MHV Lagrangian for Yang-Mills and QCD

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Conventions

Introduction

MHV Lagrangian for Pure Yang-Mills

- Light-cone Yang-Mills
- Canonical Transformation for light-cone Y-M
- CSW rules from MHV Lagrangian

MHV Lagrangian for QCD: fund. massless fermion

- Canonical Transformation
- CSW rules from MHV Lagrangian for QCD

Tree level 3-point $(+ + -)$ amplitude

Summary and Disscusion

Conventions

- Light cone co-ordinates:

$$\check{p} = \frac{1}{\sqrt{2}}(p^t + p^3)\,, \qquad \hat{p} = \frac{1}{\sqrt{2}}(p^t - p^3)\,,$$
$$\tilde{p} = -\frac{1}{\sqrt{2}}(p^1 - ip^2)\,, \qquad \bar{p} = -\frac{1}{\sqrt{2}}(p^1 + ip^2)\,.$$

- Gauge fields: Canonical A and covariant derivative:

$$D_\mu = \partial_\mu - i\frac{g}{\sqrt{2}}A_\mu\,, \quad \mathrm{tr} T^a T^b = \delta^{ab}$$

Yang-Mills lagrangian:

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Conventions

- Massless spinors:

$$p \cdot \bar{\sigma} = \sqrt{2} \begin{pmatrix} \check{p} & -p \\ -\bar{p} & \hat{p} \end{pmatrix} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}},$$

$$\lambda_\alpha = 2^{1/4} \begin{pmatrix} -p/\sqrt{\hat{p}} \\ \sqrt{\hat{p}} \end{pmatrix} \quad \text{and} \quad \tilde{\lambda}_{\dot{\alpha}} = 2^{1/4} \begin{pmatrix} -\bar{p}/\sqrt{\hat{p}} \\ \sqrt{\hat{p}} \end{pmatrix}.$$

$$(1\,2) := \hat{1}\tilde{2} - \hat{2}\tilde{1}, \quad \{1\,2\} := \hat{1}\bar{2} - \hat{2}\bar{1}.$$

$$\begin{aligned} \langle 1\,2 \rangle &:= \epsilon^{\alpha\beta} \lambda_{1\alpha} \lambda_{2\beta} = \sqrt{2} \frac{(1\,2)}{\sqrt{\hat{1}\hat{2}}} \\ [1\,2] &:= \epsilon^{\dot{\alpha}\dot{\beta}} \lambda_{1\dot{\alpha}} \lambda_{2\dot{\beta}} = \sqrt{2} \frac{\{1\,2\}}{\sqrt{\hat{1}\hat{2}}}. \end{aligned}$$

- Outgoing fermions and anti-fermions:
  $\psi^+ \sim \int u^+ \, a e^{-ik\cdot x} + v^- \, b^\dagger e^{ik\cdot x}$

$$\bar{u}^+(p) \equiv (\tilde{\lambda}_{\dot{\alpha}}, 0) \qquad \text{and} \qquad \bar{u}^-(p) \equiv (0, \lambda^\alpha)$$

$$v^+(p) \equiv \begin{pmatrix} \tilde{\lambda}^{\dot{\alpha}} \\ 0 \end{pmatrix} \qquad \text{and} \qquad v^-(p) \equiv \begin{pmatrix} 0 \\ \lambda_\alpha \end{pmatrix}$$

Conventions

- Gauge field polarizations

$$\epsilon^{+} \cdot \sigma = \sqrt{2}\, \frac{\bar{\lambda}_{\dot{\beta}}(\mu)\lambda_{\alpha}(k)}{[\mu, k]}$$

$$\epsilon^{-} \cdot \sigma = -\sqrt{2}\, \frac{\lambda^{\alpha}(\mu)\bar{\lambda}^{\dot{\beta}}(k)}{\langle \mu, k \rangle}$$

μ is an arbitary reference momentum. Changing μ is equivalent to a gauge trans.

$$\epsilon^{+}(\mu, k) - \epsilon^{+}(\nu, k) = -\sqrt{2} \frac{\langle \mu\, \nu \rangle}{\langle \mu\, k \rangle \langle k\, \nu \rangle}\, k$$

Introdution

- Difficulties in calculating the amplitudes in Yang-Mills (Y-M) theory and QCD. For multileg amplitudes
  - Too many diagrams
  - Too many terms in each diagram
  - Too many kinetic variables
- $$A(1^+, \ldots, n^+) = A(1^-, 2^+, \ldots, n^+) = 0$$
- Pure Y-M MHV (Maximal Helicity Violating) amplitude, proposed by Parke and Taylor [Phys. Lett. B **157** 81 (1985)], proved by Berends and Giele [Nucl. Phys. B **306** 759 (1988)]. : Simple & Beautiful

$$A(1^+ \cdots i^- \cdots j^- \cdots n^+) = \frac{\langle i\, j\rangle^4}{\langle 1\, 2\rangle\langle 2\, 3\rangle \cdots \langle n-1\, n\rangle}$$

Introdution

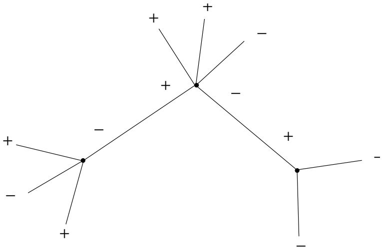
CSW rules (F. Cachazo, P. Svrcek, E. Witten, JHEP 0409:006,2004):

- Analytically continue the MHV amplitude to off-shell. For off-shell momentum define

$$\lambda_a = P_{a\dot{a}}\eta^{\dot{a}}.$$

  η is an arbitrary spinor. Use this λ to construct off-shell MHV vertices.

