{1}{32}s^2tu^3 & -\frac{1}{64}s^3t^2u^3 & -\frac{1}{64}s^3t^2u^3 & -\frac{1}{32}(-2+D)s^4tu & -\frac{1}{32}s^2t^3u & \frac{1}{64}(-2+D)s^4t^2u^2 & \frac{1}{64}s^3t^3u^2 & \frac{1}{64}s^4t^2u^2 & \frac{1}{64}s^3t^3u^2 \\
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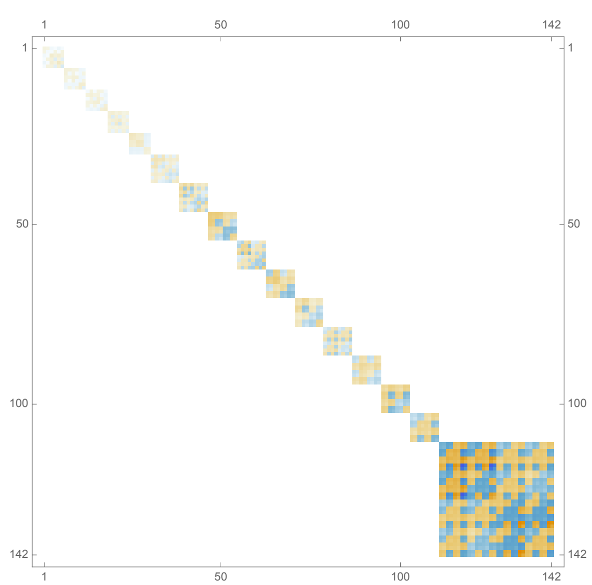
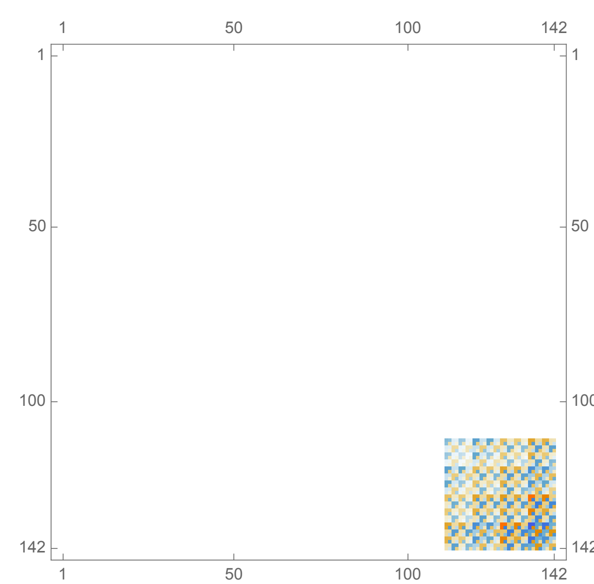
\end{pmatrix}
$$

Five gluon kinematic basis

[R. Boels, Q. Jin and HL,18']

- "Canonical" kinematic basis: 142 in total $D_{i,j} = C_{i,j} - \sum_{k,l=1}^{n-3} A_i(k)A_j(l)\left(A^m(k)A^n(l)C_{m,n}\right)$
  - 1 A + 2 D s: in total $5 \times 2 \times C_4^2/2! = 30$, eg. $A_1(1)D_{2,3}D_{4,5}$
  - 3 A s + 1 D: in total $2^3 \times C_5^2 = 80$, eg. $A_1(1)A_2(1)A_3(1)D_{4,5}$
  - 5 A s: in total $2^5 = 32$, eg. $A_1(1)A_2(1)A_3(1)A_4(1)A_5(1)$
- Inner product matrix





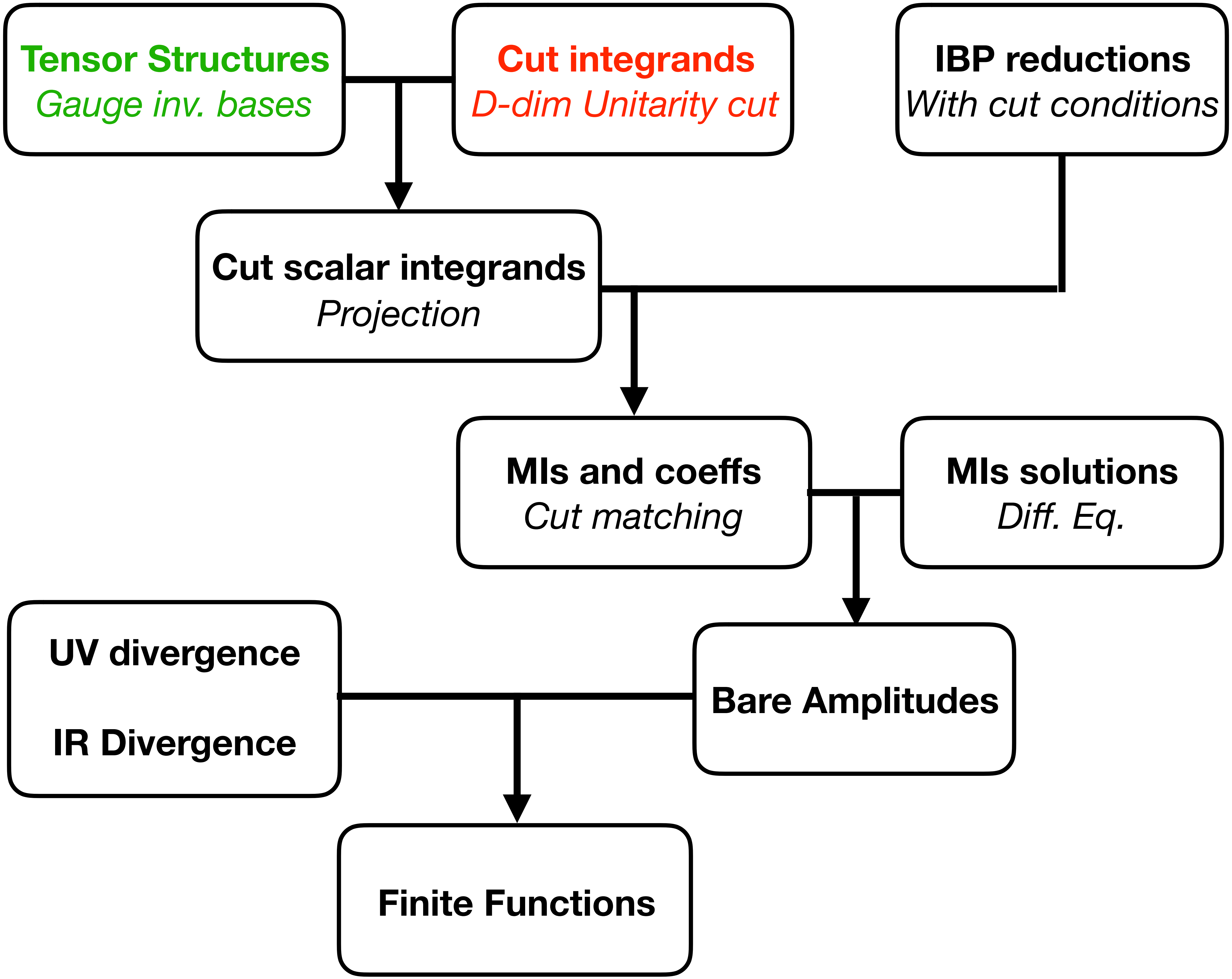
$$\sum_{h_i} A_i(k)D_{i,j} = 0 = \sum_{h_j} A_j(k)D_{i,j}, \ \forall k$$

$$\sum_{\text{helicities},i} D_{i,j}D_{i,k} = \frac{(p_i \cdot p_j)(p_i \cdot p_k)}{(p_j \cdot p_k)} D_{j,k}$$

$$\sum_{\text{helicities}} D_{i,j}D_{i,j} = (p_i \cdot p_j)^2(d-n+1)$$

$$\{d=4, n=5\} \quad \Rightarrow \sum_{\text{helicities}} D_{i,j}D_{i,j} = 0$$

- Could be used for 5 gluon any loop planar/non-planar amplitude decomposition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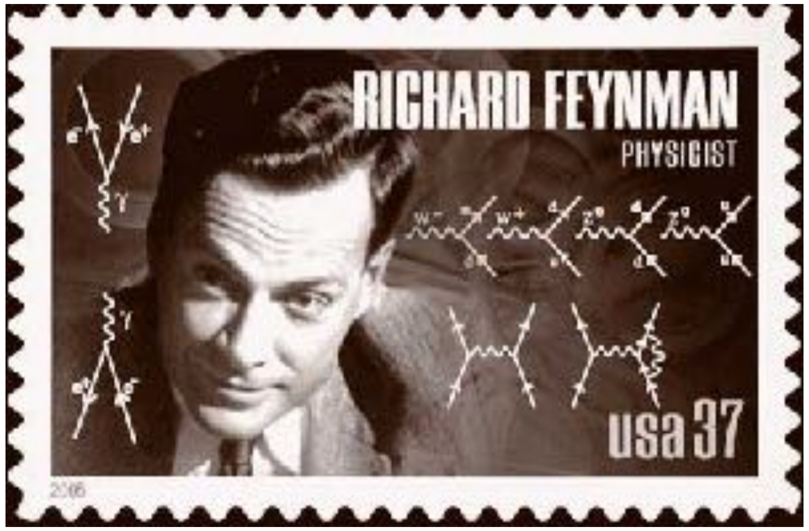
Traditional Methods: Feynman diagrams

