Renormalization-Group Approach to Matter Effects on Neutrino Oscillations

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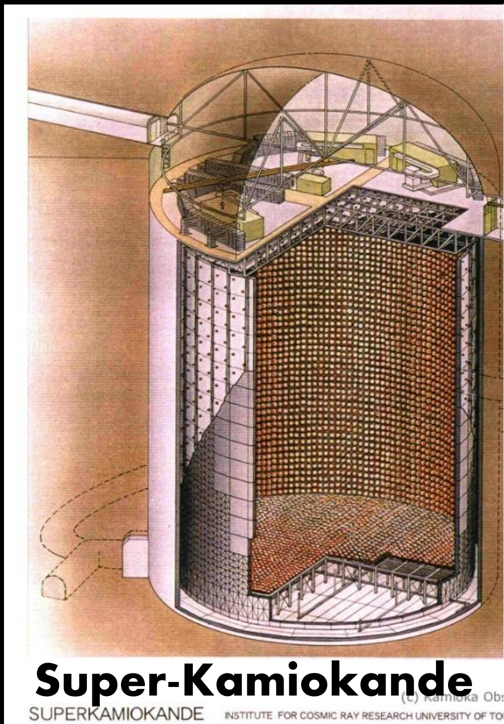
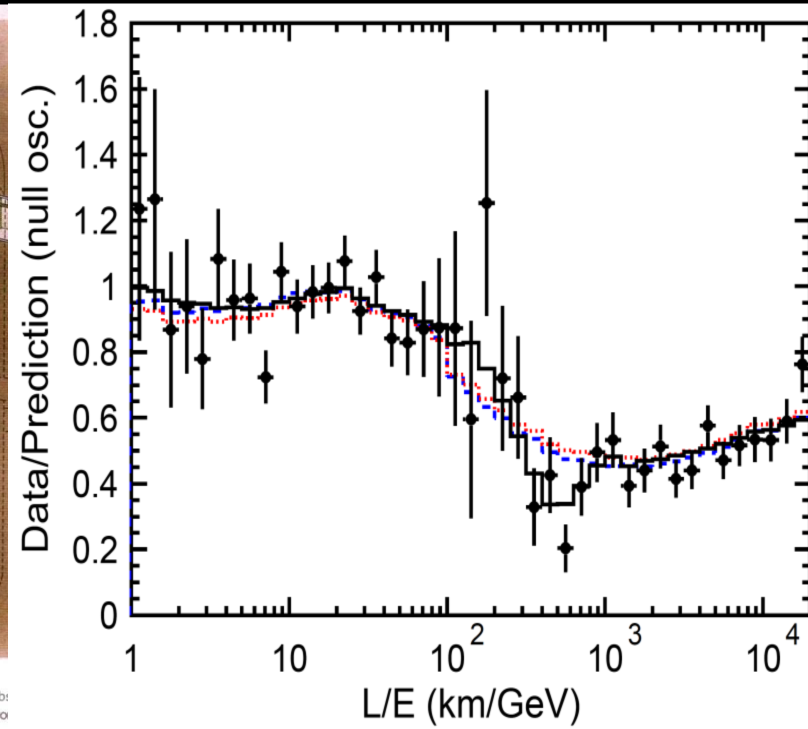
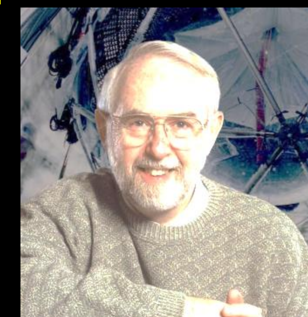
Solar & Atmospheric Neutrino Oscillations



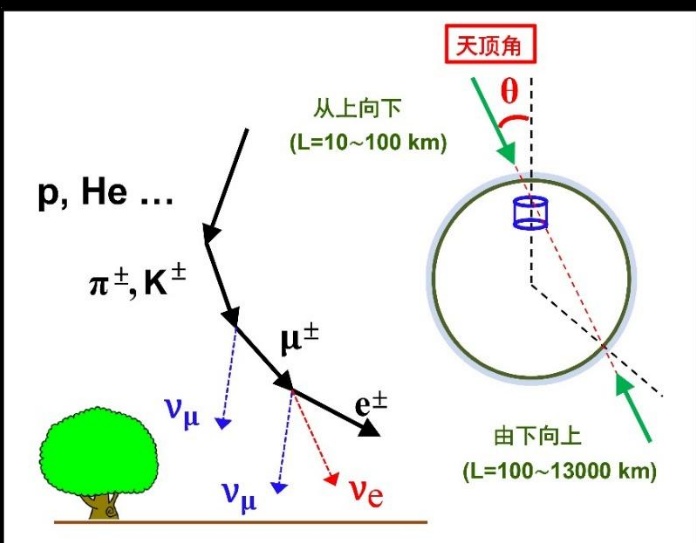


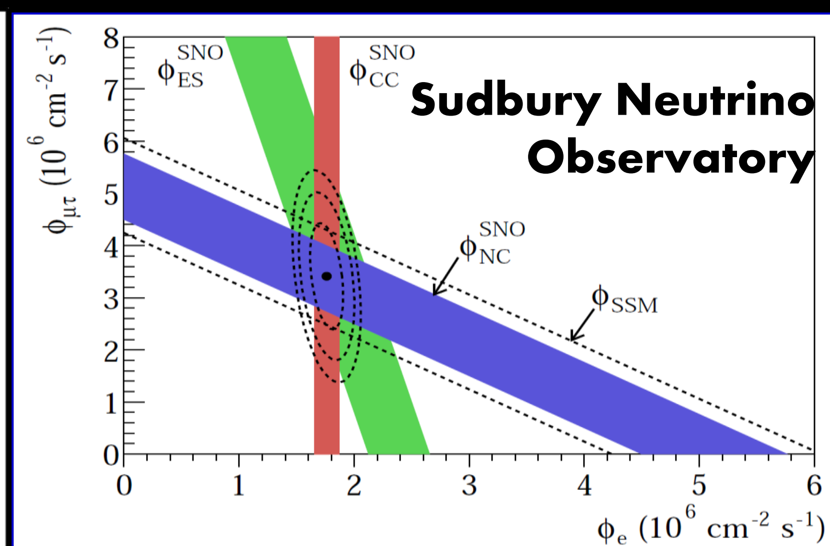
Takaaki Kajita

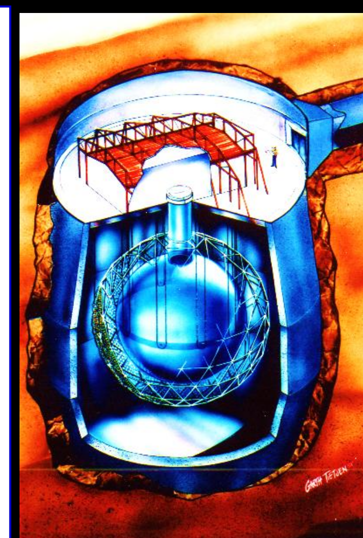
Nobel Prize in Physics 15



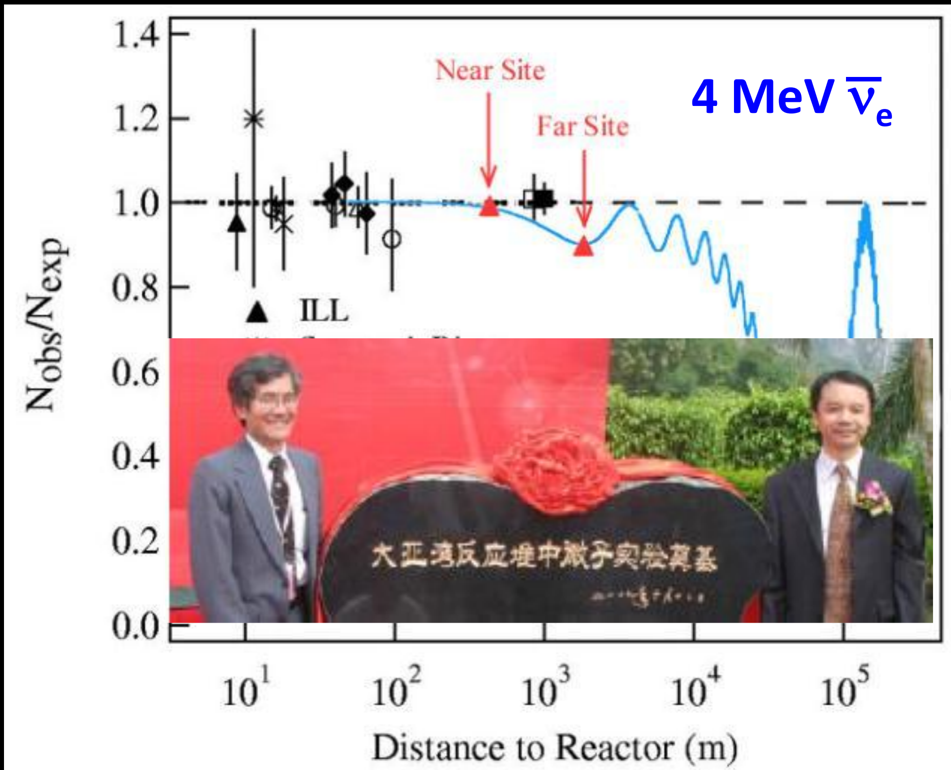
Arthur McDonald







Reactor Neutrino Oscillations

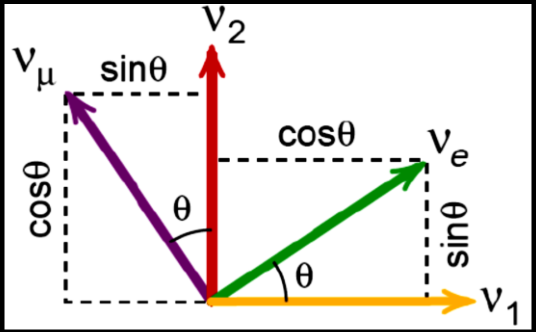


Dec. 2011	Double Chooz (far detector):	$\sin^2\theta_{13} = 0.022 \pm 0.013$	1.7σ
Mar. 2012	Daya Bay (near + far detectors):	$\sin^2\theta_{13} = 0.024 \pm 0.004$	5.2σ
Apr. 2012	RENO (near + far detectors):	$\sin^2\theta_{13} = 0.029 \pm 0.006$	4.9σ

Two-flavor Oscillations: Massive Neutrinos

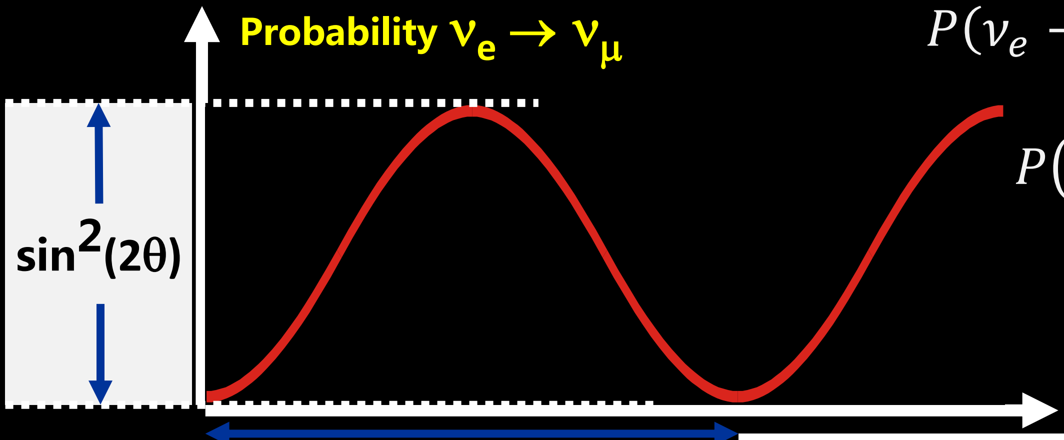
The oscillation probability for the appearance channel

$$P(\nu_e \to \nu_\mu) \equiv |\langle \nu_\mu | \nu_e(t) \rangle|^2$$
$$= 2\sin^2\theta\cos^2\theta\left(1 - \cos\frac{\Delta m^2 t}{2E}\right)$$
$$= \sin^2 2\theta \sin^2\frac{\Delta m^2 L}{4E}$$



$L \approx t$ in the relativistic limit

The survival probability for the disappearance channel



Oscillation Length

$$\frac{4\pi E}{\Delta m^2} = 2.5\,\mathrm{m}\left(\frac{E}{\mathrm{MeV}}\right)\left(\frac{\mathrm{eV}^2}{\Delta m^2}\right)$$

$$P(\nu_e \to \nu_e) \equiv 1 - \sin^2 2\theta \sin^2\frac{1.27\Delta m^2 L}{E}$$

$$P(\nu_e \to \nu_\mu) \equiv \sin^2 2\theta \sin^2\frac{1.27\Delta m^2 L}{E}$$

Quantities	Units
Δm^2	eV^2
E	GeV (MeV)
L	km (m)

Status of Neutrino Oscillations

$m_1 < m_2 < m_3$ (NO) or $m_3 < m_1 < m_2$ (IO)

NuFIT 4.1 (2019)

without SK atmospheric data		Normal Ordering (best fit)		Inverted Ordering ($\Delta\chi^2 = 6.2$)	
		bfp $\pm 1\sigma$	3σ range	bfp $\pm 1\sigma$	3σ range
	$\sin^2\theta_{12}$	$0.310^{+0.013}_{-0.012}$	$0.275 \to 0.350$	$0.310^{+0.013}_{-0.012}$	$0.275 \to 0.350$
	$\theta_{12}/^\circ$	$33.82^{+0.78}_{-0.76}$	$31.61 \to 36.27$	$33.82^{+0.78}_{-0.76}$	$31.61 \to 36.27$
	$\sin^2\theta_{23}$	$0.558^{+0.020}_{-0.033}$	$0.427 \to 0.609$	$0.563^{+0.019}_{-0.026}$	$0.430 \to 0.612$
	$\theta_{23}/^\circ$	$48.3^{+1.1}_{-1.9}$	$40.8 \to 51.3$	$48.6^{+1.1}_{-1.5}$	$41.0 \to 51.5$
	$\sin^2\theta_{13}$	$0.02241^{+0.00066}_{-0.00065}$	$0.02046 \to 0.02440$	$0.02261^{+0.00067}_{-0.00064}$	$0.02066 \to 0.02461$
	$\theta_{13}/^\circ$	$8.61^{+0.13}_{-0.13}$	$8.22 \to 8.99$	$8.65^{+0.13}_{-0.12}$	$8.26 \to 9.02$
	$\delta_{CP}/^\circ$	222^{+38}_{-28}	$141 \to 370$	285^{+24}_{-26}	$205 \to 354$
	$\frac{\Delta m^2_{21}}{10^{-5}\ \text{eV}^2}$	$7.39^{+0.21}_{-0.20}$	$6.79 \to 8.01$	$7.39^{+0.21}_{-0.20}$	$6.79 \to 8.01$
	$\frac{\Delta m^2_{3\ell}}{10^{-3}\ \text{eV}^2}$	$+2.523^{+0.032}_{-0.030}$	$+2.432 \to +2.618$	$-2.509^{+0.032}_{-0.030}$	$-2.603 \to -2.416$

Neutrino mass ordering: normal ordering favored at the 2~3σ C.L.