- Construct tree level non-MHV amplitudes using MHV vertices connected with scalar propagator, $\frac{1}{P^2}$.



Introdution

- Advantage: reduce the number of diagrams and the number of terms for each diagram
- Only in tree level amplitude and one loop supersymmetric amplitude.
- For one loop non-supersymmetric amplitude, there are some missing pieces.

Introduction

CSW rules was first proposed inspired by the considerations in twistor string theory. To derive them from field theory side

- There were some indirect proof of CSW rules.
- MHV-lagrangian approach to derive CSW rules:
  - Paul Mansfield: Propose a framework to deduce from Pure Yang-Mills to MHV lagrangian under canonical transformation of the fields (JHEP0603:037,2006). Each MHV vertex come from one term in the lagrangian

    $$L^{-+}[B,\bar{B}]+L^{--+}[B,\bar{B}]+L^{--++}[B,\bar{B}]+L^{--+++}[B,\bar{B}]\cdots$$

    The vertices in the lagrangian are MHV amplitudes continued to off-shell.
  - Tim and James: explicitly solve the canonical transformation and point out the missing pieces of the CSW rules is from the Completion vertices. (JHEP0608:003,2006, JHEP 0705:011,2007.)
  - We extend the MHV lagrangian to include fermions and to SQCD (JHEP 0808:103,2008, JHEP 0812:028,2008.) .

Basics Ideas: for Pure Yang-Mills

Gauge field $A_\mu = (\hat{A}, \check{A}, A, \bar{A})$

- Choose light-cone gauge: $\mu \cdot A = \hat{A} = 0$, $\mu = (1, 0, 0, 1)$
- Integrate out the non-dynamical variables $\check{A} \rightarrow$ Light-cone Yang-Mills—only A and $\bar{A}$ are left, positive and negative helicity. But there are still non-MHV vertices left: L^{-++} (what is MHV vertices)
- Make canonical transformations to get rid of the non-MHV vertices.
- Collect all the MHV vertices with same helicity configurations and we obtain MHV lagrangian: the interaction vertices are the off-shell continuation of the MHV amplitudes.

$$L^{-+}[B, \bar{B}] + L^{--+}[B, \bar{B}] + L^{--++}[B, \bar{B}] + L^{--+++}[B, \bar{B}] \cdots$$

Massless Light-cone Y-M Lagrangian

LCYM: Starting from massless Y-M Lagrangian in terms of $\check{A}$, $\hat{A}$, $\tilde{A}$, $\bar{A}$.

- Choose a gauge: $\mu \cdot A = \hat{A} = 0$, reference momentum $\mu^\mu = (1, 0, 0, 1)$ The lagrangian is quadratic in $\check{A}$.
- LCYM Lagrangian: Integrate out nondynamical fields: $\check{A}$. The fields left are A $\bar{A}$. For on shell polarization explicitly:

$$\epsilon^+ \cdot \sigma = \sqrt{2}\, \frac{\bar{\lambda}_{\dot{\beta}}(\mu)\lambda_\alpha(k)}{\bar{\lambda}_{\dot{\alpha}}(\mu)\bar{\lambda}^{\dot{\alpha}}(k)} = \sqrt{2} \begin{pmatrix} 0 & -1 \\ 0 & -\frac{\bar{k}}{\hat{k}} \end{pmatrix} \sim \sqrt{2} \begin{pmatrix} \hat{A} & A \\ \bar{A} & \check{A} \end{pmatrix}$$

$$\epsilon^- \cdot \sigma = -\sqrt{2}\, \frac{\lambda^\alpha(\mu)\bar{\lambda}^{\dot{\beta}}(k)}{\lambda^\alpha(\mu)\lambda_\alpha(k)} = \sqrt{2} \begin{pmatrix} 0 & 0 \\ -1 & -\frac{k}{\hat{k}} \end{pmatrix} \sim \sqrt{2} \begin{pmatrix} \hat{A} & A \\ \bar{A} & \check{A} \end{pmatrix}$$

$A \sim +$ helicity, $\bar{A} \sim -$ helicity (outgoing).

$$L_{LCYM} = L_{YM}^{-+} + L_{YM}^{++-} + L_{YM}^{--+} + L_{YM}^{--++}$$

Review: Canonical Transformation

- Canonical corrdinates: $\vec{x} = (q_1, \ldots, q_n, p_1 \ldots p_n)$

  Hamilton's equations: $\dot{\vec{x}} = J \frac{\partial H}{\partial \vec{x}}$ where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$

