Analytic Two-Loop Higgs Amplitudes and Maximal Transcendentality Principle

Gang Yang (杨刚)

Institute of Theoretical Physics,
Chinese Academy of Sciences

Based on the work with Qingjun Jin (靳庆军)
arXiv:1804.04653 and in progress

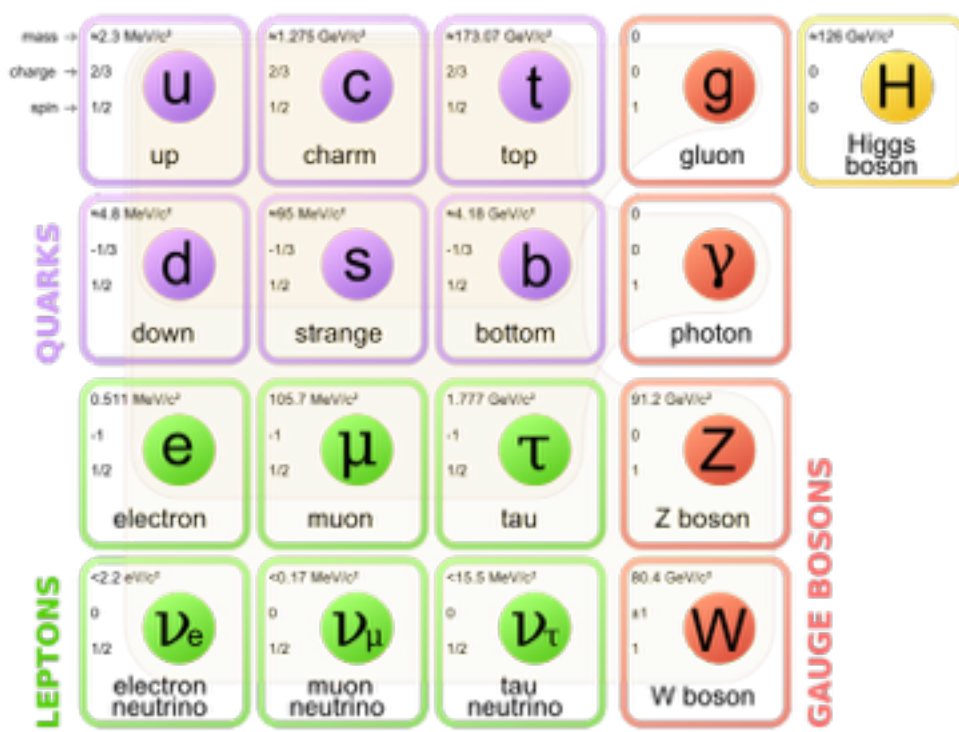
中科大交叉中心
October 11, 2018

Content

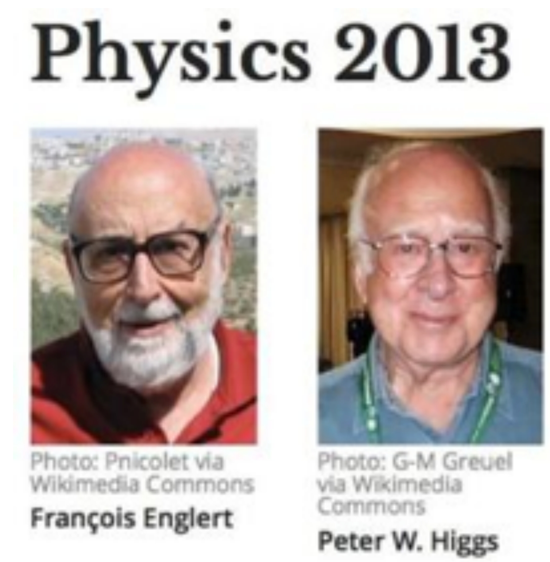
- Motivation
- Computation
- Results
- Summary and outlook

Quantum Field theory (QFT)

QFT is the foundation of modern theoretical physics:
particle physics, condensed matter, gravity and cosmology, etc.



Standard Model (SM)
of Particle Physics



SM-like Higgs
boson discovered

Challenges

Experimental

Efficient perturbative methods

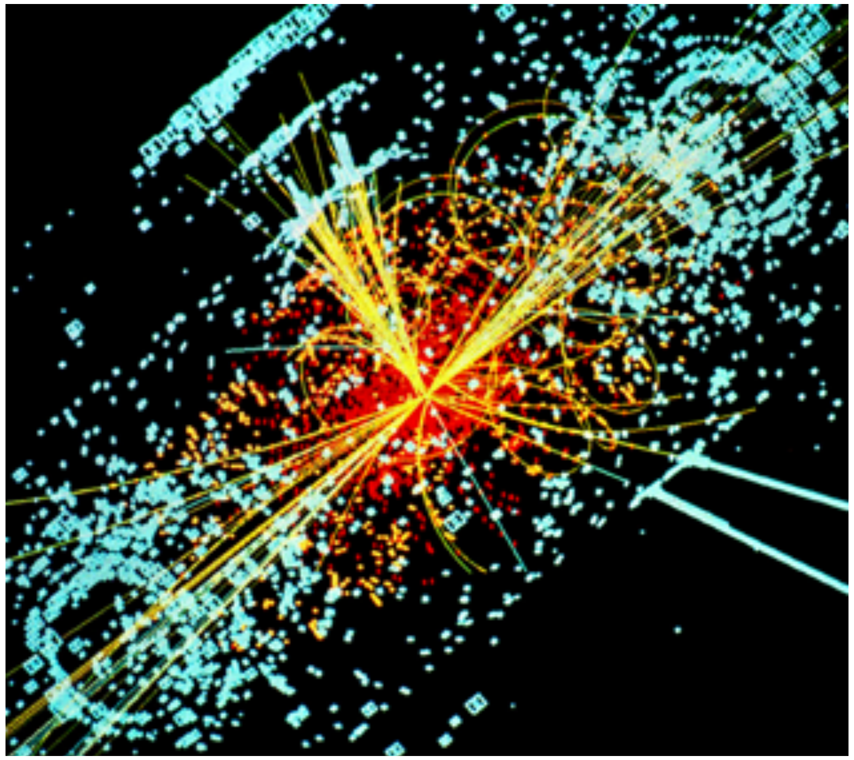
Theoretical

Non-perturbative method

Large Hadron Collider (LHC)

Precise test of SM

New physics?!



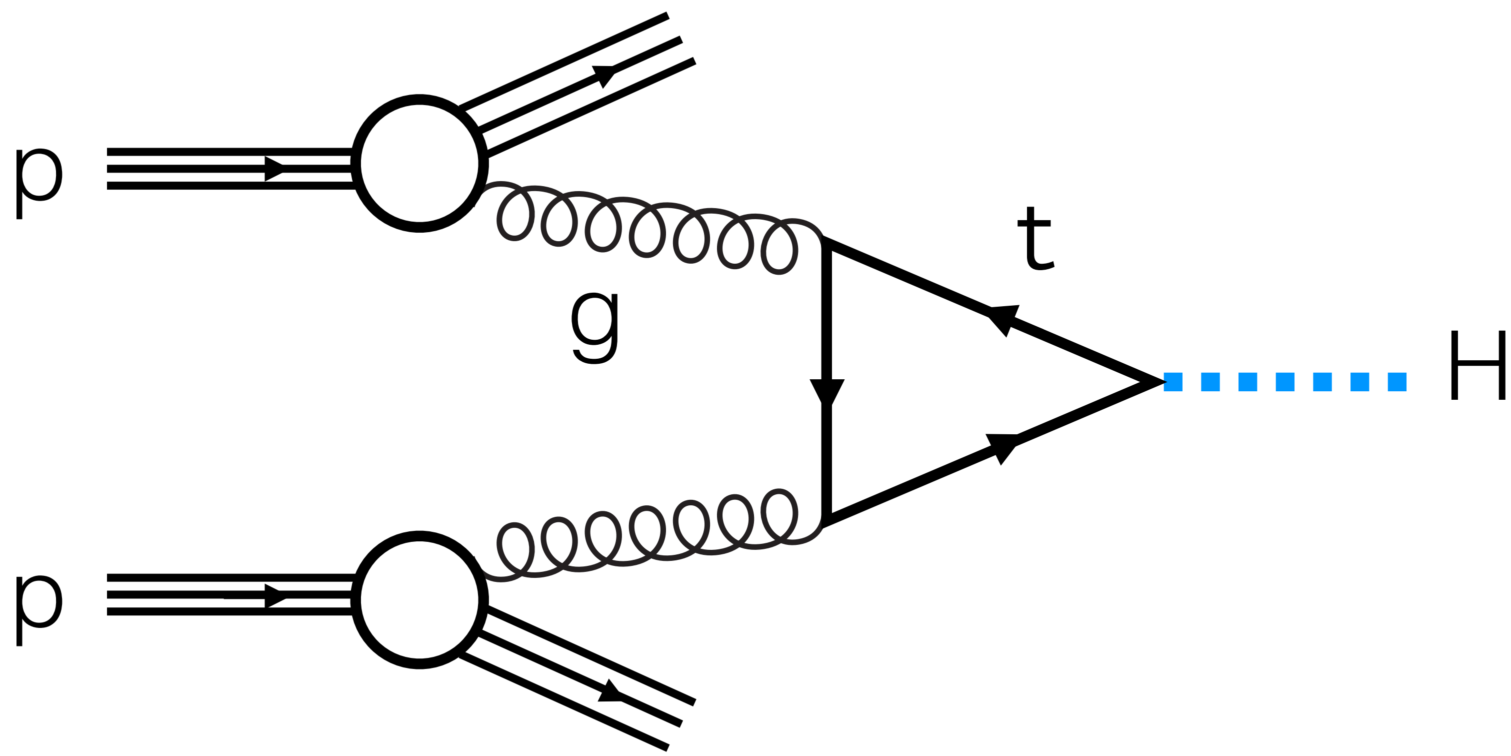
Simulated CMS event

Higher energy and luminosity -> increasing precision

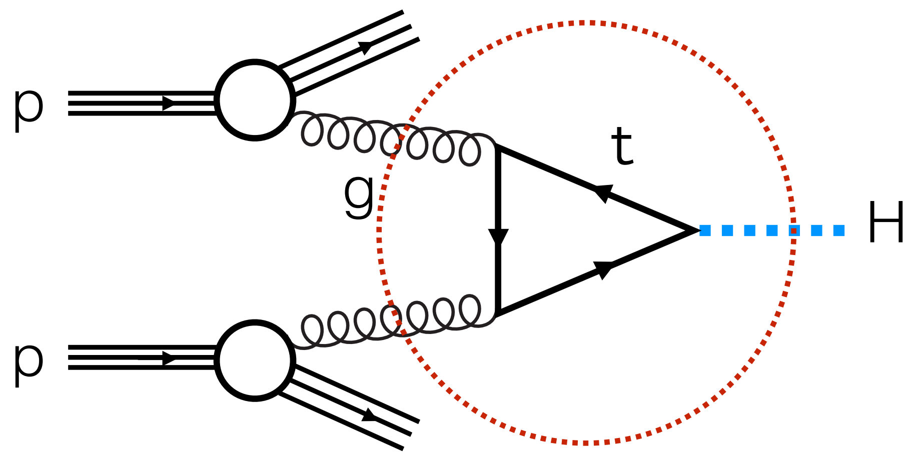
Precise theoretical prediction — at two or higher loops — of the scattering processes is mandatory.

Higgs boson @ LHC

The dominant production mechanism is the gluon fusion through a top quark loop.



Effective Field Theory (EFT)



$m_t \to \infty$

Effective gluon-Higgs vertex:

$$\mathcal{L}_{\text{eff}} = C_0 H \operatorname{tr}(G^2) + \mathcal{O}\left(\frac{1}{m_{\text{t}}^2}\right)$$

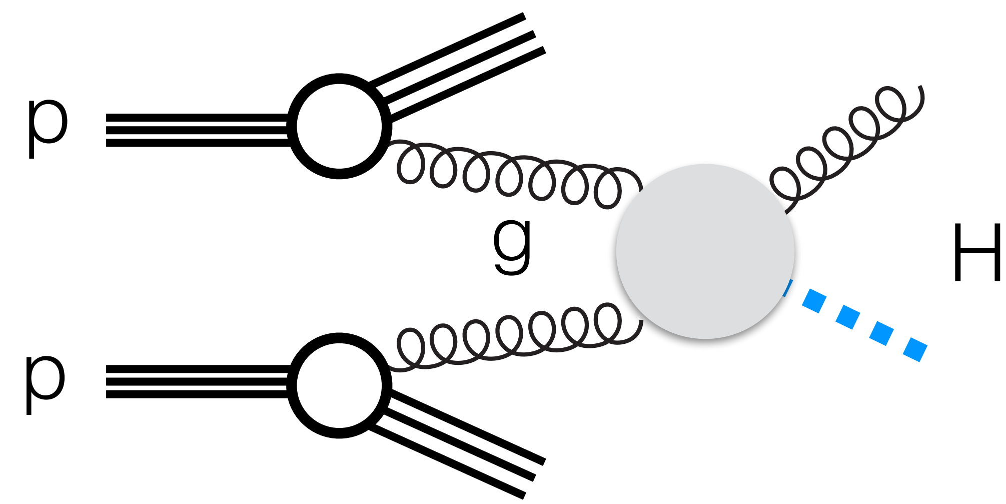
There have been computations for inclusive Higgs production to N^3LO orders in the heavy quark limit.

[Anastasiou, Duhr, Dulat, Furlan, Gehrmann, Herzog, Lazopoulos, Mistlberger 2016]

Effective Field Theory (EFT)

Higgs plus jet production
is sensitive to new physics.

EFT description is not good
when $p_T \sim 2m_t$



High dimension operators contribution are important.

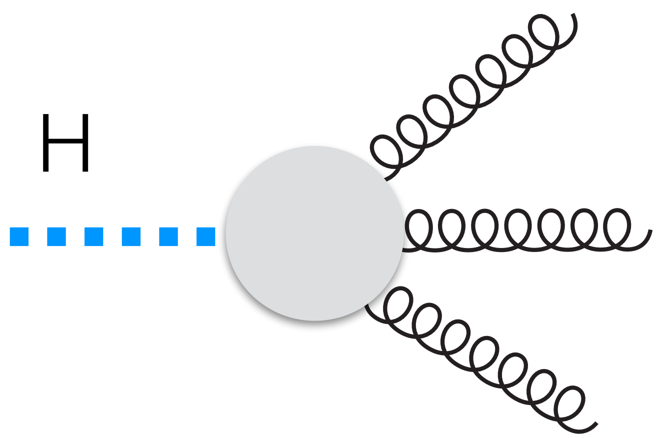
$$\mathcal{L}_{\rm eff} = C_0 O_0 + \frac{1}{m_{\rm t}^2} \sum_{i=1}^{4} C_i O_i + \mathcal{O}\left(\frac{1}{m_{\rm t}^4}\right)$$

Dimension-7 operators

$$\begin{aligned}
O_1 &= H \mathrm{Tr}(G_\mu{}^\nu G_\nu{}^\rho G_\rho{}^\mu)\,, \\
O_2 &= H \mathrm{Tr}(D_\rho G_{\mu\nu} D^\rho G^{\mu\nu})\,, \\
O_3 &= H \mathrm{Tr}(D^\rho G_{\rho\mu} D_\sigma G^{\sigma\mu})\,, \\
O_4 &= H \mathrm{Tr}(G_{\mu\rho} D^\rho D_\sigma G^{\sigma\mu})\,.
\end{aligned}$$

Goal

Compute two-loop Higgs amplitudes with dim-7 operators



$$\mathcal{L}_{\text{eff}} = C_0 O_0 + \frac{1}{m_t^2} \sum_{i=1}^{4} C_i O_i + \mathcal{O}\left(\frac{1}{m_t^4}\right)$$

$$O_0 = H\text{tr}(G^2)$$

$$O_1 = H\text{Tr}(G_\mu{}^\nu G_\nu{}^\rho G_\rho{}^\mu)\,,$$
$$O_2 = H\text{Tr}(D_\rho G_{\mu\nu} D^\rho G^{\mu\nu})\,,$$

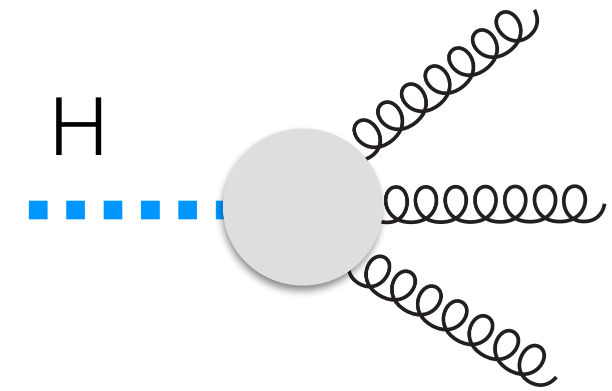
for pure YM

$$O_3 = H\text{Tr}(D^\rho G_{\rho\mu} D_\sigma G^{\sigma\mu})\,,$$
$$O_4 = H\text{Tr}(G_{\mu\rho} D^\rho D_\sigma G^{\sigma\mu})\,.$$

This provides the two-loop virtual amplitudes for the top mass correction in EFT.

Form factors

Higgs amplitudes are equivalent to **form factors**:



$$F_{\mathcal{O}_i,n} = \int d^4x\, e^{-iq\cdot x} \langle p_1, \ldots, p_n | \mathcal{O}_i(x) | 0 \rangle$$

$$\mathcal{O}_0 = \mathrm{tr}(G_{\mu\nu} G^{\mu\nu})\,.$$
$$\mathcal{O}_1 = \mathrm{tr}(G_{\mu}{}^{\nu} G_{\nu}{}^{\rho} G_{\rho}{}^{\mu})\,,$$
$$\mathcal{O}_2 = \mathrm{tr}(D_\rho G_{\mu\nu} D^\rho G^{\mu\nu})\,.$$

Linear relation:

$$\mathcal{O}_2 = \frac{1}{2}\,\partial^2 \mathcal{O}_0 - 4\, g_{\mathrm{YM}}\, \mathcal{O}_1 + 2\, \mathcal{O}_4 \longrightarrow F_{\mathcal{O}_2} = \frac{1}{2}\, q^2\, F_{\mathcal{O}_0} - 4\, g_{\mathrm{YM}}\, F_{\mathcal{O}_1}$$

Theoretical motivations

Feynman diagram?