√ Automation by Programs:
e.g. FeynTools, Qgraf, etc

√ Start from Lagrangian
Reflect the interactions intuitively



√ Already quite successful:
eg. g-2 up to 4 loops
[Laporta, 17']

X Huge number of diagrams

X Each diagram is **NOT** gauge invariant

X Significant cancellations while summing all diagrams

X Results **gauge invariant** and often **very simple**

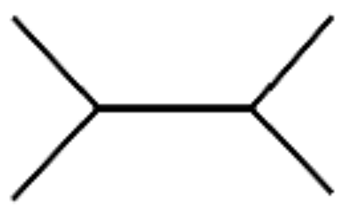
Birth of Modern Methods

- Improve the efficiency of calculation compared to Feyn. Diag.

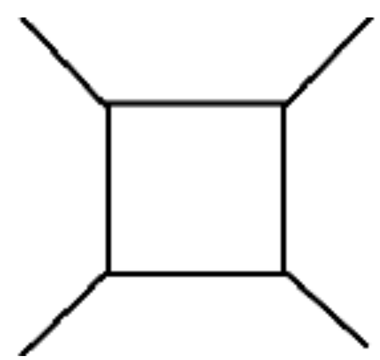
- Key ideas:

  ✦ Chops problem into on-shell gauge invariant smaller pieces, and (recursively) constructing scattering process.

  ✦ Unitarity $S^\dagger S = 1$ & physical singularities



$\sim \frac{1}{(p_1+p_2)^2}$

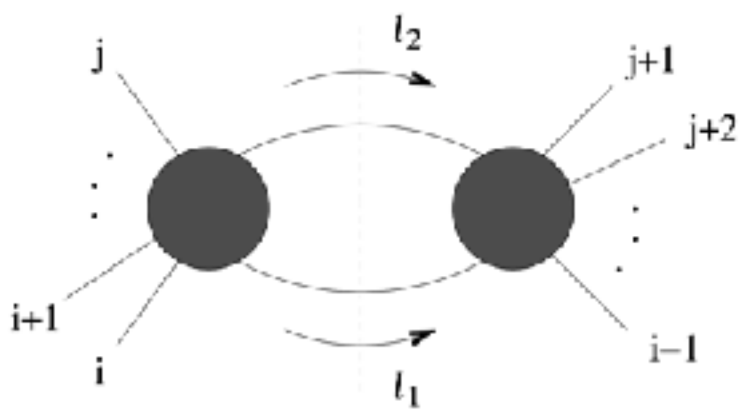


$\sim \log^2(s/t)$

- BCFW recursion relations; **[Britto et al., 03'; Britto et al., 04', etc.]**

  (generalized) unitarity cuts, etc. **[Bern et al., 92'; Bern et al., 93',07'; Bern et al., 94'; Britto et al., 04; Anastasiou et al., 06', Britto et al., 07']**

- S-matrix ($S = 1 + iT$) obeys $S^\dagger S = 1$: $\mathrm{Disc}\, T = 2\,\mathrm{Im}\, T = T^\dagger T$



$$\mathrm{Disc}\,\mathcal{A} = \sum_{P_L} \int d\mu\, A_L\, A_R$$

$$d\mu \;=\; d^d l_1\, d^d l_2\, \delta^{(d)}(l_1 + l_2 - P_L)\,\delta(l_1^2)\,\delta(l_2^2)$$

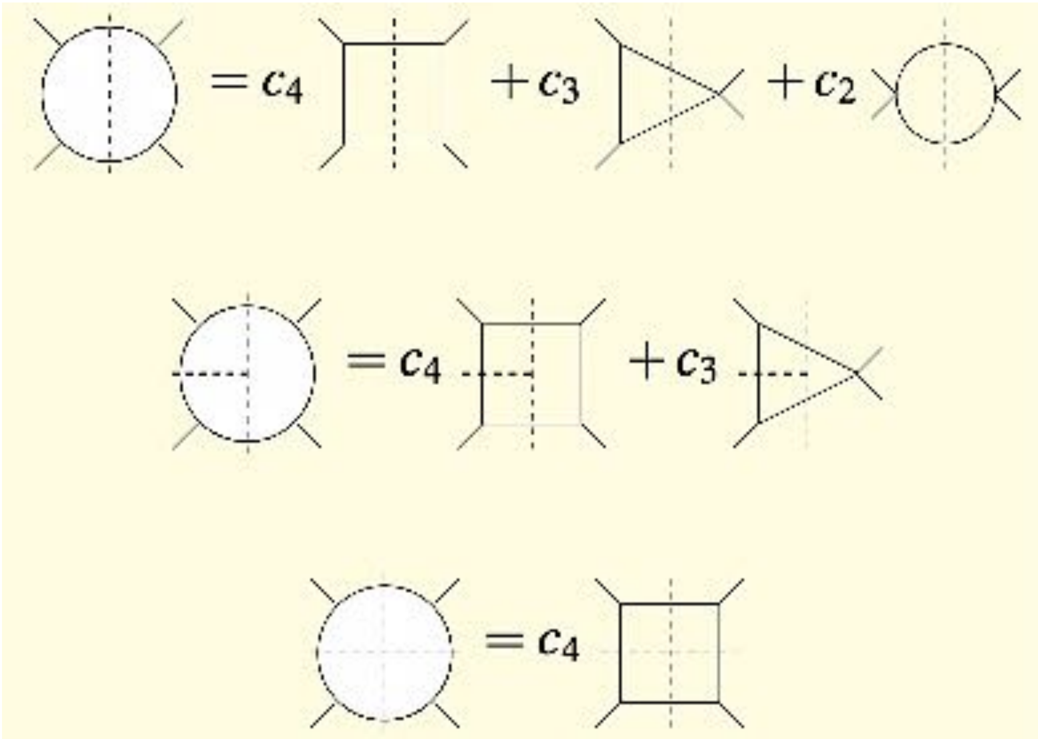
- Amplitudes in terms of master integrals

$$\mathcal{A} = \sum_i c_i\,\mathrm{MI}_i$$

- Determine coefficients of MIs:

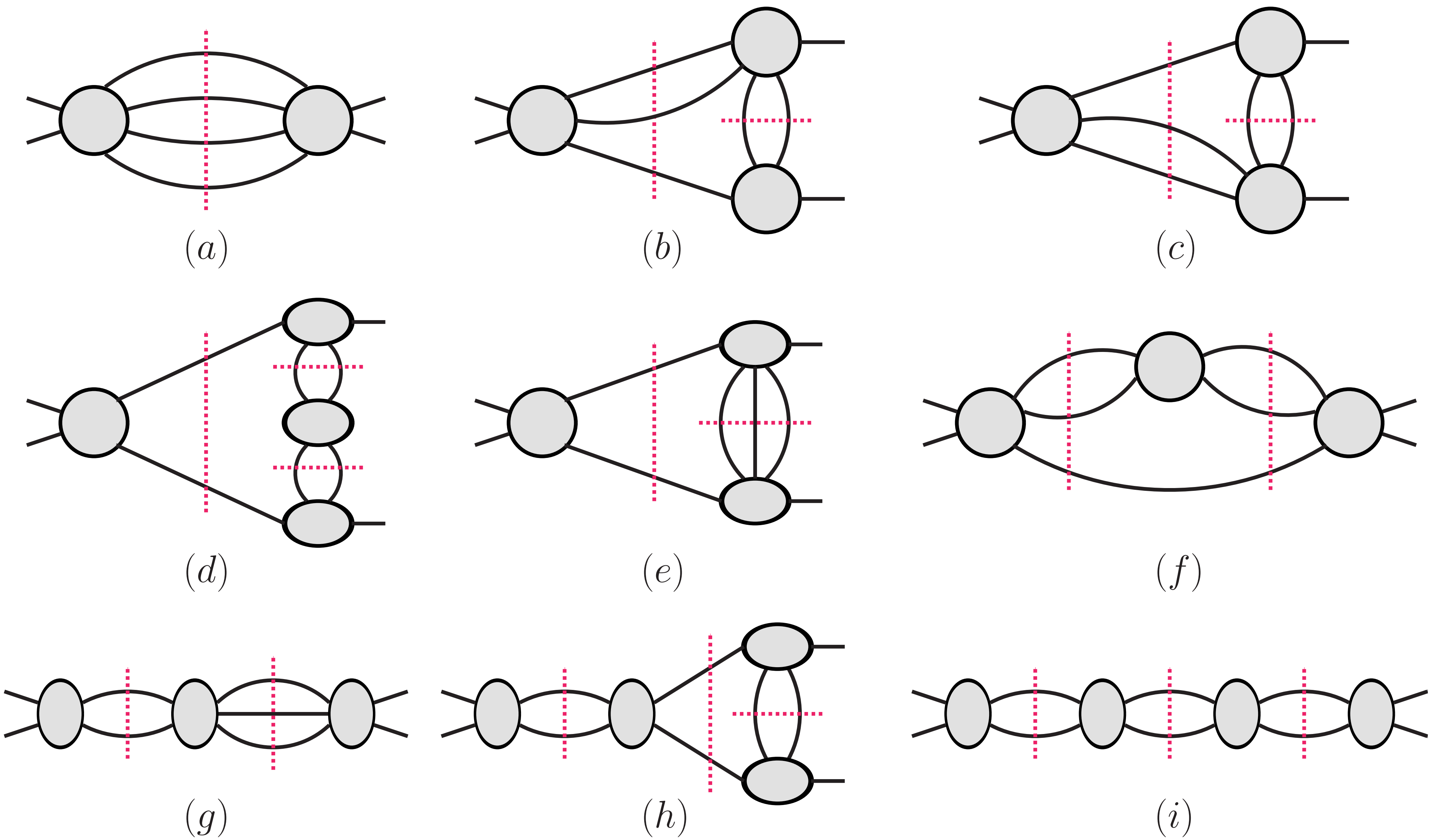
$$\mathrm{Disc}\,\mathcal{A} = \sum_i c_i\,\mathrm{Disc}\,\mathrm{MI}_i$$

- Compare cuts of amplitudes with cuts of master integrals



Cuts for 4 gluon 3 loop planar

[Q. Jin and H,19']

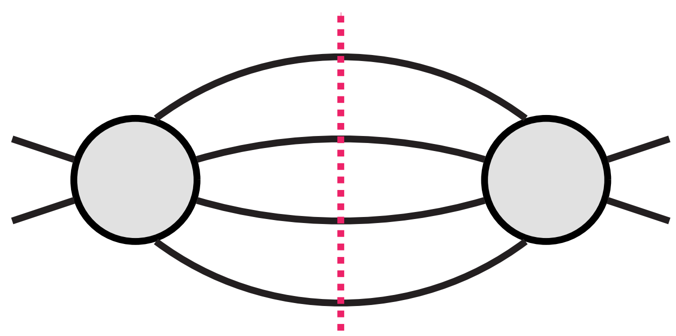


Number of cuts is reduced by Z4 rotational symmetry

An example of quadruple cuts

Propagators = $\{\ l_1^2,\, l_2^2,\, l_3^2,\, (l_1+p_1)^2,\, (l_2+p_1)^2,\, (l_3+p_1)^2, (l_1+p_{12})^2, (l_2+p_{12})^2, (l_3+p_{12})^2,$
$(l_1+p_{123})^2, (l_2+p_{123})^2, (l_3+p_{123})^2, (l_2-l_3)^2, (l_1-l_3)^2, (l_1-l_2)^2\ \}$

Permutations of $\{l_1, l_2, l_3\}$



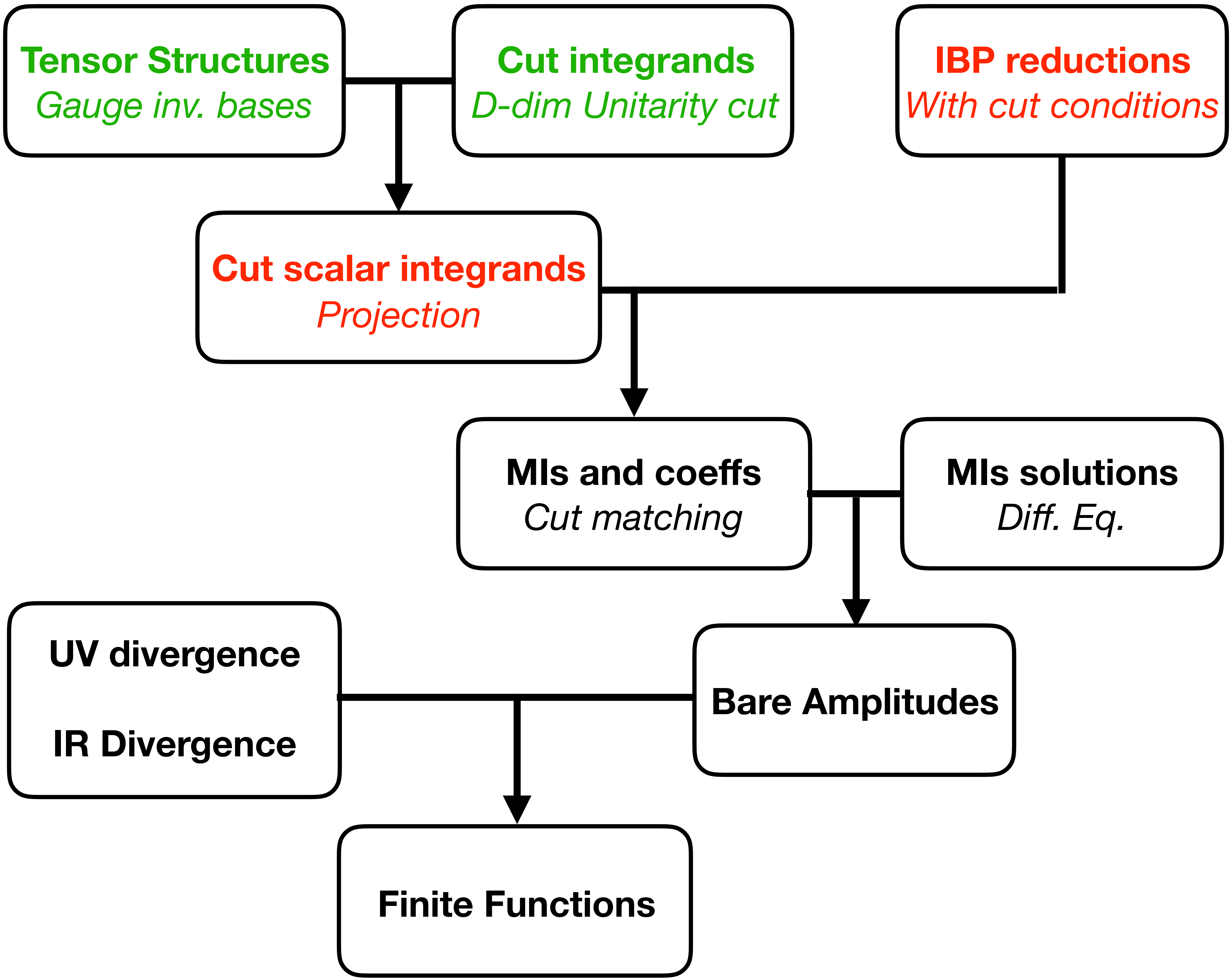
(a)

$$\sum_{\text{helicities}} \xi_\mu \xi_\nu = \eta_{\mu\nu} - \left(\frac{p_\mu q_\nu + p_\nu q_\mu}{q\cdot p}\right)$$

$$\sum_{\text{helicities}} \xi\cdot\xi = d-2$$

Ansatz of cut amplitude for this particular cut

$$\mathcal{A}^{\text{cut}} = \int d^d l_1\, d^d l_2\, d^d l_3 \boxed{\frac{1}{l_1^2\,(l_1-l_2)^2\,(l_2-l_3)^2, (l_3+p_{123})^2}}$$
$$\times \sum_{\xi_i^I}\Big(A_L^{\text{tree}}(\{\xi_1,p_1\},\{\xi_1^I,l_1\},\{\xi_2^I,-l_1+l_2\},\{\xi_3^I,-l_2+l_3\},\{\xi_4^I,-l_3-p_{12}\},\{\xi_2,p_2\})$$
$$\times A_R^{\text{tree}}(\{\xi_3,p_3\},\{\xi_4^I,l_3+p_{12}\},\{\xi_3^I,l_2-l_3\},\{\xi_2^I,l_1-l_2\},\{\xi_1^I,-l_1\},\{\xi_4,p_4\})\Big)$$