  Poisson bracket: $\{f, g\} = \frac{\partial f}{\partial x_i} J_{ij} \frac{\partial g}{\partial x_j}$

$$\{p_i, p_j\} = \{q_i, q_j\} = 0\,, \quad \{q_i, p_j\} = \delta_{ij}$$

- Canonical transformation:
  $\vec{x} = (q_1, \ldots, q_n, p_1 \ldots p_n) \to \vec{X} = (Q_1, \ldots, Q_n, P_1 \ldots P_n)$,
  $\mathcal{J} = \frac{\partial X_i}{\partial x_j}$

  preserves Poisson brackets, $\begin{array}{l}\{P_i, P_j\} = \{Q_i, Q_j\} = 0\,, \\ \{Q_i, P_j\} = \delta_{ij}\end{array}$

  $\iff$ preserves Hamilton's equations, $\mathcal{J} J \mathcal{J}^T = J$

$$\iff \quad \left.\frac{\partial P_i}{\partial p_j}\right|_q = \left.\frac{\partial q_j}{\partial Q_i}\right|_P, \quad \left.\frac{\partial P_i}{\partial q_j}\right|_p = -\left.\frac{\partial p_j}{\partial Q_i}\right|_P$$

  $\implies$ preserves phase space measure $\int \mathrm{d}p\mathrm{d}q = \int \mathrm{d}P\mathrm{d}Q\,.$

Canonical Transformation for light-cone Y-M

- From

$$L_{YM}^{-+} = \mathrm{tr}(\check{\partial} A \hat{\partial} \bar{A} - \partial A \bar{\partial} \bar{A})$$

  Canonical Fields (up to constant coeff.):

$$(A, \hat{\partial} \bar{A})$$

- Canonical Transformation: $(A, \hat{\partial} \bar{A}) \rightarrow (B, \hat{\partial} \bar{B})$, *s.t.*

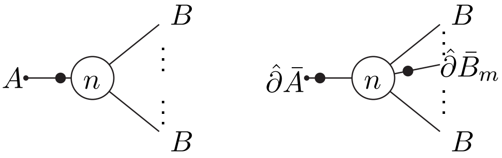
$$L_{YM}^{-+}[A, \bar{A}] + L_{YM}^{++-}[A, \bar{A}] = L_{YM}^{-+}[B, \bar{B}]$$

Canonical Transformation for light-cone Y-M

- MHV and charge conservation (preserve − helicity fields) requirements:

$$A_0 \sim \sum_n \int \Upsilon_{01\cdots n} B_1 \cdots B_n,$$

$$\hat{\partial}\bar{A}_0 \sim \sum_{m,n} \int \Xi^m_{01\cdots n} B_1 \cdots \hat{\partial}\bar{B}_m \cdots B_n$$



- If we can solve these coefficients and then we can substitute these into the LCYM and we obtain MHV Lagrangian

$$L^{-+}[B,\bar{B}]+L^{--+}[B,\bar{B}]+L^{--++}[B,\bar{B}]+L^{--+++}[B,\bar{B}]\cdots$$

Canonical Transformation

Massless MHV YM Lagrangian: Canonical Transformation

- Canonical Trans. (Cont.):$\frac{\partial p}{\partial P} = \frac{\partial Q}{\partial q}$,
  $\hat{\partial}\bar{A}$ depends linear on $\hat{\partial}\bar{B}$

$$\begin{aligned}\hat{\partial}\bar{A}^a(x^0,\vec{y}) &= \int d^3\vec{x}\frac{\delta\hat{\partial}\bar{A}^a(\vec{y})}{\delta\hat{\partial}\bar{B}^b(\vec{x})}\hat{\partial}\bar{B}^b \\ &= \int d^3\vec{x}\frac{\delta B^b(\vec{x})}{\delta A^a(\vec{y})}\hat{\partial}\bar{B}^b\end{aligned}$$

- B depends on x^0 only through A in expansion:

$$\check{\partial}B^a(x^0,\vec{x}) = \int d^3\vec{y}\,\check{\partial}A^b(x^0,\vec{y})\frac{\delta B^a(x^0,\vec{x})}{\delta A^b(x^0,\vec{y})}$$

  from above two equation we obtain:

$$\int d^3\vec{x}\,\check{\partial}B^a(x^0,\vec{x})\hat{\partial}\bar{B}^a(x^0,\vec{x}) = \int d^3\vec{x}\,\check{\partial}A^a(x^0,\vec{x})\hat{\partial}\bar{A}^a(x^0,\vec{x})$$