Feynman diagram method works in principle, but the complexity grows extremely fast with increasing number of external legs / loops.

n-gluon tree amplitudes:

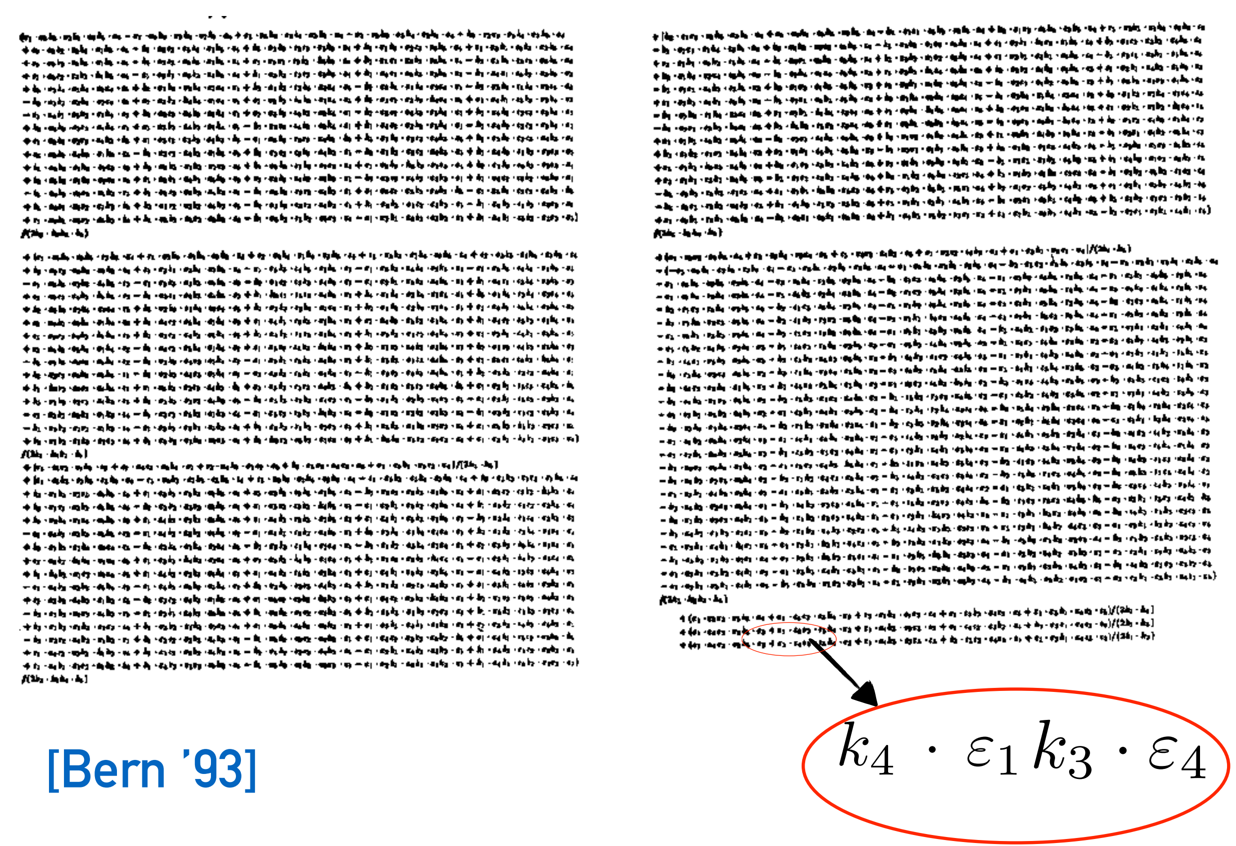
n	2	3	4	5	6	7	8
# of diagrams	4	25	220	2485	34300	559405	10525900

Surprising simplicity

MHV (Maximally-helicity-violating) amplitudes: [Parke, Taylor '86]

$$A_n^{\text{tree}}(1^+, \ldots, i^-, \ldots, j^-, \ldots, n^+) = \frac{\langle ij \rangle^4}{\langle 12 \rangle \cdots \langle n1 \rangle}$$

Comparing with result of Feynman diagrams:

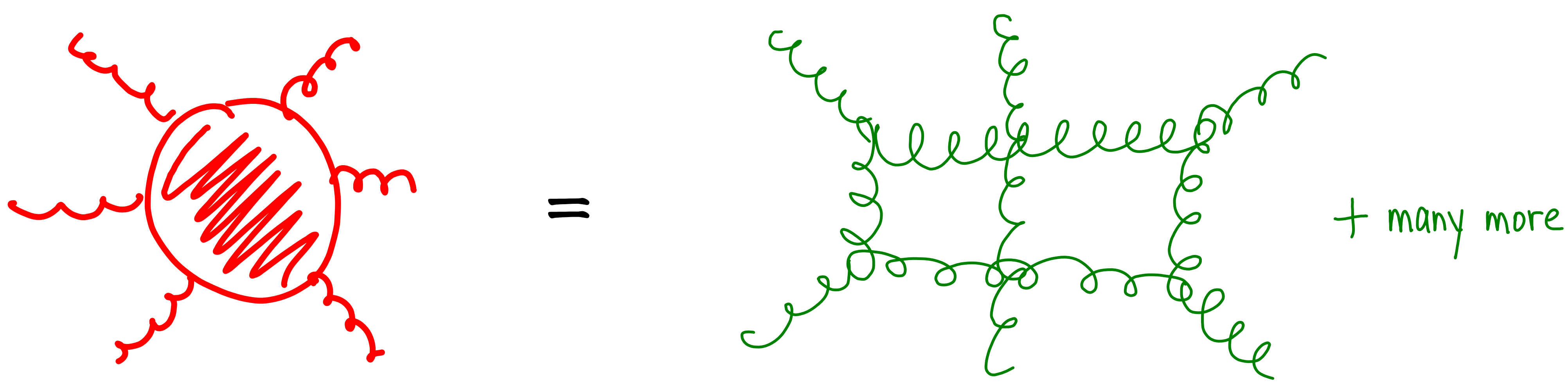


[Bern '93]

Surprising simplicity

A more non-trivial example of two-loop amplitudes:

Six-gluon MHV amplitudes in N=4 SYM



[Del Duca, Duhr, Smirnov 2010]

a heroic analytical computation

[Del Duca, Duhr, Smirnov 2010]

$$R^{(2)}_{6,WL}(u_1,u_2,u_3) = \tag{H.1}$$

$$\frac{1}{24}\pi^2 G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)+$$
$$\frac{1}{24}\pi^2 G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)+$$
$$\frac{1}{24}\pi^2 G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)+$$
$$\frac{3}{2}G\left(0,0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)+$$
$$\frac{3}{2}G\left(0,0,\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)-$$
$$\frac{1}{2}G\left(0,\frac{1}{u_1},0,\frac{1}{u_2};1\right)+G\left(0,\frac{1}{u_1},0,\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},0,\frac{1}{u_3};1\right)+$$
$$G\left(0,\frac{1}{u_1},0,\frac{1}{u_1+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)-$$
$$\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_2},0,\frac{1}{u_1};1\right)+$$
$$G\left(0,\frac{1}{u_2},0,\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_2},0,\frac{1}{u_3};1\right)+G\left(0,\frac{1}{u_2},0,\frac{1}{u_2+u_3};1\right)-$$
$$\frac{1}{2}G\left(0,\frac{1}{u_2},\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_2},\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_2},\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)-$$
$$\frac{1}{2}G\left(0,\frac{1}{u_2},\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)+\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},0,\frac{1}{1-u_1};1\right)+$$
$$\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},0;1\right)-\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},1;1\right)+$$
$$\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)-\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)-$$
$$\frac{1}{2}G\left(0,\frac{1}{u_3},0,\frac{1}{u_1};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_3},0,\frac{1}{u_2};1\right)+G\left(0,\frac{1}{u_3},0,\frac{1}{u_1+u_3};1\right)+$$
$$G\left(0,\frac{1}{u_3},0,\frac{1}{u_2+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_3},\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_3},\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)-$$
$$\frac{1}{2}G\left(0,\frac{1}{u_3},\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_3},\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)+$$
$$\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},0,\frac{1}{1-u_3};1\right)+\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},0;1\right)-$$
$$\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},1;1\right)+\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)-$$
$$\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)+\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},0,\frac{1}{1-u_2};1\right)+$$
$$\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2},0;1\right)-\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2},1;1\right)+$$
$$\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2},\frac{1}{1-u_2};1\right)-\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_1},1,\frac{1}{u_3},0;1\right)+\frac{1}{2}G\left(\frac{1}{1-u_1},\frac{1}{1-u_1},1,\frac{1}{1-u_1};1\right)+$$

Multiple polyLogarithm

$$\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},0,1;1\right)-\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},0,\frac{1}{1-u_1};1\right)+$$
$$\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},1,0;1\right)-\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},0;1\right)+$$
$$\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},1;1\right)-\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{u_2-1}{u_1+u_2-1},1;1\right)+$$
$$\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)-G\left(\frac{1}{u_1},0,0,\frac{1}{u_2};1\right)+$$
$$\frac{1}{2}G\left(\frac{1}{u_1},0,0,\frac{1}{u_1+u_2};1\right)-G\left(\frac{1}{u_1},0,0,\frac{1}{u_3};1\right)+\frac{1}{2}G\left(\frac{1}{u_1},0,0,\frac{1}{u_1+u_3};1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)-\frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)-\frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)-\frac{1}{4}G\left(\frac{1}{1-u_2},1,\frac{1}{u_1},0;1\right)+$$
$$\frac{1}{2}G\left(\frac{1}{1-u_2},\frac{1}{1-u_2},1,\frac{1}{1-u_2};1\right)+\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},0,1;1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},0,\frac{1}{1-u_2};1\right)+\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},1,0;1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2},0;1\right)+\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2},1;1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2},\frac{1}{1-u_2};1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{u_3-1}{u_2+u_3-1},1;1\right)+$$
$$\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)-G\left(\frac{1}{u_2},0,0,\frac{1}{u_1};1\right)+$$
$$\frac{1}{2}G\left(\frac{1}{u_2},0,0,\frac{1}{u_1+u_2};1\right)-G\left(\frac{1}{u_2},0,0,\frac{1}{u_3};1\right)+\frac{1}{2}G\left(\frac{1}{u_2},0,0,\frac{1}{u_2+u_3};1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)-\frac{1}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)-\frac{1}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)-\frac{1}{4}G\left(\frac{1}{1-u_3},1,\frac{1}{u_2},0;1\right)+$$
$$\frac{1}{2}G\left(\frac{1}{1-u_3},\frac{1}{1-u_3},1,\frac{1}{1-u_3};1\right)+\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},0,1;1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},0,\frac{1}{1-u_3};1\right)+\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},1,0;1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},0;1\right)+\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},1;1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{u_1-1}{u_1+u_3-1},1;1\right)-\frac{79\pi^4}{360}+$$
$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)-G\left(\frac{1}{u_3},0,0,\frac{1}{u_1};1\right)-$$

$$G\left(\frac{1}{u_3},0,0,\frac{1}{u_2};1\right)+\frac{1}{2}G\left(\frac{1}{u_3},0,0,\frac{1}{u_1+u_3};1\right)+\frac{1}{2}G\left(\frac{1}{u_3},0,0,\frac{1}{u_2+u_3};1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)-\frac{1}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)-\frac{1}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)-\frac{1}{24}\pi^2\mathcal{G}\left(\frac{1}{1-u_1},u_{123};1\right)+\frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_1},v_{123};1\right)+$$
$$\frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_1},v_{132};1\right)-\frac{1}{24}\pi^2\mathcal{G}\left(\frac{1}{1-u_2},u_{231};1\right)+\frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_2},v_{213};1\right)+$$
$$\frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_2},v_{231};1\right)-\frac{1}{24}\pi^2\mathcal{G}\left(\frac{1}{1-u_3},u_{312};1\right)+\frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_3},v_{312};1\right)+$$
$$\frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_3},v_{321};1\right)-\frac{1}{4}\mathcal{G}\left(0,0,\frac{1}{1-u_1},v_{123};1\right)-\frac{1}{4}\mathcal{G}\left(0,0,\frac{1}{1-u_1},v_{132};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,0,\frac{1}{1-u_2},v_{213};1\right)-\frac{1}{4}\mathcal{G}\left(0,0,\frac{1}{1-u_2},v_{231};1\right)-\frac{1}{4}\mathcal{G}\left(0,0,\frac{1}{1-u_3},v_{312};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,0,\frac{1}{1-u_3},v_{321};1\right)-\frac{1}{4}\mathcal{G}\left(0,0,v_{123},\frac{1}{1-u_1};1\right)+\mathcal{G}\left(0,0,v_{132},0;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,0,v_{132},\frac{1}{1-u_1};1\right)+\mathcal{G}\left(0,0,v_{213},0;1\right)-\frac{1}{4}\mathcal{G}\left(0,0,v_{213},\frac{1}{1-u_2};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,0,v_{231},\frac{1}{1-u_2};1\right)-\frac{1}{4}\mathcal{G}\left(0,0,v_{312},\frac{1}{1-u_3};1\right)+\mathcal{G}\left(0,0,v_{321},0;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,0,v_{321},\frac{1}{1-u_3};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},0,v_{123};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},0,v_{132};1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(0,\frac{1}{1-u_1},\frac{1}{1-u_1},v_{123};1\right)-\frac{1}{2}\mathcal{G}\left(0,\frac{1}{1-u_1},\frac{1}{1-u_1},v_{132};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{123},1;1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{123},\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{132},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{132},\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},0,v_{213};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},0,v_{231};1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(0,\frac{1}{1-u_2},\frac{1}{1-u_2},v_{213};1\right)-\frac{1}{2}\mathcal{G}\left(0,\frac{1}{1-u_2},\frac{1}{1-u_2},v_{231};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{213},1;1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{213},\frac{1}{1-u_2};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{231},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{231},\frac{1}{1-u_2};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},0,v_{312};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},0,v_{321};1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(0,\frac{1}{1-u_3},\frac{1}{1-u_3},v_{312};1\right)-\frac{1}{2}\mathcal{G}\left(0,\frac{1}{1-u_3},\frac{1}{1-u_3},v_{321};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{312},1;1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{312},\frac{1}{1-u_3};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{321},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3};1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{123},0,\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{1}{1-u_1},0;1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{1}{1-u_1},1;1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{u_2-1}{u_1+u_2-1},1;1\right)+\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{1}{u_3},0;1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{231},0,\frac{1}{1-u_2};1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{1}{u_1},0;1\right)-$$

$$\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{1}{1-u_2},0;1\right)+\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{1}{1-u_2},1;1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{1}{1-u_2},\frac{1}{1-u_2};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{u_3-1}{u_2+u_3-1},1;1\right)+\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,u_{312},0,\frac{1}{1-u_3};1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{312},\frac{1}{u_2},0;1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{312},\frac{1}{1-u_3},0;1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(0,u_{312},\frac{1}{1-u_3},1;1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{312},\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,u_{312},\frac{u_1-1}{u_1+u_3-1},1;1\right)+\frac{1}{4}\mathcal{G}\left(0,u_{312},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(0,v_{123},0,\frac{1}{1-u_1};1\right)-\frac{1}{2}\mathcal{G}\left(0,v_{123},1,\frac{1}{1-u_1};1\right)+\frac{1}{4}\mathcal{G}\left(0,v_{123},\frac{1}{1-u_1},0;1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(0,v_{123},\frac{1}{1-u_1},1;1\right)+\frac{1}{4}\mathcal{G}\left(0,v_{123},\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(0,v_{132},0,\frac{1}{1-u_1};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,v_{132},\frac{1}{1-u_1},0;1\right)-\frac{1}{4}\mathcal{G}\left(0,v_{132},\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(0,v_{213},0,\frac{1}{1-u_2};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,v_{213},\frac{1}{1-u_2},0;1\right)-\frac{1}{4}\mathcal{G}\left(0,v_{213},\frac{1}{1-u_2},\frac{1}{1-u_2};1\right)+\frac{1}{4}\mathcal{G}\left(0,v_{231},0,\frac{1}{1-u_2};1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(0,v_{231},1,\frac{1}{1-u_2};1\right)+\frac{1}{4}\mathcal{G}\left(0,v_{231},\frac{1}{1-u_2},0;1\right)-\frac{1}{2}\mathcal{G}\left(0,v_{231},\frac{1}{1-u_2},1;1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(0,v_{231},\frac{1}{1-u_2},\frac{1}{1-u_2};1\right)+\frac{1}{4}\mathcal{G}\left(0,v_{312},0,\frac{1}{1-u_3};1\right)-\frac{1}{2}\mathcal{G}\left(0,v_{312},1,\frac{1}{1-u_3};1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(0,v_{312},\frac{1}{1-u_3},0;1\right)-\frac{1}{2}\mathcal{G}\left(0,v_{312},\frac{1}{1-u_3},1;1\right)+\frac{1}{4}\mathcal{G}\left(0,v_{312},\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,v_{321},0,\frac{1}{1-u_3};1\right)-\frac{1}{4}\mathcal{G}\left(0,v_{321},\frac{1}{1-u_3},0;1\right)-\frac{1}{4}\mathcal{G}\left(0,v_{321},\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,0,v_{123};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,0,v_{132};1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},0,\frac{1}{1-u_1},v_{123};1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},0,\frac{1}{1-u_1},v_{132};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{123},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{123},\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{132},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{132},\frac{1}{1-u_1};1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},0,v_{123};1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},0,v_{132};1\right)-\frac{3}{4}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},\frac{1}{1-u_1},v_{123};1\right)-$$
$$\frac{3}{4}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},\frac{1}{1-u_1},v_{132};1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{123},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{123},\frac{1}{1-u_1};1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{132},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{132},\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},0,1;1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},0,\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},1,0;1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{1}{1-u_1},0;1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{1}{1-u_1},1;1\right)+$$

$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{u_2-1}{u_1+u_2-1},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{1}{u_3},0;1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},0,0;1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},0,1;1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},0,\frac{1}{1-u_1};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},1,0;1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},1,\frac{1}{1-u_1};1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},\frac{1}{1-u_1},0;1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},\frac{1}{1-u_1},1;1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},0,0;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},0,1;1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},0,\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},1,0;1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},1,\frac{1}{1-u_1};1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},\frac{1}{1-u_1},0;1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},\frac{1}{1-u_1},1;1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,0,v_{213};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,0,v_{231};1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},0,\frac{1}{1-u_2},v_{213};1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},0,\frac{1}{1-u_2},v_{231};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{213},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{213},\frac{1}{1-u_2};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{231},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{231},\frac{1}{1-u_2};1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},0,v_{213};1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},0,v_{231};1\right)-\frac{3}{4}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},\frac{1}{1-u_2},v_{213};1\right)-$$
$$\frac{3}{4}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},\frac{1}{1-u_2},v_{231};1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{213},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{213},\frac{1}{1-u_2};1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{231},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{231},\frac{1}{1-u_2};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},0,1;1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},0,\frac{1}{1-u_2};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},1,0;1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{1}{u_1},0;1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{1}{1-u_2},0;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{1}{1-u_2},1;1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{1}{1-u_2},\frac{1}{1-u_2};1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{u_3-1}{u_2+u_3-1},1;1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},0,0;1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},0,1;1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},0,\frac{1}{1-u_2};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},1,0;1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},1,\frac{1}{1-u_2};1\right)+$$

$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},\frac{1}{1-u_2},0;1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},\frac{1}{1-u_2},1;1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},\frac{1}{1-u_2},\frac{1}{1-u_2};1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},0,0;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},0,1;1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},0,\frac{1}{1-u_2};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},1,0;1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},1,\frac{1}{1-u_2};1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},\frac{1}{1-u_2},0;1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},\frac{1}{1-u_2},1;1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},\frac{1}{1-u_2},\frac{1}{1-u_2};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,0,v_{312};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,0,v_{321};1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},0,\frac{1}{1-u_3},v_{312};1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},0,\frac{1}{1-u_3},v_{321};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{312},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{312},\frac{1}{1-u_3};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{321},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{321},\frac{1}{1-u_3};1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},0,v_{312};1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},0,v_{321};1\right)-\frac{3}{4}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},\frac{1}{1-u_3},v_{312};1\right)-$$
$$\frac{3}{4}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},\frac{1}{1-u_3},v_{321};1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{312},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{312},\frac{1}{1-u_3};1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{321},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},0,1;1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},0,\frac{1}{1-u_3};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},1,0;1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{1}{u_2},0;1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{1}{1-u_3},0;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{1}{1-u_3},1;1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{u_1-1}{u_1+u_3-1},1;1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},0,0;1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},0,1;1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},0,\frac{1}{1-u_3};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},1,0;1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},1,\frac{1}{1-u_3};1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},\frac{1}{1-u_3},0;1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},\frac{1}{1-u_3},1;1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},0,0;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},0,1;1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},0,\frac{1}{1-u_3};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},1,0;1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},1,\frac{1}{1-u_3};1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3},0;1\right)-$$