Tensor Structures
Gauge inv. bases
Cut integrands
D-dim Unitarity cut
IBP reductions
With cut conditions
Cut scalar integrands
Projection
MIs and coeffs
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Diff. Eq.
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Derivation of scalar cut integrands

- Amplitude from each unitarity cut $\mathcal{A}_n^{\rm cut}$ involves terms of $\xi \cdot l$

- Projecting $\mathcal{A}_n^{\rm cut}$ ($= \sum \alpha_i^{\rm cut} B_i$) onto gauge invariant basis B_i

$$\sum_{\rm helicities} B_j \mathcal{A}_n^{\rm cut} = \sum_i \alpha_i^{\rm cut} \left(\sum_{\rm helicities} B_j B_i \right) = \sum_i P_{ji} \alpha_i^{\rm cut}$$

$$\downarrow$$

$$\alpha_i^{\rm cut} = \left(P^{-1}\right)_{ij} \sum_{\rm helicities} B_j \mathcal{A}_n^{\rm cut}$$

$$\sum_{\rm helicities} \xi_\mu \xi_\nu = \eta_{\mu\nu} - \left(\frac{p_\mu q_\nu + p_\nu q_\mu}{q \cdot p} \right)$$

$$\sum_{\rm helicities} \xi \cdot \xi = d - 2$$

- The cut coefficients

$$\alpha_i^{\rm cut}(\{p, l\}) = \sum f_\alpha(\{p_j \cdot p_k\}) \boxed{\int (d^D l)^L \frac{1}{D_1^{a_1} \cdots \widehat{D}_m^{\rm cut} \cdots D_M^{a_M} \prod_m D_m^{\rm cut}}}$$

Cut integrals

Run IBP reductions with cut conditions

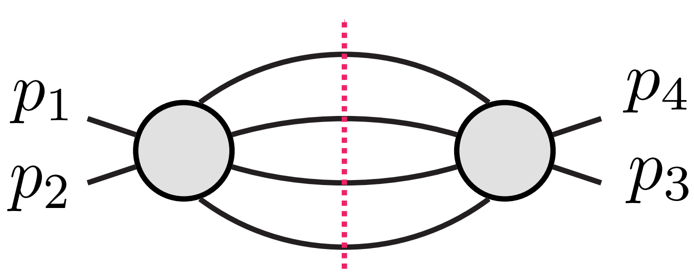
Integration by parts identities **[Chetyrkin, Tkachov 81'; Laporta 01']**

$$\int \frac{d^d l_1}{(i\pi^{d/2})} \cdots \int \frac{d^d l_L}{(i\pi^{d/2})} \frac{\partial}{\partial l_i^\mu} \frac{v_i^\mu}{D_1^{a_1} \cdots D_k^{a_k}} = 0$$

Public Programs:

- FIRE5/6 (Smirnov).
- Reduze2 (von Manteuffel, Studerus)
- LiteRed (Lee)
- Kira (Maierhofer, Usovitsch, Uwer)

FIRE6 (no LiteRed rules) with restrictions of cut conditions



(a)

- `RESTRICTIONS` (optional) — list of boundary conditions. For example if this list has an element `{-1, -1, -1, 0}`, this means that the integrals are equal to zero if the first three indices are non-positive. Since

```
RESTRICTIONS = {{-1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1}};
```

```
In[3]:= quadruplecut =
   << "/Users/huiluo/[redacted]s/IBP-settings/qcd3cut8v2/qcd3cut8v2.m";

In[4]:= Length[quadruplecut]

Out[4]= 70 206
```

SplitIntegrands45.m	39.0 kB	Objective-C source code
SplitIntegrands46.m	90.3 kB	Objective-C source code
SplitIntegrands47.m	58.2 kB	Objective-C source code
SplitIntegrands48.m	48.2 kB	Objective-C source code
SplitIntegrands49.m	45.8 kB	Objective-C source code
SplitIntegrands50.m	41.0 kB	Objective-C source code

… …

SplitIntegrands45.m	39.0 kB	Objective-C source code
SplitIntegrands46.m	90.3 kB	Objective-C source code
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SplitIntegrands48.m	48.2 kB	Objective-C source code
SplitIntegrands49.m	45.8 kB	Objective-C source code
SplitIntegrands50.m	41.0 kB	Objective-C source code

generate_config.py

generate_config.py ×

```python
for i in range(51):
    f=open('SplitIntegrands'+str(i+1)+'.config','w')
    f.write('#memory'+'\n')
    f.write('#threads           18'+'\n')
    f.write('#fthreads          36'+'\n')
    f.write('#variables         d,s1,s2'+'\n')
    f.write('#database          temp/SplitIntegrands'+'\n')
    f.write('#storage           temp/recover'+'\n')
    f.write('#start'+'\n')
    f.write('#folder            examples/'+'\n')
    f.write('#problem           '+str(i+1)+' qcd3cut8v2.start'+'\n')
    f.write('#integrals         qcd3cut8v2/SplitIntegrands'+str(i+1)+'.m'+'\n')
    f.write('#output            ../temp/SplitIntegrands'+str(i+1)+'.tables'+'\n')
    f.close()
```

... ...

SplitIntegrands43.config	340 bytes	plain text document
SplitIntegrands44.config	340 bytes	plain text document
SplitIntegrands45.config	340 bytes	plain text document
SplitIntegrands46.config	340 bytes	plain text document
SplitIntegrands47.config	340 bytes	plain text document
SplitIntegrands48.config	340 bytes	plain text document
SplitIntegrands49.config	340 bytes	plain text document
SplitIntegrands50.config	340 bytes	plain text document
SplitIntegrands51.config	340 bytes	plain text document

… …

SplitIntegrands43.config	340 bytes	plain text document
SplitIntegrands44.config	340 bytes	plain text document
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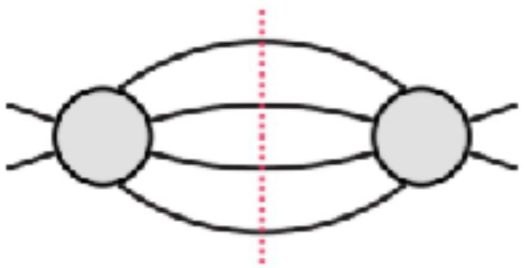
SplitIntegrands46.config	340 bytes	plain text document
SplitIntegrands47.config	340 bytes	plain text document
SplitIntegrands48.config	340 bytes	plain text document
SplitIntegrands49.config	340 bytes	plain text document
SplitIntegrands50.config	340 bytes	plain text document
SplitIntegrands51.config	340 bytes	plain text document

run_fire6.py

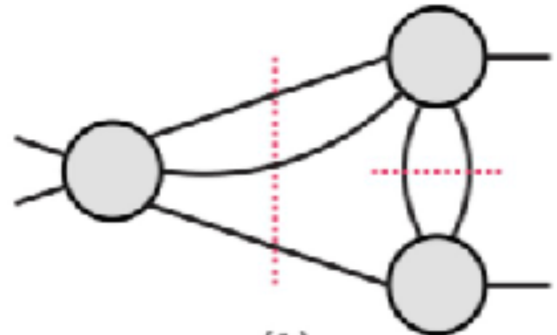
```
import os
import shutil

for i in range(3,51):
    os.system("bin/FIRE6 -c examples/qcd3cut8v2/%s" %('SplitIntegrands'+str(i+1)))
    os.system("rm -r temp/SplitIntegrands")
    os.system("rm -r temp/recover")
    print('Finish running IBP reductions for'+'SplitIntegrands'+str(i+1))
```