Integral equations for canonical transformation

From equation:

$$L_{YM}^{-+}[A,\bar{A}] + L_{YM}^{++-}[A,\bar{A}] = L_{YM}^{-+}[B,\bar{B}]$$

$$2\,\mathrm{tr}\int_{\Sigma} d^3\mathbf{x}\ \hat{\partial}\bar{A}\,(\check{\partial} - \hat{\partial}^{-1}\partial\bar{\partial})\,A - i\sqrt{2}g\,\mathrm{tr}\int_{\Sigma} d^3\mathbf{x}\ \hat{\partial}\bar{A}\,[\bar{\partial}\hat{\partial}^{-1}A,\ A] = 2\,\mathrm{tr}\int_{\Sigma} d^3\mathbf{x}\ \hat{\partial}\bar{B}\,(\check{\partial} - \hat{\partial}^{-1}\partial\bar{\partial})\,B$$

Extract $\hat{\partial}\bar{A}$ and use $\hat{\partial}\bar{A} = \int \frac{\delta B}{\delta A}\hat{\partial}\bar{B}$, we obtain an integral equation

$$\omega_{\mathbf{y}}A - i\frac{g}{\sqrt{2}}[A,\zeta_{\mathbf{y}}A](\mathbf{y}) \quad = \quad \int d\,\mathbf{x}\omega_{\mathbf{x}}B^b(\mathbf{x})\frac{\delta A(\mathbf{y})}{\delta B^b(\mathbf{x})}$$

where $\omega_{\mathbf{x}} = \frac{\partial\bar{\partial}}{\hat{\partial}}$, $\zeta_{\mathbf{x}} = \frac{\bar{\partial}}{\hat{\partial}}$

Solving the equation for A

Change to momentum space:

$$\omega_1 A_1 - i\int_{23} \zeta_3 [A_2, A_3]\,(2\pi)^3 \delta^3(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \quad = \quad \int_{\mathbf{p}} \omega_{\mathbf{p}} B^b_{\mathbf{p}} \frac{\delta A_1}{\delta B^b_{\mathbf{p}}},$$

where $\omega_i = p_i \bar{p}_i / \hat{p}_i$ and $\zeta_{i,j} = (\bar{p}_i + \cdots + \bar{p}_j)/(\hat{p}_i + \cdots + \hat{p}_j)$.

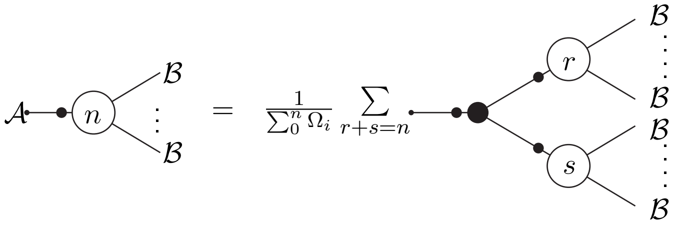
- Expand A in B

$$A_1 = \sum_{n=2}^{\infty} \int_{2\cdots n} \Upsilon_{12\cdots n}\, B_{\bar{2}} \cdots B_{\bar{n}},$$

- Substitute this into the integral equation, we have a recursion relation for Υ:

$$\begin{aligned} \Upsilon(1\cdots n) \quad &= \quad \frac{i}{\omega_1 + \cdots + \omega_n} \sum_{j=2}^{n-1} (\zeta_{j+1,n} - \zeta_{2,j}) \\ &\quad \times \Upsilon(-, 2, \cdots, j)\Upsilon(-, j+1, \cdots, n), \end{aligned}$$

Solving the equation for A



This can be solved:

$$\Upsilon(1 \cdots n) \quad = \quad -g^{n-2} \frac{\hat{1}}{\sqrt{\hat{2}\,\hat{n}} \langle 2\,3 \rangle \langle 3\,4 \rangle \cdots \langle n-1, n \rangle}, \quad n \geq 3,$$

Solving the equation for $\bar{A}$

We expand $\hat{\partial}\bar{A}$

$$\hat{1}\bar{A}_{\bar{1}} = \sum_{m=2}^{\infty} \sum_{s=2}^{m} \int_{2\cdots m} \hat{s}\; \Xi^{s-1}_{\bar{1}2\cdots m}\, B_{\bar{2}} \cdots \bar{B}_{\bar{s}} \cdots B_{\bar{m}},$$

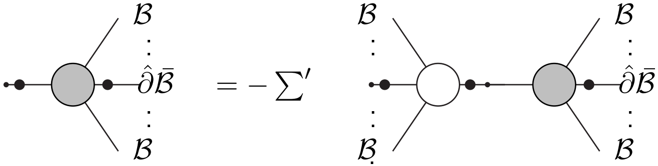
From

$$\int d^3\vec{x}\, \check{\partial} B^a(x^0,\vec{x}) \hat{\partial}\bar{B}^a(x^0,\vec{x}) = \int d^3\vec{x}\, \check{\partial} A^a(x^0,\vec{x}) \hat{\partial}\bar{A}^a(x^0,\vec{x})$$

We obtain a recursion relation for Ξ

$$\Xi^l(1\cdots n) = -\sum_{r=2}^{n+1-l} \sum_{m=\max(r,3)}^{r+l-1} \Upsilon(-, n-r+3, \cdots, m-r+1) \times \Xi^{l+r-m}(-, m-r+2, \cdots, n-r+2),$$