$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3},1;1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(v_{123},0,1,\frac{1}{1-u_1};1\right)+\frac{1}{2}\mathcal{G}\left(v_{123},0,\frac{1}{1-u_1},1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{123},1,0,\frac{1}{1-u_1};1\right)-$$
$$\frac{5}{4}\mathcal{G}\left(v_{123},1,1,\frac{1}{1-u_1};1\right)+\frac{1}{2}\mathcal{G}\left(v_{123},1,\frac{1}{1-u_1},0;1\right)-\frac{5}{4}\mathcal{G}\left(v_{123},1,\frac{1}{1-u_1},1;1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(v_{123},1,\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)+\frac{1}{2}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},0,1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},1,0;1\right)-$$
$$\frac{5}{4}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},1,1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},1,\frac{1}{1-u_1};1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},\frac{1}{1-u_1},1;1\right)-\frac{1}{4}\mathcal{G}\left(v_{132},1,1,\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(v_{132},1,\frac{1}{1-u_1},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(v_{132},\frac{1}{1-u_1},1,1;1\right)-\frac{1}{4}\mathcal{G}\left(v_{213},1,1,\frac{1}{1-u_2};1\right)-\frac{1}{4}\mathcal{G}\left(v_{213},1,\frac{1}{1-u_2},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(v_{213},\frac{1}{1-u_2},1,1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{231},0,1,\frac{1}{1-u_2};1\right)+\frac{1}{2}\mathcal{G}\left(v_{231},0,\frac{1}{1-u_2},1;1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(v_{231},1,0,\frac{1}{1-u_2};1\right)-\frac{5}{4}\mathcal{G}\left(v_{231},1,1,\frac{1}{1-u_2};1\right)+\frac{1}{2}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2},0;1\right)-$$
$$\frac{5}{4}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2},1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2},\frac{1}{1-u_2};1\right)+\frac{1}{2}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},0,1;1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},1,0;1\right)-\frac{5}{4}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},1,1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},1,\frac{1}{1-u_2};1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},\frac{1}{1-u_2},1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{312},0,1,\frac{1}{1-u_3};1\right)+\frac{1}{2}\mathcal{G}\left(v_{312},0,\frac{1}{1-u_3},1;1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(v_{312},1,0,\frac{1}{1-u_3};1\right)-\frac{5}{4}\mathcal{G}\left(v_{312},1,1,\frac{1}{1-u_3};1\right)+\frac{1}{2}\mathcal{G}\left(v_{312},1,\frac{1}{1-u_3},0;1\right)-$$
$$\frac{5}{4}\mathcal{G}\left(v_{312},1,\frac{1}{1-u_3},1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{312},1,\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)+\frac{1}{2}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},0,1;1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},1,0;1\right)-\frac{5}{4}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},1,1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},1,\frac{1}{1-u_3};1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},\frac{1}{1-u_3},1;1\right)-\frac{1}{4}\mathcal{G}\left(v_{321},1,1,\frac{1}{1-u_3};1\right)-\frac{1}{4}\mathcal{G}\left(v_{321},1,\frac{1}{1-u_3},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(v_{321},\frac{1}{1-u_3},1,1;1\right)-\frac{3}{4}G\left(0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H\left(0;u_1\right)-$$
$$\frac{3}{4}G\left(0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H\left(0;u_1\right)-\frac{1}{4}G\left(0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H\left(0;u_1\right)-$$
$$\frac{1}{4}G\left(0,\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H\left(0;u_1\right)-\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)H\left(0;u_1\right)-\frac{3}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1+u_2};1\right)H\left(0;u_1\right)-$$
$$\frac{3}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1+u_3};1\right)H\left(0;u_1\right)+\frac{1}{2}G\left(\frac{1}{u_1},\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{2}G\left(\frac{1}{u_1},\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H\left(0;u_1\right)+\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H\left(0;u_1\right)-\frac{1}{4}G\left(\frac{1}{1-u_2},1,\frac{1}{u_1};1\right)H\left(0;u_1\right)+$$

$$\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},1;1\right)H\left(0;u_1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)H\left(0;u_1\right)+\frac{1}{2}G\left(\frac{1}{u_2},0,\frac{1}{u_1};1\right)H\left(0;u_1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_1+u_2};1\right)H\left(0;u_1\right)+\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},0;1\right)H\left(0;u_1\right)+\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)H\left(0;u_1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{u_1-1}{u_1+u_3-1};1\right)H\left(0;u_1\right)+\frac{1}{2}G\left(\frac{1}{u_3},0,\frac{1}{u_1};1\right)H\left(0;u_1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_1+u_3};1\right)H\left(0;u_1\right)+\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{123};1\right)H\left(0;u_1\right)+\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{132};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{213};1\right)H\left(0;u_1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{231};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{312};1\right)H\left(0;u_1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{321};1\right)H\left(0;u_1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{1}{u_1};1\right)H\left(0;u_1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{1}{1-u_2};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(0,u_{312},\frac{1}{1-u_3};1\right)H\left(0;u_1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{312},\frac{u_1-1}{u_1+u_3-1};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(0,v_{123},\frac{1}{1-u_1};1\right)H\left(0;u_1\right)+\frac{1}{4}\mathcal{G}\left(0,v_{132},\frac{1}{1-u_1};1\right)H\left(0;u_1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(0,v_{231},\frac{1}{1-u_2};1\right)H\left(0;u_1\right)+\frac{1}{2}\mathcal{G}\left(0,v_{312},\frac{1}{1-u_3};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{123};1\right)H\left(0;u_1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{132};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{123};1\right)H\left(0;u_1\right)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{132};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},1;1\right)H\left(0;u_1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},\frac{1}{1-u_1};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},1;1\right)H\left(0;u_1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},\frac{1}{1-u_1};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{213};1\right)H\left(0;u_1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{231};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{213};1\right)H\left(0;u_1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{231};1\right)H\left(0;u_1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},1;1\right)H\left(0;u_1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{1}{u_1};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{1}{1-u_2};1\right)H\left(0;u_1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},0;1\right)H\left(0;u_1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},\frac{1}{1-u_2};1\right)H\left(0;u_1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},0;1\right)H\left(0;u_1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},\frac{1}{1-u_2};1\right)H\left(0;u_1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{312};1\right)H\left(0;u_1\right)-$$

$$
\begin{aligned}
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{321};1\right)H(0;u_1)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{312};1\right)H(0;u_1)-\\
&\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{321};1\right)H(0;u_1)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},0;1\right)H(0;u_1)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{1}{1-u_3};1\right)H(0;u_1)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{u_1-1}{u_1+u_3-1};1\right)H(0;u_1)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},0;1\right)H(0;u_1)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},\frac{1}{1-u_3};1\right)H(0;u_1)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},0;1\right)H(0;u_1)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3};1\right)H(0;u_1)+\\
&\frac{1}{4}\mathcal{G}\left(v_{123},1,\frac{1}{1-u_1};1\right)H(0;u_1)+\frac{1}{4}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},1;1\right)H(0;u_1)+\\
&\frac{1}{4}\mathcal{G}\left(v_{132},1,\frac{1}{1-u_1};1\right)H(0;u_1)+\frac{1}{4}\mathcal{G}\left(v_{132},\frac{1}{1-u_1},1;1\right)H(0;u_1)+\\
&\frac{1}{4}\mathcal{G}\left(v_{213},1,\frac{1}{1-u_2};1\right)H(0;u_1)+\frac{1}{4}\mathcal{G}\left(v_{213},\frac{1}{1-u_2},1;1\right)H(0;u_1)-\\
&\frac{3}{4}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2};1\right)H(0;u_1)-\frac{3}{4}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},1;1\right)H(0;u_1)+\\
&\frac{3}{4}\mathcal{G}\left(v_{312},1,\frac{1}{1-u_3};1\right)H(0;u_1)+\frac{3}{4}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},1;1\right)H(0;u_1)-\\
&\frac{1}{4}\mathcal{G}\left(v_{321},1,\frac{1}{1-u_3};1\right)H(0;u_1)-\frac{1}{4}\mathcal{G}\left(v_{321},\frac{1}{1-u_3},1;1\right)H(0;u_1)-\\
&\frac{1}{4}G\left(0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H(0;u_2)-\frac{3}{4}G\left(0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H(0;u_2)-\\
&\frac{3}{4}G\left(0,\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H(0;u_2)-\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)H(0;u_2)-\\
&\frac{1}{4}G\left(0,\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H(0;u_2)+\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)H(0;u_2)+\\
&\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},0;1\right)H(0;u_2)+\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)H(0;u_2)-\\
&\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{u_2-1}{u_1+u_2-1};1\right)H(0;u_2)+\frac{1}{2}G\left(\frac{1}{u_1},0,\frac{1}{u_2};1\right)H(0;u_2)-\\
&\frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1+u_2};1\right)H(0;u_2)+\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H(0;u_2)-\\
&\frac{3}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_1+u_2};1\right)H(0;u_2)-\frac{3}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_2+u_3};1\right)H(0;u_2)+\\
&\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H(0;u_2)+\frac{1}{2}G\left(\frac{1}{u_2},\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H(0;u_2)+\\
&\frac{1}{2}G\left(\frac{1}{u_2},\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H(0;u_2)+\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H(0;u_2)-\\
&\frac{1}{4}G\left(\frac{1}{1-u_3},1,\frac{1}{u_2};1\right)H(0;u_2)+\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},1;1\right)H(0;u_2)-\\
&\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)H(0;u_2)+\frac{1}{2}G\left(\frac{1}{u_3},0,\frac{1}{u_2};1\right)H(0;u_2)-\\
&\frac{1}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_2+u_3};1\right)H(0;u_2)+\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H(0;u_2)+
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{123};1\right)H(0;u_2)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{132};1\right)H(0;u_2)+\\
&\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{213};1\right)H(0;u_2)+\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{231};1\right)H(0;u_2)-\\
&\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{312};1\right)H(0;u_2)+\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{321};1\right)H(0;u_2)+\\
&\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{1}{1-u_1};1\right)H(0;u_2)-\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{u_2-1}{u_1+u_2-1};1\right)H(0;u_2)-\\
&\frac{1}{4}\mathcal{G}\left(0,u_{312},\frac{1}{u_2};1\right)H(0;u_2)-\frac{1}{4}\mathcal{G}\left(0,u_{312},\frac{1}{1-u_3};1\right)H(0;u_2)+\\
&\frac{1}{2}\mathcal{G}\left(0,v_{123},\frac{1}{1-u_1};1\right)H(0;u_2)+\frac{1}{4}\mathcal{G}\left(0,v_{213},\frac{1}{1-u_2};1\right)H(0;u_2)+\\
&\frac{1}{4}\mathcal{G}\left(0,v_{231},\frac{1}{1-u_2};1\right)H(0;u_2)-\frac{1}{2}\mathcal{G}\left(0,v_{312},\frac{1}{1-u_3};1\right)H(0;u_2)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{123};1\right)H(0;u_2)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{132};1\right)H(0;u_2)+\\
&\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{123};1\right)H(0;u_2)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{132};1\right)H(0;u_2)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},0;1\right)H(0;u_2)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{1}{1-u_1};1\right)H(0;u_2)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{u_2-1}{u_1+u_2-1};1\right)H(0;u_2)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},0;1\right)H(0;u_2)+\\
&\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},\frac{1}{1-u_1};1\right)H(0;u_2)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},0;1\right)H(0;u_2)-\\
&\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},\frac{1}{1-u_1};1\right)H(0;u_2)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{213};1\right)H(0;u_2)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{231};1\right)H(0;u_2)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{213};1\right)H(0;u_2)+\\
&\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{231};1\right)H(0;u_2)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},1;1\right)H(0;u_2)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},\frac{1}{1-u_2};1\right)H(0;u_2)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},1;1\right)H(0;u_2)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},\frac{1}{1-u_2};1\right)H(0;u_2)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{312};1\right)H(0;u_2)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{321};1\right)H(0;u_2)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{312};1\right)H(0;u_2)+\\
&\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{321};1\right)H(0;u_2)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},1;1\right)H(0;u_2)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{1}{u_2};1\right)H(0;u_2)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{1}{1-u_3};1\right)H(0;u_2)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},0;1\right)H(0;u_2)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},\frac{1}{1-u_3};1\right)H(0;u_2)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},0;1\right)H(0;u_2)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3};1\right)H(0;u_2)+\\
&\frac{3}{4}\mathcal{G}\left(v_{123},1,\frac{1}{1-u_1};1\right)H(0;u_2)+\frac{3}{4}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},1;1\right)H(0;u_2)-
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{4}\mathcal{G}\left(v_{132},1,\frac{1}{1-u_1};1\right)H\left(0;u_2\right)-\frac{1}{4}\mathcal{G}\left(v_{132},\frac{1}{1-u_1},1;1\right)H\left(0;u_2\right)+\\
&\frac{1}{4}\mathcal{G}\left(v_{213},1,\frac{1}{1-u_2};1\right)H\left(0;u_2\right)+\frac{1}{4}\mathcal{G}\left(v_{213},\frac{1}{1-u_2},1;1\right)H\left(0;u_2\right)+\\
&\frac{1}{4}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2};1\right)H\left(0;u_2\right)+\frac{1}{4}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},1;1\right)H\left(0;u_2\right)-\\
&\frac{3}{4}\mathcal{G}\left(v_{312},1,\frac{1}{1-u_3};1\right)H\left(0;u_2\right)-\frac{3}{4}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},1;1\right)H\left(0;u_2\right)+\\
&\frac{1}{4}\mathcal{G}\left(v_{321},1,\frac{1}{1-u_3};1\right)H\left(0;u_2\right)+\frac{1}{4}\mathcal{G}\left(v_{321},\frac{1}{1-u_3},1;1\right)H\left(0;u_2\right)+\\
&\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H\left(0;u_1\right)H\left(0;u_2\right)+\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H\left(0;u_1\right)H\left(0;u_2\right)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1};1\right)H\left(0;u_1\right)H\left(0;u_2\right)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312};1\right)H\left(0;u_1\right)H\left(0;u_2\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312};1\right)H\left(0;u_1\right)H\left(0;u_2\right)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321};1\right)H\left(0;u_1\right)H\left(0;u_2\right)+\frac{5}{24}\pi^2H\left(0;u_1\right)H\left(0;u_2\right)-\\
&\frac{1}{4}G\left(0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H\left(0;u_3\right)-\frac{1}{4}G\left(0,\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H\left(0;u_3\right)+\\
&\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)H\left(0;u_3\right)-\frac{3}{4}G\left(0,\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H\left(0;u_3\right)-\\
&\frac{3}{4}G\left(0,\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H\left(0;u_3\right)-\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)H\left(0;u_3\right)-\\
&\frac{1}{4}G\left(\frac{1}{1-u_1},1,\frac{1}{u_3};1\right)H\left(0;u_3\right)+\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},1;1\right)H\left(0;u_3\right)-\\
&\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)H\left(0;u_3\right)+\frac{1}{2}G\left(\frac{1}{u_1},0,\frac{1}{u_3};1\right)H\left(0;u_3\right)-\\
&\frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1+u_3};1\right)H\left(0;u_3\right)+\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H\left(0;u_3\right)+\\
&\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},0;1\right)H\left(0;u_3\right)+\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)H\left(0;u_3\right)-\\
&\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{u_3-1}{u_2+u_3-1};1\right)H\left(0;u_3\right)+\frac{1}{2}G\left(\frac{1}{u_2},0,\frac{1}{u_3};1\right)H\left(0;u_3\right)-\\
&\frac{1}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_2+u_3};1\right)H\left(0;u_3\right)+\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H\left(0;u_3\right)-\\
&\frac{3}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_1+u_3};1\right)H\left(0;u_3\right)-\frac{3}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_2+u_3};1\right)H\left(0;u_3\right)+\\
&\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H\left(0;u_3\right)+\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H\left(0;u_3\right)+\\
&\frac{1}{2}G\left(\frac{1}{u_3},\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H\left(0;u_3\right)+\frac{1}{2}G\left(\frac{1}{u_3},\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H\left(0;u_3\right)-\\
&\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{123};1\right)H\left(0;u_3\right)+\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{132};1\right)H\left(0;u_3\right)-\\
&\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{213};1\right)H\left(0;u_3\right)+\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{231};1\right)H\left(0;u_3\right)+
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{312};1\right)H\left(0;u_3\right)+\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{321};1\right)H\left(0;u_3\right)-\\
&\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{1}{1-u_1};1\right)H\left(0;u_3\right)-\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{1}{u_3};1\right)H\left(0;u_3\right)+\\
&\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{1}{1-u_2};1\right)H\left(0;u_3\right)-\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{u_3-1}{u_2+u_3-1};1\right)H\left(0;u_3\right)-\\
&\frac{1}{2}\mathcal{G}\left(0,v_{123},\frac{1}{1-u_1};1\right)H\left(0;u_3\right)+\frac{1}{2}\mathcal{G}\left(0,v_{231},\frac{1}{1-u_2};1\right)H\left(0;u_3\right)+\\
&\frac{1}{4}\mathcal{G}\left(0,v_{312},\frac{1}{1-u_3};1\right)H\left(0;u_3\right)+\frac{1}{4}\mathcal{G}\left(0,v_{321},\frac{1}{1-u_3};1\right)H\left(0;u_3\right)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{123};1\right)H\left(0;u_3\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{132};1\right)H\left(0;u_3\right)-\\
&\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{123};1\right)H\left(0;u_3\right)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{132};1\right)H\left(0;u_3\right)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},1;1\right)H\left(0;u_3\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{1}{1-u_1};1\right)H\left(0;u_3\right)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{1}{u_3};1\right)H\left(0;u_3\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},0;1\right)H\left(0;u_3\right)-\\
&\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},\frac{1}{1-u_1};1\right)H\left(0;u_3\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},0;1\right)H\left(0;u_3\right)+\\
&\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},\frac{1}{1-u_1};1\right)H\left(0;u_3\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{213};1\right)H\left(0;u_3\right)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{231};1\right)H\left(0;u_3\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{213};1\right)H\left(0;u_3\right)+\\
&\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{231};1\right)H\left(0;u_3\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},0;1\right)H\left(0;u_3\right)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{1}{1-u_2};1\right)H\left(0;u_3\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{u_3-1}{u_2+u_3-1};1\right)H\left(0;u_3\right)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},0;1\right)H\left(0;u_3\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},\frac{1}{1-u_2};1\right)H\left(0;u_3\right)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},0;1\right)H\left(0;u_3\right)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},\frac{1}{1-u_2};1\right)H\left(0;u_3\right)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{312};1\right)H\left(0;u_3\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{321};1\right)H\left(0;u_3\right)+\\
&\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{312};1\right)H\left(0;u_3\right)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{321};1\right)H\left(0;u_3\right)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},1;1\right)H\left(0;u_3\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},\frac{1}{1-u_3};1\right)H\left(0;u_3\right)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},1;1\right)H\left(0;u_3\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3};1\right)H\left(0;u_3\right)-\\
&\frac{3}{4}\mathcal{G}\left(v_{123},1,\frac{1}{1-u_1};1\right)H\left(0;u_3\right)-\frac{3}{4}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},1;1\right)H\left(0;u_3\right)+\\
&\frac{1}{4}\mathcal{G}\left(v_{132},1,\frac{1}{1-u_1};1\right)H\left(0;u_3\right)+\frac{1}{4}\mathcal{G}\left(v_{132},\frac{1}{1-u_1},1;1\right)H\left(0;u_3\right)-\\
&\frac{1}{4}\mathcal{G}\left(v_{213},1,\frac{1}{1-u_2};1\right)H\left(0;u_3\right)-\frac{1}{4}\mathcal{G}\left(v_{213},\frac{1}{1-u_2},1;1\right)H\left(0;u_3\right)+
\end{aligned}
$$

$$\frac{3}{4}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2};1\right)H\left(0;u_3\right)+\frac{3}{4}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},1;1\right)H\left(0;u_3\right)+$$
$$\frac{1}{4}\mathcal{G}\left(v_{312},1,\frac{1}{1-u_3};1\right)H\left(0;u_3\right)+\frac{1}{4}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},1;1\right)H\left(0;u_3\right)+$$
$$\frac{1}{4}\mathcal{G}\left(v_{321},1,\frac{1}{1-u_3};1\right)H\left(0;u_3\right)+\frac{1}{4}\mathcal{G}\left(v_{321},\frac{1}{1-u_3},1;1\right)H\left(0;u_3\right)+$$
$$\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H\left(0;u_1\right)H\left(0;u_3\right)+$$
$$\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1};1\right)H\left(0;u_1\right)H\left(0;u_3\right)+$$
$$\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H\left(0;u_1\right)H\left(0;u_3\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231};1\right)H\left(0;u_1\right)H\left(0;u_3\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213};1\right)H\left(0;u_1\right)H\left(0;u_3\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231};1\right)H\left(0;u_1\right)H\left(0;u_3\right)+$$
$$\frac{5}{24}\pi^2H\left(0;u_1\right)H\left(0;u_3\right)+\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1};1\right)H\left(0;u_2\right)H\left(0;u_3\right)+$$
$$\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H\left(0;u_2\right)H\left(0;u_3\right)+\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H\left(0;u_2\right)H\left(0;u_3\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123};1\right)H\left(0;u_2\right)H\left(0;u_3\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123};1\right)H\left(0;u_2\right)H\left(0;u_3\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132};1\right)H\left(0;u_2\right)H\left(0;u_3\right)+\frac{5}{24}\pi^2H\left(0;u_2\right)H\left(0;u_3\right)+$$
$$3H\left(0;u_2\right)H\left(0,0;u_1\right)H\left(0;u_3\right)+3H\left(0;u_1\right)H\left(0,0;u_2\right)H\left(0;u_3\right)+$$
$$\frac{1}{4}H\left(0;u_2\right)H\left(0,1;\frac{u_1+u_2-1}{u_2-1}\right)H\left(0;u_3\right)+\frac{1}{2}H\left(0;u_1\right)H\left(0,1;(u_1+u_3)\right)H\left(0;u_3\right)+$$
$$\frac{1}{4}H\left(0;u_1\right)H\left(0,1;\frac{u_2+u_3-1}{u_3-1}\right)H\left(0;u_3\right)+\frac{1}{2}H\left(0;u_2\right)H\left(0,1;(u_2+u_3)\right)H\left(0;u_3\right)+$$
$$\frac{3}{4}H\left(0;u_2\right)H\left(1,0;u_1\right)H\left(0;u_3\right)+\frac{3}{4}H\left(0;u_1\right)H\left(1,0;u_2\right)H\left(0;u_3\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213};1\right)H\left(0,0;u_1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231};1\right)H\left(0,0;u_1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312};1\right)H\left(0,0;u_1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321};1\right)H\left(0,0;u_1\right)-\frac{23}{24}\pi^2H\left(0,0;u_1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123};1\right)H\left(0,0;u_2\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132};1\right)H\left(0,0;u_2\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312};1\right)H\left(0,0;u_2\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321};1\right)H\left(0,0;u_2\right)-$$
$$\frac{25}{4}H\left(0,0;u_1\right)H\left(0,0;u_2\right)-\frac{23}{24}\pi^2H\left(0,0;u_2\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123};1\right)H\left(0,0;u_3\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132};1\right)H\left(0,0;u_3\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213};1\right)H\left(0,0;u_3\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231};1\right)H\left(0,0;u_3\right)+3H\left(0;u_1\right)H\left(0;u_2\right)H\left(0,0;u_3\right)-$$
$$\frac{25}{4}H\left(0,0;u_1\right)H\left(0,0;u_3\right)-\frac{25}{4}H\left(0,0;u_2\right)H\left(0,0;u_3\right)-\frac{23}{24}\pi^2H\left(0,0;u_3\right)+\frac{1}{12}\pi^2H\left(0,1;u_1\right)+$$
$$\frac{1}{12}\pi^2H\left(0,1;u_2\right)-\frac{1}{24}\pi^2H\left(0,1;\frac{u_1+u_2-1}{u_2-1}\right)+\frac{1}{2}H\left(0;u_1\right)H\left(0;u_2\right)H\left(0,1;(u_1+u_2)\right)+$$