Status of Neutrino Oscillations

$m_1 < m_2 < m_3$ (NO) or $m_3 < m_1 < m_2$ (IO)

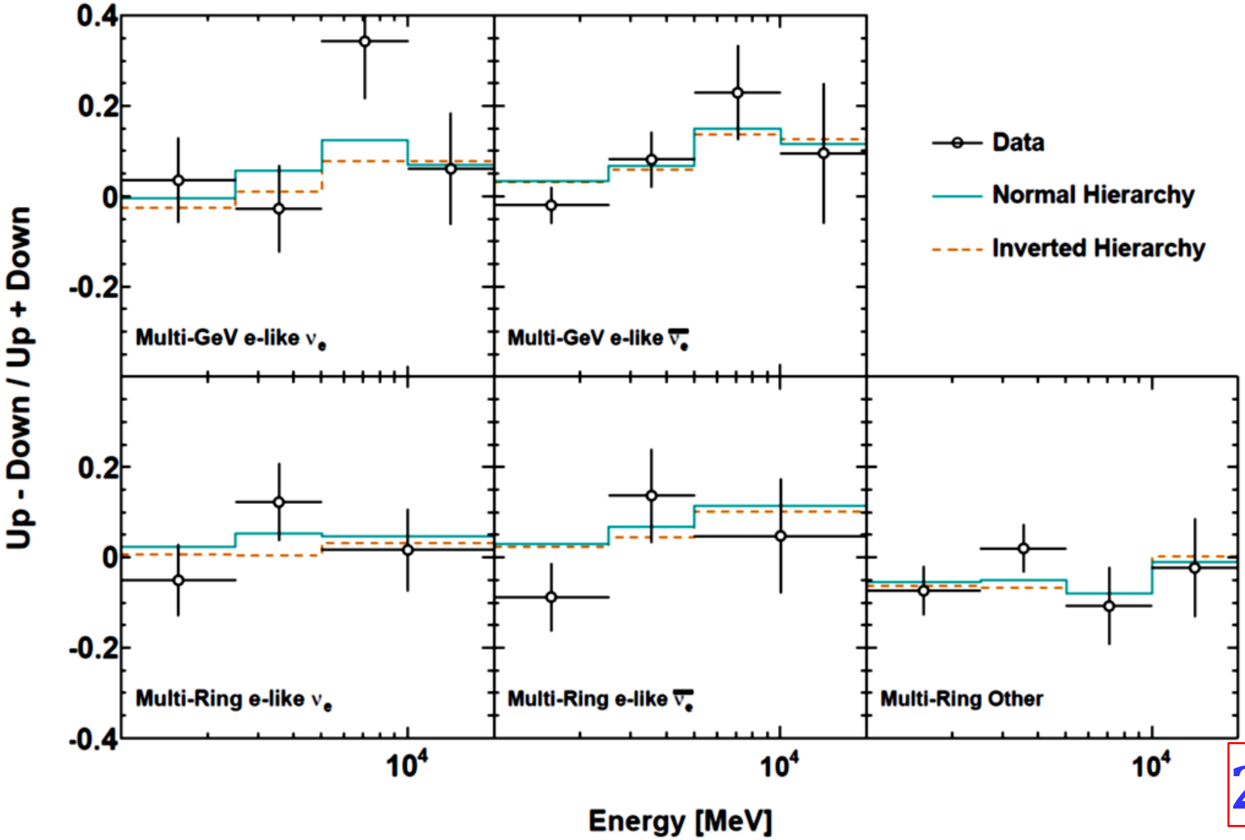
NuFIT 4.1 (2019)

with SK atmospheric data	Normal Ordering (best fit)		Inverted Ordering $\Delta\chi^2 = 10.4$	
	bfp $\pm 1\sigma$	3σ range	bfp $\pm 1\sigma$	3σ range
$\sin^2\theta_{12}$	$0.310^{+0.013}_{-0.012}$	$0.275 \to 0.350$	$0.310^{+0.013}_{-0.012}$	$0.275 \to 0.350$
$\theta_{12}/^\circ$	$33.82^{+0.78}_{-0.76}$	$31.61 \to 36.27$	$33.82^{+0.78}_{-0.75}$	$31.61 \to 36.27$
$\sin^2\theta_{23}$	$0.563^{+0.018}_{-0.024}$	$0.433 \to 0.609$	$0.565^{+0.017}_{-0.022}$	$0.436 \to 0.610$
$\theta_{23}/^\circ$	$48.6^{+1.0}_{-1.4}$	$41.1 \to 51.3$	$48.8^{+1.0}_{-1.2}$	$41.4 \to 51.3$
$\sin^2\theta_{13}$	$0.02237^{+0.00066}_{-0.00065}$	$0.02044 \to 0.02435$	$0.02259^{+0.00065}_{-0.00065}$	$0.02064 \to 0.02457$
$\theta_{13}/^\circ$	$8.60^{+0.13}_{-0.13}$	$8.22 \to 8.98$	$8.64^{+0.12}_{-0.13}$	$8.26 \to 9.02$
$\delta_{CP}/^\circ$	221^{+39}_{-28}	$144 \to 357$	282^{+23}_{-25}	$205 \to 348$
$\frac{\Delta m^2_{21}}{10^{-5}\ \text{eV}^2}$	$7.39^{+0.21}_{-0.20}$	$6.79 \to 8.01$	$7.39^{+0.21}_{-0.20}$	$6.79 \to 8.01$
$\frac{\Delta m^2_{3\ell}}{10^{-3}\ \text{eV}^2}$	$+2.528^{+0.029}_{-0.031}$	$+2.436 \to +2.618$	$-2.510^{+0.030}_{-0.031}$	$-2.601 \to -2.419$

Neutrino mass ordering: normal ordering favored at the 2~3σ C.L.

Sensitivity to Mass Ordering

SK atmospheric preference for NO due to excess of *e*-like events



C. Giunti, NNN 2018

Super-Kamiokande

$$\chi^2_{\text{IO}} - \chi^2_{\text{NO}} = 4.33$$

[PRD 97 (2018) 072001]

MSW Resonance

$$2V_eE = \Delta m^2_{31}\cos 2\theta_{13}$$

Global-fit analysis

NO: $\Delta m^2_{31} > 0$

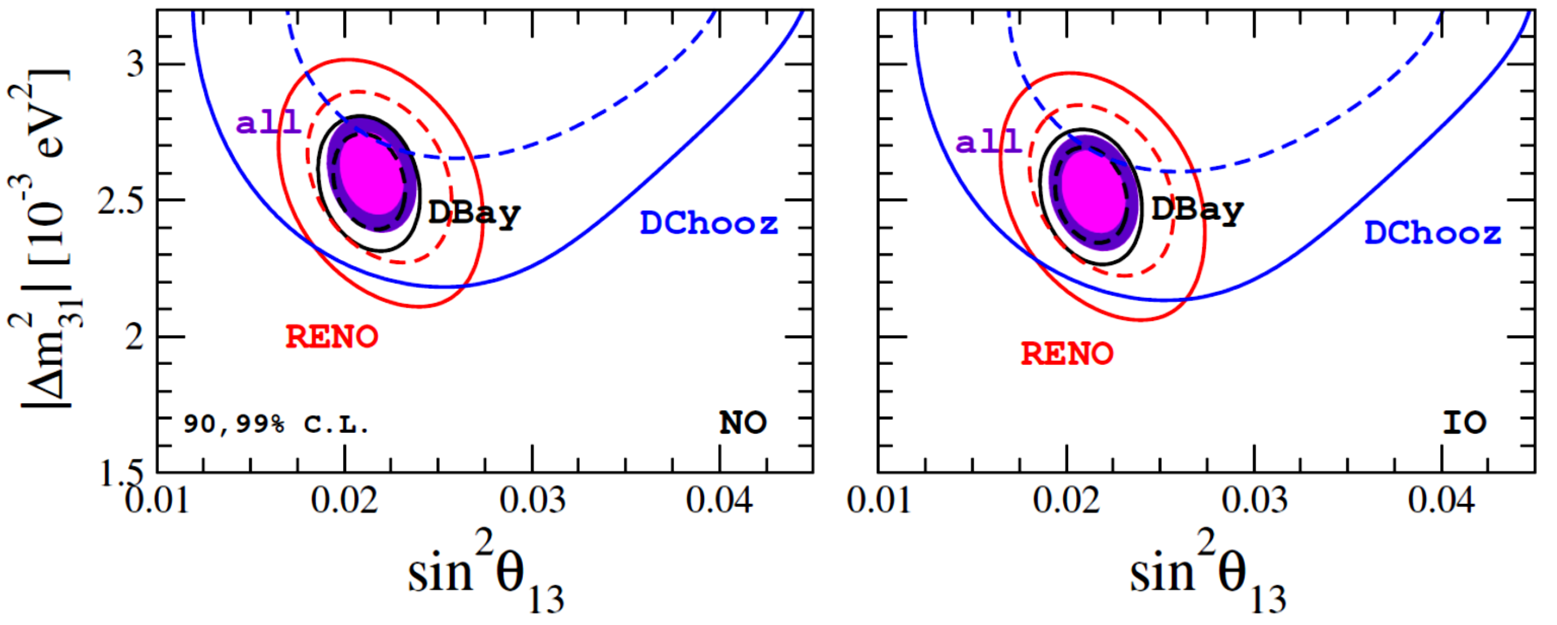
IO: $\Delta m^2_{31} < 0$

$$\nu_\mu \rightleftharpoons \nu_e$$

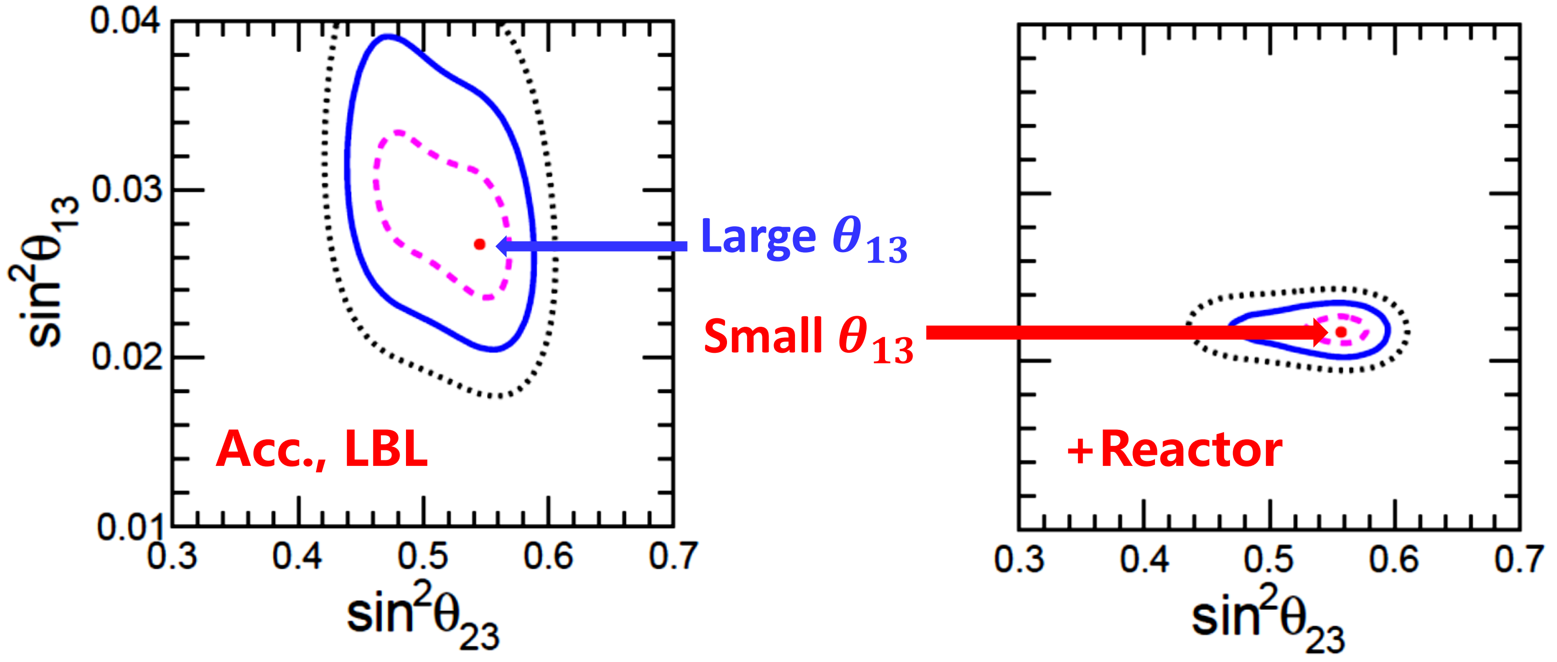
Resonant enhancement

Bari:	$\chi^2_{\text{IO}} - \chi^2_{\text{NO}} = 9.5$	$(\approx 3.1\sigma)$
NuFit:	$\chi^2_{\text{IO}} - \chi^2_{\text{NO}} = 9.1$	$(\approx 3.0\sigma)$
Valencia:	$\chi^2_{\text{IO}} - \chi^2_{\text{NO}} = 11.7$	$(\approx 3.4\sigma)$

Sensitivity to Mass Ordering



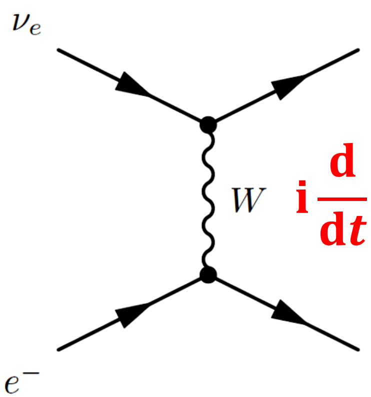
Sensitivity to θ_{13} is dominated by DYB