10 days to finish all cut IBP reductions

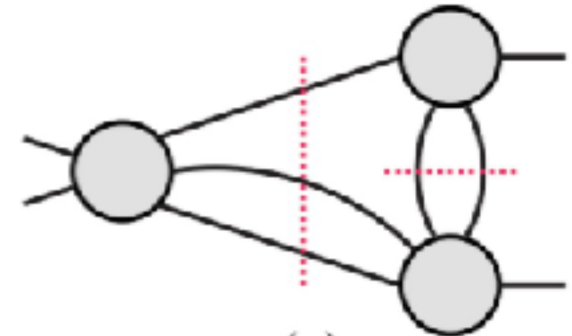
The highest memory usage is about 30% of 256 GB


(a)

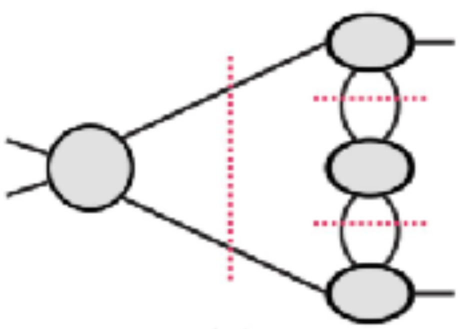
342.2 M


(b)

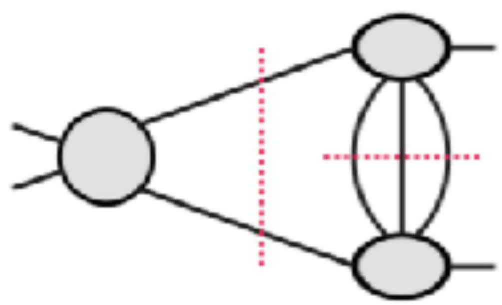
137.6 M


(c)

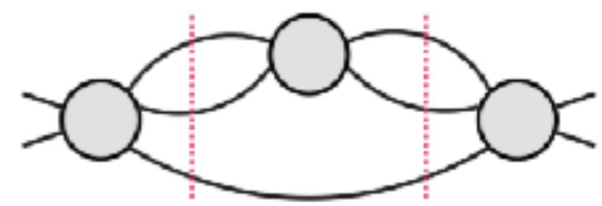
137.5 M


(d)

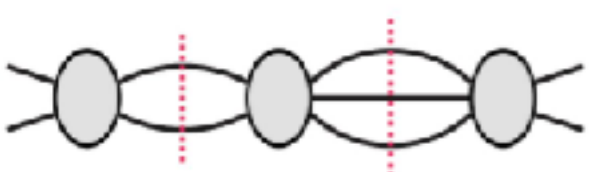
2.3 M


(e)

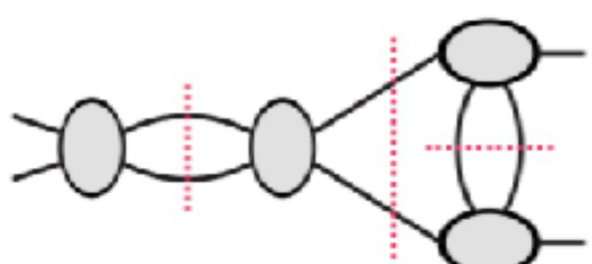
17 M


(f)

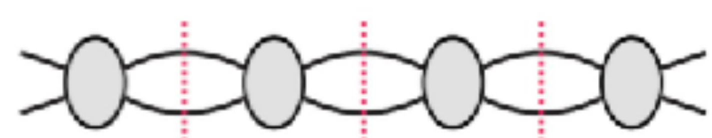
37.5 M


(g)

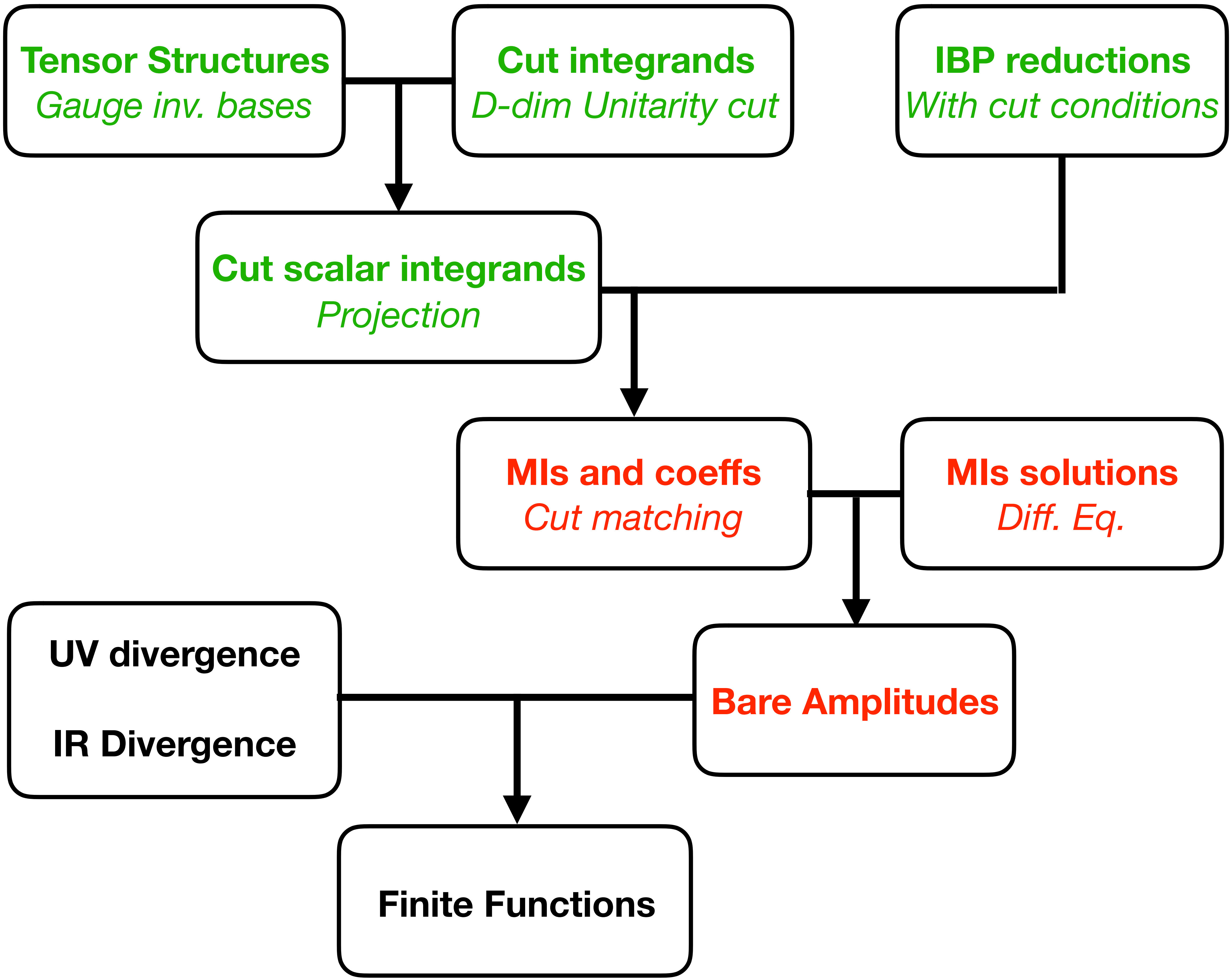
11.4 M


(h)

2 M


(i)

462 K


Tensor Structures
Gauge inv. bases
Cut integrands
D-dim Unitarity cut
IBP reductions
With cut conditions
Cut scalar integrands
Projection
MIs and coeffs
Cut matching
MIs solutions
Diff. Eq.
UV divergence
IR Divergence
Bare Amplitudes
Finite Functions

Substitute cut IBPs results into cut scalar integrals

$$\alpha_i^{\mathrm{cut}} = \sum c_{ij}^{\mathrm{cut}} \mathrm{MI}_j^{\mathrm{cut}}$$