$$\frac{1}{12}\pi^2H\left(0,1;(u_1+u_2)\right)+\frac{1}{12}\pi^2H\left(0,1;u_3\right)+\frac{1}{4}H\left(0;u_1\right)H\left(0;u_2\right)H\left(0,1;\frac{u_1+u_3-1}{u_1-1}\right)-$$
$$\frac{1}{24}\pi^2H\left(0,1;\frac{u_1+u_3-1}{u_1-1}\right)+\frac{1}{12}\pi^2H\left(0,1;(u_1+u_3)\right)-\frac{1}{24}\pi^2H\left(0,1;\frac{u_2+u_3-1}{u_3-1}\right)+$$
$$\frac{1}{12}\pi^2H\left(0,1;(u_2+u_3)\right)-\frac{1}{2}G\left(0,\frac{1}{u_1+u_2};1\right)H\left(1,0;u_1\right)-$$
$$\frac{1}{2}G\left(0,\frac{1}{u_1+u_3};1\right)H\left(1,0;u_1\right)+\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H\left(1,0;u_1\right)+$$
$$\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H\left(1,0;u_1\right)+\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H\left(1,0;u_1\right)+$$
$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1};1\right)H\left(1,0;u_1\right)+\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H\left(1,0;u_1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312};1\right)H\left(1,0;u_1\right)-\frac{3}{4}H\left(0,0;u_2\right)H\left(1,0;u_1\right)-\frac{3}{4}H\left(0,0;u_3\right)H\left(1,0;u_1\right)+$$
$$\frac{1}{4}H\left(0,1;\frac{u_1+u_3-1}{u_1-1}\right)H\left(1,0;u_1\right)-\frac{1}{3}\pi^2H\left(1,0;u_1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1+u_2};1\right)H\left(1,0;u_2\right)-$$
$$\frac{1}{2}G\left(0,\frac{1}{u_2+u_3};1\right)H\left(1,0;u_2\right)+\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1};1\right)H\left(1,0;u_2\right)+$$
$$\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H\left(1,0;u_2\right)+\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H\left(1,0;u_2\right)+$$
$$\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H\left(1,0;u_2\right)+\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H\left(1,0;u_2\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123};1\right)H\left(1,0;u_2\right)-\frac{3}{4}H\left(0,0;u_1\right)H\left(1,0;u_2\right)-\frac{3}{4}H\left(0,0;u_3\right)H\left(1,0;u_2\right)+$$
$$\frac{1}{4}H\left(0,1;\frac{u_1+u_2-1}{u_2-1}\right)H\left(1,0;u_2\right)-\frac{1}{4}H\left(1,0;u_1\right)H\left(1,0;u_2\right)-\frac{1}{3}\pi^2H\left(1,0;u_2\right)-$$
$$\frac{1}{2}G\left(0,\frac{1}{u_1+u_3};1\right)H\left(1,0;u_3\right)-\frac{1}{2}G\left(0,\frac{1}{u_2+u_3};1\right)H\left(1,0;u_3\right)+$$
$$\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H\left(1,0;u_3\right)+\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1};1\right)H\left(1,0;u_3\right)+$$
$$\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H\left(1,0;u_3\right)-\frac{1}{3}\pi^2H\left(1,0;u_3\right)+\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H\left(1,0;u_3\right)+$$
$$\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H\left(1,0;u_3\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231};1\right)H\left(1,0;u_3\right)+$$
$$\frac{3}{4}H\left(0;u_1\right)H\left(0;u_2\right)H\left(1,0;u_3\right)-\frac{3}{4}H\left(0,0;u_1\right)H\left(1,0;u_3\right)-\frac{3}{4}H\left(0,0;u_2\right)H\left(1,0;u_3\right)+$$
$$\frac{1}{4}H\left(0,1;\frac{u_2+u_3-1}{u_3-1}\right)H\left(1,0;u_3\right)-\frac{1}{4}H\left(1,0;u_1\right)H\left(1,0;u_3\right)-\frac{1}{4}H\left(1,0;u_2\right)H\left(1,0;u_3\right)+$$
$$\frac{1}{24}\pi^2H\left(1,1;u_1\right)+\frac{1}{24}\pi^2H\left(1,1;u_2\right)+\frac{1}{24}\pi^2H\left(1,1;u_3\right)+\frac{1}{2}H\left(0;u_2\right)H\left(0,0,0;u_1\right)+$$
$$\frac{1}{2}H\left(0;u_3\right)H\left(0,0,0;u_2\right)+\frac{1}{2}H\left(0;u_1\right)H\left(0,0,0;u_3\right)-\frac{1}{2}H\left(0;u_2\right)H\left(0,0,1;\frac{u_1+u_2-1}{u_2-1}\right)-$$
$$\frac{1}{2}H\left(0;u_3\right)H\left(0,0,1;\frac{u_1+u_2-1}{u_2-1}\right)-H\left(0;u_1\right)H\left(0,0,1;(u_1+u_2)\right)-$$
$$H\left(0;u_2\right)H\left(0,0,1;(u_1+u_2)\right)-\frac{1}{2}H\left(0;u_1\right)H\left(0,0,1;\frac{u_1+u_3-1}{u_1-1}\right)-$$

$$\frac{1}{2}H(0;u_2)H\left(0,0,1;\frac{u_1+u_3-1}{u_1-1}\right)-H(0;u_1)H(0,0,1;(u_1+u_3))-$$
$$H(0;u_3)H(0,0,1;(u_1+u_3))-\frac{1}{2}H(0;u_1)H\left(0,0,1;\frac{u_2+u_3-1}{u_3-1}\right)-$$
$$\frac{1}{2}H(0;u_3)H\left(0,0,1;\frac{u_2+u_3-1}{u_3-1}\right)-H(0;u_2)H(0,0,1;(u_2+u_3))-$$
$$H(0;u_3)H(0,0,1;(u_2+u_3))-\frac{1}{2}H(0;u_2)H(0,1,0;u_1)-\frac{1}{2}H(0;u_3)H(0,1,0;u_2)-$$
$$\frac{1}{2}H(0;u_1)H(0,1,0;u_3)+\frac{1}{4}H(0;u_2)H\left(0,1,1;\frac{u_1+u_2-1}{u_2-1}\right)-$$
$$\frac{1}{4}H(0;u_3)H\left(0,1,1;\frac{u_1+u_2-1}{u_2-1}\right)+\frac{1}{4}H(0;u_1)H\left(0,1,1;\frac{u_1+u_3-1}{u_1-1}\right)-$$
$$\frac{1}{4}H(0;u_2)H\left(0,1,1;\frac{u_1+u_3-1}{u_1-1}\right)-\frac{1}{4}H(0;u_1)H\left(0,1,1;\frac{u_2+u_3-1}{u_3-1}\right)+$$
$$\frac{1}{4}H(0;u_3)H\left(0,1,1;\frac{u_2+u_3-1}{u_3-1}\right)+\frac{1}{2}H(0;u_2)H(1,0,0;u_1)-\frac{1}{2}H(0;u_3)H(1,0,0;u_1)-$$
$$\frac{1}{2}H(0;u_1)H(1,0,0;u_2)+\frac{1}{2}H(0;u_3)H(1,0,0;u_2)+\frac{1}{2}H(0;u_1)H(1,0,0;u_3)-$$
$$\frac{1}{2}H(0;u_2)H(1,0,0;u_3)-\frac{1}{4}H(0;u_3)H\left(1,0,1;\frac{u_1+u_2-1}{u_2-1}\right)-$$
$$\frac{1}{4}H(0;u_2)H\left(1,0,1;\frac{u_1+u_3-1}{u_1-1}\right)-\frac{1}{4}H(0;u_1)H\left(1,0,1;\frac{u_2+u_3-1}{u_3-1}\right)-$$
$$7H(0,0,0,0;u_1)-7H(0,0,0,0;u_2)-7H(0,0,0,0;u_3)+\frac{3}{2}H\left(0,0,0,1;\frac{u_1+u_2-1}{u_2-1}\right)+$$
$$3H(0,0,0,1;(u_1+u_2))+\frac{3}{2}H\left(0,0,0,1;\frac{u_1+u_3-1}{u_1-1}\right)+3H(0,0,0,1;(u_1+u_3))+$$
$$\frac{3}{2}H\left(0,0,0,1;\frac{u_2+u_3-1}{u_3-1}\right)+3H(0,0,0,1;(u_2+u_3))+\frac{9}{4}H(0,0,1,0;u_1)+$$
$$\frac{9}{4}H(0,0,1,0;u_2)+\frac{9}{4}H(0,0,1,0;u_3)-\frac{1}{2}H(0,1,0,0;u_1)-\frac{1}{2}H(0,1,0,0;u_2)-$$
$$\frac{1}{2}H(0,1,0,0;u_3)+\frac{1}{2}H\left(0,1,0,1;\frac{u_1+u_2-1}{u_2-1}\right)+\frac{1}{2}H\left(0,1,0,1;\frac{u_1+u_3-1}{u_1-1}\right)+$$
$$\frac{1}{2}H\left(0,1,0,1;\frac{u_2+u_3-1}{u_3-1}\right)+H(0,1,1,0;u_1)+H(0,1,1,0;u_2)+H(0,1,1,0;u_3)-$$
$$\frac{1}{4}H\left(0,1,1,1;\frac{u_1+u_2-1}{u_2-1}\right)-\frac{1}{4}H\left(0,1,1,1;\frac{u_1+u_3-1}{u_1-1}\right)-$$
$$\frac{1}{4}H\left(0,1,1,1;\frac{u_2+u_3-1}{u_3-1}\right)+H\left(1,0,0,1;\frac{u_1+u_2-1}{u_2-1}\right)+H\left(1,0,0,1;\frac{u_1+u_3-1}{u_1-1}\right)+$$
$$H\left(1,0,0,1;\frac{u_2+u_3-1}{u_3-1}\right)+2H(1,0,1,0;u_1)+2H(1,0,1,0;u_2)+2H(1,0,1,0;u_3)+$$
$$\frac{1}{4}H\left(1,1,0,1;\frac{u_1+u_2-1}{u_2-1}\right)+\frac{1}{4}H\left(1,1,0,1;\frac{u_1+u_3-1}{u_1-1}\right)+$$
$$\frac{1}{4}H\left(1,1,0,1;\frac{u_2+u_3-1}{u_3-1}\right)+\frac{1}{2}H(1,1,1,0;u_1)+\frac{1}{2}H(1,1,1,0;u_2)+\frac{1}{2}H(1,1,1,0;u_3)-$$
$$\frac{1}{24}\pi^2H(0;u_3)\mathcal{H}\left(1;\frac{1}{u_{123}}\right)-\frac{1}{24}\pi^2H(0;u_1)\mathcal{H}\left(1;\frac{1}{u_{231}}\right)-\frac{1}{24}\pi^2H(0;u_2)\mathcal{H}\left(1;\frac{1}{u_{312}}\right)+$$
$$\frac{1}{8}\pi^2H(0;u_2)\mathcal{H}\left(1;\frac{1}{v_{123}}\right)-\frac{1}{8}\pi^2H(0;u_3)\mathcal{H}\left(1;\frac{1}{v_{123}}\right)+\frac{1}{24}\pi^2H(0;u_2)\mathcal{H}\left(1;\frac{1}{v_{132}}\right)-$$

$$\frac{1}{24}\pi^2H(0;u_3)\mathcal{H}\left(1;\frac{1}{v_{132}}\right)-\frac{1}{24}\pi^2H(0;u_1)\mathcal{H}\left(1;\frac{1}{v_{213}}\right)+\frac{1}{24}\pi^2H(0;u_3)\mathcal{H}\left(1;\frac{1}{v_{213}}\right)-$$
$$\frac{1}{8}\pi^2H(0;u_1)\mathcal{H}\left(1;\frac{1}{v_{231}}\right)+\frac{1}{8}\pi^2H(0;u_3)\mathcal{H}\left(1;\frac{1}{v_{231}}\right)+\frac{1}{8}\pi^2H(0;u_1)\mathcal{H}\left(1;\frac{1}{v_{312}}\right)-$$
$$\frac{1}{8}\pi^2H(0;u_2)\mathcal{H}\left(1;\frac{1}{v_{312}}\right)+\frac{1}{24}\pi^2H(0;u_1)\mathcal{H}\left(1;\frac{1}{v_{321}}\right)-\frac{1}{24}\pi^2H(0;u_2)\mathcal{H}\left(1;\frac{1}{v_{321}}\right)-$$
$$\frac{1}{4}H(0;u_2)H(0;u_3)\mathcal{H}\left(0,1;\frac{1}{u_{123}}\right)-\frac{1}{4}H(1,0;u_2)\mathcal{H}\left(0,1;\frac{1}{u_{123}}\right)+\frac{1}{24}\pi^2\mathcal{H}\left(0,1;\frac{1}{u_{123}}\right)+$$
$$\frac{1}{24}\pi^2\mathcal{H}\left(0,1;\frac{1}{u_{231}}\right)-\frac{1}{4}H(0;u_1)H(0;u_3)\mathcal{H}\left(0,1;\frac{1}{u_{231}}\right)-\frac{1}{4}H(1,0;u_3)\mathcal{H}\left(0,1;\frac{1}{u_{231}}\right)-$$
$$\frac{1}{4}H(0;u_1)H(0;u_2)\mathcal{H}\left(0,1;\frac{1}{u_{312}}\right)-\frac{1}{4}H(1,0;u_1)\mathcal{H}\left(0,1;\frac{1}{u_{312}}\right)+\frac{1}{24}\pi^2\mathcal{H}\left(0,1;\frac{1}{u_{312}}\right)-$$
$$\frac{1}{4}H(0;u_2)H(0;u_3)\mathcal{H}\left(0,1;\frac{1}{v_{123}}\right)+\frac{1}{4}H(0,0;u_2)\mathcal{H}\left(0,1;\frac{1}{v_{123}}\right)+$$
$$\frac{1}{4}H(0,0;u_3)\mathcal{H}\left(0,1;\frac{1}{v_{123}}\right)+\frac{1}{6}\pi^2\mathcal{H}\left(0,1;\frac{1}{v_{123}}\right)-\frac{1}{4}H(0;u_2)H(0;u_3)\mathcal{H}\left(0,1;\frac{1}{v_{132}}\right)+$$
$$\frac{1}{4}H(0,0;u_2)\mathcal{H}\left(0,1;\frac{1}{v_{132}}\right)+\frac{1}{4}H(0,0;u_3)\mathcal{H}\left(0,1;\frac{1}{v_{132}}\right)+\frac{1}{6}\pi^2\mathcal{H}\left(0,1;\frac{1}{v_{132}}\right)-$$
$$\frac{1}{4}H(0;u_1)H(0;u_3)\mathcal{H}\left(0,1;\frac{1}{v_{213}}\right)+\frac{1}{4}H(0,0;u_1)\mathcal{H}\left(0,1;\frac{1}{v_{213}}\right)+$$
$$\frac{1}{4}H(0,0;u_3)\mathcal{H}\left(0,1;\frac{1}{v_{213}}\right)+\frac{1}{6}\pi^2\mathcal{H}\left(0,1;\frac{1}{v_{213}}\right)-\frac{1}{4}H(0;u_1)H(0;u_3)\mathcal{H}\left(0,1;\frac{1}{v_{231}}\right)+$$
$$\frac{1}{4}H(0,0;u_1)\mathcal{H}\left(0,1;\frac{1}{v_{231}}\right)+\frac{1}{4}H(0,0;u_3)\mathcal{H}\left(0,1;\frac{1}{v_{231}}\right)+\frac{1}{6}\pi^2\mathcal{H}\left(0,1;\frac{1}{v_{231}}\right)-$$
$$\frac{1}{4}H(0;u_1)H(0;u_2)\mathcal{H}\left(0,1;\frac{1}{v_{312}}\right)+\frac{1}{4}H(0,0;u_1)\mathcal{H}\left(0,1;\frac{1}{v_{312}}\right)+$$
$$\frac{1}{4}H(0,0;u_2)\mathcal{H}\left(0,1;\frac{1}{v_{312}}\right)+\frac{1}{6}\pi^2\mathcal{H}\left(0,1;\frac{1}{v_{312}}\right)-\frac{1}{4}H(0;u_1)H(0;u_2)\mathcal{H}\left(0,1;\frac{1}{v_{321}}\right)+$$
$$\frac{1}{4}H(0,0;u_1)\mathcal{H}\left(0,1;\frac{1}{v_{321}}\right)+\frac{1}{4}H(0,0;u_2)\mathcal{H}\left(0,1;\frac{1}{v_{321}}\right)+\frac{1}{6}\pi^2\mathcal{H}\left(0,1;\frac{1}{v_{321}}\right)-$$
$$\frac{1}{2}H(0;u_2)H(0;u_3)\mathcal{H}\left(1,1;\frac{1}{v_{123}}\right)+\frac{1}{2}H(0,0;u_2)\mathcal{H}\left(1,1;\frac{1}{v_{123}}\right)+$$
$$\frac{1}{2}H(0,0;u_3)\mathcal{H}\left(1,1;\frac{1}{v_{123}}\right)+\frac{11}{24}\pi^2\mathcal{H}\left(1,1;\frac{1}{v_{123}}\right)-\frac{1}{24}\pi^2\mathcal{H}\left(1,1;\frac{1}{v_{132}}\right)-$$
$$\frac{1}{24}\pi^2\mathcal{H}\left(1,1;\frac{1}{v_{213}}\right)-\frac{1}{2}H(0;u_1)H(0;u_3)\mathcal{H}\left(1,1;\frac{1}{v_{231}}\right)+\frac{1}{2}H(0,0;u_1)\mathcal{H}\left(1,1;\frac{1}{v_{231}}\right)+$$
$$\frac{1}{2}H(0,0;u_3)\mathcal{H}\left(1,1;\frac{1}{v_{231}}\right)+\frac{11}{24}\pi^2\mathcal{H}\left(1,1;\frac{1}{v_{231}}\right)-\frac{1}{2}H(0;u_1)H(0;u_2)\mathcal{H}\left(1,1;\frac{1}{v_{312}}\right)+$$
$$\frac{1}{2}H(0,0;u_1)\mathcal{H}\left(1,1;\frac{1}{v_{312}}\right)+\frac{1}{2}H(0,0;u_2)\mathcal{H}\left(1,1;\frac{1}{v_{312}}\right)+\frac{11}{24}\pi^2\mathcal{H}\left(1,1;\frac{1}{v_{312}}\right)-$$
$$\frac{1}{24}\pi^2\mathcal{H}\left(1,1;\frac{1}{v_{321}}\right)+\frac{1}{2}H(0;u_2)\mathcal{H}\left(0,0,1;\frac{1}{u_{123}}\right)+\frac{1}{2}H(0;u_3)\mathcal{H}\left(0,0,1;\frac{1}{u_{123}}\right)+$$
$$\frac{1}{2}H(0;u_1)\mathcal{H}\left(0,0,1;\frac{1}{u_{231}}\right)+\frac{1}{2}H(0;u_3)\mathcal{H}\left(0,0,1;\frac{1}{u_{231}}\right)+\frac{1}{2}H(0;u_1)\mathcal{H}\left(0,0,1;\frac{1}{u_{312}}\right)+$$
$$\frac{1}{2}H(0;u_2)\mathcal{H}\left(0,0,1;\frac{1}{u_{312}}\right)+\frac{1}{4}H(0;u_3)\mathcal{H}\left(0,1,1;\frac{1}{u_{123}}\right)+\frac{1}{4}H(0;u_1)\mathcal{H}\left(0,1,1;\frac{1}{u_{231}}\right)+$$