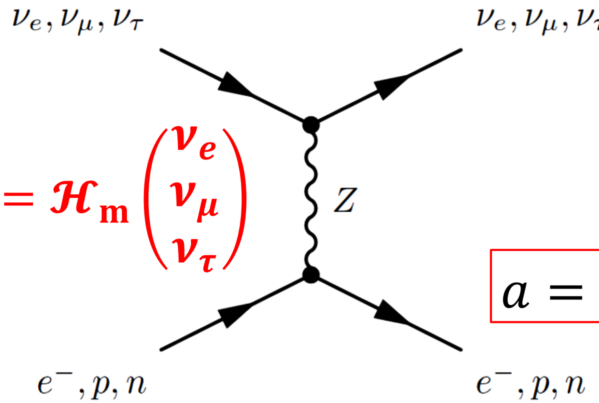
Large θ_{13} is preferred by T2K/NOvA

Small θ_{13} is preferred by DYB/RENO

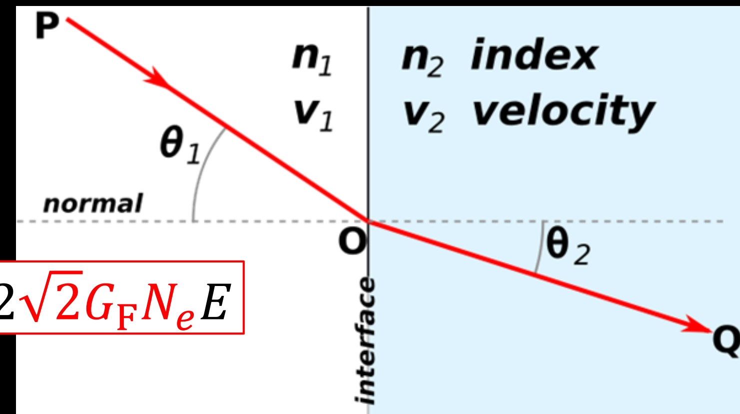
Importance of Matter Effects



$$i\frac{d}{dt}\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix} = \mathcal{H}_m \begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix}$$



$$a = 2\sqrt{2}G_F N_e E$$



$$\mathcal{H}_m = \frac{1}{2E}\left[U\begin{pmatrix}m_1^2 & 0 & 0\\ 0 & m_2^2 & 0\\ 0 & 0 & m_3^2\end{pmatrix}U^\dagger + \begin{pmatrix}a & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right] \Rightarrow V^\dagger \mathcal{H}_m V = \frac{1}{2E}\begin{pmatrix}\tilde{m}_1^2 & 0 & 0\\ 0 & \tilde{m}_2^2 & 0\\ 0 & 0 & \tilde{m}_3^2\end{pmatrix}$$

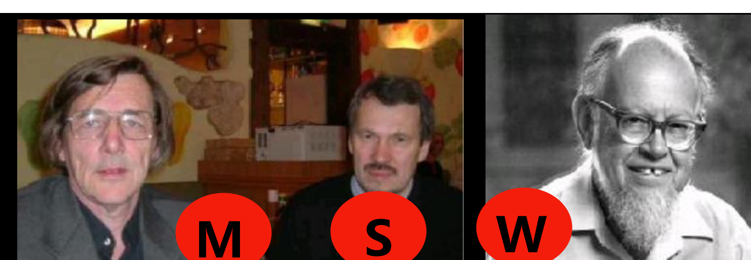
$$P(\nu_\mu \to \nu_e) \simeq \sin^2 2\theta_{13}\sin^2\theta_{23}\frac{\sin^2[(1-\hat{A})\Delta]}{(1-\hat{A})^2} \qquad \Delta \equiv \frac{\Delta m_{31}^2 L}{4E} \qquad \hat{A} \equiv \frac{a}{\Delta m_{31}^2}$$

Large θ_{13} from T2K/NOvA

$$+\frac{\Delta m_{21}^2}{\Delta m_{31}^2}\sin 2\theta_{13}\sin 2\theta_{12}\sin 2\theta_{23}\cos(\Delta+\delta)\frac{\sin(\hat{A}\Delta)}{\hat{A}}\frac{\sin[(1-\hat{A})\Delta]}{1-\hat{A}}$$

Sensitivity to Mass Ordering

When other parameters are fixed, the NO will be favored to realize a smaller value of θ_{13}



Matter Effects on Neutrino Oscillations

An example for two-flavor neutrino mixing

$$\mathcal{H}_{\rm m} = \frac{1}{2E} U \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix} U^\dagger + \begin{pmatrix} V_e & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2E} \left[U \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix} U^\dagger + \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \right]$$

The effective Hamiltonian in a more compact form:

$$\mathcal{H}_{\rm m} = \frac{1}{4E} \left[U \begin{pmatrix} -\Delta m_{21}^2 & 0 \\ 0 & +\Delta m_{21}^2 \end{pmatrix} U^\dagger + \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \right] + \frac{m_1^2 + m_2^2 + a}{4E}$$

$$\Delta m_{21}^2 \equiv m_2^2 - m_1^2$$

$$\Delta \widetilde{m}_{21}^2 \equiv \widetilde{m}_2^2 - \widetilde{m}_1^2$$

$$\mathcal{H}_{\rm m} = \frac{1}{4E} \begin{pmatrix} a - \Delta m_{21}^2 c_{2\theta} & \Delta m_{21}^2 s_{2\theta} \\ \Delta m_{21}^2 s_{2\theta} & \Delta m_{21}^2 c_{2\theta} - a \end{pmatrix}$$

Converted into the mass basis

$$\mathcal{H}_{\rm m} = \frac{1}{4E} \left[V \begin{pmatrix} -\Delta \widetilde{m}_{21}^2 & 0 \\ 0 & +\Delta \widetilde{m}_{21}^2 \end{pmatrix} V^\dagger \right]$$

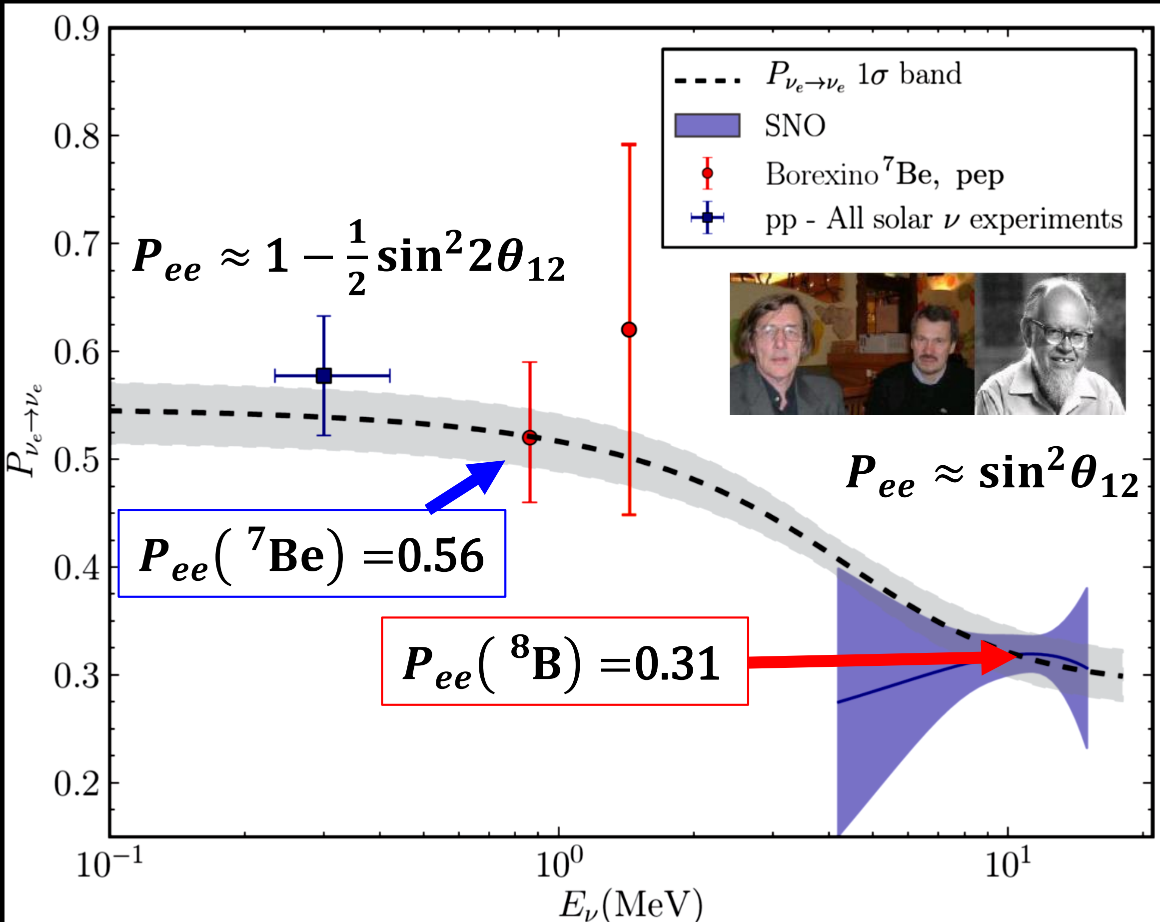
Mixing matrix & mass states in matter

$$\begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix} = \begin{pmatrix} c_{\widetilde{\theta}} & s_{\widetilde{\theta}} \\ -s_{\widetilde{\theta}} & c_{\widetilde{\theta}} \end{pmatrix} \begin{pmatrix} |\widetilde{\nu}_1\rangle \\ |\widetilde{\nu}_2\rangle \end{pmatrix} \equiv \widetilde{U} \begin{pmatrix} |\widetilde{\nu}_1\rangle \\ |\widetilde{\nu}_2\rangle \end{pmatrix}$$

$$\Delta \widetilde{m}_{21}^2 = \sqrt{\left(\Delta m_{21}^2 c_{2\theta} - a\right)^2 + \left(\Delta m_{21}^2 s_{2\theta}\right)^2} \qquad s_{2\widetilde{\theta}}^2 = \frac{\Delta m_{21}^2 s_{2\theta}^2}{\left(\Delta m_{21}^2 c_{2\theta} - a\right)^2 + \left(\Delta m_{21}^2 s_{2\theta}\right)^2}$$

Relationship between the mixing angle (mass difference) in vacuum and that in matter

Importance of Matter Effects



For high-energy ^{8}B neutrinos

at production $r = 0$

$$\begin{pmatrix} |\tilde{\nu}_1(0)\rangle \\ |\tilde{\nu}_2(0)\rangle \end{pmatrix} = \begin{pmatrix} c_{\hat{\theta}} & -s_{\hat{\theta}} \\ s_{\hat{\theta}} & c_{\hat{\theta}} \end{pmatrix} \begin{pmatrix} |\nu_e(0)\rangle \\ |\nu_\mu(0)\rangle \end{pmatrix}$$

adiabatic evolution

$$\begin{pmatrix} |\tilde{\nu}_1(R)\rangle \\ |\tilde{\nu}_2(R)\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} |\tilde{\nu}_1(0)\rangle \\ |\tilde{\nu}_2(0)\rangle \end{pmatrix}$$

on the solar surface $r = R$

$$\begin{pmatrix} |\nu_e(R)\rangle \\ |\nu_\mu(R)\rangle \end{pmatrix} = \begin{pmatrix} c_\theta & s_\theta \\ -s_\theta & c_\theta \end{pmatrix} \begin{pmatrix} |\tilde{\nu}_1(R)\rangle \\ |\tilde{\nu}_2(R)\rangle \end{pmatrix}$$

survival probability

$$P_{ee} = c^2_{\hat{\theta}} c^2_\theta + s^2_{\hat{\theta}} s^2_\theta \longrightarrow \sin^2\theta$$

$$\hat{\theta} \longrightarrow \pi/2 \qquad \text{as } A \gg \Delta m^2$$

For the MSW resonance to happen

$\theta \in [0, \frac{\pi}{2}]$ $\Delta m^2_{21} > 0$ ⟷ $\theta \in [0, \frac{\pi}{4}]$ Δm^2_{21}

$\theta_{12} = 34^\circ$

Normal neutrino mass ordering

For low-energy ^{7}Be neutrinos

$$P_{ee} \approx 1 - \frac{1}{2}\sin^2 2\theta$$

Oscillations in vacuum

Three-flavor Oscillations in Vacuum

The general formula of oscillation probabilities

Jarlskog Invariant

$$\mathrm{Im}\left[U_{\alpha i}U_{\beta j}U^*_{\alpha j}U^*_{\beta i}\right] \equiv \mathcal{J}\sum_{\gamma,k}\varepsilon_{\alpha\beta\gamma}\,\varepsilon_{ijk}$$

$$P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - 4\sum_{i<j}^{3}\mathrm{Re}\left[U_{\alpha i}U_{\beta j}U^*_{\alpha j}U^*_{\beta i}\right]\sin^2\frac{\Delta m^2_{ji}L}{4E}$$

$$+ 8\mathcal{J}\sum_{\gamma}\varepsilon_{\alpha\beta\gamma}\sin\frac{\Delta m^2_{21}L}{4E}\sin\frac{\Delta m^2_{32}L}{4E}\sin\frac{\Delta m^2_{31}L}{4E}$$

$$P(\bar\nu_\alpha \to \bar\nu_\beta) = \delta_{\alpha\beta} - 4\sum_{i<j}^{3}\mathrm{Re}\left[U_{\alpha i}U_{\beta j}U^*_{\alpha j}U^*_{\beta i}\right]\sin^2\frac{\Delta m^2_{ji}L}{4E}$$

$$- 8\mathcal{J}\sum_{\gamma}\varepsilon_{\alpha\beta\gamma}\sin\frac{\Delta m^2_{21}L}{4E}\sin\frac{\Delta m^2_{32}L}{4E}\sin\frac{\Delta m^2_{31}L}{4E}$$

$$\boldsymbol{U} \Rightarrow \boldsymbol{U}^*$$

$$\mathcal{J} \Rightarrow -\mathcal{J}$$

CP violation in neutrino oscillations

$$A_{\mathrm{CP}} = P(\nu_\alpha \to \nu_\beta) - P(\bar\nu_\alpha \to \bar\nu_\beta) = 16\mathcal{J}\sum_{\gamma}\varepsilon_{\alpha\beta\gamma}\sin\frac{\Delta m^2_{21}L}{4E}\sin\frac{\Delta m^2_{32}L}{4E}\sin\frac{\Delta m^2_{31}L}{4E}$$

