# Warped Symmetries from Rotating Black Holes

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# Content

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- The asymptotic symmetry analysis helps to reveal potential dual field theories of black holes.
- Known examples: (W)AdS/(W)CFT, Kerr/CFT, $\cdots$

# Introduction (extremal Kerr)

- The near horizon extreme Kerr scaling geometry (NHEK)

$$d\bar{s}^2 = 2GJ\Omega^2 \left( -(1+r^2)d\tau^2 + \frac{dr^2}{1+r^2} + d\theta^2 + \Lambda^2(d\varphi + rd\tau)^2 \right) .$$

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The non-trivial asymptotic symmetry can be specified by the following boundary conditions [Guica, Hartman, Song and Strominger, 2009]

$$\begin{pmatrix} h_{\tau\tau} = \mathcal{O}(r^2) & h_{\tau\varphi} = \mathcal{O}(1) & h_{\tau\theta} = \mathcal{O}(r^{-1}) & h_{\tau r} = \mathcal{O}(r^{-2}) \\ h_{\varphi\tau} = h_{\tau\varphi} & h_{\varphi\varphi} = \mathcal{O}(1) & h_{\varphi\theta} = \mathcal{O}(r^{-1}) & h_{\varphi r} = \mathcal{O}(r^{-1}) \\ h_{\theta\tau} = h_{\tau\theta} & h_{\theta\varphi} = h_{\varphi\theta} & h_{\theta\theta} = \mathcal{O}(r^{-1}) & h_{\theta r} = \mathcal{O}(r^{-2}) \\ h_{r\tau} = h_{\tau r} & h_{r\varphi} = h_{\varphi r} & h_{r\theta} = h_{\theta r} & h_{rr} = \mathcal{O}(r^{-3}) \end{pmatrix},$$

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The most general diffeomorphisms which preserve the falloffs are of the form

$$\zeta = (-r\epsilon'(\varphi) + \mathcal{O}(1))\partial_r + (C + \mathcal{O}(r^{-3}))\partial_\tau + (\epsilon(\varphi) + \mathcal{O}(r^{-2}))\partial_\varphi + \mathcal{O}(r^{-1})\partial_\theta$$

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The asymptotic symmetry contains one copy of the conformal group

$$\zeta_\epsilon = \epsilon(\varphi)\partial_\varphi - r\epsilon'(\varphi)\partial_r\,, \qquad i[\zeta_m, \zeta_n] = (m-n)\zeta_{m+n}\,.$$

# Introduction (extremal Kerr)

- The central term appears when considering the conserved charge algebra under Dirac bracket $\{\cdot,\cdot\}_{D.B.} \to -\frac{i}{\hbar}[\cdot,\cdot]$

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{J}{\hbar} m(m^2-1)\delta_{m,-n}\,, \qquad \hbar L_n = Q_{\zeta_n} + \frac{3J}{2}\delta_n\,.$$

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The Cardy formula with left-moving central charge $c_L = \frac{12J}{\hbar}$ and temperature $T_L = \frac{1}{2\pi}$ reproduces the black hole entropy

$$S = \frac{\pi^2}{3} c_L T_L = \frac{2\pi J}{\hbar} = S_{EKBH}\,.$$

# Extremal Kerr-like spacetimes/WCFT

- The most general Kerr-like spacetimes in four dimensions [Griffiths and Podolsky, 2006]

$$
\begin{aligned}
ds^2 = \frac{1}{\Omega^2}\Bigg\{ & -\frac{Q}{\rho^2}\left[dt - \left(a(1-x^2) + 2l(1+x)\right) d\phi\right]^2 + \rho^2\left[\frac{dr^2}{Q} + \frac{dx^2}{(1-x^2)P}\right] \\
& + \frac{a^2(1-x^2)P}{\rho^2}\left[dt - \frac{r^2+(a+l)^2}{a} d\phi\right]^2 \Bigg\},
\end{aligned}
$$

where

$$
\Omega = 1 - \frac{\alpha}{\lambda}(l - ax)r, \qquad \rho^2 = r^2 + (l-ax)^2, \qquad P = 1 + a_3 x + a_4 x^2\,,
$$

$$
Q = (\lambda^2 k + e^2 + g^2) - 2Mr + \epsilon r^2 - 2\frac{\alpha n}{\lambda} r^3 - \left(\alpha^2 k + \frac{\Lambda}{3}\right) r^4\,,
$$

are metric functions characterized by $M, e, g, a, \alpha, l, \Lambda$.

# Extremal Kerr-like spacetimes/WCFT

- The horizon polynomial generically takes the form

$$Q = -\left(\alpha^2 k + \frac{\Lambda}{3}\right)(r - r_0)(r - r_1)(r - r_2)(r - r_3)\,.$$

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Approximate $Q$ to quadratic order

$$\begin{aligned} Q &\approx Q(r_+) + \frac{dQ}{dr}\Big|_{r=r_+}(r-r_+) + \frac{1}{2}\frac{d^2Q}{dr^2}\Big|_{r=r_+}(r-r_+)^2 + \mathcal{O}\left((r-r_+)^3\right) \\ &\approx k_+(r-r_+)(r-r_s) + \mathcal{O}\left((r-r_+)^3\right)\,, \end{aligned}$$

where

$$\begin{aligned} k_+ &= \frac{1}{2}\frac{d^2Q}{dr^2}\Big|_{r=r_+} \\ &= \epsilon - 6r_+\frac{\alpha n}{\lambda} - 6r_+^2(\alpha^2 k + \frac{\Lambda}{3})\,, \\ r_s &= r_+ - \frac{1}{k_+}\frac{dQ}{dr}\Big|_{r=r_+} \\ &= r_+ - \frac{1}{k_+}\left[2r_+\epsilon - 2M - 6r_+^2\frac{\alpha n}{\lambda} - 4r_+^3\left(\alpha^2 k + \frac{\Lambda}{3}\right)\right]\,. \end{aligned}$$