Solving the equation for $\bar{A}$



At least two left legs on the white blob. Solution:

$$\Xi^{s-1}(1\cdots n) \quad = \quad -\frac{\hat{s}}{\hat{1}}\Upsilon(1\cdots n), \qquad (s = 2, \cdots, n \text{ and } n \geq 2).$$

CSW rules from MHV Lagrangian

- We have obtained the canonical transformation for A and $\hat{\partial}\bar{A}$
- Substituting these into the LCYM and collecting the similar helicity terms, we would obtain MHV lagrangian:

  $$L^{-+}[B,\bar{B}]+L^{--+}[B,\bar{B}]+L^{--++}[B,\bar{B}]+L^{--+++}[B,\bar{B}]\cdots$$

  the vertices for B fields are the off-shell continuation of the MHV amplitudes. This is proved in paper arXiv:0908.0020, Fu.
- The propagators of B are only the scalar propagators.

  $$L^{-+}_{YM} = \mathrm{tr}(\check{\partial}A\hat{\partial}\bar{A} - \partial A\bar{\partial}\bar{A})$$

  $$\langle A(p)\bar{A}(-p)\rangle = \frac{i}{p^2}$$

  This is CSW rule.

Extend to QCD with fundamental massless fermions

- Massless QCD lagrangian:

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\not{D}\psi$$

- Fermions:

$$\psi = (\alpha^+, \beta^+, \beta^-, \alpha^-)^{\mathrm{T}}, \quad \bar{\psi} = (\bar{\beta}^+, \bar{\alpha}^+, \bar{\alpha}^-, \bar{\beta}^-),$$

The $\pm$ denote the outgoing chirality, ψ ~anti-fermions, $\bar{\psi}$ ~fermions. For massless fermions, helicity=chirality.

Light-cone QCD

- LCQCD Lagrangian: Integrate out nondynamical fields: $\check{A}$ and $\beta^{\pm}$. The fields left are A $\bar{A}$ and $\alpha^{\pm}$ $\bar{\alpha}^{\pm}$. $\bar{\alpha}^{+}\alpha^{-}$, $\bar{\alpha}^{-}\alpha^{+}$ always appear together.

$$
\begin{aligned}
L_{LCQCD} &= L_{YM} + L_F \\
L_{YM} &= L_{YM}^{-+} + L_{YM}^{++-} + L_{YM}^{--+} + L_{YM}^{--++} \\
L_F &= L_F^{-+} + L_F^{++-} + L_F^{--+} + L_F^{-+-+}
\end{aligned}
$$

$$
\begin{aligned}
L_F^{+-} &= i(\bar{\alpha}^{+}\check{\partial}\alpha^{-} + \bar{\alpha}^{-}\check{\partial}\alpha^{+} - \bar{\alpha}^{+}\bar{\partial}\hat{\partial}^{-1}\partial\alpha^{-} - \bar{\alpha}^{-}\partial\hat{\partial}^{-1}\bar{\partial}\alpha^{+}) \\
L_F^{++-} &= i(\bar{\alpha}^{+}(\hat{\partial}^{-1}\bar{\partial}A)\alpha^{-} + \bar{\alpha}^{-}(\hat{\partial}^{-1}\bar{\partial}A)\alpha^{+} \\
&\quad -\bar{\alpha}^{+}(\bar{\partial}\hat{\partial}^{-1}(A\alpha^{-})) - \bar{\alpha}^{-}A\hat{\partial}^{-1}\bar{\partial}\alpha^{+}))
\end{aligned}
$$

Massless MHV QCD Lagrangian: Canonical Trans.

- Canonical Fields (up to constant coeff.):

$$(A, \hat{\partial}\bar{A}), (\alpha^{\pm}, \bar{\alpha}^{\mp})$$

- Canonical Transformation: $(A, \hat{\partial}\bar{A}) \to (B, \hat{\partial}\bar{B})$, $(\alpha^{\pm}, \bar{\alpha}^{\mp}) \to (\xi^{\pm}, \bar{\xi}^{\mp})$ *s.t.*

$$L_{YM}^{-+}[A, \bar{A}] + L_{YM}^{++-}[A, \bar{A}] + L_{F}^{-+}[\alpha, \bar{\alpha}] + L_{F}^{++-}[A, \alpha, \bar{\alpha}] = L_{YM}^{-+}[B, \bar{B}] + L_{F}^{-+}[\xi, \bar{\xi}]$$

- MHV and charge conservation requirements:

$$\begin{aligned} A &\sim B \cdots B, \\ \bar{A} &\sim B \cdots \bar{B} \cdots B \text{, or } B \cdots \bar{\xi}^{\pm}\xi^{\mp} \cdots B \\ \alpha^{\pm} &\sim B \cdots B\xi^{\pm}, \\ \bar{\alpha}^{\pm} &\sim \bar{\xi}^{\pm} B \cdots B \end{aligned}$$