Relations between Vacuum and Matter Cases

A: Establish the relationship between intrinsic and effective parameters

$$\mathcal{H}_{\rm m} = \frac{1}{2E}\left[U\begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix}U^\dagger + \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] \equiv \frac{\Omega_{\rm m}}{2E}$$

$$\Omega_{\rm v} = U\begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix}U^\dagger$$

$$\boldsymbol{D_l} = \begin{pmatrix} m_e^2 & 0 & 0 \\ 0 & m_\mu^2 & 0 \\ 0 & 0 & m_\tau^2 \end{pmatrix}$$

$$\Omega_{\rm m} = V\begin{pmatrix} \widetilde{m}_1^2 & 0 & 0 \\ 0 & \widetilde{m}_2^2 & 0 \\ 0 & 0 & \widetilde{m}_3^2 \end{pmatrix}V^\dagger$$

Calculate the commutators:

$$[D_l, \Omega_{\rm m}] \equiv i\widetilde{X}_l \quad [D_l, \Omega_{\rm v}] \equiv iX_l$$

Harrison & Scott, PLB, 00;
Xing, PLB, 00

$$X_l = i\begin{pmatrix} 0 & \Delta_{e\mu}Z_{\mu e} & \Delta_{\tau e}Z_{e\tau} \\ \Delta_{\mu e}Z_{e\mu} & 0 & \Delta_{\mu\tau}Z_{\tau\mu} \\ \Delta_{e\tau}Z_{\tau e} & \Delta_{\tau\mu}Z_{\mu\tau} & 0 \end{pmatrix} \qquad \widetilde{X}_l = i\begin{pmatrix} 0 & \Delta_{e\mu}\widetilde{Z}_{\mu e} & \Delta_{\tau e}\widetilde{Z}_{e\tau} \\ \Delta_{\mu e}\widetilde{Z}_{e\mu} & 0 & \Delta_{\mu\tau}\widetilde{Z}_{\tau\mu} \\ \Delta_{e\tau}\widetilde{Z}_{\tau e} & \Delta_{\tau\mu}\widetilde{Z}_{\mu\tau} & 0 \end{pmatrix}$$

$$\Delta_{\alpha\beta} \equiv m_\alpha^2 - m_\beta^2 \qquad \Delta_{ij} \equiv m_i^2 - m_j^2$$

$$\tilde{Z}_{\alpha\beta} \equiv \sum_i \widetilde{m}_i^2\, V_{\alpha i}V_{\beta i}^* \qquad Z_{\alpha\beta} \equiv \sum_i m_i^2\, U_{\alpha i}U_{\beta i}^*$$

$$\text{Det}[X_l] = 2i\Delta_{\mu e}\Delta_{\tau e}\Delta_{\tau\mu}\text{Im}[Z_{\mu e}Z_{\tau e}Z_{\mu\tau}] = 2i\Delta_{\mu e}\Delta_{\tau e}\Delta_{\tau\mu}\Delta_{21}\Delta_{31}\Delta_{32}\mathcal{J}$$

$$\text{Det}[\widetilde{X}_l] = 2i\Delta_{\mu e}\Delta_{\tau e}\Delta_{\tau\mu}\text{Im}[\tilde{Z}_{\mu e}\tilde{Z}_{\tau e}\tilde{Z}_{\mu\tau}] = 2i\Delta_{\mu e}\Delta_{\tau e}\Delta_{\tau\mu}\widetilde{\Delta}_{21}\widetilde{\Delta}_{31}\widetilde{\Delta}_{32}\tilde{\mathcal{J}}$$

Naumov relation, 92

$$\frac{\tilde{\mathcal{J}}}{\mathcal{J}} = \frac{\Delta_{21}\Delta_{31}\Delta_{32}}{\widetilde{\Delta}_{21}\widetilde{\Delta}_{31}\widetilde{\Delta}_{32}}$$

Relations between Vacuum and Matter Cases

B: Series expansion of the effective parameters

Freund, PRD, 01; Akhmedov et al., JHEP, 04

Eigenvalues:

$$\tilde{m}_1^2 = m_1^2 + \Delta_{31}\left(\hat{A} + \alpha s_{12}^2 + s_{13}^2 \frac{\hat{A}}{\hat{A}-1} + \alpha^2 \frac{\sin^2 2\theta_{12}}{4\hat{A}}\right)$$

$$\tilde{m}_2^2 = m_1^2 + \Delta_{31}\left(\alpha c_{12}^2 - \alpha^2 \frac{\sin^2 2\theta_{12}}{4\hat{A}}\right)$$

$$\alpha \equiv \frac{\Delta_{21}}{\Delta_{31}} \approx 0.03$$

$$s_{13}^2 \approx 0.02$$

up to the second order of α and s_{13}^2

$$\tilde{m}_3^2 = m_1^2 + \Delta_{31}\left(1 - s_{13}^2 \frac{\hat{A}}{\hat{A}-1}\right)$$

Divergent in the limits of $\hat{A} \to 0$ and $\hat{A} \to 1$

Oscillation Probabilities:

$$\tilde{P}_{ee} = 1 - \alpha^2 \sin^2 2\theta_{12} \frac{\sin^2 \hat{A}\Delta}{\hat{A}^2} - 4 s_{13}^2 \frac{\sin^2(\hat{A}-1)\Delta}{(\hat{A}-1)^2}$$

Finite in the limits $\hat{A} \to 0$ and $\hat{A} \to 1$

$$\tilde{P}_{e\mu} = \alpha^2 \sin^2 2\theta_{12} c_{23}^2 \frac{\sin^2 \hat{A}\Delta}{\hat{A}^2} + 4 s_{13}^2 s_{23}^2 \frac{\sin^2(\hat{A}-1)\Delta}{(\hat{A}-1)^2}$$

$$+ 2\alpha\, s_{13} \sin 2\theta_{12} \sin 2\theta_{23} \cos(\Delta - \delta) \frac{\sin \hat{A}\Delta}{\hat{A}} \frac{\sin(\hat{A}-1)\Delta}{(\hat{A}-1)}$$

Valid only when Δ_{21}-driven osci. are small $\alpha\Delta \ll 1$

$$\Delta \equiv \frac{\Delta_{31} L}{4E}$$

A Differential Way to Understand Matter Effects

Look at the effective Hamiltonian in matter once again

$$H_{\rm m} = \frac{1}{2E}\left[U\begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix}U^\dagger + \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] \equiv \frac{1}{2E} V\begin{pmatrix} \widetilde{m}_1^2 & 0 & 0 \\ 0 & \widetilde{m}_2^2 & 0 \\ 0 & 0 & \widetilde{m}_3^2 \end{pmatrix}V^\dagger$$

- $H_{\rm m}$: Effective Hamiltonian
- U: Mixing matrix in vacuum
- a: Matter parameter, $a \equiv 2\sqrt{2}\, G_{\rm F} N_e E$
- V: Mixing matrix in matter
- $\widetilde{m}_i^2$: Effective masses

Oscillation probabilities in vacuum

$$P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - 4\sum_{i<j}^{3} {\rm Re}\left[U_{\alpha i}U_{\beta j}U_{\alpha j}^* U_{\beta i}^*\right]\sin^2\frac{\Delta m_{ji}^2 L}{4E} + 8\mathcal{J}\sum_{\gamma}\varepsilon_{\alpha\beta\gamma}\sin\frac{\Delta m_{21}^2 L}{4E}\sin\frac{\Delta m_{32}^2 L}{4E}\sin\frac{\Delta m_{31}^2 L}{4E}$$

Oscillation probabilities in matter

$$\tilde{P}(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - 4\sum_{i<j}^{3} {\rm Re}\left[V_{\alpha i}V_{\beta j}V_{\alpha j}^* V_{\beta i}^*\right]\sin^2\frac{\Delta \widetilde{m}_{ji}^2 L}{4E} + 8\tilde{\mathcal{J}}\sum_{\gamma}\varepsilon_{\alpha\beta\gamma}\sin\frac{\Delta \widetilde{m}_{21}^2 L}{4E}\sin\frac{\Delta \widetilde{m}_{32}^2 L}{4E}\sin\frac{\Delta \widetilde{m}_{31}^2 L}{4E}$$

Form invariance of oscillation probabilities under: $m_i^2 \leftrightarrow \widetilde{m}_i^2 \quad U_{ij} \leftrightarrow V_{ij}$

A Differential Way to Understand Matter Effects

$$\mathcal{H}_{\rm m} = \frac{1}{2E}\left[U\begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix}U^\dagger + \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] \Rightarrow \mathcal{H}_{\rm m} = \frac{1}{2E}\left[V\begin{pmatrix} \widetilde{m}_1^2 & 0 & 0 \\ 0 & \widetilde{m}_2^2 & 0 \\ 0 & 0 & \widetilde{m}_3^2 \end{pmatrix}V^\dagger\right]$$

Take the derivative of the effective Hamiltonian with respect to ***a***

$$\dot{D} + \left[V^\dagger \dot{V}, D\right] = V^\dagger \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V = \begin{pmatrix} |V_{e1}|^2 & V_{e1}^* V_{e2} & V_{e1}^* V_{e3} \\ V_{e2}^* V_{e1} & |V_{e2}|^2 & V_{e2}^* V_{e3} \\ V_{e3}^* V_{e1} & V_{e3}^* V_{e2} & |V_{e3}|^2 \end{pmatrix}$$

$$D \equiv \begin{pmatrix} \widetilde{m}_1^2 & 0 & 0 \\ 0 & \widetilde{m}_2^2 & 0 \\ 0 & 0 & \widetilde{m}_3^2 \end{pmatrix}$$

A complete set of differential equations (“RGEs”)

$$\widetilde{\Delta}_{ij} \equiv \widetilde{m}_i^2 - \widetilde{m}_j^2$$

$$\frac{\rm d}{{\rm d}a}|V_{e1}|^2 = 2|V_{e1}|^2\left(|V_{e2}|^2\widetilde{\Delta}_{12}^{-1} - |V_{e3}|^2\widetilde{\Delta}_{31}^{-1}\right)$$
$$\frac{\rm d}{{\rm d}a}|V_{e2}|^2 = 2|V_{e2}|^2\left(|V_{e3}|^2\widetilde{\Delta}_{23}^{-1} - |V_{e1}|^2\widetilde{\Delta}_{12}^{-1}\right)$$
$$\frac{\rm d}{{\rm d}a}|V_{e3}|^2 = 2|V_{e3}|^2\left(|V_{e1}|^2\widetilde{\Delta}_{31}^{-1} - |V_{e2}|^2\widetilde{\Delta}_{23}^{-1}\right)$$

Electron flavor is SPECIAL

$$\frac{\rm d}{{\rm d}a}\widetilde{\Delta}_{12} = |V_{e1}|^2 - |V_{e2}|^2$$
$$\frac{\rm d}{{\rm d}a}\widetilde{\Delta}_{23} = |V_{e2}|^2 - |V_{e3}|^2$$
$$\frac{\rm d}{{\rm d}a}\widetilde{\Delta}_{31} = |V_{e3}|^2 - |V_{e1}|^2$$

S.H. Chiu & T.K. Kuo, PRD 2018; Z.Z. Xing, S. Zhou & Y.L. Zhou, JHEP 2018

A Differential Way to Understand Matter Effects

Diff. Invariant

$$\frac{\mathrm{d}}{\mathrm{d}a}\left[\ln\left(|V_{e1}|^2|V_{e2}|^2|V_{e3}|^2\widetilde{\Delta}^2_{12}\widetilde{\Delta}^2_{23}\widetilde{\Delta}^2_{31}\right)\right] = \sum_{i=1}^{3}\frac{\mathrm{d}}{\mathrm{d}a}\left(\ln|V_{ei}|^2\right) + \sum_{j>k}\frac{\mathrm{d}}{\mathrm{d}a}\left(\ln\widetilde{\Delta}^2_{jk}\right) = 0$$

$$\widetilde{\mathcal{J}} = \mathrm{Im}\left[V_{e1}V_{\mu 2}V^*_{e2}V^*_{\mu 1}\right]$$

Jarlskog rephrasing invariant

$$\frac{\mathrm{d}}{\mathrm{d}a}\ln\left[\widetilde{\mathcal{J}}\widetilde{\Delta}_{12}\widetilde{\Delta}_{23}\widetilde{\Delta}_{31}\right] = 0$$

Differential Invariant

$$\dot{V}_{e1} = |V_{e2}|^2V_{e1}\widetilde{\Delta}^{-1}_{12} - |V_{e3}|^2V_{e1}\widetilde{\Delta}^{-1}_{31} + \sum_{\alpha}\dot{V}_{\alpha i}V^*_{\alpha i}V_{e1} ,$$

$$\dot{V}^*_{e2} = |V_{e3}|^2V^*_{e2}\widetilde{\Delta}^{-1}_{23} - |V_{e1}|^2V^*_{e2}\widetilde{\Delta}^{-1}_{12} + \sum_{\alpha}\dot{V}_{\alpha i}V^*_{\alpha i}V^*_{e2} ,$$

$$\dot{V}^*_{\mu 1} = V^*_{\mu 2}V^*_{e1}V_{e2}\widetilde{\Delta}^{-1}_{12} - V^*_{\mu 3}V^*_{e1}V_{e3}\widetilde{\Delta}^{-1}_{31} + \sum_{\alpha}\dot{V}_{\alpha i}V^*_{\alpha i}V^*_{\mu 1} ,$$