$$\frac{1}{4}H\left(0;u_2\right)\mathcal{H}\left(0,1,1;\frac{1}{u_{312}}\right)+\frac{1}{4}H\left(0;u_2\right)\mathcal{H}\left(0,1,1;\frac{1}{v_{123}}\right)-\frac{1}{4}H\left(0;u_3\right)\mathcal{H}\left(0,1,1;\frac{1}{v_{123}}\right)-$$
$$\frac{1}{4}H\left(0;u_2\right)\mathcal{H}\left(0,1,1;\frac{1}{v_{132}}\right)+\frac{1}{4}H\left(0;u_3\right)\mathcal{H}\left(0,1,1;\frac{1}{v_{132}}\right)+\frac{1}{4}H\left(0;u_1\right)\mathcal{H}\left(0,1,1;\frac{1}{v_{213}}\right)-$$
$$\frac{1}{4}H\left(0;u_3\right)\mathcal{H}\left(0,1,1;\frac{1}{v_{213}}\right)-\frac{1}{4}H\left(0;u_1\right)\mathcal{H}\left(0,1,1;\frac{1}{v_{231}}\right)+\frac{1}{4}H\left(0;u_3\right)\mathcal{H}\left(0,1,1;\frac{1}{v_{231}}\right)+$$
$$\frac{1}{4}H\left(0;u_1\right)\mathcal{H}\left(0,1,1;\frac{1}{v_{312}}\right)-\frac{1}{4}H\left(0;u_2\right)\mathcal{H}\left(0,1,1;\frac{1}{v_{312}}\right)-\frac{1}{4}H\left(0;u_1\right)\mathcal{H}\left(0,1,1;\frac{1}{v_{321}}\right)+$$
$$\frac{1}{4}H\left(0;u_2\right)\mathcal{H}\left(0,1,1;\frac{1}{v_{321}}\right)+\frac{1}{4}H\left(0;u_3\right)\mathcal{H}\left(1,0,1;\frac{1}{u_{123}}\right)+\frac{1}{4}H\left(0;u_1\right)\mathcal{H}\left(1,0,1;\frac{1}{u_{231}}\right)+$$
$$\frac{1}{4}H\left(0;u_2\right)\mathcal{H}\left(1,0,1;\frac{1}{u_{312}}\right)+\frac{1}{4}H\left(0;u_2\right)\mathcal{H}\left(1,0,1;\frac{1}{v_{123}}\right)-\frac{1}{4}H\left(0;u_3\right)\mathcal{H}\left(1,0,1;\frac{1}{v_{123}}\right)-$$
$$\frac{1}{4}H\left(0;u_2\right)\mathcal{H}\left(1,0,1;\frac{1}{v_{132}}\right)+\frac{1}{4}H\left(0;u_3\right)\mathcal{H}\left(1,0,1;\frac{1}{v_{132}}\right)+\frac{1}{4}H\left(0;u_1\right)\mathcal{H}\left(1,0,1;\frac{1}{v_{213}}\right)-$$
$$\frac{1}{4}H\left(0;u_3\right)\mathcal{H}\left(1,0,1;\frac{1}{v_{213}}\right)-\frac{1}{4}H\left(0;u_1\right)\mathcal{H}\left(1,0,1;\frac{1}{v_{231}}\right)+\frac{1}{4}H\left(0;u_3\right)\mathcal{H}\left(1,0,1;\frac{1}{v_{231}}\right)+$$
$$\frac{1}{4}H\left(0;u_1\right)\mathcal{H}\left(1,0,1;\frac{1}{v_{312}}\right)-\frac{1}{4}H\left(0;u_2\right)\mathcal{H}\left(1,0,1;\frac{1}{v_{312}}\right)-\frac{1}{4}H\left(0;u_1\right)\mathcal{H}\left(1,0,1;\frac{1}{v_{321}}\right)+$$
$$\frac{1}{4}H\left(0;u_2\right)\mathcal{H}\left(1,0,1;\frac{1}{v_{321}}\right)+H\left(0;u_2\right)\mathcal{H}\left(1,1,1;\frac{1}{v_{123}}\right)-H\left(0;u_3\right)\mathcal{H}\left(1,1,1;\frac{1}{v_{123}}\right)-$$
$$H\left(0;u_1\right)\mathcal{H}\left(1,1,1;\frac{1}{v_{231}}\right)+H\left(0;u_3\right)\mathcal{H}\left(1,1,1;\frac{1}{v_{231}}\right)+H\left(0;u_1\right)\mathcal{H}\left(1,1,1;\frac{1}{v_{312}}\right)-$$
$$H\left(0;u_2\right)\mathcal{H}\left(1,1,1;\frac{1}{v_{312}}\right)-\frac{3}{2}\mathcal{H}\left(0,0,0,1;\frac{1}{u_{123}}\right)-\frac{3}{2}\mathcal{H}\left(0,0,0,1;\frac{1}{u_{231}}\right)-$$
$$\frac{3}{2}\mathcal{H}\left(0,0,0,1;\frac{1}{u_{312}}\right)-3\mathcal{H}\left(0,0,0,1;\frac{1}{v_{132}}\right)-3\mathcal{H}\left(0,0,0,1;\frac{1}{v_{213}}\right)-3\mathcal{H}\left(0,0,0,1;\frac{1}{v_{321}}\right)-$$
$$\frac{1}{2}\mathcal{H}\left(0,0,1,1;\frac{1}{u_{123}}\right)-\frac{1}{2}\mathcal{H}\left(0,0,1,1;\frac{1}{u_{231}}\right)-\frac{1}{2}\mathcal{H}\left(0,0,1,1;\frac{1}{u_{312}}\right)-$$
$$\frac{1}{2}\mathcal{H}\left(0,1,0,1;\frac{1}{u_{123}}\right)-\frac{1}{2}\mathcal{H}\left(0,1,0,1;\frac{1}{u_{231}}\right)-\frac{1}{2}\mathcal{H}\left(0,1,0,1;\frac{1}{u_{312}}\right)+$$
$$\frac{1}{4}\mathcal{H}\left(0,1,1,1;\frac{1}{v_{123}}\right)+\frac{1}{4}\mathcal{H}\left(0,1,1,1;\frac{1}{v_{132}}\right)+\zeta_3H\left(0;u_1\right)+\zeta_3H\left(0;u_2\right)+\zeta_3H\left(0;u_3\right)+$$
$$\frac{5}{2}\zeta_3H\left(1;u_1\right)+\frac{5}{2}\zeta_3H\left(1;u_2\right)+\frac{5}{2}\zeta_3H\left(1;u_3\right)+\frac{1}{2}\zeta_3\mathcal{H}\left(1;\frac{1}{u_{123}}\right)+\frac{1}{2}\zeta_3\mathcal{H}\left(1;\frac{1}{u_{231}}\right)+$$
$$\frac{1}{2}\zeta_3\mathcal{H}\left(1;\frac{1}{u_{312}}\right)-\frac{1}{2}\mathcal{H}\left(1,0,0,1;\frac{1}{u_{123}}\right)-\frac{1}{2}\mathcal{H}\left(1,0,0,1;\frac{1}{u_{231}}\right)-\frac{1}{2}\mathcal{H}\left(1,0,0,1;\frac{1}{u_{312}}\right)+$$
$$\frac{1}{4}\zeta_3\mathcal{H}\left(1;\frac{1}{v_{123}}\right)+\frac{1}{4}\zeta_3\mathcal{H}\left(1;\frac{1}{v_{132}}\right)+\frac{1}{4}\zeta_3\mathcal{H}\left(1;\frac{1}{v_{213}}\right)+\frac{1}{4}\zeta_3\mathcal{H}\left(1;\frac{1}{v_{231}}\right)+\frac{1}{4}\zeta_3\mathcal{H}\left(1;\frac{1}{v_{312}}\right)+$$
$$\frac{1}{4}\zeta_3\mathcal{H}\left(1;\frac{1}{v_{321}}\right)+\frac{1}{4}\mathcal{H}\left(0,1,1,1;\frac{1}{v_{213}}\right)+\frac{1}{4}\mathcal{H}\left(0,1,1,1;\frac{1}{v_{231}}\right)+\frac{1}{4}\mathcal{H}\left(0,1,1,1;\frac{1}{v_{312}}\right)+$$
$$\frac{1}{4}\mathcal{H}\left(0,1,1,1;\frac{1}{v_{321}}\right)+\frac{1}{4}\mathcal{H}\left(1,0,1,1;\frac{1}{v_{123}}\right)+\frac{1}{4}\mathcal{H}\left(1,0,1,1;\frac{1}{v_{132}}\right)+\frac{1}{4}\mathcal{H}\left(1,0,1,1;\frac{1}{v_{213}}\right)+$$
$$\frac{1}{4}\mathcal{H}\left(1,0,1,1;\frac{1}{v_{231}}\right)+\frac{1}{4}\mathcal{H}\left(1,0,1,1;\frac{1}{v_{312}}\right)+\frac{1}{4}\mathcal{H}\left(1,0,1,1;\frac{1}{v_{321}}\right)+\frac{1}{4}\mathcal{H}\left(1,1,0,1;\frac{1}{v_{123}}\right)+$$
$$\frac{1}{4}\mathcal{H}\left(1,1,0,1;\frac{1}{v_{132}}\right)+\frac{1}{4}\mathcal{H}\left(1,1,0,1;\frac{1}{v_{213}}\right)+\frac{1}{4}\mathcal{H}\left(1,1,0,1;\frac{1}{v_{231}}\right)+\frac{1}{4}\mathcal{H}\left(1,1,0,1;\frac{1}{v_{312}}\right)+$$

$$\frac{1}{4}\mathcal{H}\left(1,1,0,1;\frac{1}{v_{321}}\right)+\frac{3}{2}\mathcal{H}\left(1,1,1,1;\frac{1}{v_{123}}\right)+\frac{3}{2}\mathcal{H}\left(1,1,1,1;\frac{1}{v_{231}}\right)+\frac{3}{2}\mathcal{H}\left(1,1,1,1;\frac{1}{v_{312}}\right)$$

References

[1] C. Anastasiou, Z. Bern, L. J. Dixon and D. A. Kosower, "Planar amplitudes in maximally supersymmetric Yang-Mills theory," Phys. Rev. Lett. **91** (2003) 251602 [arXiv:hep-th/0309040].

[2] Z. Bern, J. S. Rozowsky and B. Yan, "Two-loop four-gluon amplitudes in N = 4 super-Yang-Mills," Phys. Lett. B **401** (1997) 273 [arXiv:hep-ph/9702424].

[3] Z. Bern, M. Czakon, D. A. Kosower, R. Roiban and V. A. Smirnov, "Two-loop iteration of five-point N = 4 super-Yang-Mills amplitudes," Phys. Rev. Lett. **97** (2006) 181601 [arXiv:hep-th/0604074].

[4] F. Cachazo, M. Spradlin and A. Volovich, "Iterative structure within the five-particle two-loop amplitude," Phys. Rev. D **74**, 045020 (2006) [arXiv:hep-th/0602228].

[5] V. Del Duca, C. Duhr, E. W. Nigel Glover and V. A. Smirnov, "The one-loop pentagon to higher orders in epsilon," JHEP **1001** (2010) 042 [arXiv:0905.0097 [hep-th]].

[6] V. Del Duca, C. Duhr and E. W. Nigel Glover, "The five-gluon amplitude in the high-energy limit," JHEP **0912** (2009) 023 [arXiv:0905.0100 [hep-th]].

[7] L. F. Alday, J. M. Henn, J. Plefka and T. Schuster, "Scattering into the fifth dimension of N=4 super Yang-Mills," JHEP **1001** (2010) 077 [arXiv:0908.0684 [hep-th]].

[8] J. M. Henn, S. G. Naculich, H. J. Schnitzer and M. Spradlin, "Higgs-regularized three-loop four-gluon amplitude in N=4 SYM: exponentiation and Regge limits," arXiv:1001.1358 [hep-th].

[9] Z. Bern, L. J. Dixon and V. A. Smirnov, "Iteration of planar amplitudes in maximally supersymmetric Yang-Mills theory at three loops and beyond," Phys. Rev. D **72** (2005) 085001 [arXiv:hep-th/0505205].

[10] Z. Bern, L. J. Dixon, D. A. Kosower, R. Roiban, M. Spradlin, C. Vergu and A. Volovich, "The Two-Loop Six-Gluon MHV Amplitude in Maximally Supersymmetric Yang-Mills Theory," Phys. Rev. D **78** (2008) 045007 [arXiv:0803.1465 [hep-th]].

[11] L. F. Alday and J. Maldacena, "Comments on gluon scattering amplitudes via AdS/CFT," JHEP **0711** (2007) 068 [arXiv:0710.1060 [hep-th]].

[12] J. M. Drummond, J. Henn, G. P. Korchemsky and E. Sokatchev, "The hexagon Wilson loop and the BDS ansatz for the six-gluon amplitude," Phys. Lett. B **662** (2008) 456 [arXiv:0712.4138 [hep-th]].

[13] J. Bartels, L. N. Lipatov and A. Sabio Vera, "BFKL Pomeron, Reggeized gluons and Bern-Dixon-Smirnov amplitudes," Phys. Rev. D **80** (2009) 045002 [arXiv:0802.2065 [hep-th]].

[14] J. Bartels, L. N. Lipatov and A. Sabio Vera, "N=4 supersymmetric Yang Mills scattering amplitudes at high energies: the Regge cut contribution," arXiv:0807.0894 [hep-th].

[15] R. M. Schabinger, "The Imaginary Part of the N = 4 Super-Yang-Mills Two-Loop Six-Point MHV Amplitude in Multi-Regge Kinematics," JHEP **0911** (2009) 108 [arXiv:0910.3933 [hep-th]].

Result can be remarkably simple

17 pages =

[Goncharov, Spradlin, Vergu, Volovich 2010]

$$\sum_{i=1}^{3}\left(L_4(x_i^+, x_i^-) - \frac{1}{2}\mathrm{Li}_4(1-1/u_i)\right) - \frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2 + \frac{1}{24}J^4 + \frac{\pi^2}{12}J^2 + \frac{\pi^4}{72}$$

which is a single line of only classical polylogarithms!

→ require advanced mathematical tools： ”Symbol”

Why so simple?

Many examples show that the final result can be put in a form which is far simpler than the intermediate steps !

Surprising simplicity → Hidden structure

“Theoretical experiment”:
looking into the theoretical data, and try to find hidden structures

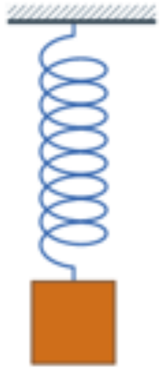
A toy model

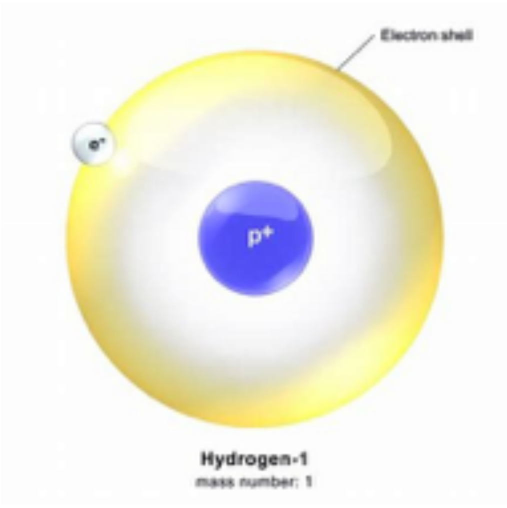
N=4 SYM theory : -> QCD's maximally supersymmetric cousin

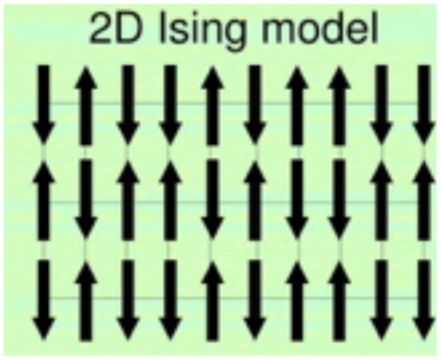
$$\mathcal{L} = -\frac{1}{g_{\mathrm{YM}}^2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + \mathrm{fermions} + \mathrm{scalars}$$

where all fields are the in the adjoint representation of the gauge group SU(Nc).

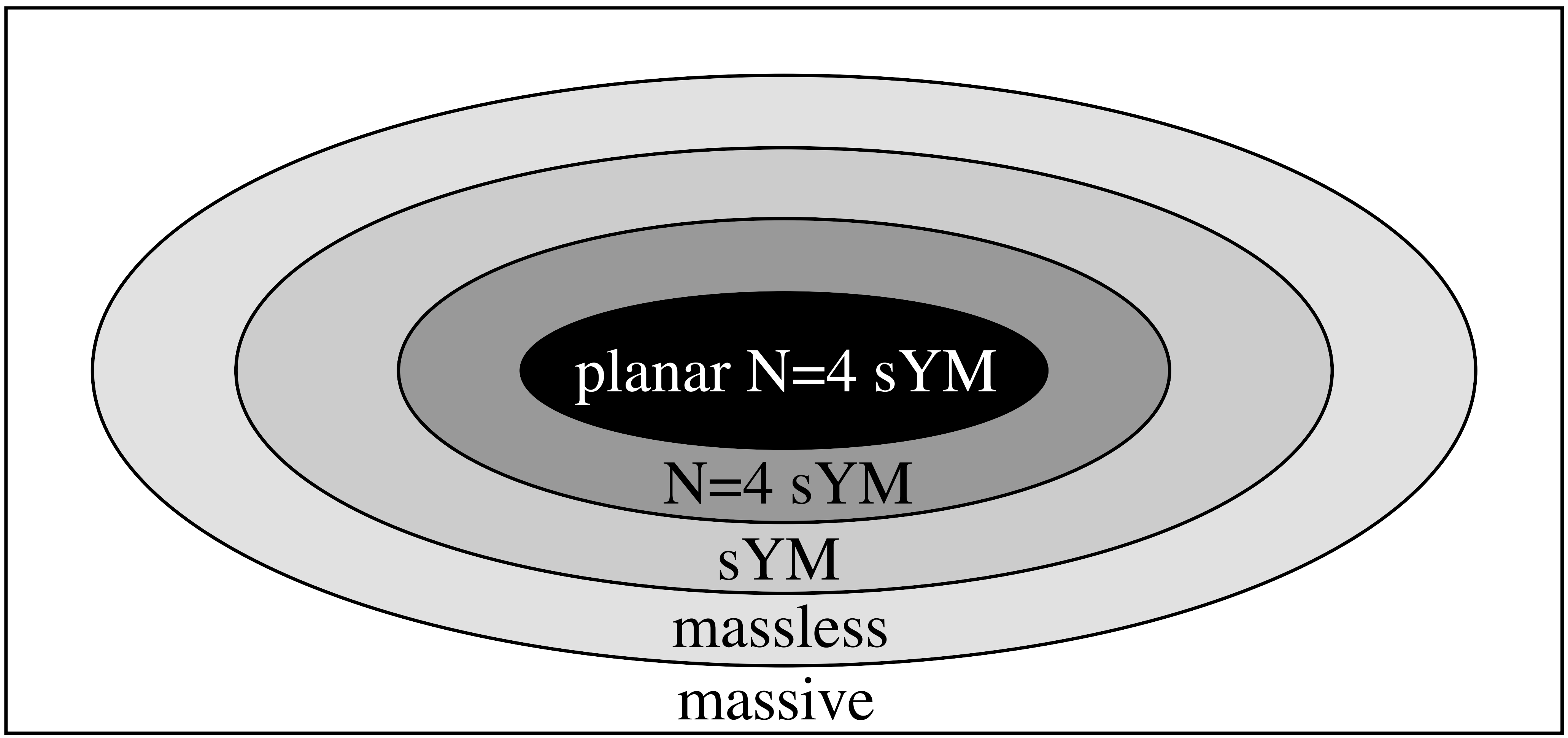
Exactly solvable in planar limit!







Hierarchy of simplicity

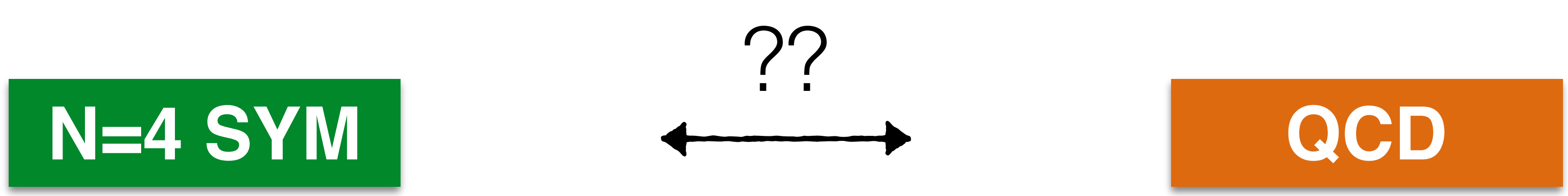


Lance Dixon 1105.0771

N=4 SYM

Techniques first developed by studying this toy model are used in general theories such as QCD, e.g.: BCFW recursion relations, unitarity on-shell method.

Are there direct connections between the two theories?



Maximal Transcendentality Principle

N=4 SYM ⟷ **QCD**

Maximally transcendental parts are equal between two theories!?