- For an extremal black hole $r_+ = r_s$, the near horizon region can be infinitely scaled to the following geometry (with $AdS_2 \times S^2$)

$$r = r_+ + \delta r_0 \tilde{r}, \qquad t = \frac{r_0}{\delta}\left(u - \frac{1}{\tilde{r}}\right), \cdots$$

$$ds^2 = \Gamma(x)\left(-\tilde{r}^2 du^2 - 2du d\tilde{r} + \sigma^2(x)dx^2 + \gamma^2(x)b(d\varphi + \tilde{r}d\tilde{u})^2\right) \; ,$$

where

$$r_0 = \sqrt{\frac{r_+^2 + (a+l)^2}{k_+}}, \qquad \Gamma(x) = \frac{\lambda^2(r_+^2 + (l-ax)^2)}{k_+(\lambda - r_+\alpha(l-ax))^2}, \qquad \sigma^2(x) = \frac{k_+}{(1-x^2)P} \; ,$$

$$b = \frac{2r_+ a}{k_+(r_+^2 + (a+l)^2)}, \qquad \gamma^2(x) = \frac{k_+(r_+^2 + (a+l)^2)^2(1-x^2)P}{(r_+^2 + (l-ax)^2)^2} \; ,$$

# Extremal Kerr-like spacetime/WCFT

- We impose the following boundary conditions [X. Jiang and JX, 2024]

$$\begin{pmatrix} h_{uu} = \mathcal{O}(\tilde{r}) & h_{u\tilde{r}} = \mathcal{O}(\tilde{r}^{-1}) & h_{ux} = \mathcal{O}(\tilde{r}^{-1}) & h_{u\varphi} = \mathcal{O}(\tilde{r}^{0}) \\ h_{\tilde{r}u} = h_{u\tilde{r}} & h_{\tilde{r}\tilde{r}} = \mathcal{O}(\tilde{r}^{-3}) & h_{\tilde{r}x} = \mathcal{O}(\tilde{r}^{-2}) & h_{\tilde{r}\varphi} = \mathcal{O}(\tilde{r}^{-2}) \\ h_{xu} = h_{ux} & h_{x\tilde{r}} = h_{\tilde{r}x} & h_{xx} = \mathcal{O}(\tilde{r}^{-1}) & h_{x\varphi} = \mathcal{O}(\tilde{r}^{-1}) \\ h_{\varphi u} = h_{u\varphi} & h_{\varphi\tilde{r}} = h_{\tilde{r}\varphi} & h_{\varphi x} = h_{x\varphi} & h_{\varphi\varphi} = \mathcal{O}(\tilde{r}^{-1}) \end{pmatrix} .$$

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The most general asymptotic Killing vector that preserves this falloffs can be found out

$$\begin{aligned}\eta &= \left(f(u)+\mathcal{O}(\tilde{r}^{-3})\right)\frac{\partial}{\partial u}+\left(-\tilde{r}f'(u)-g'(u)+\mathcal{O}(\tilde{r}^{-1})\right)\frac{\partial}{\partial\tilde{r}} \\ &\quad +\mathcal{O}(\tilde{r}^{-1})\frac{\partial}{\partial x}+\left(g(u)+\mathcal{O}(\tilde{r}^{-2})\right)\frac{\partial}{\partial\varphi}.\end{aligned}$$

# Extremal Kerr-like spacetimes/WCFT

- Alternatively, these B.C.s can be viewed as uplifted from the new boundary conditions for $AdS_2$ [Godet and Marteau, 2020],[Detournay, Smoes, and Wutte, 2023]

$$ds^2 = \Gamma(x)\Big((-\tilde{r}^2 + 2P(u)\tilde{r} + 2T(u))du^2 - 2dud\tilde{r} + \sigma^2(x)dx^2 + \gamma^2(x)b(d\varphi + \tilde{r}d\tilde{u})^2\Big)\,,$$

which is preserved by the leading asymptotic Killing vector

$$\eta = f(u)\frac{\partial}{\partial u} + (-\tilde{r}f'(u) - g'(u))\frac{\partial}{\partial \tilde{r}} + g(u)\frac{\partial}{\partial \varphi}\,.$$

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Suppose the time coordinate $u$ is periodic under length $\tau$, then the modes $f_n = \eta\left(f(u) = \frac{\tau}{2\pi}e^{\frac{2\pi i n u}{\tau}}\right)$ and $g_n = \eta\left(g(u) = \frac{\tau}{2\pi i}e^{\frac{2\pi i n u}{\tau}}\right)$ satisfy the classical Virasoro and $U(1)$ Kac-Moody algebra

$$i[f_m, f_n] = (m-n)f_{m+n}\,,$$
$$i[f_m, g_n] = -n g_{m+n}\,,$$
$$i[g_m, g_n] = 0\,.$$

# Extremal Kerr-like spacetimes/WCFT

- The central extension terms appear in the covariant charge algebra. The variation of charge associated to a vector $\zeta$ is given by [Iyer and Wald, 1994]

$$\delta \mathcal{Q}(\xi, h; g) = \frac{1}{16\pi} \int_{\partial\Sigma} *F_{IW} \,,$$

with

$$(F_{IW})_{\mu\nu} = \frac{1}{2}\nabla_\mu \zeta_\nu h + \nabla_\mu h^\sigma{}_\nu \zeta_\sigma + \nabla_\sigma \zeta_\mu h^\sigma{}_\nu + \nabla_\sigma h^\sigma{}_\mu \zeta_\nu - \nabla_\mu h \zeta_\nu - (\mu \leftrightarrow \nu) \,.$$

For any two vector fields $\xi$ and $\zeta$, the charges satisfy an algebra under Dirac bracket [Compère and Fiorucci, 2018]

$$\{\mathcal{Q}_\xi, \mathcal{Q}_\zeta\} = \mathcal{Q}_{[\xi,\zeta]} + K_{\zeta,\xi}, \qquad K_{\zeta,\xi} = \delta \mathcal{Q}(\xi, \mathcal{L}_\zeta g; g) \,.$$