Cyclic permutation $r : s \leftrightarrow t,$

$$\begin{aligned} &B_1 \to B_1,\ B_2 \to B_2,\ B_3 \leftrightarrow B_4,\ B_5 \leftrightarrow B_6, \\ &B_7 \to B_8,\ B_8 \to B_9,\ B_9 \to B_{10},\ B_{10} \to B_7\ . \end{aligned}$$

Subtleties of double counting

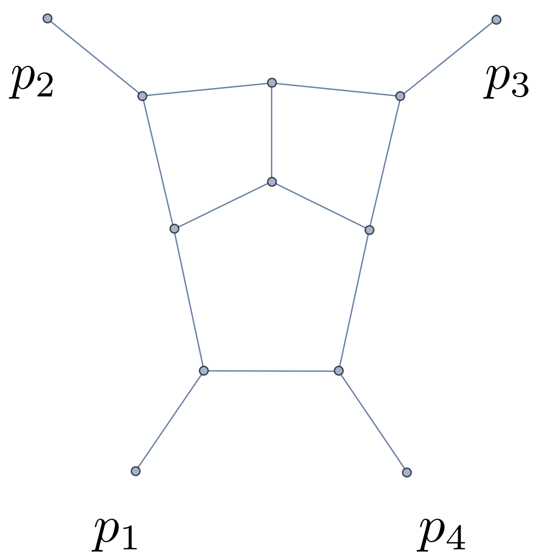
- Amplitude at MI level:

$$\mathcal{A}_n = \sum_i \left(\sum_j c_{ij}\, \mathrm{MI}_j \right) B_i$$

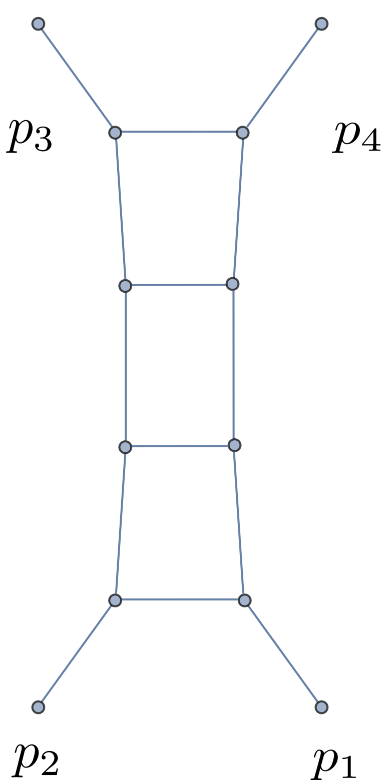
81 MIs in total if including cyclic permutations for 4pt 3 loop

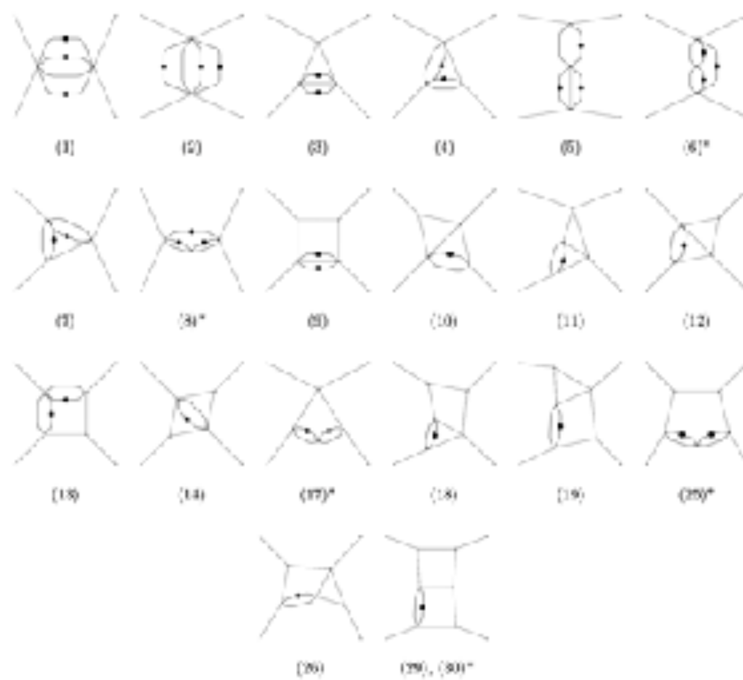
[J. Henn, A.V. Smirnov and V. A. Smirnov,13']

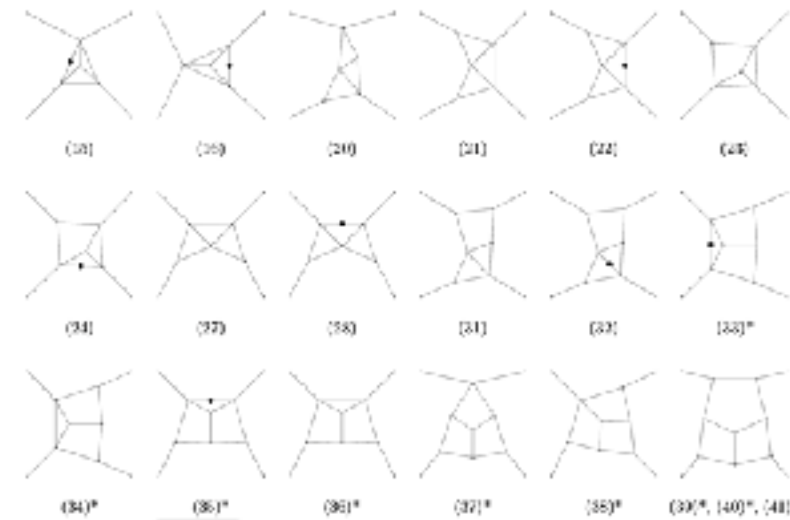
UT MIs with transcendental degree 6



+



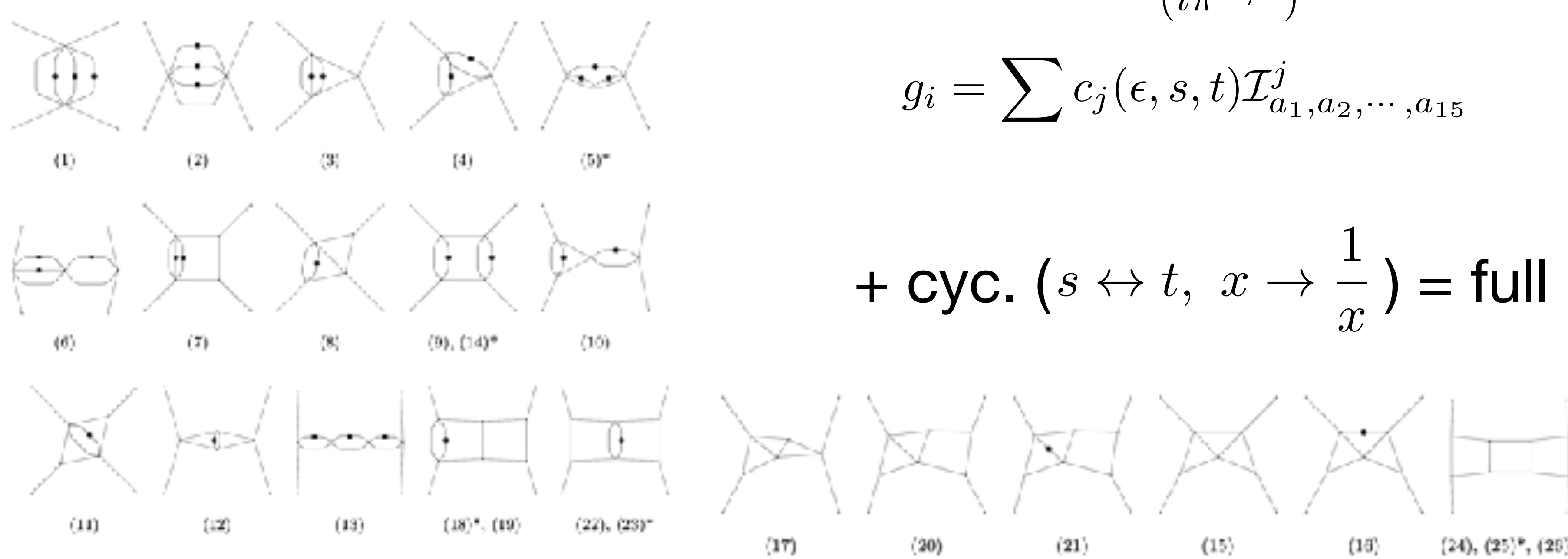




$$\partial_x f(x,\epsilon) = \epsilon \left[\frac{a}{x} + \frac{b}{1+x} \right] f(x,\epsilon)$$

$$f_i = \epsilon^3 (-s)^{3\epsilon} \frac{e^{3\epsilon\gamma_E}}{(i\pi^{D/2})^3} g_i \qquad x = \frac{t}{s}$$

$$g_i = \sum c_j(\epsilon, s, t) \mathcal{I}^j_{a_1, a_2, \cdots, a_{15}}$$



+ cyc. ($s \leftrightarrow t,\ x \to \frac{1}{x}$) = full MIs

Substitute analytical solutions of MIs into amplitudes

$$\mathcal{A}_n^{(L)} = \sum_i \left(\sum_j c_{ij}^{(L)} \, \mathrm{MI}_j^{(L)} \right) B_i$$