Number	Function	Transcendentality degree
2/3, $\sqrt{2}$	rational function	0
π	Log(x)	1
Riemann zeta value $\zeta(n)$	Polylog function $Li_n(x)$	n

Known examples

N=4 SYM $\overset{!}{\longleftrightarrow}$ QCD

- Anomalous dimension of twist-2 operators

$$\gamma^{\mathcal{N}=4}(j) = \gamma^{\rm QCD}(j)|_{\rm max.\ trans}$$

[Kotikov, Lipatov, Onishchenko, Velizhanin 2004]

- Two-loop remainder function (kinematic dependent functions!):

[Brandhuber, Travaglini, GY 2012]

N=4 SYM 1/2 BPS form factor $\longleftrightarrow$ Higgs plus 3-gluon amplitudes $m_t \to \infty$

- Also for certain Wilson lines [Li, Manteuffel, Schabinger, Zhu 2014]

Two-loop Higgs to 3-gluon

$$
\begin{gathered}
-2G(0,0,1,0,u)+G(0,0,1-v,1-v,u)+2G(0,0,-v,1-v,u)-G(0,1,0,1-v,u)+4G(0,1,1,0,u)-G(0,1,1-v,0,u)+G(0,1-v,0,1-v,u)\\
+G(0,1-v,1-v,0,u)-G(0,1-v,-v,1-v,u)+2G(0,-v,0,1-v,u)+2G(0,-v,1-v,0,u)-2G(0,-v,1-v,1-v,u)-2G(1,0,0,1-v,u)\\
-2G(1,0,1-v,0,u)+4G(1,1,0,0,u)-4G(1,1,1,0,u)-2G(1,1-v,0,0,u)+G(1-v,0,0,1-v,u)-G(1-v,0,1,0,u)-2G(-v,1-v,1-v,u)H(0,v)\\
-2G(1-v,1,0,0,u)+2G(1-v,1,0,1-v,u)+2G(1-v,1,1-v,0,u)+G(1-v,1-v,0,0,u)+2G(1-v,1-v,1,0,u)-2G(1-v,1-v,-v,1-v,u)\\
-G(1-v,-v,1-v,0,u)+4G(1-v,-v,-v,1-v,u)-2G(-v,0,1-v,1-v,u)-2G(-v,1-v,0,1-v,u)-2G(-v,1-v,1-v,0,u)+4G(1,0,1,0,u)\\
+4G(-v,-v,1-v,1-v,u)-4G(-v,-v,-v,1-v,u)-G(0,0,1-v,u)H(0,v)-G(0,1,0,u)H(0,v)-G(0,1-v,0,u)H(0,v)+G(0,1-v,1-v,u)H(0,v)\\
-G(0,-v,1-v,u)H(0,v)-2G(1,0,0,u)H(0,v)+G(1,0,1-v,u)H(0,v)+G(1,1-v,0,u)H(0,v)+G(1-v,0,0,u)H(0,v)-G(1-v,0,1-v,u)H(0,v)\\
-G(1-v,1,0,u)H(0,v)-G(1-v,1-v,0,u)H(0,v)-G(1-v,-v,1-v,u)H(0,v)+G(-v,0,1-v,u)H(0,v)+G(-v,1-v,0,u)H(0,v)+H(1,0,0,1,v)\\
-G(0,0,1-v,u)H(1,v)-G(0,0,-v,u)H(1,v)+G(0,1,0,u)H(1,v)-G(0,1-v,0,u)H(1,v)+G(0,1-v,-v,u)H(1,v)-2G(0,-v,0,u)H(1,v)\\
+2G(0,-v,1-v,u)H(1,v)+2G(1,0,0,u)H(1,v)-G(1-v,0,0,u)H(1,v)+G(1-v,0,-v,u)H(1,v)-2G(1-v,1,0,u)H(1,v)-G(1-v,0,-v,1-v,u)\\
+G(1-v,-v,0,u)H(1,v)-4G(1-v,-v,-v,u)H(1,v)+2G(-v,0,1-v,u)H(1,v)+2G(-v,1-v,0,u)H(1,v)-4G(-v,1-v,-v,u)H(1,v)\\
-4G(-v,-v,1-v,u)H(1,v)+4G(-v,-v,-v,u)H(1,v)+G(0,0,u)H(0,0,v)+G(0,1-v,u)H(0,0,v)+G(1-v,0,u)H(0,0,v)+H(1,0,1,0,v)\\
-G(0,0,u)H(0,1,v)+G(0,-v,u)H(0,1,v)-G(1 \ldots G(1-v,0,u)H(0,1,v)+2G(1-v,1-v,u)H(0,1,v)-3G(1-v,-v,u)H(0,1,v)\\
-G(-v,0,u)H(0,1,v)-2G(-v,1-v,u)H(0,1,v) \ldots 0,1,v)-G(0,0,u)H(1,0,v)+G(0,-v,u)H(1,0,v)-G(1,0,u)H(1,0,v)\\
+2G(1-v,0,u)H(1,0,v)-2G(1-v,1-v,u)H(1 \ldots ,u)H(1,0,v)-G(-v,0,u)H(1,0,v)+2G(-v,1-v,u)H(1,0,v)+G(0,0,u)H(1,1,v)\\
-2G(0,-v,u)H(1,1,v)-2G(-v,0,u)H(1,1,v)+4G(-v,-v,u)H(1,1,v)+G(0,u)H(0,0,1,v)-3G(1-v,u)H(0,0,1,v)+4G(-v,u)H(0,0,1,v)\\
+G(0,u)H(0,1,0,v)+G(1-v,u)H(0,1,0,v)-G(0,u)H(0,1,1,v)+2G(-v,u)H(0,1,1,v)+G(0,u)H(1,0,0,v)+G(1-v,u)H(1,0,0,v)+H(1,1,0,0,v)\\
-G(0,u)H(1,0,1,v)+2G(-v,u)H(1,0,1,v)-G(0,u)H(1,1,0,v)+4G(1-v,u)H(1,1,0,v)-2G(-v,u)H(1,1,0,v)+H(0,0,1,1,v)+H(0,1,0,1,v)\\
+G(1-v,1-v,u)H(0,0,v)+2G(1-v,1-v,-v,u)H(1,v)-G(1-v,-v,0,1-v,u)+H(0,1,1,0,v)+G(1-v,0,1-v,0,u)-G(0,1-v,1,0,u)\\
+4G(-v,1-v,-v,1-v,u)
\end{gathered}
$$

QCD

[Gehrmann, Jaquier, Glover, Koukoutsakis 2011]

$$G(1-v,1-v,v,1-v;u)$$

Multiple polyLogarithm

↕

$$
-2\left[\mathrm{J}_4\left(-\frac{uv}{w}\right)+\mathrm{J}_4\left(-\frac{vw}{u}\right)+\mathrm{J}_4\left(-\frac{wu}{v}\right)\right]-8\sum_{i=1}^{3}\left[\mathrm{Li}_4\left(1-u_i^{-1}\right)+\frac{\log^4 u_i}{4!}\right]
$$
$$
-2\left[\sum_{i=1}^{3}\mathrm{Li}_2(1-u_i)+\frac{\log^2 u_i}{2!}\right]^2+\frac{1}{2}\left[\sum_{i=1}^{3}\log^2 u_i\right]^2-\frac{\log^4(uvw)}{4!}-\frac{23}{2}\zeta_4
$$

[Brandhuber, Travaglini, GY 2012]

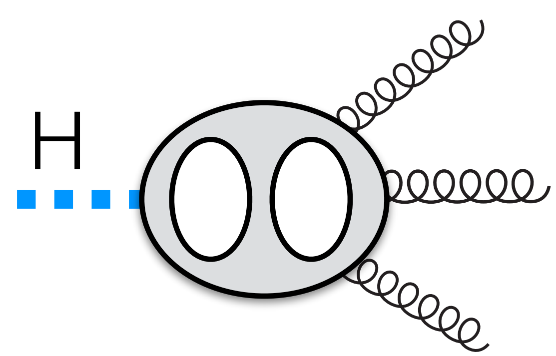
N=4

Maximal Transcendentality Principle

N=4 SYM ⟷ **QCD**

To which extend is this correspondence between correct?

Require more two-loop computations.



Two-loop QCD computation

Still a very challenging problem!

- 4-gluon amplitudes known many years ago

[Glover, Oleari, Tejeda-Yeomans 2001]
[Bern, De Freitas, Dixon 2002]

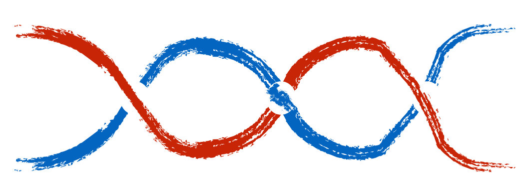
- 5-gluon results are still not fully known; numerical planar results last year

[Badger, Brønnum-Hansen, Hartanto, Peraro 2017]
[Abreua, Corderoa, Ita, Pagea, Zeng 2017]
……

Our problem also involves high dimensional operators.

Strategy:

On-shell unitarity



Integration by parts

Content

- Motivations
- **Computation**
- Results
- Summary and outlook

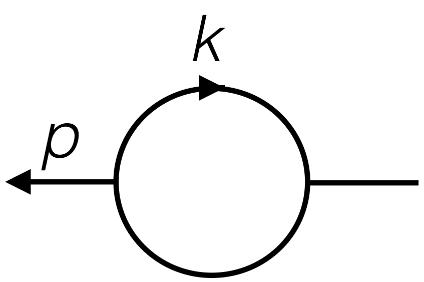
Integration by part reduction

[Chetyrkin, Tkachov 1981]

Integration by part (IBP): $\int d^D l_1 \ldots d^D l_L \, \frac{\partial}{\partial l_i^\mu} (\text{integrand}) = 0$

Solve a set of linear relations between different integrals.

Example: $J(a_1, a_2) := \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{(-k^2+m^2)^{a_1}(-(k+p)^2+m^2)^{a_2}}$



$$\longrightarrow \quad 0 = \int \frac{d^D k}{i\pi^{D/2}} \frac{\partial}{\partial k^\mu} \left(k^\mu \frac{1}{(-k^2+m^2)^{a_1}(-(k+p)^2+m^2)^{a_2}} \right)$$

$$\longrightarrow \quad 0 = (D-2a_1-a_2) J(a_1,a_2) - a_2 J(a_1-1, a_2+1) + 2m^2 a_1 J(a_1+1, a_2) + (2m^2-p^2) a_2 J(a_1, a_2+1)$$

$$\xrightarrow{a_2=0} \quad 0 = (D-2a_1) J(a_1, 0) + 2m^2 a_1 J(a_1+1, 0)$$

$$J(1,2) = \frac{(D-2)}{2m^2(4m^2-p^2)} J(1,0) + \frac{(D-3)}{4m^2-p^2} J(1,1)$$

2 Master Integral

Public packages:
Reduze 2, FIRE, LiteRed, etc

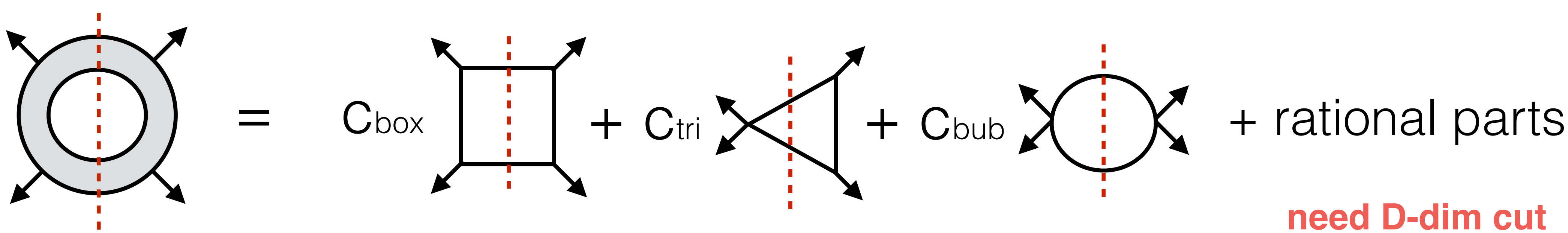
Unitarity method

As a replacement of Feynman diagram method:

construct the integrand from physical singularities,
i.e. poles or branch-cuts.

[Bern, Dixon, Dunbar, Kosower 1994]
[Britto, Cachazo, Feng 2004]

- At one-loop:



Unitarity method

Challenges for higher loop QCD:

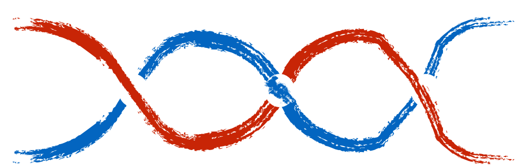
- need D-dimensional cuts (rational term issue)
- non-trivial to reconstruct full integrand (non-planar)
- need to further reduce the integrand, such as via IBP (sometimes IBP is the bottleneck)

Try new strategy

IBP for cut integrand

$$F^{(l)}|_{\text{cut}} = \sum_{\text{helicities}} F^{\text{tree}} \prod_j A_j^{\text{tree}} = \sum_i c_i\, M_i|_{\text{cut}}$$

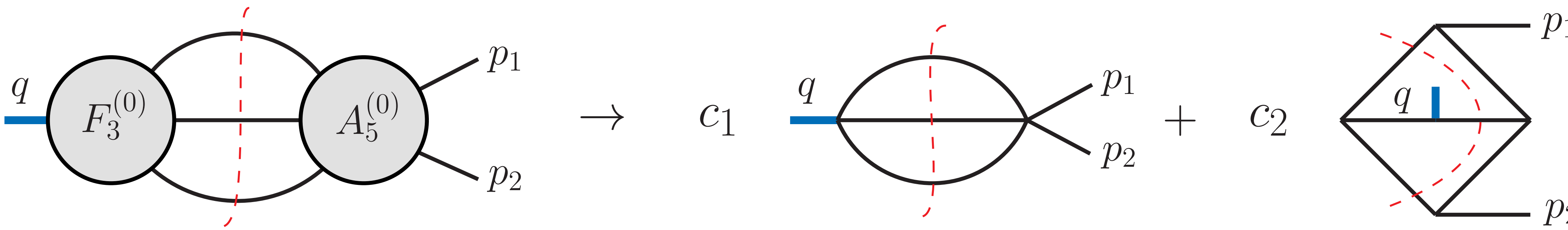
On-shell unitarity



Integration by parts

- D-dimensional cuts
- no need to reconstruct full integrand
- IBP is simplified

Example



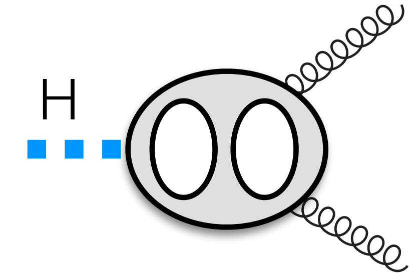
- Tree by Feynman rules in D dimensions
- Helicity sum via contraction rule:

$$\sum_{\text{helicities}} \varepsilon_i^\mu \varepsilon_i^\nu = \eta^{\mu\nu} - \frac{q^\mu p_i^\nu + q^\nu p_i^\mu}{q \cdot p_i}$$

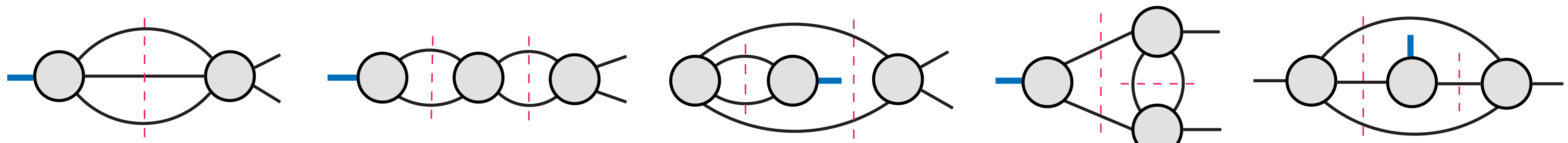
→ spinor helicity formalism for N=4 SYM

- Project to gauge invariant basis
- IBP reduction

2-loop 2-gluon

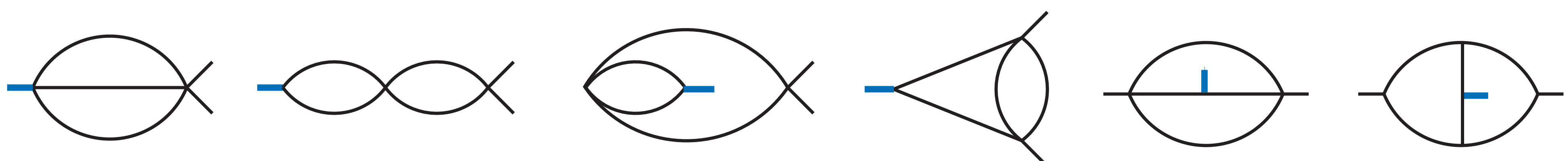


Cuts:



(a) (b) (c) (d) (e)

Master integrals:

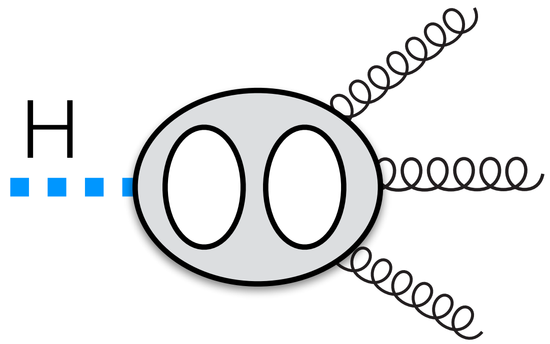


(1) (2) (3) (4) (5) (6)

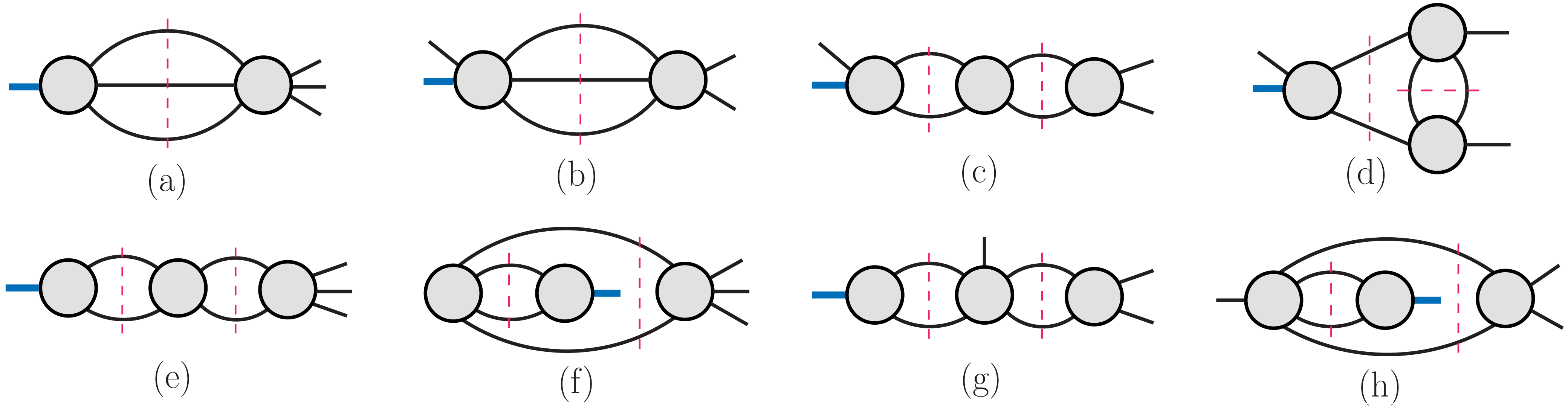
New feature (complication) of form factor:
'non-planar type' cuts appear for colour-planar part.

$$\text{Full result} \quad = \quad \left(\sum_{i=1}^{4} c_i M_i + \frac{1}{2} \sum_{i=5,6} c_i M_i \right) + \text{perms}(p_1, p_2)$$

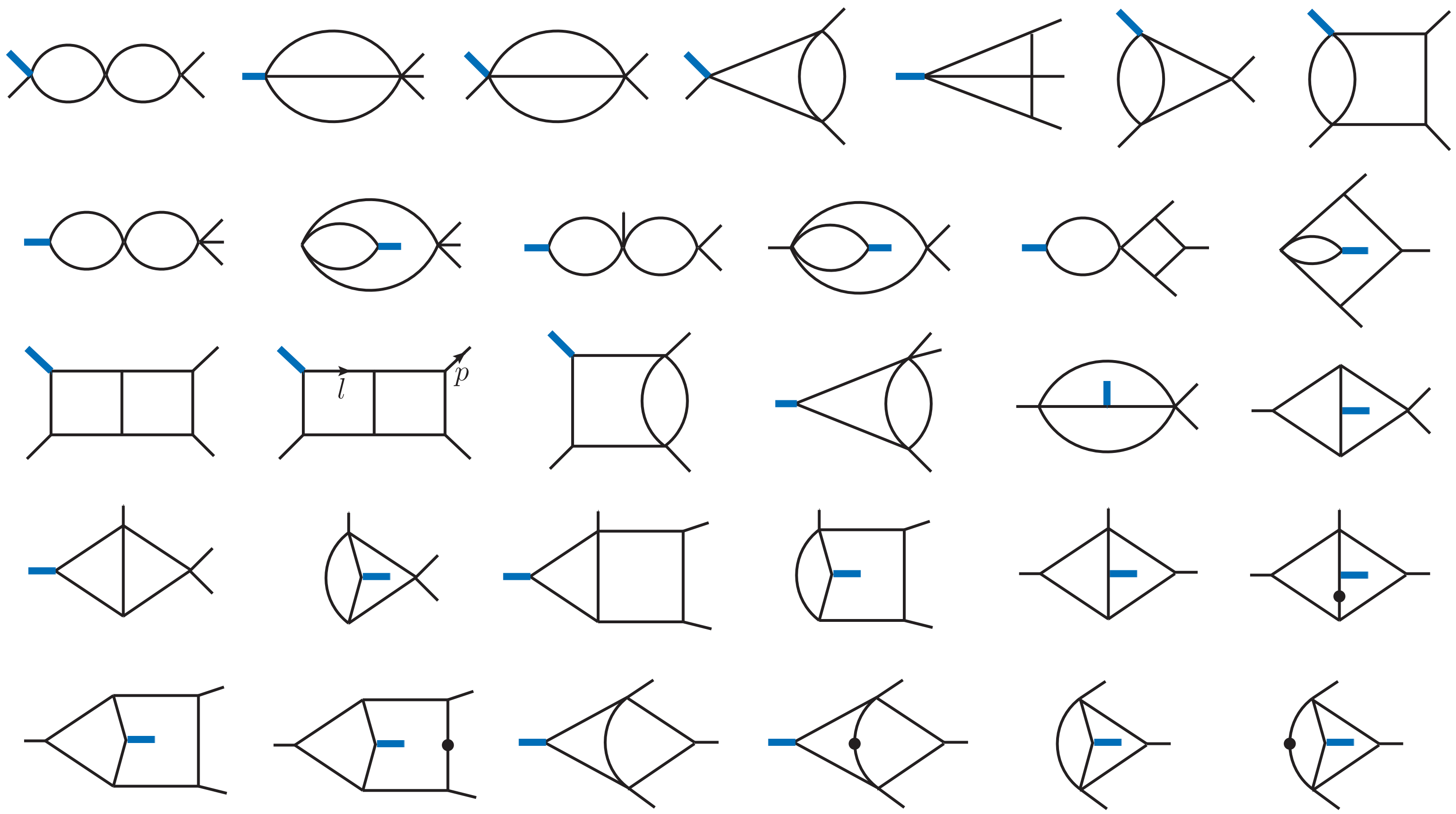
2-loop 3-gluon



All cuts that are needed:



Master integrals



All analytic results are known, given in terms of
2d Harmonic polylogarithms. [Gehrmann, Remiddi 2001]

Thus the results are given in terms of explicit functions.