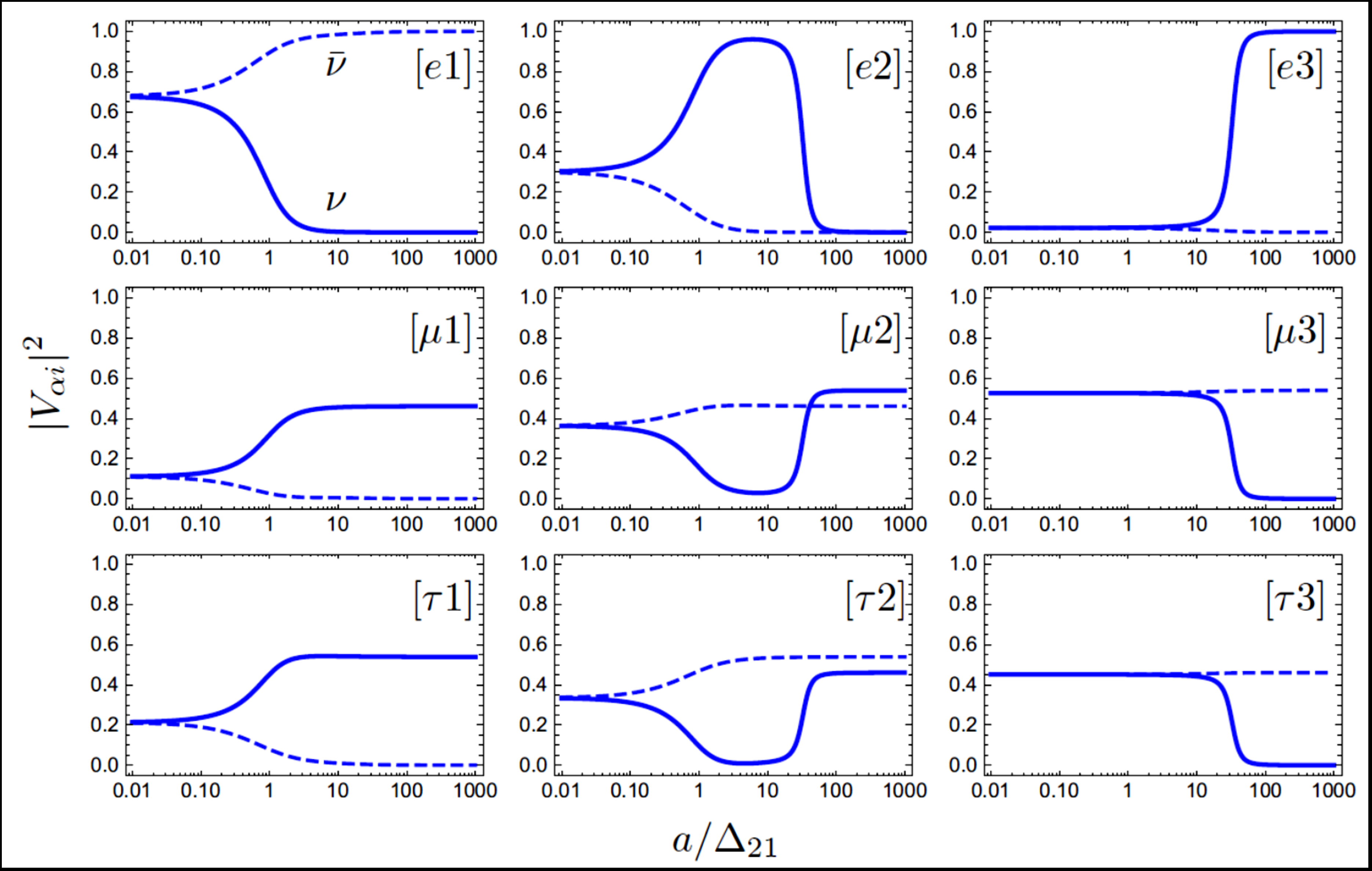
$$\dot{V}_{\mu 2} = V_{\mu 3}V_{e2}V^*_{e3}\widetilde{\Delta}^{-1}_{23} - V_{\mu 1}V_{e2}V^*_{e1}\widetilde{\Delta}^{-1}_{12} + \sum_{\alpha}\dot{V}_{\alpha i}V^*_{\alpha i}V_{\mu 2} .$$

$$\frac{\mathrm{d}}{\mathrm{d}a}|V_{\mu 1}|^2 = |V_{\mu 1}|^2\left[\frac{|V_{e2}|^2}{\widetilde{\Delta}_{12}} - \frac{|V_{e3}|^2}{\widetilde{\Delta}_{31}}\right] + |V_{e1}|^2\left[\frac{|V_{\mu 2}|^2}{\widetilde{\Delta}_{12}} - \frac{|V_{\mu 3}|^2}{\widetilde{\Delta}_{31}}\right] - \left[\frac{|V_{\tau 3}|^2}{\widetilde{\Delta}_{12}} - \frac{|V_{\tau 2}|^2}{\widetilde{\Delta}_{31}}\right] ,$$

$$\frac{\mathrm{d}}{\mathrm{d}a}|V_{\mu 2}|^2 = |V_{\mu 2}|^2\left[\frac{|V_{e3}|^2}{\widetilde{\Delta}_{23}} - \frac{|V_{e1}|^2}{\widetilde{\Delta}_{12}}\right] + |V_{e2}|^2\left[\frac{|V_{\mu 3}|^2}{\widetilde{\Delta}_{23}} - \frac{|V_{\mu 1}|^2}{\widetilde{\Delta}_{12}}\right] - \left[\frac{|V_{\tau 1}|^2}{\widetilde{\Delta}_{23}} - \frac{|V_{\tau 3}|^2}{\widetilde{\Delta}_{12}}\right] ,$$

$$\frac{\mathrm{d}}{\mathrm{d}a}|V_{\mu 3}|^2 = |V_{\mu 3}|^2\left[\frac{|V_{e1}|^2}{\widetilde{\Delta}_{31}} - \frac{|V_{e2}|^2}{\widetilde{\Delta}_{23}}\right] + |V_{e3}|^2\left[\frac{|V_{\mu 1}|^2}{\widetilde{\Delta}_{31}} - \frac{|V_{\mu 2}|^2}{\widetilde{\Delta}_{23}}\right] - \left[\frac{|V_{\tau 2}|^2}{\widetilde{\Delta}_{31}} - \frac{|V_{\tau 1}|^2}{\widetilde{\Delta}_{23}}\right] .$$

Numerical Solutions to Mixing Matrix Elements



Evolution of the mixing matrix elements in the NO case (input global-fit data)

Numerical Solutions to Mixing Parameters

Standard Parametrization

$$V = \begin{pmatrix} \tilde{c}_{13}\tilde{c}_{12} & \tilde{c}_{13}\tilde{s}_{12} & \tilde{s}_{13}e^{-i\tilde{\delta}} \\ -\tilde{s}_{12}\tilde{c}_{23} - \tilde{c}_{12}\tilde{s}_{23}\tilde{s}_{13}e^{i\tilde{\delta}} & +\tilde{c}_{12}\tilde{c}_{23} - \tilde{s}_{12}\tilde{s}_{23}\tilde{s}_{13}e^{i\tilde{\delta}} & \tilde{c}_{13}\tilde{s}_{23} \\ +\tilde{s}_{12}\tilde{s}_{23} - \tilde{c}_{12}\tilde{c}_{23}\tilde{s}_{13}e^{i\tilde{\delta}} & -\tilde{c}_{12}\tilde{s}_{23} - \tilde{s}_{12}\tilde{c}_{23}\tilde{s}_{13}e^{i\tilde{\delta}} & \tilde{c}_{13}\tilde{c}_{23} \end{pmatrix}$$

$$\frac{\mathrm{d}\widetilde{\theta}_{12}}{\mathrm{d}a} = \frac{1}{2}\sin 2\widetilde{\theta}_{12}\left(\cos^2\widetilde{\theta}_{13}\widetilde{\Delta}_{21}^{-1} - \sin^2\widetilde{\theta}_{13}\widetilde{\Delta}_{21}\widetilde{\Delta}_{31}^{-1}\widetilde{\Delta}_{32}^{-1}\right) ,$$

$$\frac{\mathrm{d}\widetilde{\theta}_{13}}{\mathrm{d}a} = \frac{1}{2}\sin 2\widetilde{\theta}_{13}\left(\cos^2\widetilde{\theta}_{12}\widetilde{\Delta}_{31}^{-1} + \sin^2\widetilde{\theta}_{12}\widetilde{\Delta}_{32}^{-1}\right) ,$$

$$\frac{\mathrm{d}\widetilde{\theta}_{23}}{\mathrm{d}a} = \frac{1}{2}\sin 2\widetilde{\theta}_{12}\sin\widetilde{\theta}_{13}\cos\widetilde{\delta}\,\widetilde{\Delta}_{21}\widetilde{\Delta}_{31}^{-1}\widetilde{\Delta}_{32}^{-1} ,$$

$$\frac{\mathrm{d}\widetilde{\delta}}{\mathrm{d}a} = -\sin 2\widetilde{\theta}_{12}\sin\widetilde{\theta}_{13}\sin\widetilde{\delta}\cot 2\widetilde{\theta}_{23}\widetilde{\Delta}_{21}\widetilde{\Delta}_{31}^{-1}\widetilde{\Delta}_{32}^{-1} ;$$

$$\frac{\mathrm{d}\widetilde{\Delta}_{21}}{\mathrm{d}a} = -\cos^2\widetilde{\theta}_{13}\cos 2\widetilde{\theta}_{12} ,$$

$$\frac{\mathrm{d}\widetilde{\Delta}_{31}}{\mathrm{d}a} = \sin^2\widetilde{\theta}_{13} - \cos^2\widetilde{\theta}_{13}\cos^2\widetilde{\theta}_{12} ,$$

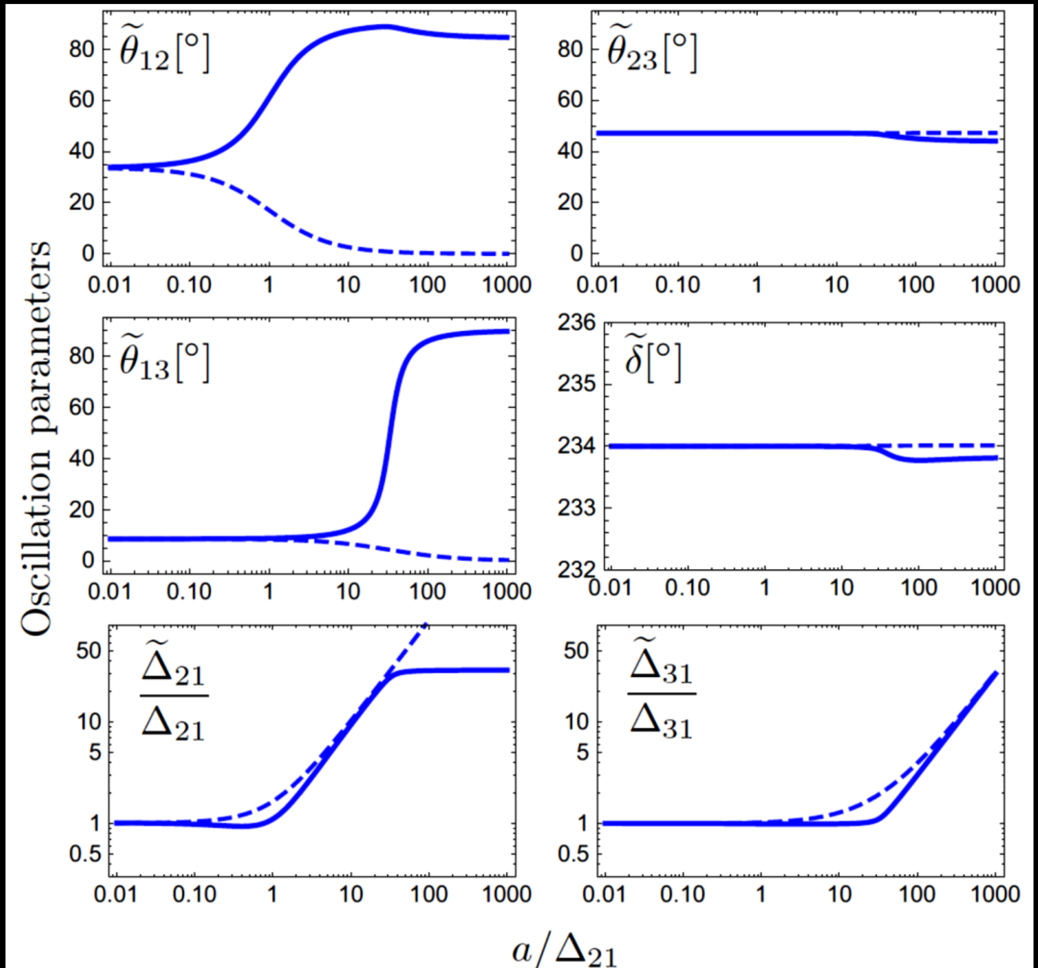
$$\frac{\mathrm{d}\widetilde{\Delta}_{32}}{\mathrm{d}a} = \sin^2\widetilde{\theta}_{13} - \cos^2\widetilde{\theta}_{13}\sin^2\widetilde{\theta}_{12} ,$$

$\{\widetilde{\theta}_{13}, \widetilde{\theta}_{12}\}$

$\{\widetilde{\Delta}_{21}, \widetilde{\Delta}_{31}, \widetilde{\Delta}_{32}\}$

⬇

$\{\widetilde{\theta}_{23}, \widetilde{\delta}\}$



Evolution of $\{\widetilde{\theta}_{13}, \widetilde{\theta}_{12}\}$ and $\{\widetilde{\Delta}_{21}, \widetilde{\Delta}_{31}, \widetilde{\Delta}_{32}\}$ is independent of $\widetilde{\theta}_{23}$ and $\widetilde{\delta}$

Z.Z. Xing, S. Zhou & Y.L. Zhou, JHEP, 18

Are analytical solutions possible ???