- Charges can be shown to be integrable on the backgrounds satisfying the new B.C.s. One can define the variation of the metric according to

$$h^{\mu\nu} = \frac{\delta g^{\mu\nu}}{\delta P(u)}\delta P(u) + \frac{\delta g^{\mu\nu}}{\delta T(u)}\delta T(u)\,.$$

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Then the variation of charge associated to the asymptotic Killing vector can be written as

$$\delta Q(\eta, h; g) = \frac{1}{8\pi}\int_{-1}^{1}\int_{0}^{\tau} b\Gamma(x)\sigma(x)\gamma(x)\left(g(u)\delta P(u) - f(u)\delta T(u)\right) dx du\,,$$

which is explicitly integrable with the finite charge

$$Q(\eta, h; g) = \frac{1}{8\pi}\int_{-1}^{1}\int_{0}^{\tau} b\Gamma(x)\sigma(x)\gamma(x)\left(g(u)P(u) - f(u)T(u)\right) dx du\,.$$

- The charge variations associated to the mode vectors

$$\delta L_n^{ext} = \delta \mathcal{Q}(f_n, h; g), \qquad \delta P_n^{ext} = \delta \mathcal{Q}(g_n, h; g)\,,$$

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$$\delta L_n^{ext} = \delta\mathcal{Q}(f_n, h; g), \qquad \delta P_n^{ext} = \delta\mathcal{Q}(g_n, h; g)\,,$$

will form the warped conformal algebra

$$\begin{aligned}
\{L_n^{ext}, L_m^{ext}\} &= (m-n)L_{m+n}^{ext} + K_{m,n}^{ext}, \qquad K_{m,n}^{ext} = \delta\mathcal{Q}(f_n, \mathcal{L}_{f_m} g; g)\,,\\
\{L_n^{ext}, P_m^{ext}\} &= mP_{m+n}^{ext} + \mathfrak{R}_{m,n}^{ext}, \qquad \mathfrak{R}_{m,n}^{ext} = \delta Q(f_n, \mathcal{L}_{g_m} g; g)\,,\\
\{P_n^{ext}, P_m^{ext}\} &= k_{m,n}^{ext}, \qquad k_{m,n}^{ext} = \delta\mathcal{Q}(g_n, \mathcal{L}_{g_m} g; g)\,,
\end{aligned}$$

with central extensions

$$\begin{aligned}
\delta\mathcal{Q}(f_n, \mathcal{L}_{f_m} g; g) &= 0\,,\\
\delta Q(f_n, \mathcal{L}_{g_m} g; g) &= i\,\frac{\lambda^2 r_+ a\tau}{2\pi k_+(\lambda - r_+\alpha(a+l))(\lambda + r_+\alpha(a-l))}\,m^2\delta_{m,-n}\,,\\
\delta\mathcal{Q}(g_n, \mathcal{L}_{g_m} g; g) &= i\,\frac{\lambda^2 r_+ a\tau^2}{4\pi^2 k_+(\lambda - r_+\alpha(a+l))(\lambda + r_+\alpha(a-l))}\,m\delta_{m,-n}\,.
\end{aligned}$$

# Extremal Kerr-like spacetimes/WCFT

- In a standard form of the most general warped conformal algebra

$$
\begin{aligned}
\{L_n^{ext}, L_m^{ext}\} &= (m-n)L_{m+n}^{ext} + i\frac{c^{ext}}{12}m^3\delta_{m,-n}\,,\\
\{L_n^{ext}, P_m^{ext}\} &= mP_{m+n}^{ext} + i\varkappa^{ext}m^2\delta_{m,-n}\,,\\
\{P_n^{ext}, P_m^{ext}\} &= i\frac{\kappa^{ext}}{2}m\delta_{m,-n}\,,
\end{aligned}
$$

the central charges now read as

$$
\begin{aligned}
c^{ext} &= 0,\\
\varkappa^{ext} &= \frac{\lambda^2 r_+ a\tau}{2\pi k_+(\lambda - r_+\alpha(a+l))(\lambda + r_+\alpha(a-l))}\,,\\
\kappa^{ext} &= \frac{\lambda^2 r_+ a\tau^2}{2\pi^2 k_+(\lambda - r_+\alpha(a+l))(\lambda + r_+\alpha(a-l))}\,.
\end{aligned}
$$

# Extremal Kerr-like spacetimes/WCFT

- The mixed central terms can be eliminated by a charge redefinition

$$L_n^{*ext} = L_n^{ext} + \frac{2\varkappa^{ext}}{\kappa^{ext}} n P_n^{ext}, \qquad P_n^{*ext} = P_n^{ext} \, .$$

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$$L_n^{*ext} = L_n^{ext} + \frac{2\varkappa^{ext}}{\kappa^{ext}} n P_n^{ext}, \qquad P_n^{*ext} = P_n^{ext} .$$

which leads to the standard warped conformal algebra

$$\begin{aligned} \{L_n^{*ext}, L_m^{*ext}\} &= (m-n) L_{m+n}^{*ext} + i\frac{c^{*ext}}{12} m^3 \delta_{m,-n} , \\ \{L_n^{*ext}, P_m^{*ext}\} &= m P_{m+n}^{*ext} , \\ \{P_n^{*ext}, P_m^{*ext}\} &= i\frac{\kappa^{*ext}}{2} m \delta_{m,-n} . \end{aligned}$$

with

$$\begin{aligned} c^{*ext} &= \frac{12\lambda^2 r_+ a}{k_+(\lambda - r_+\alpha(a+l))(\lambda + r_+\alpha(a-l))} , \\ \kappa^{*ext} &= \frac{\lambda^2 r_+ a \tau^2}{2\pi^2 k_+(\lambda - r_+\alpha(a+l))(\lambda + r_+\alpha(a-l))} . \end{aligned}$$

# Non-extremal Kerr-like spacetimes/WCFT

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- In the non-extremal case, the conformal symmetry hides in the solution space of perturbations.