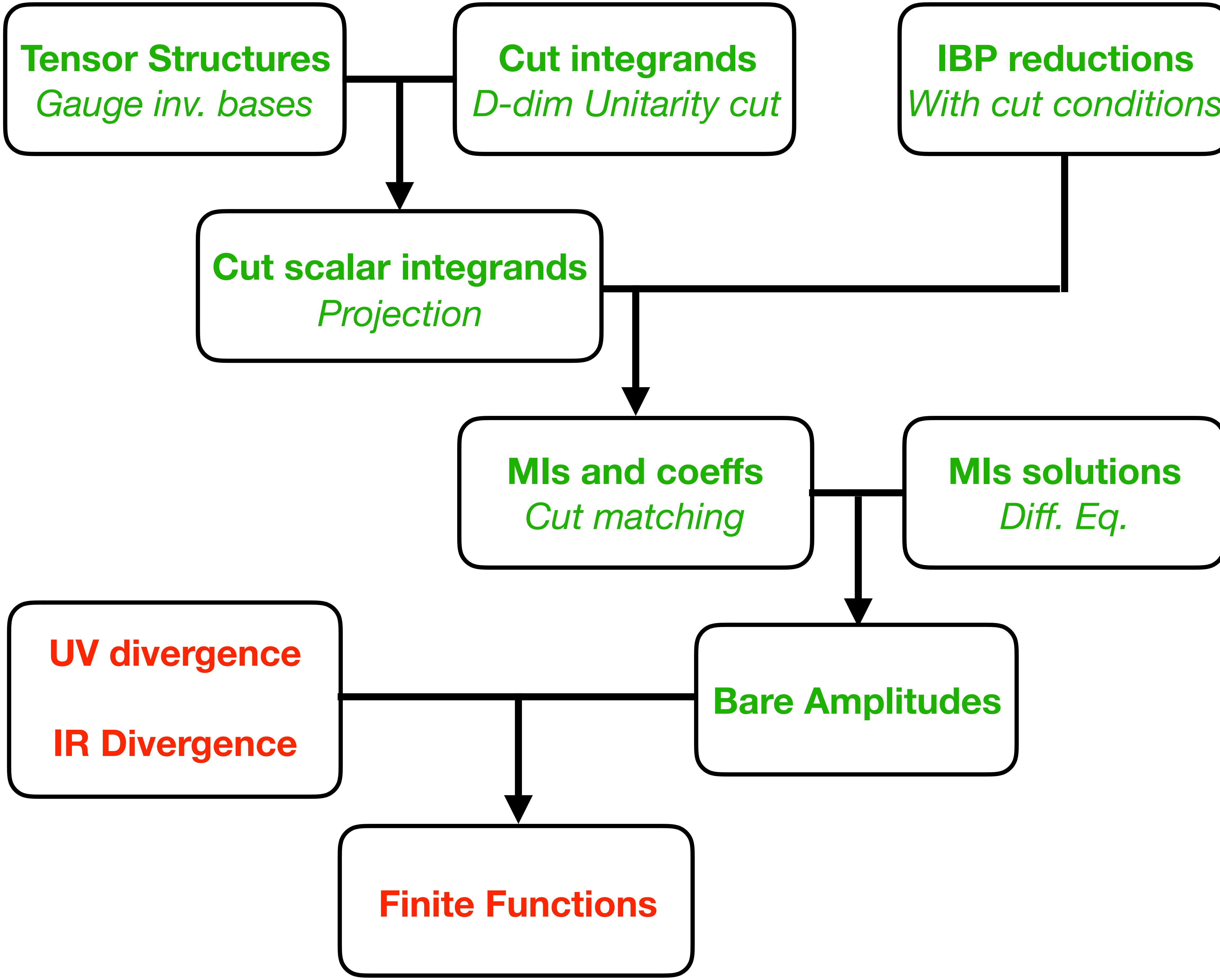
Most divergent part of 3 loop bare amplitudes

(- - - -)	(- - - +)	(+ + - -)	(+ - + -)
$\frac{8}{3\,\mathrm{ep}^4\,s\,t}$	$-\frac{8\,(s+t)}{3\,\mathrm{ep}^4\,s\,t}$	$-\frac{32}{3\,\mathrm{ep}^6}$	$-\frac{32}{3\,\mathrm{ep}^6}$

Solve UT MIs of 4-gluon one- and two-loop amplitude up to transcendental 6 **[J. Henn, 13’ & 14’; R. Boels and HL, 17’]**

Bare amplitudes up to three loop

$$\mathcal{A} = g_0^2 \sum_{L=0,1,2,3} \left(\frac{\alpha_0}{4\pi}\right)^L C_A^L \mathcal{A}^{(L)}$$



Tensor Structures
Gauge inv. bases
Cut integrands
D-dim Unitarity cut
IBP reductions
With cut conditions
Cut scalar integrands
Projection
MIs and coeffs
Cut matching
MIs solutions
Diff. Eq.
UV divergence
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- Ultraviolet divergence

Renormalized amplitudes via $\alpha_0 = (4\pi)^{-\epsilon} e^{\epsilon\gamma_E} \alpha_s \mu^{2\epsilon} Z_\alpha(\alpha,\epsilon)$ in $\overline{\mathrm{MS}}$

$$Z_\alpha(\alpha,\epsilon) = 1 - \frac{\alpha_s}{4\pi}\frac{\beta_0}{\epsilon} + \left(\frac{\alpha_s}{4\pi}\right)^2 \left(\frac{\beta_0^2}{\epsilon^2} - \frac{\beta_1}{2\epsilon}\right) + \left(\frac{\alpha_s}{4\pi}\right)^3 \left(-\frac{\beta_0^3}{\epsilon^3} + \frac{7\beta_0\beta_1}{6\epsilon^2} - \frac{\beta_2}{3\epsilon}\right),$$

$$\beta_0 = \frac{11}{3} C_A, \quad \beta_1 = \frac{34}{3} C_A^2, \quad \beta_2 = \frac{2857}{54} C_A^3.$$

$$\mathcal{A}_R = g_s^2 \sum_{L=0,1,2,3} \left(\frac{\alpha_s}{4\pi}\right)^L C_A^L \mathcal{A}_R^{(L)}$$

- Infrared divergence

$$\mathbf{Z}(\{p_i\},\epsilon,\mu) = \mathbf{P}\exp\int_\mu^\infty \frac{d\mu'}{\mu'} \mathbf{\Gamma}(\{p_i\},\mu') \qquad \mathbf{\Gamma}(s,t;\mu) = \gamma_{\text{cusp}}(\alpha_s)\left(\ln\frac{\mu^2}{-s} + \ln\frac{\mu^2}{-t}\right) C_A + 4\gamma_g(\alpha_s).$$

[Becher, 09'*2 & 14']

$$\mathcal{H} = \lim_{\epsilon\to 0} \mathbf{Z}^{-1} \mathcal{A}_R$$

Hard Functions expressions in ancillary files 1910.05889

名称	修改日期	大小	种类
pt4L3mmmmHardEu.m	2019/10/13	2 KB	Objec...ource
pt4L3mmmmHardPhy.m	2019/10/13	2 KB	Objec...ource
pt4L3mmmpHardEu.m	2019/10/13	4 KB	Objec...ource
pt4L3mmmpHardPhy.m	2019/10/13	6 KB	Objec...ource
pt4L3pmpmHardEu.m	2019/10/13	14 KB	Objec...ource
pt4L3pmpmHardPhy.m	2019/10/13	22 KB	Objec...ource
pt4L3ppmmHardEu.m	2019/10/13	21 KB	Objec...ource
pt4L3ppmmHardPhy.m	2019/10/13	34 KB	Objec...ource