Analytical Solutions to RGEs

Series expansion of mass eigenvalues

$$\widetilde{\Delta}_{21} \approx \Delta_{31}\left[\frac{1}{2}(1+A-C_{13}) + \alpha\left(\frac{C_{13}+1-A\cos 2\theta_{13}}{2C_{13}}\sin^2\theta_{12} - \cos^2\theta_{12}\right)\right] ,$$

$$\widetilde{\Delta}_{31} \approx \Delta_{31}\left[\frac{1}{2}(1+A+C_{13}) + \alpha\left(\frac{C_{13}-1+A\cos 2\theta_{13}}{2C_{13}}\sin^2\theta_{12} - \cos^2\theta_{12}\right)\right] ,$$

$$\widetilde{\Delta}_{32} \approx \Delta_{31}\left[C_{13} + \alpha\sin^2\theta_{12}\left(\frac{A\cos 2\theta_{13}-1}{C_{13}}\right)\right] ,$$

Freund, PRD, 01; Akhmedov et al., JHEP, 04

As a starting point

X. Wang, S.Z, JHEP, 19

Introduce a new mass-squared difference

$\Delta_c \equiv \Delta_{31}\cos^2\theta_{12} + \Delta_{32}\sin^2\theta_{12}$ $\quad \alpha_c = \Delta_{21}/\Delta_c$

Minakata, Parke, JHEP, 16;
Y.F. Li, J. Zhang, S. Zhou, J.Y. Zhu, JHEP, 16

$$\widetilde{\Delta}_{21} \approx \Delta_c\left[\frac{1}{2}\left(1+A_c-\widehat{C}_{13}\right) - \alpha_c\cos 2\theta_{12}\right] ,$$

$$\widetilde{\Delta}_{31} \approx \Delta_c\left[\frac{1}{2}\left(1+A_c+\widehat{C}_{13}\right) - \alpha_c\cos 2\theta_{12}\right] ,$$

$$\widetilde{\Delta}_{32} \approx \Delta_c\widehat{C}_{13} , \quad \boxed{\widehat{C}_{13} \equiv \sqrt{1-2A_c\cos 2\theta_{13}+A_c^2}}$$

$$\frac{\mathrm{d}\widetilde{\Delta}_{31}}{\mathrm{d}a} = \sin^2\widetilde{\theta}_{13} - \cos^2\widetilde{\theta}_{13}\cos^2\widetilde{\theta}_{12}$$

$$\frac{\mathrm{d}\widetilde{\Delta}_{32}}{\mathrm{d}a} = \sin^2\widetilde{\theta}_{13} - \cos^2\widetilde{\theta}_{13}\sin^2\widetilde{\theta}_{12}$$

$\Rightarrow$

$$\frac{1}{2}\left(1+\frac{A_c-\cos 2\theta_{13}}{\widehat{C}_{13}}\right) = \sin^2\widetilde{\theta}_{13} - \cos^2\widetilde{\theta}_{13}\cos^2\widetilde{\theta}_{12}$$

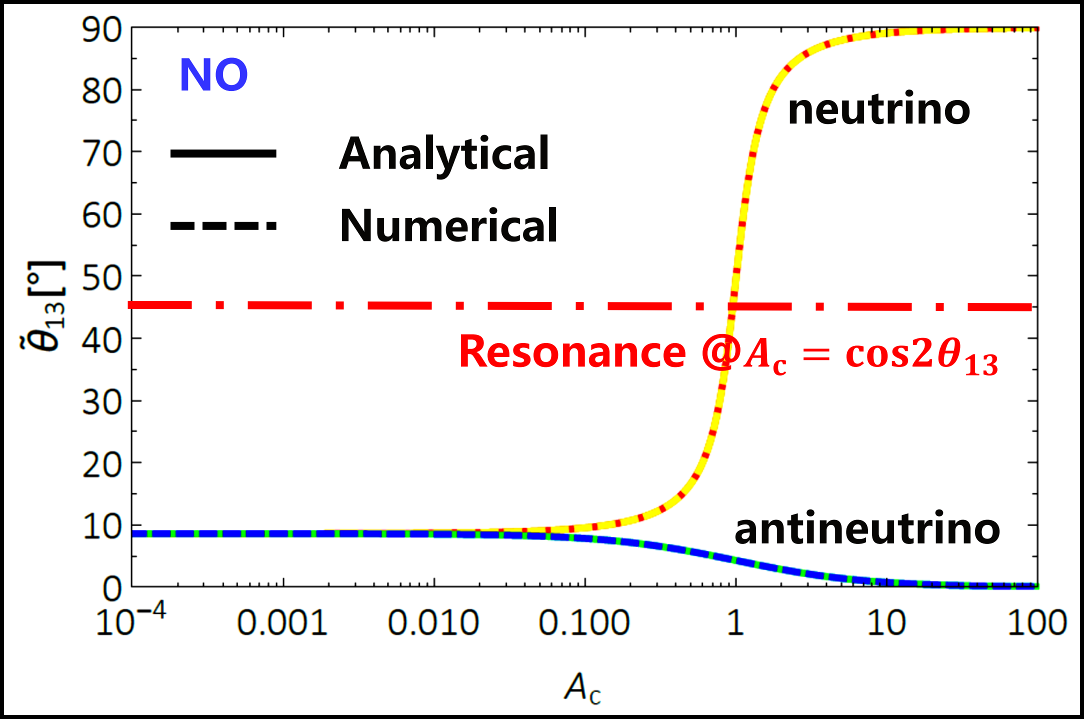
$$\frac{A_c-\cos 2\theta_{13}}{\widehat{C}_{13}} = \sin^2\widetilde{\theta}_{13} - \cos^2\widetilde{\theta}_{13}\sin^2\widetilde{\theta}_{12}$$

$$\cos^2\widetilde{\theta}_{13} = \frac{1}{2}\left(1-\frac{A_c-\cos 2\theta_{13}}{\widehat{C}_{13}}\right)$$

$$\sin^2 2\widetilde{\theta}_{13} = 1 - \frac{(A_c-\cos 2\theta_{13})^2}{\widehat{C}_{13}^2} = \frac{\sin^2 2\theta_{13}}{(A_c-\cos 2\theta_{13})^2+\sin^2 2\theta_{13}}$$

Note: $\widetilde{\theta}_{13}$ is given by the formula in the limit of two-flavor neutrino mixing

Analytical Solutions to RGEs



Analytical result

$$\cos^2 \widetilde{\theta}_{13} = \frac{1}{2}\left(1 - \frac{A_c - \cos 2\theta_{13}}{\widehat{C}_{13}}\right)$$

$$\widehat{C}_{13} \equiv \sqrt{1 - 2A_c \cos 2\theta_{13} + A_c^2}$$

$$A_c = \frac{a}{\Delta_c}$$

Order of magnitude

$$= 6 \times 10^{-4} \left(\frac{N_e}{N_A\ \mathrm{cm}^{-3}}\right)\left(\frac{E}{10\ \mathrm{MeV}}\right)$$

The Earth, A_c~1 for E = 10 GeV

Analytical solution to $\widetilde{\theta}_{12}$

- Series expansion invalid for small values of A_c
- Introduce two functions of A_c to be solved from RGEs

$$\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}A_c} = -\frac{(\cos 2\theta_{12} + \mathcal{F})\cos^2\theta_{13}}{(\cos 2\theta_{12} + 2\mathcal{F})\alpha_c + A_c}$$

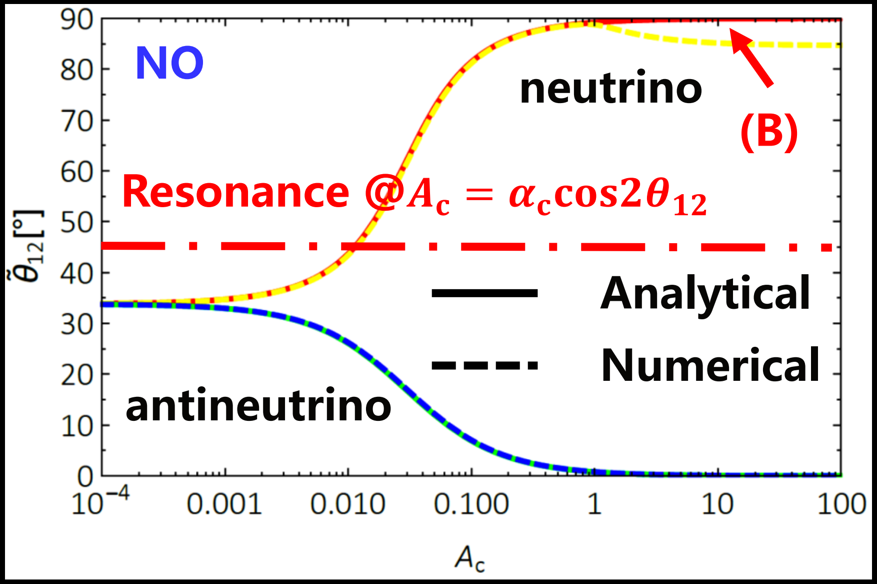
$$\widetilde{\Delta}_{21} = \Delta_c \left[\frac{1}{2}(1 + A_c - \widehat{C}_{13}) + \alpha_c(\mathcal{F} - \mathcal{G})\right],$$

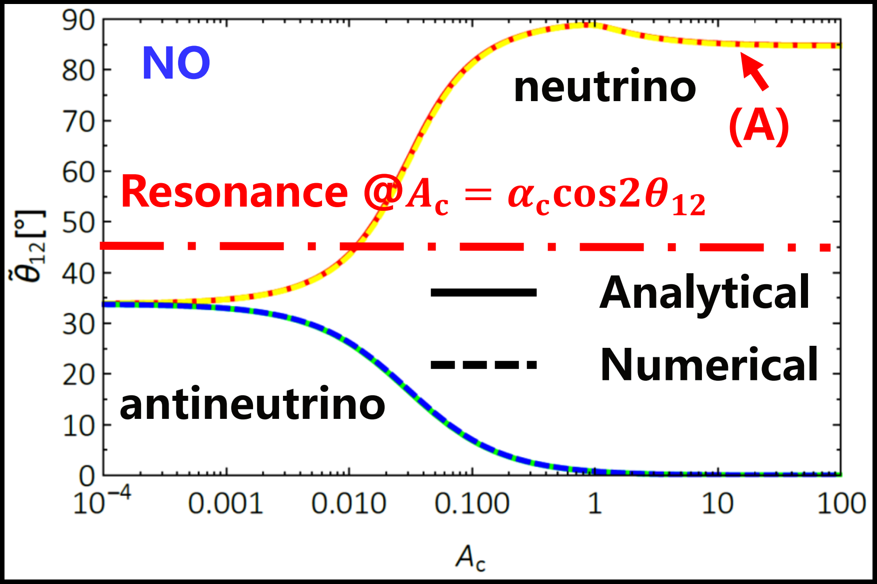
$$\widetilde{\Delta}_{31} = \Delta_c \left[\frac{1}{2}(1 + A_c + \widehat{C}_{13}) + \alpha_c\mathcal{F}\right],$$

$$\widetilde{\Delta}_{32} = \Delta_c \left(\widehat{C}_{13} + \alpha_c\mathcal{G}\right),$$

$$\mathrm{d}\mathcal{F}/\mathrm{d}A_c + \mathrm{d}\mathcal{G}/\mathrm{d}A_c = 0$$
$$\mathcal{F} + \mathcal{G} = -\cos 2\theta_{12}$$

Analytical Solutions to RGEs





Analytical Solution:

$$\mathcal{F}(A_c) = \frac{1}{2\alpha_c}\left[\sqrt{(A_c - \alpha_c \cos 2\theta_{12})^2 + \alpha_c^2 \sin^2 2\theta_{12}} - (A_c + \alpha_c \cos 2\theta_{12})\right]$$

(A)
$$\cos^2 \tilde{\theta}_{12} = \frac{1}{2}\left(1 - \frac{A_* - \cos 2\theta_{12}}{\widehat{C}_{12}}\right) \frac{2\widehat{C}_{13} \cos^2 \theta_{13}}{\widehat{C}_{13} - A_c + \cos 2\theta_{13}}$$

Define $A_* \equiv A_c/\alpha_c = a/\Delta_{21}$

$$\widehat{C}_{12} \equiv \sqrt{1 - 2A_* \cos 2\theta_{12} + A_*^2}$$

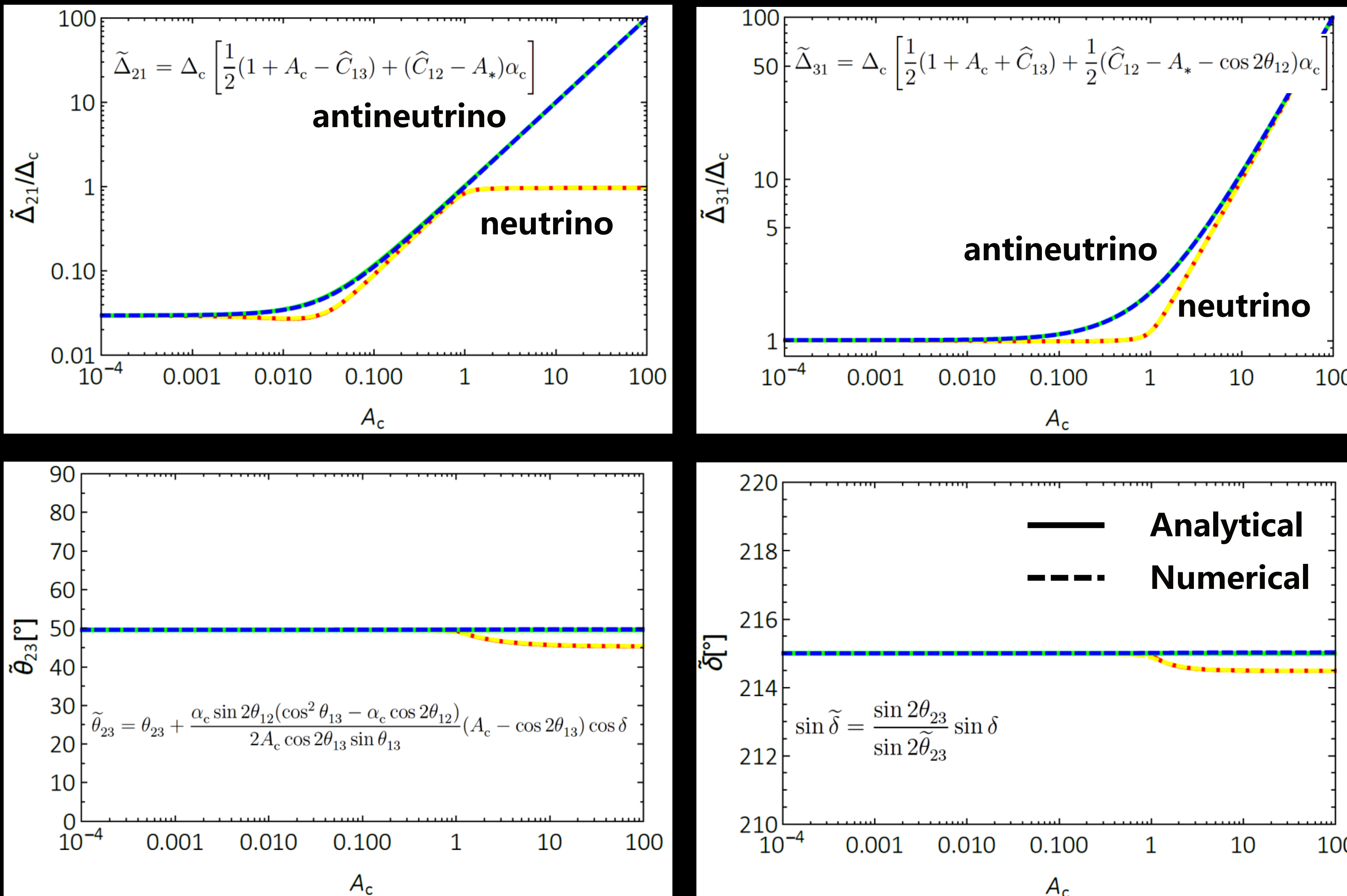
(B)
$$\cos^2 \tilde{\theta}_{12} = \frac{1}{2}\left(1 - \frac{A_* - \cos 2\theta_{12}}{\widehat{C}_{12}}\right)$$

$$\frac{\cos^2\theta_{13}}{\cos^2\tilde{\theta}_{13}}$$

$\tilde{\theta}_{12}$ **given by the formula in the limit of two-flavor mixing**

- $\tilde{\theta}_{12}$ **can be understood by two-flavor neutrino mixing in matter, which will receive corrections near the resonance at** $A_c = \cos 2\theta_{13}$ **and for large values of** A_c