# Non-extremal Kerr-like spacetimes/WCFT

- For a non-extreme case, the near horizon geometry is a Rindler space without conformal symmetry.
- A geometry with conformal symmetry is not a **necessary condition** for the interactions to exhibit conformal invariance.
- In the non-extremal case, the conformal symmetry hides in the solution space of perturbations.
- Consider a scalar perturbation $\Phi = e^{-i\omega t + im\phi} S(\theta) R(r)$ on Kerr. In the low frequency limit $\omega \ll 1/M$, divide the radial coordinate into near and far regions, the hidden conformal symmetry acts on the near region field solution space. [Castro, Maloney and Strominger, 2010]

far region

$r \gg M$ matching region $r \ll 1/\omega$

near region

- The hidden conformal symmetry from conformally coupled scalar perturbations. The KG equation

$$(\nabla_\mu \nabla^\mu - \chi R)\Phi = 0\,,$$

is not separable due to the metric conformal factor.

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is not separable due to the metric conformal factor. Therefor we perform a Weyl transformation $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ and $\tilde{\Phi} = \Omega^{-1}\Phi$, so that the transformed scalar equation [X. Jiang and JX, 2024]

$$(\tilde{\nabla}_\mu \tilde{\nabla}^\mu - \chi \tilde{R})\tilde{\Phi} = 0\,,$$

is separable and the separated angular and radial equations under the ansatz $\tilde{\Phi} = e^{-i\omega t + im\phi} R(r) S(x)$ are

# Non-extremal Kerr-like spacetimes/WCFT

- $$\frac{d}{dr}\left(Q\frac{dR(r)}{dr}\right)+\left[\frac{\left((r^2+(a+l)^2)\omega-am\right)^2}{Q}+\frac{Q''}{6}\right]R(r)=\mathcal{K}R(r)\,,$$

$$\frac{d}{dx}\left((1-x^2)P\frac{dS(x)}{dx}\right)-\left[\frac{\left(\left(a(1-x^2)+2l(1+x)\right)\omega-m\right)^2}{(1-x^2)P}\right.$$
$$\left.+\frac{2P+4xP'-(1-x^2)P''}{6}\right]S(x)=-\mathcal{K}S(x)\,,$$

which are all Heun-type equations.

# Non-extremal Kerr-like spacetimes/WCFT

- Near the horizon, the radial equation can be approximately written as

$$
\begin{aligned}
&\frac{d}{dr}\left((r-r_+)(r-r_s)\frac{dR(r)}{dr}\right)+\left[\frac{\left((r_+^2+(a+l)^2)\omega-am\right)^2}{(r-r_+)(r_+-r_s)k_+^2}-\frac{\left((r_s^2+(a+l)^2)\omega-am\right)^2}{(r-r_s)(r_+-r_s)k_+^2}\right.\\
&\left.+\left(r^2+(r_++r_s)r+r_+^2+r_s^2+r_+r_s+2(a+l)^2\right)\frac{\omega^2}{k_+^2}-\frac{2am\omega}{k_+^2}+\frac{1}{3k_+}+\mathcal{O}(r-r_+)\right]R(r)\\
&=\frac{\mathcal{K}}{k_+}R(r)\,.
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$$\frac{d}{dr}\left((r-r_+)(r-r_s)\frac{dR(r)}{dr}\right)+\left[\frac{\left((r_+^2+(a+l)^2)\omega-am\right)^2}{(r-r_+)(r_+-r_s)k_+^2}-\frac{\left((r_s^2+(a+l)^2)\omega-am\right)^2}{(r-r_s)(r_+-r_s)k_+^2}\right.$$
$$\left.+\left(r^2+(r_++r_s)r+r_+^2+r_s^2+r_+r_s+2(a+l)^2\right)\frac{\omega^2}{k_+^2}-\frac{2am\omega}{k_+^2}+\frac{1}{3k_+}+\mathcal{O}(r-r_+)\right]R(r)$$
$$=\frac{\mathcal{K}}{k_+}R(r)\,.$$

The potential term can be further simplified by imposing the near region and low frequency condition $r \ll 1/\omega$

$$\frac{d}{dr}\left(\Delta\frac{dR(r)}{dr}\right)+\left[\frac{\left((r_+^2+(a+l)^2)\omega-am\right)^2}{(r-r_+)(r_+-r_s)k_+^2}-\frac{\left((r_s^2+(a+l)^2)\omega-am\right)^2}{(r-r_s)(r_+-r_s)k_+^2}\right]R(r)$$
$$=\mathcal{K}'R(r)\,.$$

# Non-extremal Kerr-like spacetimes/WCFT

- The reduced radial equation can be mapped to the Casimir equation of $SL(2, R)$ satisfied by $\Psi = e^{-i\omega t + im\phi} R(r)$

$$\left(-H_0^2 + \frac{1}{2}(H_+ H_- + H_- H_+)\right) \Psi = \mathcal{K}' \Psi \,,$$

# Non-extremal Kerr-like spacetimes/WCFT

- The reduced radial equation can be mapped to the Casimir equation of $SL(2,R)$ satisfied by $\Psi = e^{-i\omega t + im\phi} R(r)$

$$\left(-H_0^2 + \frac{1}{2}(H_+ H_- + H_- H_+)\right)\Psi = \mathcal{K}'\Psi\,,$$

The conformal generators are defined in terms of conformal coordinates

$$H_+ = i\frac{\partial}{\partial\omega^+}, \qquad \bar{H}_+ = i\frac{\partial}{\partial\omega^-}\,,$$

$$H_0 = i\left(\omega^+\frac{\partial}{\partial\omega^+} + \frac{y}{2}\frac{\partial}{\partial y}\right), \qquad \bar{H}_0 = i\left(\omega^-\frac{\partial}{\partial\omega^-} + \frac{y}{2}\frac{\partial}{\partial y}\right)\,,$$

$$H_- = i\left((\omega^+)^2\frac{\partial}{\partial\omega^+} + \omega^+ y\frac{\partial}{\partial y} - y^2\frac{\partial}{\partial\omega^-}\right), \quad \bar{H}_- = i\left((\omega^-)^2\frac{\partial}{\partial\omega^-} + \omega^- y\frac{\partial}{\partial y} - y^2\frac{\partial}{\partial\omega^+}\right).$$