Massless MHV QCD Lagrangian: Canonical Transformation

- Canonical Trans. (Cont.): $\frac{\partial p}{\partial P} = \frac{\partial Q}{\partial q}$

$$\begin{aligned}
\bar{\alpha}^{\pm}(x^0, \vec{x}) &= \int d^3\vec{y}\, \bar{\xi}^{\pm}(x^0, \vec{y}) R^{\pm}(\vec{y}, \vec{x}) \\
\xi^{\pm}(x^0, \vec{x}) &= \int d^3\vec{y}\, R^{\mp}(\vec{x}, \vec{y}) \alpha^{\pm}(x^0, \vec{y})
\end{aligned}$$

$$\begin{aligned}
\hat{\partial}\bar{A}^a(x^0, \vec{y}) &= \int d^3\vec{x} \frac{\delta B^b(\vec{x})}{\delta A^a(\vec{y})} \hat{\partial}\bar{B}^b \\
&\quad + i\frac{1}{\sqrt{2}} \int d^3\vec{x}\, d^3\vec{x}' \Bigg(\bar{\xi}^{+}(x^0, \vec{x}) \frac{\delta R^{+}(\vec{x}, \vec{x}')}{\delta A^a(\vec{y})} \alpha^{-}(x^0, \vec{x}') \\
&\qquad + \bar{\xi}^{-}(x^0, \vec{x}) \frac{\delta R^{-}(\vec{x}, \vec{x}')}{\delta A^a(\vec{y})} \alpha^{+}(x^0, \vec{x}') \Bigg)
\end{aligned}$$

where $R^{\mp} = \delta\xi^{\pm}/\delta\alpha^{\pm}$ is only the functional of A.

Massless MHV QCD Lagrangian: Integral equations

- Integral equations from canonical trans.:

$$\int_\Sigma d^3\vec{x}\, \omega_{\vec{x}}[B^b(\vec{x})] \frac{\delta A^a(\vec{y})}{\delta B^b(\vec{x})} = \left[D, \zeta_{\vec{y}}[A]\right]^a(\vec{y})$$

$$\int_\Sigma d^3\vec{z}\, \omega_{\vec{z}}[B^b(\vec{z})]\, \frac{\delta R^-(\vec{x},\vec{y})}{\delta B^b(\vec{z})} = (\omega_{\vec{x}} + \omega_{\vec{y}})\, R^-(\vec{x},\vec{y})$$
$$-i\frac{g}{\sqrt{2}}\Big(R^-(\vec{x},\vec{y})\zeta_{\vec{y}}[A(\vec{y})] - \zeta_{\vec{y}}[R^-(\vec{x},\vec{y})A(\vec{y})]\Big)$$

$$\int_\Sigma d^3\vec{z}\, \omega_{\vec{z}}[B^b(\vec{z})]\, \frac{\delta R^+(\vec{x},\vec{y})}{\delta B^b(\vec{z})} = (\omega_{\vec{x}} + \omega_{\vec{y}})\, R^+(\vec{x},\vec{y})$$
$$-i\frac{g}{\sqrt{2}}\Big(R^+(\vec{x},\vec{y})\zeta_{\vec{y}}[A(\vec{y})] - \zeta_{\vec{y}}[R^+(\vec{x},\vec{y})]A(\vec{y})\Big),$$

The first for A is the same as in the pure Yang-Mills case.

Massless MHV QCD Lagrangian: Solutions

- Solutions for A is the same as in Pure YM case:

$$A_1 = \sum_{n=2}^{\infty} \int_{2\cdots n} \Upsilon_{12\cdots n}\, B_{\bar{2}} \cdots B_{\bar{n}},$$

-

$$\bar{\alpha}_1^{\pm} \;=\; \int_q \bar{\xi}_{\bar{q}}^{\pm} R^{\pm}(q,1) = \sum_{n=2}^{\infty} \int_{q,3,\cdots,n} \Upsilon^{\pm}_{q,1,3,4,\cdots,n}\, \bar{\xi}_{\bar{q}}^{\pm} B_{\bar{3}} \cdots B_{\bar{n}},$$

Solving the recursion relations from the integral equations:

$$\Upsilon(1\cdots n) \;=\; \left(-i\frac{g}{\sqrt{2}}\right)^{n-2} i^n \frac{\hat{1}\hat{3}\hat{4}\cdots\widehat{n-1}}{(2\,3)(3\,4)\cdots(n-1,n)}, \quad n \geq 3,$$

$$\Upsilon^{+}(1\cdots n) \;=\; \Upsilon(2,1,3\cdots n), \quad \Upsilon^{-}(1\cdots n) = -\frac{\hat{1}}{\hat{2}}\Upsilon^{+}(1\cdots n),$$

Expansions for $\alpha^{\pm}$

- The expansion of $\alpha^{\pm}$

$$\alpha_1^{\pm} = \int_q Q^{\pm}(1,q)\xi_{\bar{q}}^{\pm} = \sum_{n=2}^{\infty} \int_{q,3\cdots n} \Xi^{\pm}{}_{1,q,3\cdots n}\, B_{\bar{3}} \cdots B_{\bar{n}} \xi_{\bar{q}}^{\pm},$$