Analytical Solutions to RGEs



Applications: the Jarlskog Invariant

X. Wang & S.Z., arXiv: 1901.10882

$$\frac{\widetilde{\mathcal{J}}}{\mathcal{J}} = \frac{\sin\widetilde{\theta}_{12}\cos\widetilde{\theta}_{12}\sin\widetilde{\theta}_{13}\cos^2\widetilde{\theta}_{13}}{\sin\theta_{12}\cos\theta_{12}\sin\theta_{13}\cos^2\theta_{13}} \approx \frac{1}{\widehat{C}_{12}\widehat{C}_{13}}$$

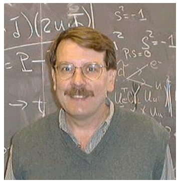
where

$$\widehat{C}_{12} \equiv \sqrt{1 - 2A_* \cos 2\theta_{12} + A_*^2} \qquad A_* \equiv a/\Delta_{21}$$

$$\widehat{C}_{13} \equiv \sqrt{1 - 2A_c \cos 2\theta_{13} + A_c^2} \qquad A_c \equiv a/\Delta_c$$

P. Denton & S. Parke, arXiv: 1902.07185v1

$$\widehat{J} \approx \frac{J}{\mathcal{S}_{12}\mathcal{S}_{13}}\,, \qquad (5)$$



where

$$\mathcal{S}_{12} = \sqrt{1 - 2\cos 2\theta_{12}(c_{13}^2 a/\Delta m_{21}^2) + (c_{13}^2 a/\Delta m_{21}^2)^2}\,,$$
$$\mathcal{S}_{13} = \sqrt{1 - 2\cos 2\theta_{13}(a/\Delta m_{ee}^2) + (a/\Delta m_{ee}^2)^2}\,. \qquad (6)$$

X. Wang & S.Z., arXiv: 1908.07304

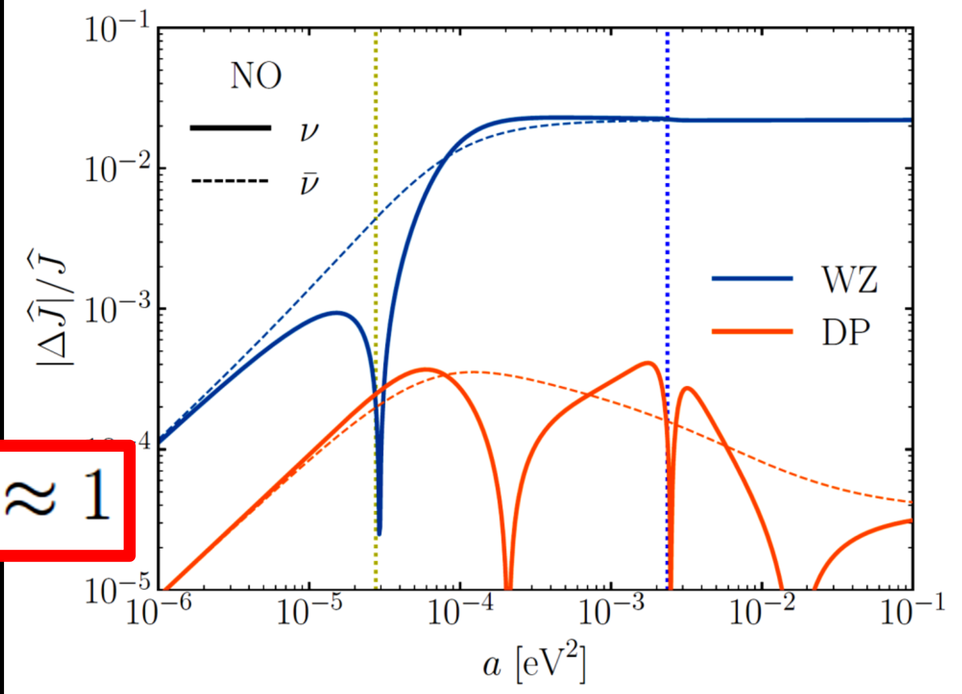
$$\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}(A_c\cos^2\theta_{13})} = -\frac{\cos 2\theta_{12} + \mathcal{F}}{(\cos 2\theta_{12} + 2\mathcal{F})\alpha_c + A_c\cos^2\theta_{13}}$$

$$\cos^2\widetilde{\theta}_{12} = \frac{1}{2}\left(1 - \frac{\widehat{A}_* - \cos 2\theta_{12}}{\widehat{C}_{12}}\right)\frac{2\widehat{C}_{13}\cos^2\theta_{13}}{\widehat{C}_{13} - A_c + \cos 2\theta_{13}}$$

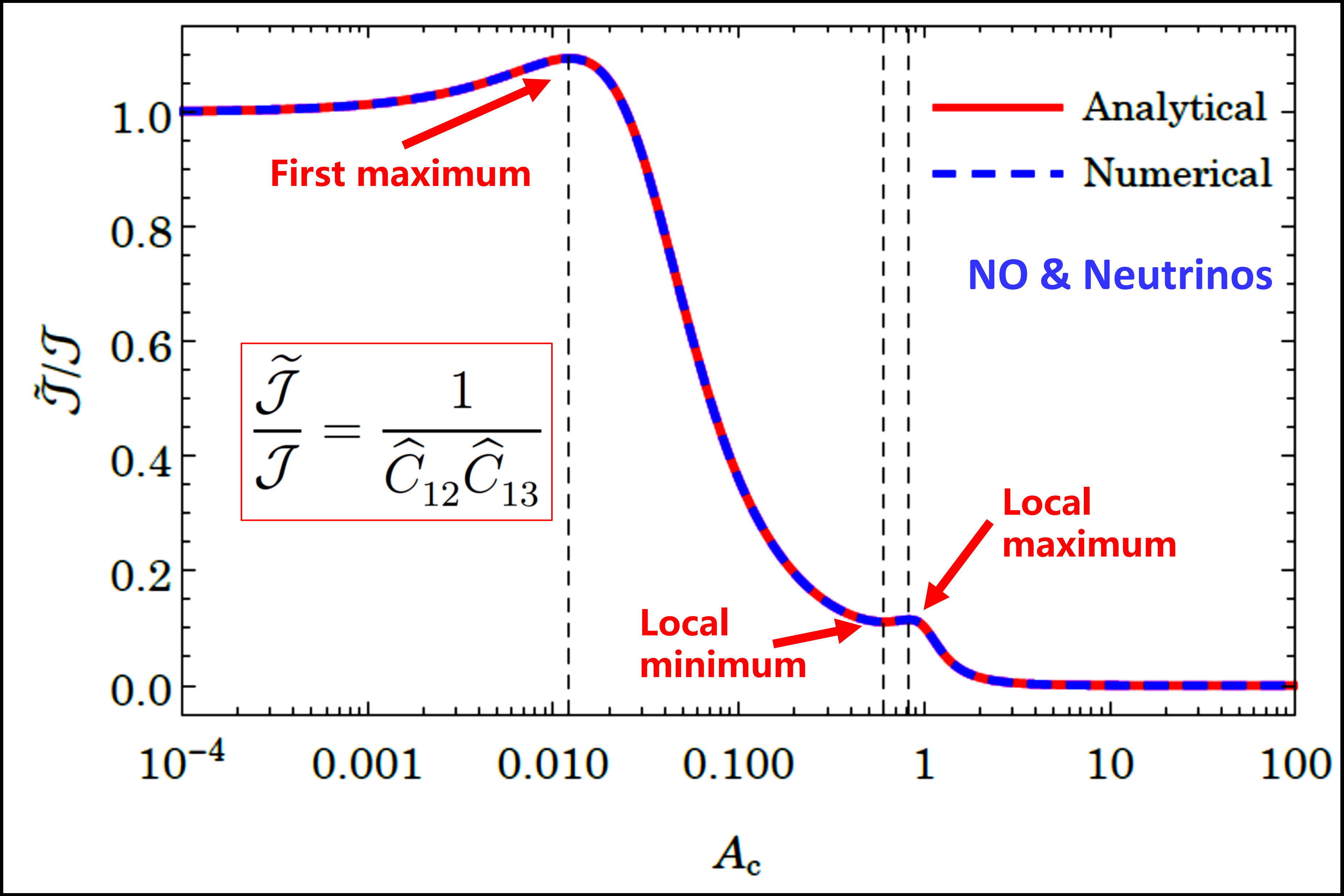
$$\widehat{C}_{12} \equiv \sqrt{1 - 2\widehat{A}_*\cos 2\theta_{12} + \widehat{A}_*^2} \qquad \cos^2\theta_{13} \approx 1$$

$$\widehat{A}_* \equiv A_c\cos^2\theta_{13}/\alpha_c = a\cos^2\theta_{13}/\Delta_{21}$$

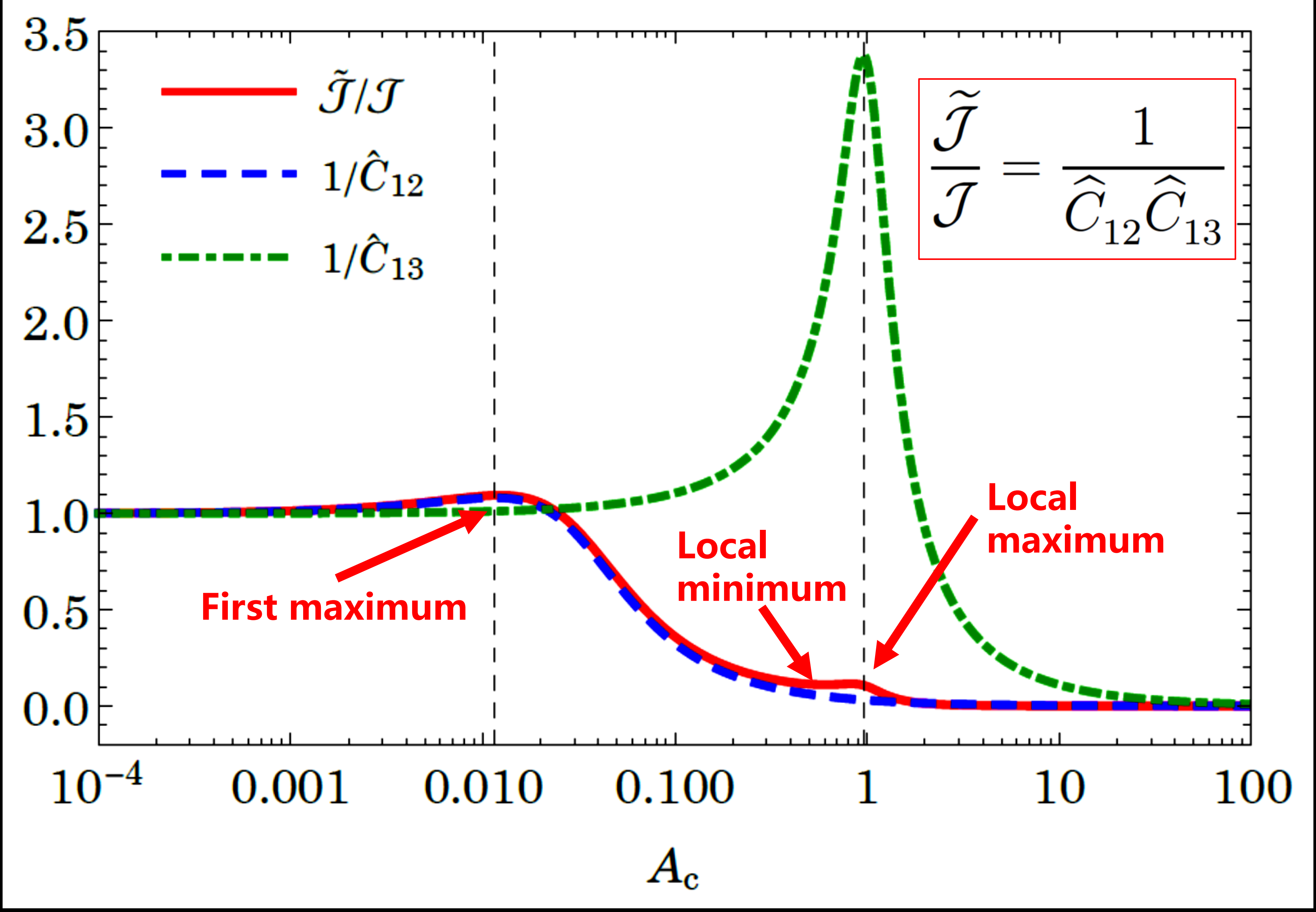
Analytical solution to $\widetilde{\theta}_{12}$ improved



Applications: the Jarlskog Invariant



Applications: the Jarlskog Invariant



Applications: the Jarlskog Invariant

Find the extrema

$$\frac{\mathrm{d}}{\mathrm{d}A_{\mathrm{c}}}\left(\frac{\widetilde{\mathcal{J}}}{\mathcal{J}}\right) = \frac{\mathrm{d}}{\mathrm{d}A_{\mathrm{c}}}\left(\frac{1}{\widehat{C}_{12}\widehat{C}_{13}}\right) = -\frac{1}{\widehat{C}_{12}^2\widehat{C}_{13}^2}\left[\widehat{C}_{13}\left(\frac{\mathrm{d}\widehat{C}_{12}}{\mathrm{d}A_{\mathrm{c}}}\right) + \widehat{C}_{12}\left(\frac{\mathrm{d}\widehat{C}_{13}}{\mathrm{d}A_{\mathrm{c}}}\right)\right] = 0$$

$$A_{\mathrm{c}}^{(1)} = \frac{\cos 2\theta_{12}}{\cos^2\theta_{13}}\alpha_{\mathrm{c}} \,,$$

$$A_{\mathrm{c}}^{(2)} = \frac{1}{4}\left(3\cos 2\theta_{13} + \frac{\cos 2\theta_{12}}{\cos^2\theta_{13}}\alpha_{\mathrm{c}} - \sqrt{1 - 9\sin^2 2\theta_{13} - \frac{2\cos 2\theta_{12}\cos 2\theta_{13}}{\cos^2\theta_{13}}\alpha_{\mathrm{c}}}\right)$$

$$A_{\mathrm{c}}^{(3)} = \frac{1}{4}\left(3\cos 2\theta_{13} + \frac{\cos 2\theta_{12}}{\cos^2\theta_{13}}\alpha_{\mathrm{c}} + \sqrt{1 - 9\sin^2 2\theta_{13} - \frac{2\cos 2\theta_{12}\cos 2\theta_{13}}{\cos^2\theta_{13}}\alpha_{\mathrm{c}}}\right)$$