# Non-extremal Kerr-like spacetimes/WCFT

- The conformal coordinates are defined as

$$\omega^+ = \sqrt{\frac{r-r_+}{r-r_s}}e^{2\pi T_R\phi+2n_Rt}, \quad T_R = \frac{k_+(r_+-r_s)}{4\pi a}, \quad n_R = 0\,,$$
$$\omega^- = \sqrt{\frac{r-r_+}{r-r_s}}e^{2\pi T_L\phi+2n_Lt}, \quad T_L = \frac{k_+(r_+^2+r_s^2+2(a+l)^2)}{4\pi a(r_++r_s)}, \quad n_L = -\frac{k_+}{2(r_++r_s)}\,,$$
$$y = \sqrt{\frac{r_+-r_s}{r-r_s}}e^{\pi(T_R+T_L)\phi+(n_R+n_L)t}\,,$$

# Non-extremal Kerr-like spacetimes/WCFT

- The conformal coordinates are defined as

$$
\begin{aligned}
\omega^+ &= \sqrt{\frac{r-r_+}{r-r_s}}\, e^{2\pi T_R \phi + 2 n_R t}, \quad T_R = \frac{k_+(r_+ - r_s)}{4\pi a}, \quad n_R = 0\,, \\
\omega^- &= \sqrt{\frac{r-r_+}{r-r_s}}\, e^{2\pi T_L \phi + 2 n_L t}, \quad T_L = \frac{k_+(r_+^2 + r_s^2 + 2(a+l)^2)}{4\pi a (r_+ + r_s)}, \quad n_L = -\frac{k_+}{2(r_+ + r_s)}\,, \\
y &= \sqrt{\frac{r_+ - r_s}{r - r_s}}\, e^{\pi (T_R + T_L)\phi + (n_R + n_L) t}\,,
\end{aligned}
$$

The allowed generators for warped conformal symmetry

$$
\begin{aligned}
H_+ &= i \frac{\partial}{\partial \omega^+}\,, \\
H_0 &= i\left(\omega^+ \frac{\partial}{\partial \omega^+} + \frac{y}{2}\frac{\partial}{\partial y}\right), \qquad \bar{H}_0 = i\left(\omega^- \frac{\partial}{\partial \omega^-} + \frac{y}{2}\frac{\partial}{\partial y}\right)\,, \\
H_- &= i\left((\omega^+)^2 \frac{\partial}{\partial \omega^+} + \omega^+ y \frac{\partial}{\partial y} - y^2 \frac{\partial}{\partial \omega^-}\right)\,.
\end{aligned}
$$

# Non-extremal Kerr-like spacetimes/WCFT

- The local symmetries are represented by the vector fields which preserve the Casimir operator and $\bar{H}_0$

$$\xi(l(\omega^+)) = l(\omega^+)\frac{\partial}{\partial\omega^+} + \frac{\partial l(\omega^+)}{\partial\omega^+}\frac{y}{2}\frac{\partial}{\partial y}\,,$$
$$\zeta(p(\omega^+)) = p(\omega^+)\left(\omega^-\frac{\partial}{\partial\omega^-} + \frac{y}{2}\frac{\partial}{\partial y}\right)\,,$$

# Non-extremal Kerr-like spacetimes/WCFT

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$$\zeta(p(\omega^+)) = p(\omega^+)\left(\omega^-\frac{\partial}{\partial\omega^-} + \frac{y}{2}\frac{\partial}{\partial y}\right),$$

Their Fourier modes satisfy the classical Virasoro and $U(1)$ Kac-Moody algebra

$$i[l_m, l_n] = (m-n)l_{m+n}, \qquad l_n = \xi\left(l(\omega^+) = 2\pi T_R(\omega^+)^{1+\frac{in}{2\pi T_R}}\right),$$
$$i[l_m, p_n] = -np_{m+n}, \qquad p_n = \zeta\left(p(\omega^+) = (\omega^+)^{\frac{in}{2\pi T_R}}\right),$$
$$i[p_m, p_n] = 0.$$

# Non-extremal Kerr-like spacetimes/WCFT

- In contrast to the extremal case, we need an additional counterterm in defining the covariant charge variation to carry out the local symmetry algebra with central extensions in the non-extremal case [X. Jiang and JX, 2024]

$$\delta\mathcal{Q} = \delta\mathcal{Q}_{IW} + \delta\mathcal{Q}_{ct}\,.$$

with

$$\delta\mathcal{Q}_{ct}(\xi, h; g) = \frac{1}{16\pi}\int_{\partial\Sigma} i_\xi \cdot (\,^*X), \qquad X = 2h^\nu{}_\mu \nabla_\rho N^\rho{}_\nu dx^\mu\,,$$

# Non-extremal Kerr-like spacetimes/WCFT

- Define the charge variations with respect to the mode vectors

$$\delta L_n = \delta \mathcal{Q}(l_n, h; g), \qquad \delta P_n = \delta \mathcal{Q}(p_n, h; g)\,,$$

# Non-extremal Kerr-like spacetimes/WCFT

- Define the charge variations with respect to the mode vectors

$$\delta L_n = \delta \mathcal{Q}(l_n, h; g), \qquad \delta P_n = \delta \mathcal{Q}(p_n, h; g)\,,$$

the Dirac brackets among these charges form the standard warped conformal algebra

$$\begin{aligned}
\{L_n, L_m\} &= (m-n)L_{m+n} + i\frac{c}{12}m^3\delta_{m,-n}\,, \\
\{L_n, P_m\} &= mP_{m+n}\,, \\
\{P_n, P_m\} &= i\frac{\kappa}{2}m\delta_{m,-n}\,,
\end{aligned}$$

with

$$\begin{aligned}
c &= \frac{6\lambda^2(r_+ + r_s)a}{k_+\left(\lambda - r_+\alpha(a+l)\right)\left(\lambda + r_+\alpha(a-l)\right)}\,, \\
\kappa &= -\frac{\lambda^2(r_+ + r_s)a}{k_+\left(\lambda - r_+\alpha(a+l)\right)\left(\lambda + r_+\alpha(a-l)\right)}\,.
\end{aligned}$$