Content

- Motivation
- Computation
- **Results**
- Summary and outlook

Divergence structure

UV divergences

both gauge coupling and local operator require renormalization

IR divergences

soft and collinear singularities from massless particle; universal!

Checks

- Consistent divergence structure: $\frac{1}{\epsilon^4}, \frac{1}{\epsilon^3}, \frac{1}{\epsilon^2}, \frac{1}{\epsilon}$

  UV renormalisation (operator mixing) and Universal IR

  $$Z^{(2)}_{\tilde{\mathcal{O}}_1} = -\frac{19}{24\epsilon^2} + \frac{25}{12\epsilon}\,, \qquad \gamma^{(2)}_{\tilde{\mathcal{O}}_1} = \frac{25}{3}$$ (new result)

- Reproduce all known results, including the 2-loop Higgs to 3-gluon amplitudes in the heavy top limit $\mathcal{O}_0 = \mathrm{tr}(G_{\mu\nu}G^{\mu\nu})$

- Results satisfy the linear relation:

$$F_{\mathcal{O}_2} = \frac{1}{2} q^2 F_{\mathcal{O}_0} - 4\, g_{\mathrm{YM}}\, F_{\mathcal{O}_1}$$

$$\mathcal{O}_0 = \mathrm{tr}(G_{\mu\nu}G^{\mu\nu})\,.$$
$$\mathcal{O}_1 = \mathrm{tr}(G_\mu{}^\nu G_\nu{}^\rho G_\rho{}^\mu)\,,$$
$$\mathcal{O}_2 = \mathrm{tr}(D_\rho G_{\mu\nu} D^\rho G^{\mu\nu})\,.$$

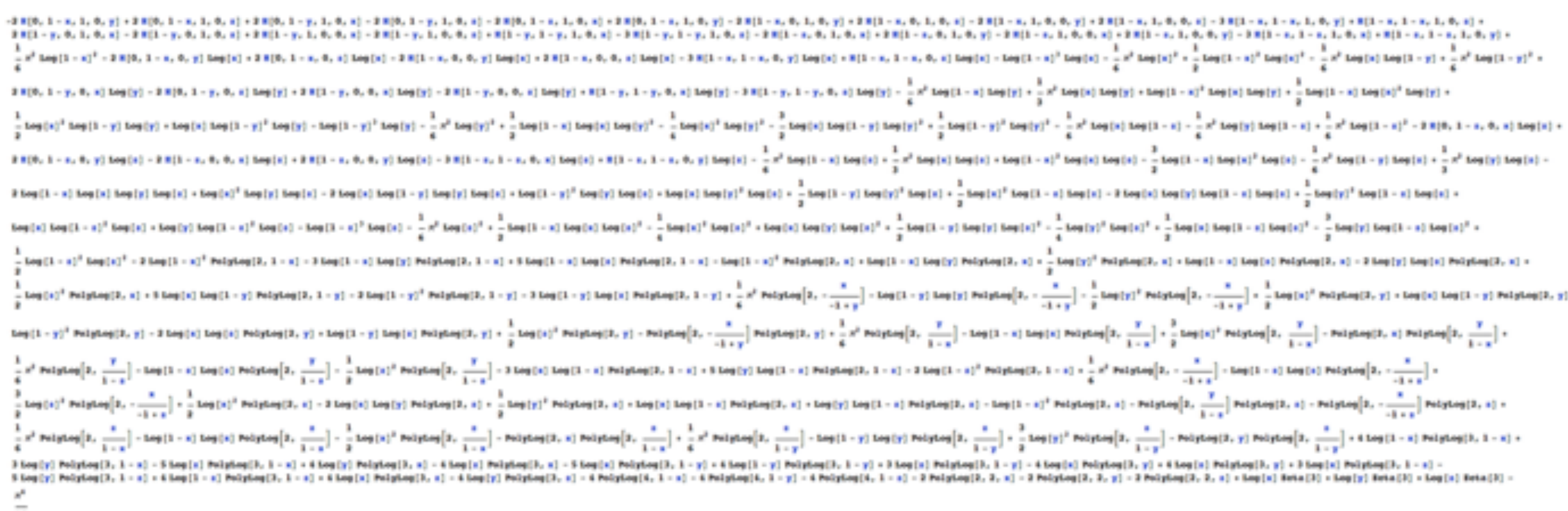
Finite remainder

Apply UV renormalisation and IR subtraction, the **finite remainder function** can be organized according to **transcendentality degree**:

$$F^{(2),\mathrm{fin}}_{\mathrm{R},\mathcal{O}_1} = F^{(0)}_{\mathcal{O}_1} \sum_{i=0}^{4} \Omega^{(2)}_{\mathcal{O}_1;i}$$

Weight-4 part:

$$\Omega^{(2)}_{\mathcal{O}_2;4} =$$



Degree 4:

$$-2\,\mathrm{H}[0, 1-x, 1, 0, y] + 2\,\mathrm{H}[0, 1-x, 1, 0, z] + 2\,\mathrm{H}[0, 1-y, 1, 0, x] - 2\,\mathrm{H}[0, 1-y, 1, 0, z] - 2\,\mathrm{H}[0, 1-z, 1, 0, x] + 2\,\mathrm{H}[0, 1-z, 1, 0, y] - 2\,\mathrm{H}[1-x, 0, 1, 0, y] + 2\,\mathrm{H}[1-x, 0, 1, 0, z] -$$

$$2\,\mathrm{H}[1-x, 1, 0, 0, y] + 2\,\mathrm{H}[1-x, 1, 0, 0, z] - 3\,\mathrm{H}[1-x, 1-x, 1, 0, y] + \mathrm{H}[1-x, 1-x, 1, 0, z] + 2\,\mathrm{H}[1-y, 0, 1, 0, x] - 2\,\mathrm{H}[1-y, 0, 1, 0, z] + 2\,\mathrm{H}[1-y, 1, 0, 0, x] - 2\,\mathrm{H}[1-y, 1, 0, 0, z] +$$

$$\mathrm{H}[1-y, 1-y, 1, 0, x] - 3\,\mathrm{H}[1-y, 1-y, 1, 0, z] - 2\,\mathrm{H}[1-z, 0, 1, 0, x] + 2\,\mathrm{H}[1-z, 0, 1, 0, y] - 2\,\mathrm{H}[1-z, 1, 0, 0, x] + 2\,\mathrm{H}[1-z, 1, 0, 0, y] - 3\,\mathrm{H}[1-z, 1-z, 1, 0, x] + \mathrm{H}[1-z, 1-z, 1, 0, y] +$$

$$\frac{1}{6}\pi^2 \mathrm{Log}[1-x]^2 - 2\,\mathrm{H}[0, 1-x, 0, y]\,\mathrm{Log}[x] + 2\,\mathrm{H}[0, 1-x, 0, z]\,\mathrm{Log}[x] - 2\,\mathrm{H}[1-x, 0, 0, y]\,\mathrm{Log}[x] + 2\,\mathrm{H}[1-x, 0, 0, z]\,\mathrm{Log}[x] - 3\,\mathrm{H}[1-x, 1-x, 0, y]\,\mathrm{Log}[x] + \mathrm{H}[1-x, 1-x, 0, z]\,\mathrm{Log}[x] -$$

$$\mathrm{Log}[1-x]^3 \mathrm{Log}[x] - \frac{1}{6}\pi^2 \mathrm{Log}[x]^2 + \frac{1}{2}\mathrm{Log}[1-x]^2 \mathrm{Log}[x]^2 - \frac{1}{6}\pi^2 \mathrm{Log}[x]\,\mathrm{Log}[1-y] + \frac{1}{6}\pi^2 \mathrm{Log}[1-y]^2 + 2\,\mathrm{H}[0, 1-y, 0, x]\,\mathrm{Log}[y] - 2\,\mathrm{H}[0, 1-y, 0, z]\,\mathrm{Log}[y] + 2\,\mathrm{H}[1-y, 0, 0, x]\,\mathrm{Log}[y] -$$

$$2\,\mathrm{H}[1-y, 0, 0, z]\,\mathrm{Log}[y] + \mathrm{H}[1-y, 1-y, 0, x]\,\mathrm{Log}[y] - 3\,\mathrm{H}[1-y, 1-y, 0, z]\,\mathrm{Log}[y] - \frac{1}{6}\pi^2 \mathrm{Log}[1-x]\,\mathrm{Log}[y] + \frac{1}{3}\pi^2 \mathrm{Log}[x]\,\mathrm{Log}[y] + \mathrm{Log}[1-x]^2 \mathrm{Log}[x]\,\mathrm{Log}[y] + \frac{1}{2}\mathrm{Log}[1-x]\,\mathrm{Log}[x]^2 \mathrm{Log}[y] +$$

$$\frac{1}{2}\mathrm{Log}[x]^2 \mathrm{Log}[1-y]\,\mathrm{Log}[y] + \mathrm{Log}[x]\,\mathrm{Log}[1-y]^2 \mathrm{Log}[y] - \mathrm{Log}[1-y]^3 \mathrm{Log}[y] - \frac{1}{6}\pi^2 \mathrm{Log}[y]^2 + \frac{1}{2}\mathrm{Log}[1-x]\,\mathrm{Log}[x]\,\mathrm{Log}[y]^2 - \frac{1}{4}\mathrm{Log}[x]^2 \mathrm{Log}[y]^2 - \frac{3}{2}\mathrm{Log}[x]\,\mathrm{Log}[1-y]\,\mathrm{Log}[y]^2 + \frac{1}{2}\mathrm{Log}[1-y]^2 \mathrm{Log}[y]^2 -$$

$$\frac{1}{6}\pi^2 \mathrm{Log}[x]\,\mathrm{Log}[1-z] - \frac{1}{6}\pi^2 \mathrm{Log}[y]\,\mathrm{Log}[1-z] + \frac{1}{6}\pi^2 \mathrm{Log}[1-z]^2 - 2\,\mathrm{H}[0, 1-z, 0, x]\,\mathrm{Log}[z] + 2\,\mathrm{H}[0, 1-z, 0, y]\,\mathrm{Log}[z] - 2\,\mathrm{H}[1-z, 0, 0, x]\,\mathrm{Log}[z] + 2\,\mathrm{H}[1-z, 0, 0, y]\,\mathrm{Log}[z] -$$

$$3\,\mathrm{H}[1-z, 1-z, 0, x]\,\mathrm{Log}[z] + \mathrm{H}[1-z, 1-z, 0, y]\,\mathrm{Log}[z] - \frac{1}{6}\pi^2 \mathrm{Log}[1-x]\,\mathrm{Log}[z] + \frac{1}{3}\pi^2 \mathrm{Log}[x]\,\mathrm{Log}[z] + \mathrm{Log}[1-x]^2 \mathrm{Log}[x]\,\mathrm{Log}[z] - \frac{3}{2}\mathrm{Log}[1-x]\,\mathrm{Log}[x]^2 \mathrm{Log}[z] - \frac{1}{6}\pi^2 \mathrm{Log}[1-y]\,\mathrm{Log}[z] +$$

$$\frac{1}{3}\pi^2 \mathrm{Log}[y]\,\mathrm{Log}[z] - 2\,\mathrm{Log}[1-x]\,\mathrm{Log}[x]\,\mathrm{Log}[y]\,\mathrm{Log}[z] + \mathrm{Log}[x]^2 \mathrm{Log}[y]\,\mathrm{Log}[z] - 2\,\mathrm{Log}[x]\,\mathrm{Log}[1-y]\,\mathrm{Log}[y]\,\mathrm{Log}[z] + \mathrm{Log}[1-y]^2 \mathrm{Log}[y]\,\mathrm{Log}[z] + \mathrm{Log}[x]\,\mathrm{Log}[y]^2 \mathrm{Log}[z] + \frac{1}{2}\mathrm{Log}[1-y]\,\mathrm{Log}[y]^2 \mathrm{Log}[z] +$$

$$\frac{1}{2}\mathrm{Log}[x]^2 \mathrm{Log}[1-z]\,\mathrm{Log}[z] - 2\,\mathrm{Log}[x]\,\mathrm{Log}[y]\,\mathrm{Log}[1-z]\,\mathrm{Log}[z] + \frac{1}{2}\mathrm{Log}[y]^2 \mathrm{Log}[1-z]\,\mathrm{Log}[z] + \mathrm{Log}[x]\,\mathrm{Log}[1-z]^2 \mathrm{Log}[z] + \mathrm{Log}[y]\,\mathrm{Log}[1-z]^2 \mathrm{Log}[z] - \mathrm{Log}[1-z]^3 \mathrm{Log}[z] - \frac{1}{6}\pi^2 \mathrm{Log}[z]^2 +$$

$$\frac{1}{2}\mathrm{Log}[1-x]\,\mathrm{Log}[x]\,\mathrm{Log}[z]^2 - \frac{1}{4}\mathrm{Log}[x]^2 \mathrm{Log}[z]^2 + \mathrm{Log}[x]\,\mathrm{Log}[y]\,\mathrm{Log}[z]^2 + \frac{1}{2}\mathrm{Log}[1-y]\,\mathrm{Log}[y]\,\mathrm{Log}[z]^2 - \frac{1}{4}\mathrm{Log}[y]^2 \mathrm{Log}[z]^2 + \frac{1}{2}\mathrm{Log}[x]\,\mathrm{Log}[1-z]\,\mathrm{Log}[z]^2 - \frac{3}{2}\mathrm{Log}[y]\,\mathrm{Log}[1-z]\,\mathrm{Log}[z]^2 +$$

$$\frac{1}{2}\mathrm{Log}[1-z]^2 \mathrm{Log}[z]^2 - 2\,\mathrm{Log}[1-x]^2 \mathrm{PolyLog}[2, 1-x] - 3\,\mathrm{Log}[1-x]\,\mathrm{Log}[y]\,\mathrm{PolyLog}[2, 1-x] + 5\,\mathrm{Log}[1-x]\,\mathrm{Log}[z]\,\mathrm{PolyLog}[2, 1-x] - \mathrm{Log}[1-x]^2 \mathrm{PolyLog}[2, x] + \mathrm{Log}[1-x]\,\mathrm{Log}[y]\,\mathrm{PolyLog}[2, x] +$$

$$\frac{1}{2}\mathrm{Log}[y]^2 \mathrm{PolyLog}[2, x] + \mathrm{Log}[1-x]\,\mathrm{Log}[z]\,\mathrm{PolyLog}[2, x] - 2\,\mathrm{Log}[y]\,\mathrm{Log}[z]\,\mathrm{PolyLog}[2, x] + \frac{1}{2}\mathrm{Log}[z]^2 \mathrm{PolyLog}[2, x] + 5\,\mathrm{Log}[x]\,\mathrm{Log}[1-y]\,\mathrm{PolyLog}[2, 1-y] - 2\,\mathrm{Log}[1-y]^2 \mathrm{PolyLog}[2, 1-y] -$$

$$3\,\mathrm{Log}[1-y]\,\mathrm{Log}[z]\,\mathrm{PolyLog}[2, 1-y] + \frac{1}{6}\pi^2 \mathrm{PolyLog}\left[2, -\frac{x}{-1+y}\right] - \mathrm{Log}[1-y]\,\mathrm{Log}[y]\,\mathrm{PolyLog}\left[2, -\frac{x}{-1+y}\right] - \frac{1}{2}\mathrm{Log}[y]^2 \mathrm{PolyLog}\left[2, -\frac{x}{-1+y}\right] + \frac{1}{2}\mathrm{Log}[x]^2 \mathrm{PolyLog}[2, y] + \mathrm{Log}[x]\,\mathrm{Log}[1-y]\,\mathrm{PolyLog}[2, y]$$

$$\mathrm{Log}[1-y]^2 \mathrm{PolyLog}[2, y] - 2\,\mathrm{Log}[x]\,\mathrm{Log}[z]\,\mathrm{PolyLog}[2, y] + \mathrm{Log}[1-y]\,\mathrm{Log}[z]\,\mathrm{PolyLog}[2, y] + \frac{1}{2}\mathrm{Log}[z]^2 \mathrm{PolyLog}[2, y] - \mathrm{PolyLog}\left[2, -\frac{x}{-1+y}\right] \mathrm{PolyLog}[2, y] + \frac{1}{6}\pi^2 \mathrm{PolyLog}\left[2, \frac{y}{1-x}\right] -$$

$$\mathrm{Log}[1-x]\,\mathrm{Log}[x]\,\mathrm{PolyLog}\left[2, \frac{y}{1-x}\right] + \frac{3}{2}\mathrm{Log}[x]^2 \mathrm{PolyLog}\left[2, \frac{y}{1-x}\right] - \mathrm{PolyLog}[2, x]\,\mathrm{PolyLog}\left[2, \frac{y}{1-x}\right] + \frac{1}{6}\pi^2 \mathrm{PolyLog}\left[2, \frac{y}{1-z}\right] - \mathrm{Log}[1-z]\,\mathrm{Log}[z]\,\mathrm{PolyLog}\left[2, \frac{y}{1-z}\right] - \frac{1}{2}\mathrm{Log}[z]^2 \mathrm{PolyLog}\left[2, \frac{y}{1-z}\right] -$$

$$3\,\mathrm{Log}[x]\,\mathrm{Log}[1-z]\,\mathrm{PolyLog}[2, 1-z] + 5\,\mathrm{Log}[y]\,\mathrm{Log}[1-z]\,\mathrm{PolyLog}[2, 1-z] - 2\,\mathrm{Log}[1-z]^2 \mathrm{PolyLog}[2, 1-z] + \frac{1}{6}\pi^2 \mathrm{PolyLog}\left[2, -\frac{x}{-1+z}\right] - \mathrm{Log}[1-z]\,\mathrm{Log}[z]\,\mathrm{PolyLog}\left[2, -\frac{x}{-1+z}\right] +$$

$$\frac{3}{2}\mathrm{Log}[z]^2 \mathrm{PolyLog}\left[2, -\frac{x}{-1+z}\right] + \frac{1}{2}\mathrm{Log}[x]^2 \mathrm{PolyLog}[2, z] - 2\,\mathrm{Log}[x]\,\mathrm{Log}[y]\,\mathrm{PolyLog}[2, z] + \frac{1}{2}\mathrm{Log}[y]^2 \mathrm{PolyLog}[2, z] + \mathrm{Log}[x]\,\mathrm{Log}[1-z]\,\mathrm{PolyLog}[2, z] + \mathrm{Log}[y]\,\mathrm{Log}[1-z]\,\mathrm{PolyLog}[2, z] -$$

$$\mathrm{Log}[1-z]^2 \mathrm{PolyLog}[2, z] - \mathrm{PolyLog}\left[2, \frac{y}{1-z}\right] \mathrm{PolyLog}[2, z] - \mathrm{PolyLog}\left[2, -\frac{x}{-1+z}\right] \mathrm{PolyLog}[2, z] + \frac{1}{6}\pi^2 \mathrm{PolyLog}\left[2, \frac{z}{1-x}\right] - \mathrm{Log}[1-x]\,\mathrm{Log}[x]\,\mathrm{PolyLog}\left[2, \frac{z}{1-x}\right] - \frac{1}{2}\mathrm{Log}[x]^2 \mathrm{PolyLog}\left[2, \frac{z}{1-x}\right] -$$

$$\mathrm{PolyLog}[2, x]\,\mathrm{PolyLog}\left[2, \frac{z}{1-x}\right] + \frac{1}{6}\pi^2 \mathrm{PolyLog}\left[2, \frac{z}{1-y}\right] - \mathrm{Log}[1-y]\,\mathrm{Log}[y]\,\mathrm{PolyLog}\left[2, \frac{z}{1-y}\right] + \frac{3}{2}\mathrm{Log}[y]^2 \mathrm{PolyLog}\left[2, \frac{z}{1-y}\right] - \mathrm{PolyLog}[2, y]\,\mathrm{PolyLog}\left[2, \frac{z}{1-y}\right] + 4\,\mathrm{Log}[1-x]\,\mathrm{PolyLog}[3, 1-x] +$$

$$3\,\mathrm{Log}[y]\,\mathrm{PolyLog}[3, 1-x] - 5\,\mathrm{Log}[z]\,\mathrm{PolyLog}[3, 1-x] + 4\,\mathrm{Log}[y]\,\mathrm{PolyLog}[3, x] - 4\,\mathrm{Log}[z]\,\mathrm{PolyLog}[3, x] - 5\,\mathrm{Log}[x]\,\mathrm{PolyLog}[3, 1-y] + 4\,\mathrm{Log}[1-y]\,\mathrm{PolyLog}[3, 1-y] + 3\,\mathrm{Log}[z]\,\mathrm{PolyLog}[3, 1-y] -$$

$$4\,\mathrm{Log}[x]\,\mathrm{PolyLog}[3, y] + 4\,\mathrm{Log}[z]\,\mathrm{PolyLog}[3, y] + 3\,\mathrm{Log}[x]\,\mathrm{PolyLog}[3, 1-z] - 5\,\mathrm{Log}[y]\,\mathrm{PolyLog}[3, 1-z] + 4\,\mathrm{Log}[1-z]\,\mathrm{PolyLog}[3, 1-z] + 4\,\mathrm{Log}[x]\,\mathrm{PolyLog}[3, z] - 4\,\mathrm{Log}[y]\,\mathrm{PolyLog}[3, z] -$$

$$4\,\mathrm{PolyLog}[4, 1-x] - 4\,\mathrm{PolyLog}[4, 1-y] - 4\,\mathrm{PolyLog}[4, 1-z] - 2\,\mathrm{PolyLog}[2, 2, x] - 2\,\mathrm{PolyLog}[2, 2, y] - 2\,\mathrm{PolyLog}[2, 2, z] + \mathrm{Log}[x]\,\mathrm{Zeta}[3] + \mathrm{Log}[y]\,\mathrm{Zeta}[3] + \mathrm{Log}[z]\,\mathrm{Zeta}[3] - \frac{\pi^4}{16}$$