- **We find excellent agreement between analytical & numerical calculations**

$$\left(\frac{\widetilde{\mathcal{J}}}{\mathcal{J}}\right)\Bigg|_{(1)}^{\max} = \frac{1}{\sin 2\theta_{12}}\left(1 + \cos 2\theta_{12}\cos 2\theta_{13}\sec^2\theta_{13}\alpha_{\mathrm{c}}\right) \,,$$

$$\left(\frac{\widetilde{\mathcal{J}}}{\mathcal{J}}\right)\Bigg|_{(2)}^{\min} = \frac{4\sqrt{2}\sec^2\theta_{13}\alpha_{\mathrm{c}}}{\sqrt{4 - 3(1-3\sin^2 2\theta_{13})^2 + \cos 2\theta_{13}(1-9\sin^2 2\theta_{13})^{3/2}}} \,,$$

$$\left(\frac{\widetilde{\mathcal{J}}}{\mathcal{J}}\right)\Bigg|_{(3)}^{\max} = \frac{4\sqrt{2}\sec^2\theta_{13}\alpha_{\mathrm{c}}}{\sqrt{4 - 3(1-3\sin^2 2\theta_{13})^2 - \cos 2\theta_{13}(1-9\sin^2 2\theta_{13})^{3/2}}} \,,$$

- **The extrema are associated with two resonances corresponding to Δ_{21} & Δ_{c}**
- **A new way to understand matter effects on neutrino oscillations!**

An Integral Way to Understand RG Running

One-loop RGEs for the quark and lepton Yukawa coupling matrices

$$16\pi^2 \frac{\mathrm{d}Y_\mathrm{u}}{\mathrm{d}t} = \left[\alpha_\mathrm{u} + \frac{3}{2}(Y_\mathrm{u}Y_\mathrm{u}^\dagger) - \frac{3}{2}(Y_\mathrm{d}Y_\mathrm{d}^\dagger)\right] Y_\mathrm{u},$$

$$16\pi^2 \frac{\mathrm{d}Y_\mathrm{d}}{\mathrm{d}t} = \left[\alpha_\mathrm{d} - \frac{3}{2}(Y_\mathrm{u}Y_\mathrm{u}^\dagger) + \frac{3}{2}(Y_\mathrm{d}Y_\mathrm{d}^\dagger)\right] Y_\mathrm{d},$$

$$16\pi^2 \frac{\mathrm{d}Y_\nu}{\mathrm{d}t} = \left[\alpha_\nu + \frac{3}{2}(Y_\nu Y_\nu^\dagger) - \frac{3}{2}(Y_l Y_l^\dagger)\right] Y_\nu,$$

$$16\pi^2 \frac{\mathrm{d}Y_l}{\mathrm{d}t} = \left[\alpha_l - \frac{3}{2}(Y_\nu Y_\nu^\dagger) + \frac{3}{2}(Y_l Y_l^\dagger)\right] Y_l,$$

$$\alpha_\mathrm{u} = -\frac{17}{20}g_1^2 - \frac{9}{4}g_2^2 - 8g_3^2 + \chi,$$

$$\alpha_\mathrm{d} = -\frac{1}{4}g_1^2 - \frac{9}{4}g_2^2 - 8g_3^2 + \chi,$$

$$\alpha_\nu = -\frac{9}{20}g_1^2 - \frac{9}{4}g_2^2 + \chi,$$

$$\alpha_l = -\frac{9}{4}g_1^2 - \frac{9}{4}g_2^2 + \chi,$$

Standard Model with three massive Dirac neutrinos

$$t \equiv \ln(\mu/\Lambda_\mathrm{EW})$$

$$\chi \equiv \mathrm{Tr}\left[3(Y_\mathrm{u}Y_\mathrm{u}^\dagger) + 3(Y_\mathrm{d}Y_\mathrm{d}^\dagger) + (Y_\nu Y_\nu^\dagger) + (Y_l Y_l^\dagger)\right]$$

Gauge couplings

$$16\pi^2 (\mathrm{d}g_i/\mathrm{d}t) = b_i g_i^3 \qquad \{b_3, b_2, b_1\} = \{-7, -19/6, 41/10\}$$

$$16\pi^2 \frac{\mathrm{d}D_\mathrm{u}}{\mathrm{d}t} = \left[\alpha_\mathrm{u} + \frac{3}{2}D_\mathrm{u}^2\right] D_\mathrm{u},$$

$$16\pi^2 \frac{\mathrm{d}Y_\mathrm{d}}{\mathrm{d}t} = \left[\alpha_\mathrm{d} - \frac{3}{2}D_\mathrm{u}^2\right] Y_\mathrm{d},$$

$$16\pi^2 \frac{\mathrm{d}Y_\nu}{\mathrm{d}t} = \left[\alpha_\nu - \frac{3}{2}D_l^2\right] Y_\nu,$$

$$16\pi^2 \frac{\mathrm{d}D_l}{\mathrm{d}t} = \left[\alpha_l + \frac{3}{2}D_l^2\right] D_l,$$

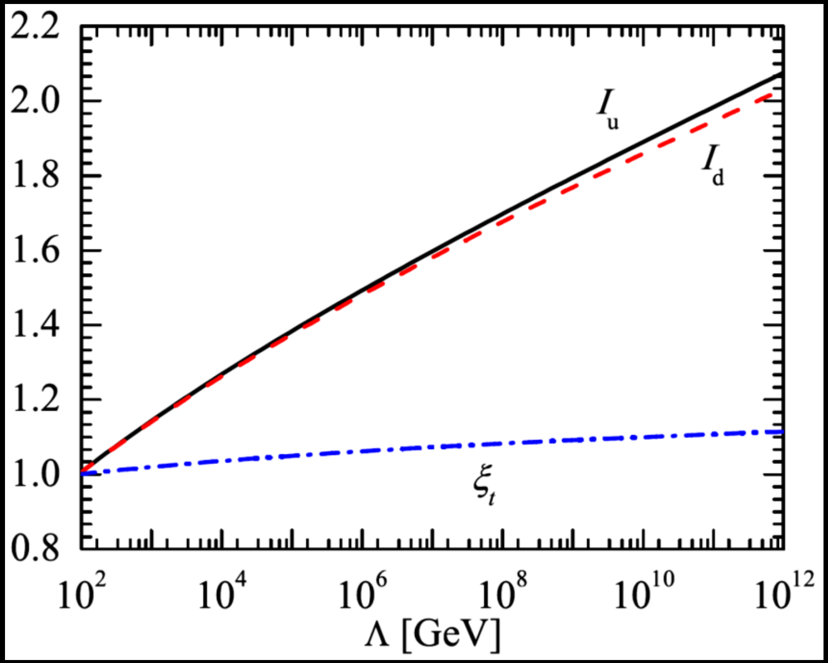
Choose the flavor basis where Y_u and Y_l are diagonal:

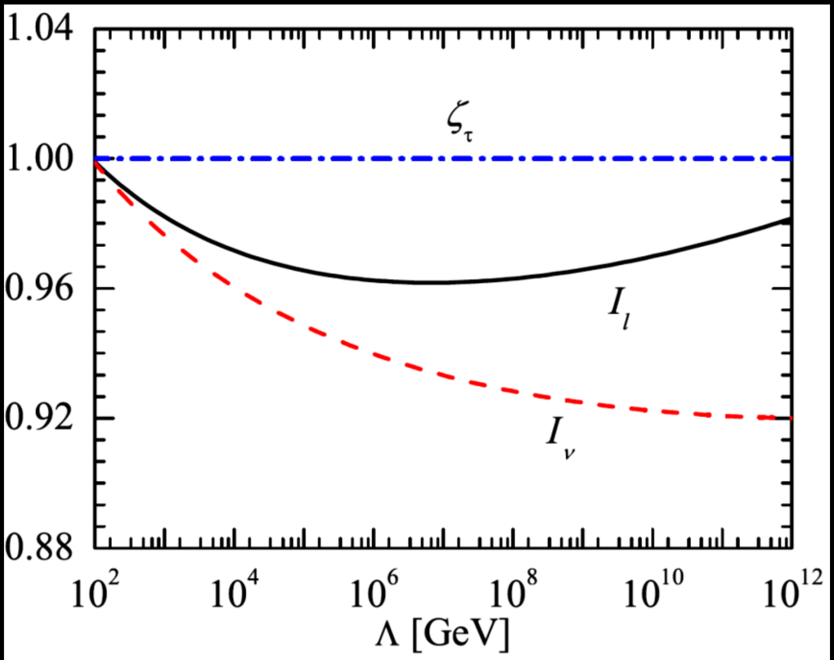
$$Y_\mathrm{u} = \mathrm{diag}\{y_u, y_c, y_t\} \equiv D_\mathrm{u}$$

$$Y_l = \mathrm{diag}\{y_e, y_\mu, y_\tau\} \equiv D_l$$

In consideration of the strong hierarchy of fermion Yukawa couplings, the down-type quark and neutrino Yukawa terms on the right-hand side are safely omitted

An Integral Way to Understand RG Running





Superhigh scale Λ

$$t_0 \equiv \ln(\Lambda/\Lambda_{\rm EW})$$

$$I_{\rm f} = \exp\left[-\frac{1}{16\pi^2}\int_0^{t_0} \alpha_{\rm f}(t)\,{\rm d}t\right]$$

$$\xi_q = \exp\left[+\frac{3}{32\pi^2}\int_0^{t_0} y_q^2(t)\,{\rm d}t\right]$$

$$\zeta_\alpha = \exp\left[+\frac{3}{32\pi^2}\int_0^{t_0} y_\alpha^2(t)\,{\rm d}t\right]$$

$$D'_{\rm u} = I_{\rm u} T_{\rm u} D_{\rm u} \qquad Y'_{\rm d} = I_{\rm d} T_{\rm d} Y_{\rm d}$$

$$D'_l = I_l T_l D_l \qquad Y'_\nu = I_\nu T_\nu Y_\nu$$

Yukawa couplings at the electroweak scale $\Lambda_{\rm EW}$

$$T_{\rm d} = {\rm diag}\{\xi_u, \xi_c, \xi_t\} = T_{\rm u}^{-1}$$

$$T_\nu = {\rm diag}\{\zeta_e, \zeta_\mu, \zeta_\tau\} \equiv T_l^{-1}$$

Relationship between mixing parameters at different scales

$$H'_{\rm d} \equiv Y'_{\rm d} Y'^{\dagger}_{\rm d} = V' D'^2_{\rm d} V'^{\dagger}$$

$$H'_\nu \equiv Y'_\nu Y'^{\dagger}_\nu = U' D'^2_\nu U'^{\dagger}$$

$$[D'^2_{\rm u}, H'_{\rm d}] \equiv {\rm i} X'_{\rm q}$$

$$[D^2_{\rm u}, H_{\rm d}] \equiv {\rm i} X_{\rm q}$$

RG Running vs. Matter Effects on Flavor Mixing

RG running of flavor mixing parameters

$$X_{\rm q}' = {\rm i}\begin{pmatrix} 0 & \Delta_{cu}' Z_{uc}' & \Delta_{tu}' Z_{ut}' \\ \Delta_{uc}' Z_{cu}' & 0 & \Delta_{tc}' Z_{ct}' \\ \Delta_{ut}' Z_{tu}' & \Delta_{ct}' Z_{tc}' & 0 \end{pmatrix}$$

$$X_{\rm q} = {\rm i}\begin{pmatrix} 0 & \Delta_{cu} Z_{uc} & \Delta_{tu} Z_{ut} \\ \Delta_{uc} Z_{cu} & 0 & \Delta_{tc} Z_{ct} \\ \Delta_{ut} Z_{tu} & \Delta_{ct} Z_{tc} & 0 \end{pmatrix}$$

$$\mathcal{J}_{\rm q}' \Delta_{sd}' \Delta_{bd}' \Delta_{bs}' = I_{\rm d}^6 \xi_u^2 \xi_c^2 \xi_t^2 \mathcal{J}_{\rm q} \Delta_{sd} \Delta_{bd} \Delta_{bs}$$

$$\mathcal{J}_\ell' \Delta_{21}' \Delta_{31}' \Delta_{32}' = I_\nu^6 \zeta_e^{-2} \zeta_\mu^{-2} \zeta_\tau^{-2} \mathcal{J}_\ell \Delta_{21} \Delta_{31} \Delta_{32}$$

Matter effects on flavor mixing parameters

$$X_l = i\begin{pmatrix} 0 & \Delta_{e\mu} Z_{\mu e} & \Delta_{\tau e} Z_{e\tau} \\ \Delta_{\mu e} Z_{e\mu} & 0 & \Delta_{\mu\tau} Z_{\tau\mu} \\ \Delta_{e\tau} Z_{\tau e} & \Delta_{\tau\mu} Z_{\mu\tau} & 0 \end{pmatrix}$$

$$\widetilde{X}_l = i\begin{pmatrix} 0 & \Delta_{e\mu} \widetilde{Z}_{\mu e} & \Delta_{\tau e} \widetilde{Z}_{e\tau} \\ \Delta_{\mu e} \widetilde{Z}_{e\mu} & 0 & \Delta_{\mu\tau} \widetilde{Z}_{\tau\mu} \\ \Delta_{e\tau} \widetilde{Z}_{\tau e} & \Delta_{\tau\mu} \widetilde{Z}_{\mu\tau} & 0 \end{pmatrix}$$

For quark flavor mixing in media?

$$\widetilde{\mathcal{J}}_\ell \widetilde{\Delta}_{21} \widetilde{\Delta}_{31} \widetilde{\Delta}_{32} = \mathcal{J}_\ell \Delta_{21} \Delta_{31} \Delta_{32}$$

- **A complete set of differential equations have been derived and applied to understand matter effects on effective neutrino mixing parameters**

- **Inspired by the matter effects on neutrino oscillations, we attempt to study RG running of quark and lepton flavor mixing in an integral way and find the integral invariants: direct relations between parameters at low- and high-energy scales**