# Non-extremal Kerr-like spacetimes/WCFT

- Comparing the central charges

$$c = \frac{6\lambda^2 (r_+ + r_s) a}{k_+ \left(\lambda - r_+ \alpha (a + l)\right) \left(\lambda + r_+ \alpha (a - l)\right)},$$

$$\kappa = -\frac{\lambda^2 (r_+ + r_s) a}{k_+ \left(\lambda - r_+ \alpha (a + l)\right) \left(\lambda + r_+ \alpha (a - l)\right)},$$

to the extremal case

$$c^{*ext} = \frac{12\lambda^2 r_+ a}{k_+ (\lambda - r_+ \alpha (a + l))(\lambda + r_+ \alpha (a - l))},$$

$$\kappa^{*ext} = \frac{\lambda^2 r_+ a \tau^2}{2\pi^2 k_+ (\lambda - r_+ \alpha (a + l))(\lambda + r_+ \alpha (a - l))},$$

# Non-extremal Kerr-like spacetimes/WCFT

- Comparing the central charges

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\end{aligned}
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\end{aligned}
$$

one find the time periodicity $\tau = 2\pi i$, which is also a requirement from the horizon smoothness condition in the Euclidean signature.

# Horizon entropy from WCFT modularity

- The global warped conformal symmetry is spontaneously broken to $U(1) \times U(1)$ due to the $2\pi$ identification along $\phi$, these are the translational symmetries in the dual WCFT with coordinates

$$t^{+} = 2\pi T_R \phi + 2 n_R t, \qquad t^{-} = -2\pi T_L \phi - 2 n_L t \,.$$

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The WCFT is defined on a torus

$$(t, \phi) \sim (t, \phi + 2\pi) \sim (t + i\beta, \phi + i\beta\Omega_H)\,,$$

$$(t^{+}, t^{-}) \sim (t^{+} + 4\pi^2 T_R, t^{-} - 4\pi^2 T_L) \sim (t^{+} + 2\pi i, t^{-} + 2\pi i)\,,$$

on which the partition functions have modular properties.

# Horizon entropy from WCFT modularity

- Generically, the WCFT torus partition function can be written as

$$Z_{\bar{\ell}|\ell}(\bar{\tau}|\tau) = \mathrm{Tr}_{\bar{\ell}|\ell}\left(e^{2\pi i\bar{\tau}P_0}e^{-2\pi i\tau L_0}\right) ,$$

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which is related to the canonical partition function through

$$\begin{aligned} Z_{\bar{\ell}|\ell}(\bar{\tau}|\tau) &= e^{\pi i\kappa\bar{\ell}\left(\bar{\tau}-\frac{\bar{\ell}\tau}{2\ell}\right)} Z_{0|1}\left(\bar{\tau}-\frac{\bar{\ell}\tau}{\ell}\Big|\frac{\tau}{\ell}\right), \qquad \text{WCT} \\ &= e^{\pi i\kappa\frac{\ell\bar{\tau}^2}{2\tau}} Z_{0|1}\left(\frac{\ell\bar{\tau}}{\tau}-\bar{\ell}\Big|-\frac{\ell}{\tau}\right), \qquad \text{ST} \end{aligned}$$

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In the $\tau \to -i0$ limit, the thermal entropy can be calculated by the vacuum charges

$$\begin{aligned} S &= (1-\tau\partial_\tau-\bar{\tau}\partial_{\bar{\tau}})\log Z \\ &= 4\pi^2 i(T_R+T_L)\hat{P}_0^{vac} + 8\pi^2 T_R\hat{L}_0^{vac}. \end{aligned}$$

# Horizon entropy from WCFT modularity

- The vacuum charges are related by the WCFT spectral flow [Detournay, Hartman, and Hofman, 2012]

$$\hat{L}_0^{vac} = -\frac{c}{24} + \frac{(\hat{P}_0^{vac})^2}{\kappa},$$

# Horizon entropy from WCFT modularity

- The vacuum charges are related by the WCFT spectral flow [Detournay, Hartman, and Hofman, 2012]

$$\hat{L}_0^{vac} = -\frac{c}{24} + \frac{(\hat{P}_0^{vac})^2}{\kappa} \,,$$

$\hat{L}_0$ is proportional to the angular momentum thus has vanishing vacuum value

$$\hat{L}_0^{vac} = 0, \quad (\hat{P}_0^{vac})^2 = -\left(\frac{\lambda^2(r_+ + r_s)a}{2k_+\left(\lambda - r_+\alpha(a+l)\right)\left(\lambda + r_+\alpha(a-l)\right)}\right)^2 \,.$$

The WCFT entropy match the horizon entropy by the area law [X. Jiang and JX, 2024]

$$S = 4\pi^2|\hat{P}_0^{vac}|(T_R + T_L) = \frac{\pi\lambda^2(r_+^2 + (a+l)^2)}{(\lambda - r_+\alpha(a+l))(\lambda + r_+\alpha(a-l))} = S_{BH} \,.$$

# Bulk absorption probability from WCFT

- Consider the scattering process of a scalar filed $\Phi$ originating in the asymptotically flat region of a Kerr black hole. The Klein-Gordon particle number flux

$$\mathcal{F} = \int \sqrt{-g} J^r d\theta d\phi \,, \qquad \text{with } J^\mu = \frac{i}{8\pi}(\Phi^* \nabla^\mu \Phi - \Phi \nabla^\mu \Phi^*) \,.$$

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The absorption cross section of the scalar scattering process

$$\sigma_{abs}^{bulk} = \frac{\mathcal{F}_{in} - \mathcal{F}_{out}}{\mathcal{F}_{in}} \,.$$