Inverse the expansion: using $\xi(\vec{x}) \sim \int_{\vec{y}} R_{\vec{x},\vec{y}} \alpha_{\vec{y}} \sim \int_{\vec{y}\vec{z}} R_{\vec{x}\vec{y}} Q_{\vec{y},\vec{z}} \xi_{\vec{z}} \Rightarrow \int_{\vec{y}} R_{\vec{x}\vec{y}} Q_{\vec{y}\vec{z}} = \delta_{\vec{x}\vec{z}}$, we can find a recursion relation for $\Xi^{\pm}$ and solve it:

$$\Xi^{+}(1\cdots n) = \left(-i\frac{g}{\sqrt{2}}\right)^{n-2} (-i^{n}) \frac{\hat{1}\hat{4}\hat{5}\cdots\widehat{n}}{(2\,n)(3\,4)\cdots(n-1,n)}, \qquad n$$

$$\Xi^{-}(1\cdots n) = -\frac{\hat{2}}{\hat{1}}\Xi^{+}(1\cdots n),$$

Solutions of the canonical transformations: $\bar{A}$

We have relation:

$$2\mathrm{tr}\left[\int_1 \check{\partial} A_1 \hat{1} \bar{A}_{\bar{1}}\right] + \frac{g^2}{8}\int_1 \bar{\alpha}_1^- \check{\partial}\alpha_{\bar{1}}^+ + \frac{g^2}{8}\int_1 \bar{\alpha}_1^+ \check{\partial}\alpha_{\bar{1}}^- \quad = \quad 2\mathrm{tr}\left[\int_1 \check{\partial} B_1 \, \hat{1} \bar{B}_{\bar{1}}\right] + \frac{g^2}{8}\int_1 \bar{\xi}_1^- \check{\partial}\xi_{\bar{1}}^+ + \frac{g^2}{8}\int_1 \bar{\xi}_1^+ \check{\partial}\xi_{\bar{1}}^-$$

- We separate $\hat{1}\bar{A}_{\bar{1}} = \hat{1}\bar{A}_{\bar{1}}^{old} + \hat{1}\bar{A}_{\bar{1}}^{F}$, $A_{\bar{1}}^{old}$ contains the pure Y-M part and A^F contains the fermionic fields. Since the old $\bar{A}^{old}$ already satisfies

$$2\mathrm{tr}\left[\int_1 \check{\partial} A_1 \hat{1} \bar{A}_{\bar{1}}^{old}\right] = 2\mathrm{tr}\left[\int_1 \check{\partial} B_1 \, \hat{1} \bar{B}_{\bar{1}}\right] ,$$

We have

$$2\mathrm{tr}\left[\int_1 \check{\partial} A_1 \hat{1} \bar{A}_{\bar{1}}^{F}\right] + \frac{g^2}{8}\int_1 \bar{\alpha}_1^- \check{\partial}\alpha_{\bar{1}}^+ + \frac{g^2}{8}\int_1 \bar{\alpha}_1^+ \check{\partial}\alpha_{\bar{1}}^- \quad = \quad \frac{g^2}{8}\int_1 \bar{\xi}_1^- \check{\partial}\xi_{\bar{1}}^+ + \frac{g^2}{8}\int_1 \bar{\xi}_1^+ \check{\partial}\xi_{\bar{1}}^- ,$$

Solutions of the canonical transformations: $\bar{A}$

1) $\bar{A}^{old}$ has the same solution as in the pure Y-M case.
2) $\bar{A}_F$:

$$
\begin{aligned}
\hat{1}\bar{A}^F_{\bar{1}} \quad &= \quad \frac{g^2}{2\sqrt{2}} \sum_{n=1}^{\infty} \sum_{d=1}^{n} \int_{2\cdots n,q,p} \Big\{ K^{+,d}_{\bar{1},2,\cdots,d,q,p,d+1,\cdots,n} \\
\times \Bigg(&- (B_{\bar{2}}, \cdots, B_{\bar{d}} \xi^{+}_{\bar{q}} \bar{\xi}^{-}_{\bar{p}} B_{\overline{d+1}} \cdots B_{\bar{n}})_{\alpha\beta} - \frac{\delta_{\alpha\beta}}{N_c} (\bar{\xi}^{-}_{\bar{p}} B_{\overline{d+1}} \cdots B_{\bar{n}} B_{\bar{2}}, \cdots, B_{\bar{d}} \xi^{+}_{\bar{q}}) \Bigg) \\
&\qquad + (K^{+} \to K^{-}, \xi^{+} \leftrightarrow \xi^{-}) \Big\},
\end{aligned}
$$

$$
\begin{aligned}
K^{+,d}(1\cdots n) \quad &= \quad \left(-i \frac{g}{\sqrt{2}} \right)^{n-4} (-i^n) \frac{\widehat{d+2}\,\hat{3}\hat{4}\cdots\widehat{n-1}}{(2\,3)(3\,4)\cdots(n{-}1,n)}, \\
K^{-,d}(1\cdots n) \quad &= \quad -\frac{\widehat{d}+1}{\widehat{d}+2} K^{+,d}(1\cdots n), \\
&\qquad (d = 1, \cdots, n-2 \text{ and } n \geq 3).
\end{aligned}
$$