Finite remainder

Simplify via “symbol” for transcendental functions

$$\Omega^{(2)}_{\mathcal{O}_1;4} = -\frac{3}{2}\mathrm{Li}_4(u) + \frac{3}{4}\mathrm{Li}_4\left(-\frac{uv}{w}\right) - \frac{3}{2}\log(w)\mathrm{Li}_3\left(-\frac{u}{v}\right) + \frac{\zeta_2}{8}\left[5\log^2(u) - 2\log(v)\log(w)\right]$$
$$+ \frac{\log^2(u)}{32}\left[\log^2(u) + 2\log^2(v) - 4\log(v)\log(w)\right] - \frac{1}{4}\zeta_4 - \frac{1}{2}\zeta_3\log(-q^2) + \mathrm{perms}(u,v,w)$$

$$\rightarrow \quad \boxed{\Omega^{(2)}_{\mathcal{O}_1;4} = \Omega^{(2),\mathcal{N}=4}_{\mathcal{O}_1;4}}$$

for N=4 result see:
Brandhuber, Kostacinska, Penante, Travaglini 2017]

It also appears as a universal function for length-3 operators.

[Brandhuber, Kostacinska, Penante, Travaglini, Wen, Young 2014, 2016]

[Loebbert, Nandan, Sieg, Wilhelm, GY 2015, 2016]

Finite remainder

Weight-3 part:

$$\Omega^{(2)}_{\mathcal{O}_1;3} = \left(1 + \frac{u}{w}\right) T_3 + \frac{143}{72}\zeta_3 - \frac{11}{24}\zeta_2 \log(-u\, q^2) + \text{perms}(u,v,w)$$

$$\Omega^{(2),\mathcal{N}=4}_{\mathcal{O}_1;3} = \left(1 + \frac{u}{w}\right) T_3 + \text{perms}(u,v,w)$$

$$T_3 := \Big[-\text{Li}_3\left(-\frac{u}{w}\right) + \log(u)\text{Li}_2\left(\frac{v}{1-u}\right) - \frac{1}{2}\log(1-u)\log(u)\log\left(\frac{w^2}{1-u}\right) + \frac{1}{2}\text{Li}_3\left(-\frac{uv}{w}\right) + \frac{1}{12}\log^3(w)$$
$$+ \frac{1}{2}\log(u)\log(v)\log(w) + (u \leftrightarrow v)\Big] + \text{Li}_3(1-v) - \text{Li}_3(u) + \frac{1}{2}\log^2(v)\log\left(\frac{1-v}{u}\right) - \zeta_2 \log\left(\frac{uv}{w}\right) .$$

T_3 function is also a building block appearing in many form factors in N=4 SYM: $T_3 = -(R_i^{(2)})^{XYX}_{XXY}\Big|_3 - \zeta_2 \log(u)$

[Loebbert, Nandan, Sieg, Wilhelm, GY 2015]

Finite remainder

Degree 2 to 0:

$$\Omega^{(2)}_{\mathcal{O}_1;2} = \Big\{ \Big(\frac{u^2}{w^2} + \frac{v^2}{w^2} - 1\Big) \Big[\mathrm{Li}_2(1-u) + \frac{1}{2}\log(u)\log(v) - \frac{1}{2}\zeta_2\Big] - \frac{55}{48}\log^2(u) + \frac{73}{72}\log(u)\log(v) + \frac{23}{6}\zeta_2 + \mathrm{perms}(u,v,w)\Big\} - \frac{19}{36}\log(uvw)\log(-q^2) - \frac{19}{24}\log^2(-q^2)\,.$$

$$\Omega^{(2)}_{\mathcal{O}_1;1} = \Big(\frac{119}{18} + \frac{v}{w} + \frac{u^2}{2vw}\Big)\log(u) + \Big(\frac{119}{18} - \frac{1}{3uvw}\Big)\log(-q^2) + \mathrm{perms}(u,v,w)$$

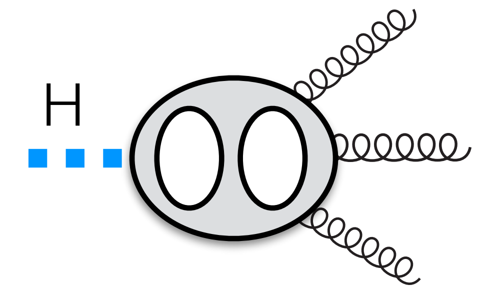
$$\Omega^{(2)}_{\mathcal{O}_2;0} = \frac{487}{72}\frac{1}{uvw} - \frac{14075}{216}\,.$$

Lower transcendental terms with rational kinematic coefficients are also identical to the N=4 results.

Content

- Motivation
- Computation
- Results
- **Summary and outlook**

Summary



- Two-loop Higgs amplitudes with dim-7 operators
- Efficient method based on on-shell unitarity and IBP
- Simple analytic result which provide further evidence of transcendentality principle

Outlook

- Form factors with more general operators and more gluons

- Understand better the maximal transcendentality principle (more examples)

- Origin of the simplicity?
  Is there a way to understand it directly? Bootstraps??

Thank you for your attention!

From function to “Symbol”

Recursion definition of “Symbol”:

$$\mathsf{d}f_k = \sum_i f^i_{k-1}\,\mathsf{dLog}(R_i), \qquad \mathsf{Symbol}(f_k) = \sum_i \mathsf{Symbol}(f^i_{k-1}) \otimes R_i$$

Some examples:

Function	Differential	**symbol**
R	d R	0
log(R)	d log(R)	R
log(R1)log(R2)	logR1 dlogR2+logR2 dlogR1	R1 ⊗ R2 + R2 ⊗ R1
Li_2(R)	Li_1(R) dlogR	-(1-R) ⊗ R

Symbol contains analytics properties of functions, e.g. branch cuts.

Symbol

Properties:

$$R_1 \otimes \ldots \otimes (c\, R_i) \otimes \ldots \otimes R_n = R_1 \otimes \ldots \otimes R_i \otimes \ldots \otimes R_n$$ c=const

$$R_1 \otimes \ldots \otimes (R_i\, R_j) \otimes \ldots \otimes R_n = R_1 \otimes \ldots \otimes R_i \otimes \ldots \otimes R_n + R_1 \otimes \ldots \otimes R_j \otimes \ldots \otimes R_n$$

Make it easy to prove non-trivial identities, e.g.:

$$\mathrm{Li}_2\left(\frac{x}{1-y}\right) + \mathrm{Li}_2\left(\frac{y}{1-x}\right) - \mathrm{Li}_2(x) - \mathrm{Li}_2(y) - \mathrm{Li}_2\left(\frac{xy}{(1-x)(1-y)}\right) = \mathrm{Log}(1-x)\mathrm{Log}(1-y)$$

$(x < 1 \text{ and } y < 1)$

$$\mathrm{Symbol}(\mathrm{LHS}) = (1-x) \otimes (1-y) + (1-y) \otimes (1-x)$$

back to main content

应用

复杂结果 ⟶ symbol ⟶ 简单结果

更进一步：

从一些基本约束出发，
直接求解symbol ⟶ 简单结果

(这类方法称为Bootstrap)

Gauge invariant basis projection

$$F_n(\varepsilon_i, p_i, l_a)|_{\text{cut}} = \sum_\alpha f_n^\alpha(p_i, l_a) B_\alpha$$

$$f_n^\alpha(p_i, l_a) = B^\alpha \circ F_n(\varepsilon_i, p_i, l_a)$$

$$\varepsilon_i^\mu \circ \varepsilon_i^\nu \equiv \sum_{\text{helicities}} \varepsilon_i^\mu \varepsilon_i^\nu = \eta^{\mu\nu} - \frac{q^\mu p_i^\nu + q^\nu p_i^\mu}{q \cdot p_i}$$

$$B^\alpha \circ B_\beta = \delta_\beta^\alpha, \; B_\alpha = G_{\alpha\beta} B^\beta, \; G_{\alpha\beta} = B_\alpha \circ B_\beta$$

Three gluon case:

$$B_1 = A_1 C_{23}\,, \quad B_2 = A_2 C_{31}\,, \quad B_3 = A_3 C_{12}\,, \quad B_4 = A_1 A_2 A_3$$

where $A_i = \frac{\varepsilon_i \cdot p_j}{p_i \cdot p_j} - \frac{\varepsilon_i \cdot p_k}{p_i \cdot p_k}\,, \quad C_{ij} = \varepsilon_i \cdot \varepsilon_j - \frac{(p_i \cdot \varepsilon_j)(p_j \cdot \varepsilon_i)}{p_i \cdot p_j}$

Two gluon case:

$$B_0 = C_{12}$$

UV renormalization

Coupling constant renormalisation:

$$\alpha_0 = \alpha_s S_\epsilon^{-1} \frac{\mu^{2\epsilon}}{\mu_0^{2\epsilon}} \left[1 - \frac{\beta_0}{\epsilon} \frac{\alpha_s}{4\pi} + \left(\frac{\beta_0^2}{\epsilon^2} - \frac{\beta_1}{2\epsilon} \right) \left(\frac{\alpha_s}{4\pi} \right)^2 + \mathcal{O}(\alpha_s^3) \right]$$

Renormalisation constant Z for the operators:

anomalous dimension

$$\mathcal{O}_I^b \longrightarrow Z_{IJ} \mathcal{O}_I^b \qquad Z = 1 + \sum_{l=1}^{\infty} \left(\frac{\alpha_s}{4\pi} \right)^l Z^{(l)} \longrightarrow \gamma = \mu \frac{\partial}{\partial \mu} \log Z$$

Renormalized form factor: $F = g_s^x \, S_\epsilon^{-x/2} \sum_{l=0}^{\infty} \left(\frac{\alpha_s}{4\pi} \right)^l F^{(l)}$

$$\begin{aligned}
F^{(0)} &= F_{\mathrm{b}}^{(0)} \,, \\
F^{(1)} &= S_\epsilon^{-1} F_{\mathrm{b}}^{(1)} + \left(Z^{(1)} - \frac{x}{2} \frac{\beta_0}{\epsilon} \right) F_{\mathrm{b}}^{(0)} \,, \\
F^{(2)} &= S_\epsilon^{-2} F_{\mathrm{b}}^{(2)} + S_\epsilon^{-1} \left[Z^{(1)} - \left(1 + \frac{x}{2} \right) \frac{\beta_0}{\epsilon} \right] F_{\mathrm{b}}^{(1)} \\
&\quad + \left[Z^{(2)} - \frac{x}{2} \frac{\beta_0}{\epsilon} Z^{(1)} + \frac{x^2 + 2x}{8} \frac{\beta_0^2}{\epsilon^2} - \frac{x}{4} \frac{\beta_1}{\epsilon} \right] F_{\mathrm{b}}^{(0)}
\end{aligned}$$

IR subtraction

Universal IR structure: [Catani 1998]

$$F^{(1)} = I^{(1)}(\epsilon) F^{(0)} + F^{(1),\text{fin}} + \mathcal{O}(\epsilon)\,,$$
$$F^{(2)} = I^{(2)}(\epsilon) F^{(0)} + I^{(1)}(\epsilon) F^{(1)} + F^{(2),\text{fin}} + \mathcal{O}(\epsilon)$$

where

$$I^{(1)}(\epsilon) = -\frac{e^{\gamma_E \epsilon}}{\Gamma(1-\epsilon)} \left(\frac{N_c}{\epsilon^2} + \frac{\beta_0}{2\epsilon} \right) \sum_{i=1}^{n} (-s_{i,i+1})^{-\epsilon}\,,$$

$$\begin{aligned} I^{(2)}(\epsilon) = & -\frac{1}{2}\left[I^{(1)}(\epsilon) \right]^2 - \frac{\beta_0}{\epsilon} I^{(1)}(\epsilon) \\ & + \frac{e^{-\gamma_E \epsilon}\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)} \left[\frac{\beta_0}{\epsilon} + \left(\frac{67}{9} - \frac{\pi^2}{3} \right) N_c \right] I^{(1)}(2\epsilon) \\ & + n \frac{e^{\gamma_E \epsilon}}{\epsilon \Gamma(1-\epsilon)} \left[\left(\frac{\zeta_3}{2} + \frac{5}{12} + \frac{11\pi^2}{144} \right) N_c^2 \right]\,. \end{aligned}$$

Anomalous dimension

Operator mixing:

$$F^{(2)}_{\mathcal{O}_1}(1^-,2^-,3^-)\big|_{Z^{(2)}\text{-part}} = F^{(0)}_{\mathcal{O}_1}(1^-,2^-,3^-)\left(-\frac{19}{24\epsilon^2}+\frac{25}{12\epsilon}-\frac{1}{uvw}\frac{1}{\epsilon}\right)$$

$$= \left(-\frac{19}{24\epsilon^2}+\frac{25}{12\epsilon}\right)F^{(0)}_{\mathcal{O}_1}(1^-,2^-,3^-) - \frac{F^{(0)}_{\tilde{\mathcal{O}}_2}(1^-,2^-,3^-)}{\epsilon}$$

$$F^{(2)}_{\mathcal{O}_1}(1^-,2^-) = F^{(0)}_{\tilde{\mathcal{O}}_2}(1^-,2^-)\left(-\frac{1}{\epsilon}+2\log s_{12}-\frac{487}{72}\right)+\mathcal{O}(\epsilon^1)$$

Eigen-operators:

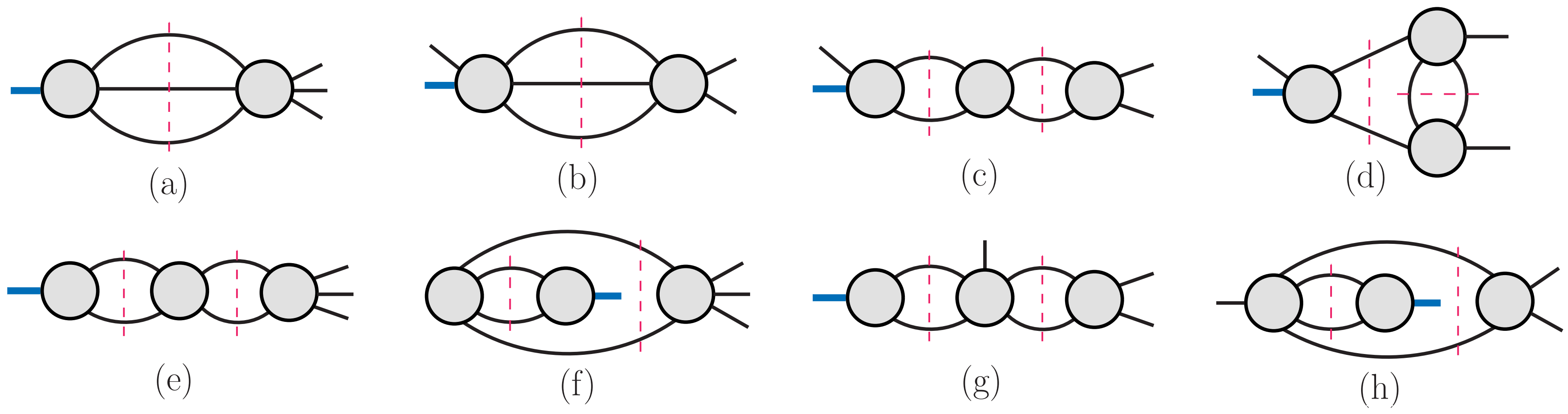
$$\tilde{\mathcal{O}}_2 = -\frac{3}{2}(\mathcal{O}_2 + 8g_{\rm YM}\,\mathcal{O}_1) = -\frac{3}{4}\,\partial^2\mathcal{O}_0$$

$$\tilde{\mathcal{O}}_1 = \mathcal{O}_1 + \frac{1}{\epsilon}\frac{1}{g_{\rm YM}}\left(\frac{\alpha_s}{4\pi}\right)^2\tilde{\mathcal{O}}_2$$

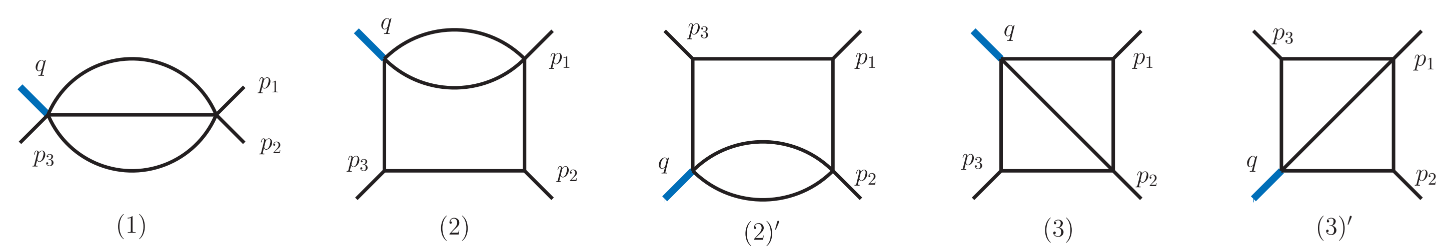
Anomalous dimension: $\gamma = \mu\frac{\partial}{\partial\mu}\log Z$

$$Z^{(2)}_{\tilde{\mathcal{O}}_1} = -\frac{19}{24\epsilon^2}+\frac{25}{12\epsilon}\,, \qquad \gamma^{(2)}_{\tilde{\mathcal{O}}_1} = \frac{25}{3}$$

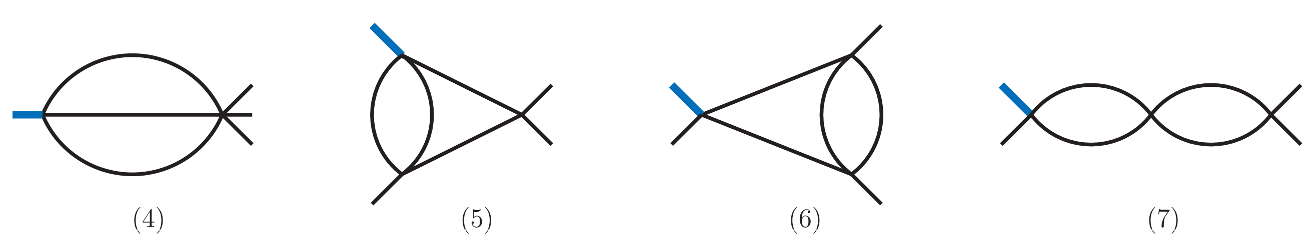
2-loop 3-gluon



$\mathrm{Tr}(F^3)$ cut (b):



Other MIs:



Full result $= \frac{1}{2}\left(\sum_{i=1}^{7} c_i M_i + \sum_{i=2,5} c_i M_i\right) + \mathrm{perms}(p_1, p_2, p_3)$