# Bulk absorption probability from WCFT

- On the Kerr background, the scalar wave equation is separable

$$\left[\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta) - \frac{m^2}{\sin^2\theta} + \omega^2 a^2 \cos^2\theta\right] S(\theta) = -K_\ell\,,$$

$$\left[\frac{d}{dr}\Delta\frac{d}{dr} + \frac{(2Mr_+\omega - am)^2}{(r-r_+)(r_+-r_-)} - \frac{(2Mr_-\omega - am)^2}{(r-r_-)(r_+-r_-)} \right.$$
$$\left. + (r^2 + 2M(r+2M))\omega^2\right] R(r) = K_\ell R(r)\,.$$

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$$\Bigg[\frac{d}{dr}\Delta\frac{d}{dr} + \frac{(2Mr_+\omega - am)^2}{(r-r_+)(r_+-r_-)} - \frac{(2Mr_-\omega - am)^2}{(r-r_-)(r_+-r_-)} + (r^2 + 2M(r+2M))\omega^2\Bigg] R(r) = K_\ell R(r)\,.$$

Consider the low frequency limit when $\omega \ll 1/M$ and in this case $K_\ell = \ell(\ell-1), \ell = -m+1, \cdots, m+1$. In the far region $r \gg M$, the radial wave equation is the spherical Bessel's equation

$$\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right) + r^2\omega^2 R(r) = \ell(\ell-1)R(r)\,.$$

# Bulk absorption probability from WCFT

- There are two linearly independent solutions of the far region radial equation [Nian and Tian, 2023]

$$R_{far}(r) = A_{far}\frac{1}{\sqrt{\omega r}}\mathrm{J}_{-\ell+\frac{1}{2}}(\omega r) + B_{far}\frac{1}{\sqrt{\omega r}}\mathrm{J}_{\ell-\frac{1}{2}}(\omega r)\,,$$

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Separate the pure outgoing and pure ingoing parts

$$\begin{aligned} R_{far}(r) = {} & Z_{out}\left(ie^{i\pi\ell}\frac{1}{\sqrt{\omega r}}\mathrm{J}_{-\ell+\frac{1}{2}}(\omega r) + \frac{1}{\sqrt{\omega r}}\mathrm{J}_{\ell-\frac{1}{2}}(\omega r)\right) \\ & + Z_{in}\left(-ie^{-i\pi\ell}\frac{1}{\sqrt{\omega r}}\mathrm{J}_{-\ell+\frac{1}{2}}(\omega r) + \frac{1}{\sqrt{\omega r}}\mathrm{J}_{\ell-\frac{1}{2}}(\omega r)\right)\,, \end{aligned}$$

where

$$\begin{aligned} Z_{out} &= \frac{1}{2\cos(\pi\ell)}(-iA_{far} + e^{-i\pi\ell}B_{far})\,, \\ Z_{in} &= \frac{1}{2\cos(\pi\ell)}(iA_{far} + e^{i\pi\ell}B_{far})\,. \end{aligned}$$

# Bulk absorption probability from WCFT

- The outgoing and ingoing scalar modes at the outer boundary of the far region

$$\Phi_{out}(r \to \infty) = \sum_{\ell,m} e^{-i\omega t + im\phi} S_\ell(\theta) Z_{out} \left( \sqrt{\frac{2}{\pi}} \frac{e^{i\omega r + i\pi\ell/2}}{\omega r} \cos(\pi\ell) \right) ,$$

$$\Phi_{in}(r \to \infty) = \sum_{\ell,m} e^{-i\omega t + im\phi} S_\ell(\theta) Z_{in} \left( \sqrt{\frac{2}{\pi}} \frac{e^{-i\omega r - i\pi\ell/2}}{\omega r} \cos(\pi\ell) \right) .$$

# Bulk absorption probability from WCFT

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The outgoing and ingoing Klein-Gordon particle fluxes

$$\mathcal{F}_{out} \propto |Z_{out}|^2 , \quad \mathcal{F}_{in} \propto |Z_{in}|^2 ,$$

# Bulk absorption probability from WCFT

- So the absorption cross section is fixed by the coefficients

$$\sigma_{abs}^{bulk} = \frac{\mathcal{F}_{in} - \mathcal{F}_{out}}{\mathcal{F}_{in}} = 1 - \frac{|Z_{out}|^2}{|Z_{in}|^2}$$
$$= \frac{2i\cos(\pi\ell)(A_{far}B^*_{far} - B_{far}A^*_{far})}{|iA_{far} + e^{i\pi\ell}B_{far}|^2}\,.$$

# Bulk absorption probability from WCFT

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What is the relation between these coefficients and the black hole parameters?

# Bulk absorption probability from WCFT

- The coefficients can be determined by the near-far matching.

# Bulk absorption probability from WCFT

- The coefficients can be determined by the near-far matching.
- In the near region $r \ll 1/\omega$, the radial wave equation becomes a hypergeometric equation

$$\left[\frac{d}{dr}\Delta\frac{d}{dr}+\frac{(2Mr_+\omega-am)^2}{(r-r_+)(r_+-r_-)}-\frac{(2Mr_-\omega-am)^2}{(r-r_-)(r_+-r_-)}\right]R(r)=\ell(\ell-1)R(r)\,,$$

with solution (imposing ingoing B.C. at horizon)

$$R_{near}(r)=\left(\frac{r-r_+}{r-r_-}\right)^{-i\frac{2Mr_+\omega-am}{r_+-r_-}}(r-r_-)^{-\ell}$$
$$\times_2F_1\left(\ell-i\frac{4M^2\omega-2am}{r_+-r_-},\ell-i2M\omega;1-i\frac{4Mr_+\omega-2am}{r_+-r_-};\frac{r-r_+}{r-r_-}\right).$$

# Bulk absorption probability from WCFT

- The near region radial wave solution has the following large $r$ behaviour

$$R_{near}(r \gg M) \sim A_{near} r^{-\ell} + B_{near}(r_+ - r_-)^{-2\ell+1} r^{\ell-1} \, ,$$

where

$$A_{near} = \frac{\Gamma\left(1 - i\frac{4Mr_+\omega - 2am}{r_+ - r_-}\right)\Gamma(-2\ell+1)}{\Gamma(1-\ell-i2M\omega)\Gamma\left(1-\ell-i\frac{4M^2\omega-2am}{r_+-r_-}\right)} \, ,$$

$$B_{near} = \frac{\Gamma\left(1 - i\frac{4Mr_+\omega - 2am}{r_+ - r_-}\right)\Gamma(2\ell-1)}{\Gamma(\ell-i2M\omega)\Gamma\left(\ell-i\frac{4M^2\omega-2am}{r_+-r_-}\right)} \, .$$