- Euclidean Region ($s<0$, $t<0$) < Physical Region ($s>0$, $t<0$)
- (- - - -) < (- - - +) < (+ - + -) < (+ + - -)

eg. Hard functions of (- - - -) in Euclidean region at three loop

weight 4

$$\frac{1}{90 s^2 x}\Big(240\,\mathrm{Li}_4(-x)+240\,\mathrm{Li}_2(-x)\log(x+1)\log(x)+240\,\mathrm{Li}_2(x+1)\log(x+1)\log(x)-$$
$$120\,\mathrm{Li}_3(-x)\log(x)-240\,\mathrm{Li}_3(x+1)\log(x)-240\,\mathrm{Li}_3(-x)\log(x+1)-240\,S_{2,2}(-x)+$$
$$60\log^4\left(-\frac{s}{\mu^2}\right)+120\log(x)\log^3\left(-\frac{s}{\mu^2}\right)+120\log^2(x)\log^2\left(-\frac{s}{\mu^2}\right)+60\log\left(-\frac{s}{\mu^2}\right)\left(\log^3(x)-2\,\zeta(3)\right)+$$
$$60\,\zeta(3)\log(x)+240\,\zeta(3)\log(x+1)+15\log^4(x)-20\log(x+1)\log^3(x)+60\log^2(x+1)\log^2(x)+$$
$$20\pi^2\log^2(x)+120\log(-x)\log^2(x+1)\log(x)+60\pi^2\log^2(x+1)-60\pi^2\log(x+1)\log(x)+8\pi^4\Big)$$

weight 3

$$\frac{22\log^3\left(-\frac{s}{\mu^2}\right)+33\log(x)\log^2\left(-\frac{s}{\mu^2}\right)+\left(33\log^2(x)-22\pi^2\right)\log\left(-\frac{s}{\mu^2}\right)+11\left(\log^3(x)-\pi^2\log(x)+10\,\zeta(3)\right)}{27 s^2 x}$$

weight 2

$$\frac{1}{54 s^2 x^2 (x+1)^2}\Big(-4\left(3x^2+97x+3\right)(x+1)^2\log^2\left(-\frac{s}{\mu^2}\right)-4\left(3x^2+97x+3\right)(x+1)^2\log(x)\log\left(-\frac{s}{\mu^2}\right)-$$
$$2x\left(109x^2+17x+109\right)\log^2(x)+\pi^2\left(8x^4+79x^3+544x^2+79x+8\right)\Big)$$

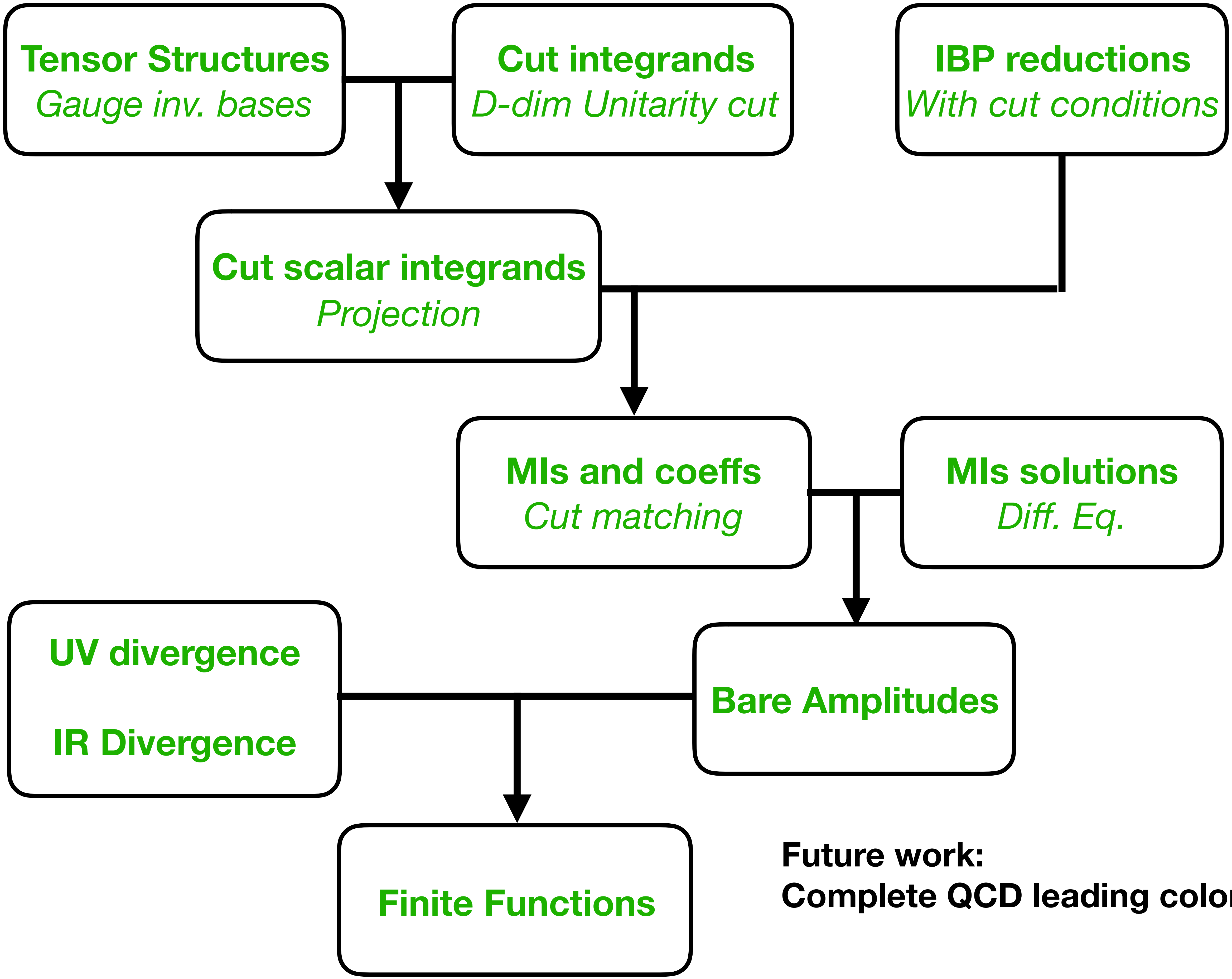
weight 1

$$\frac{\left(-33x^3+409x^2+409x-33\right)\log\left(-\frac{s}{\mu^2}\right)+\left(824x^2-415x-33\right)\log(x)}{81 s^2 x^2 (x+1)}$$

weight 0

$$\frac{1011x^2+1346x+1011}{243 s^2 x^2}$$

x= t/s



Future work:
Complete QCD leading color !

Thank you for your attention !

Kinematic Basis Construction for Pure-YM

[R. Boels, Q. Jin and HL,18']

"Canonical" kinematic basis construction

- Given >=3 gluon particles in the process, kinematic basis B can be constructed from multi-copies of all possible A and C/D types

  Linearly independent and complete in general dimensions

- The total number of basis elements with n gluons and no scalars is

$$N_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!(n-2)^{(n-2k)}}{2^k k!(n-2k)!}$$

  Compare with brute-force derivation up to seven gluons

Example: 5 gluon 2 loop planar

- Calculations in HV scheme: D-dim for internal particles and 4-dim for external particles

  - Number of kinematic basis reduces from 142 to 32

    Only kinematic basis from 5 As (eg. $A_1(2)A_2(3)A_3(4)A_4(5)A_5(1)$) remains

  - No $\sum_{\text{hels}\, i,j} (\xi_i^{\text{EX}} \cdot \xi_j^{\text{EX}})(\xi_i^{\text{EX}} \cdot l_a)(\xi_j^{\text{EX}} \cdot l_b) \sim l_a^{[4]} \cdot l_b^{[4]}$

    Only establish in 5pt case !

  - Relatively straight forward for HV scheme

  - Cross checked with numerical results

    [Badger et al., 17'; Abreu et al., 17']