# Bulk absorption probability from WCFT

- The near-far matching:

far region

$r \gg M$ matching region $r \ll 1/\omega$

near region

Compering the $r$ dependencies of the near and far region solutions in the matching region. They are the two limiting cases of a single full solution which are required to be equal in the matching region

$$R_{near}(r \gg M) \sim R_{far}(r \to 0)\,.$$

# Bulk absorption probability from WCFT

- Thus the near and far asymptotic expansion coefficients are related

$$A_{far} = 2^{-\ell+\frac{1}{2}}\Gamma\left(-\ell+\frac{3}{2}\right)\omega^{\ell}A_{near}\,,$$
$$B_{far} = 2^{\ell-\frac{1}{2}}\Gamma\left(\ell+\frac{1}{2}\right)(r_{+}-r_{-})^{-2\ell+1}\omega^{-\ell+1}B_{near}\,.$$

# Bulk absorption probability from WCFT

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$$B_{far} = 2^{\ell-\frac{1}{2}}\Gamma\left(\ell+\frac{1}{2}\right)(r_+ - r_-)^{-2\ell+1}\omega^{-\ell+1}B_{near}\,.$$

With the above relations and $\omega \ll 1/M$, the absorption cross section finally can be written as (superradiant scattering $0<\omega<m\frac{a}{2Mr_+}$)

$$\begin{aligned}\sigma^{bulk}_{abs} &= \frac{2i\cos(\pi\ell)(A_{far}B^*_{far} - B_{far}A^*_{far})}{|B_{far}|^2} \\ &= \frac{(\omega/2)^{2\ell-1}(r_+ - r_-)^{2\ell-1}}{\Gamma\left(\ell+\frac{1}{2}\right)^2\Gamma(2\ell-1)^2}\sinh\left(\pi\frac{4Mr_+\omega - 2am}{r_+ - r_-}\right) \\ &\times|\Gamma(\ell - i2M\omega)|^2\left|\Gamma\left(\ell - i\frac{4M^2\omega - 2am}{r_+ - r_-}\right)\right|^2.\end{aligned}$$

# Bulk absorption probability from WCFT

- The finite temperature warped CFT coordinates in the Kerr case

$$X = 2\pi T_R \phi, \qquad Y = \frac{1}{2M} t - 2\pi(T_L + \bar{q} T_R)\phi, \qquad T_{R,L} = \frac{r_+ \mp r_-}{4\pi a}.$$

# Bulk absorption probability from WCFT

- The finite temperature warped CFT coordinates in the Kerr case

$$X = 2\pi T_R \phi, \qquad Y = \frac{1}{2M} t - 2\pi (T_L + \bar{q} T_R)\phi, \qquad T_{R,L} = \frac{r_+ \mp r_-}{4\pi a}\,.$$

- In a field theory, the Fermis Golden rule gives the transition rate out of the thermal states

$$\mathcal{R} = 2\pi \sum_{\ell m} |J_{\ell m}|^2 \int dX dY e^{i\Omega X - iQY} G(X, Y)\,,$$

where

$$\Omega = \frac{4M^2\omega - 2am}{r_+ - r_-} + 2M\omega\bar{q}, \qquad Q = -2M\omega\,,$$

and $G(X, Y)$ is the two point correlation function. For the warped CFT [W. Song and JX, 2018]

$$G(X, Y) \sim \mathcal{C}_{\mathcal{O}} (-1)^{\delta} e^{iQ(Y + \bar{q}X)} \left( \frac{\beta}{\pi} \sinh \frac{\pi X}{\beta} \right)^{-2\delta}\,.$$

# Bulk absorption probability from WCFT

- The absorption cross section for the thermal states in the warped CFT

$$\sigma_{abs} \sim \int dXdYe^{i\Omega X - iQY}\left[G(X - i\epsilon, Y - i\epsilon) - G(X + i\epsilon, Y + i\epsilon)\right],$$

$\mp i\epsilon$ correspond to absorption and emission [S. Gubser, 1997].

# Bulk absorption probability from WCFT

- The absorption cross section for the thermal states in the warped CFT

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$\mp i\epsilon$ correspond to absorption and emission [S. Gubser, 1997].
The form of the absorption cross section in warped CFT

$$\sigma_{abs} \sim \frac{e^{\pi Q}}{\Gamma(2\delta)} \sinh(\pi(\Omega + Q\bar{q} - Q))|\Gamma(\delta - i(\Omega + Q\bar{q}))|^2,$$

matches the bulk absorption cross section in the $\Omega$ dependence [JX, 2023]

$$\sigma_{abs}^{bulk} \sim \sinh\left(\pi\frac{4Mr_+\omega - 2am}{r_+ - r_-}\right)\left|\Gamma\left(\ell - i\frac{4M^2\omega - 2am}{r_+ - r_-}\right)\right|^2,$$

given $\delta = \ell$.

# Summary

- The Kerr-like spacetimes with extremal horizons have infinite scaling regions near their horizons where consistent boundary conditions can be imposed to manifest the asymptotic symmetry group as warped conformal symmetries.
- In the non-extremal case, these spacetimes have hidden conformal symmetry from which the local warped conformal symmetries can be recovered from the vector fields that keep the Casimir operator and scalar frequency invariant. The covariant charges with additional counterterms form the symmetry algebra with well defined central charges.
- The black hole entropy as well as the absorption probability in a bulk scattering process can be shown to be reproduced from warped CFT calculations.
- Future directions include relating the Heun-type equations to the Virasoro and Kac-Moody blocks' equations, evaluating black hole QNM using warped CFT methods, higher dimensional black holes, $\cdots$

Thank You