MHV Lagrangian for QCD

$$
\begin{aligned}
L_F^{MHV} \quad = \quad & \text{kinetic term} \\
& + \sum_{n=3}^{\infty} \sum_{s=2}^{n-1} \int_{1\cdots n} \Big[V_F^{s,+-}(1\cdots n)\bar{\xi}_1^{+} \mathcal{B}_2 \cdots \bar{\mathcal{B}}_s \cdots \mathcal{B}_{n-1} \xi_n^{-} \\
& + V_F^{s,-+}(1\cdots n)\bar{\xi}_1^{-} \mathcal{B}_2 \cdots \bar{\mathcal{B}}_s \cdots \mathcal{B}_{n-1} \xi_n^{+} \Big] \\
+ \sum_{n=4}^{\infty} \sum_{s=2}^{n-2} \int_{1\cdots n} & \Big[\frac{1}{2} V_F^{s,+-+-}(1\cdots n)\bar{\xi}_1^{+} \mathcal{B}_2 \cdots \mathcal{B}_{s-1} \xi_s^{-} \bar{\xi}_{s+1}^{+} \mathcal{B}_{s+2} \cdots \mathcal{B}_{n-1} \xi_n^{-} \\
& + \frac{1}{2} V_F^{s,-+-+}(1\cdots n)\bar{\xi}_1^{-} \mathcal{B}_2 \cdots \mathcal{B}_{s-1} \xi_s^{+} \bar{\xi}_{s+1}^{-} \mathcal{B}_{s+2} \cdots \mathcal{B}_{n-1} \xi_n^{+} \\
& + V_F^{s,++--}(1\cdots n)\bar{\xi}_1^{+} \mathcal{B}_2 \cdots \mathcal{B}_{s-1} \xi_s^{+} \bar{\xi}_{s+1}^{-} \mathcal{B}_{s+2} \cdots \mathcal{B}_{n-1} \xi_n^{-} \\
& + V_F^{s,+--+}(1\cdots n)\bar{\xi}_1^{+} \mathcal{B}_2 \cdots \mathcal{B}_{s-1} \xi_s^{-} \bar{\xi}_{s+1}^{-} \mathcal{B}_{s+2} \cdots \mathcal{B}_{n-1} \xi_n^{+} \Big]
\end{aligned}
$$

External polarization factor for fermions

LSZ:

- Source terms: $\kappa^{\pm}$ source for $\beta^{\pm}$

$$L_{src} = \bar{\kappa}_1^+ \hat{\partial}^{-1} \bar{D} \alpha^+ - \bar{\kappa}_1^- \hat{\partial}^{-1} D \alpha^- + \hat{\partial}^{-1} D \bar{\alpha}^- \kappa_1^- - \hat{\partial}^{-1} \bar{D} \bar{\alpha}^+ \kappa_1^+$$

- Decouple of the nondynamical fermion component β off-shell

$$\bar{u}^+(p)(-i\not{p})\left\langle \cdots \begin{pmatrix} \alpha^+ \\ \beta^+ \\ \beta^- \\ \alpha^- \end{pmatrix} \cdots \right\rangle$$

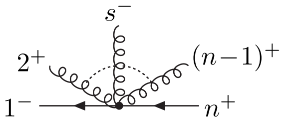
$$\rightarrow \quad -\frac{i}{2^{1/4}\sqrt{\hat{p}}}(0,0,0,p^2)\left\langle \cdots \begin{pmatrix} \alpha^+ \\ \beta^+ \\ \beta^- \\ \alpha^- \end{pmatrix} \cdots \right\rangle$$

$$\rightarrow \quad 2^{1/4}\sqrt{\hat{p}}\,\mathcal{V}(\cdots \bar{\alpha}^+ \cdots)$$

Propagator $\langle \alpha \bar{\alpha} \rangle = \frac{i\sqrt{2}\hat{p}}{p^2}$

CSW rules from MHV Lagrangian for QCD

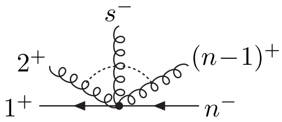
- The MHV vertices can be directly read off from the MHV Lagrangian. Absorbing the external polariztion factor to the V we get the MHV amplitudes. One pair of fermions:



$$= A(1_{\mathrm{q}}^-, 2^+, \ldots, s^-, \ldots, (n-1)^+, n_{\bar{\mathrm{q}}}^+)$$

$$= ig^{n-2} \frac{\langle 1\, s\rangle^3 \langle s\, n\rangle}{\langle 1\, 2\rangle\langle 2\, 3\rangle \cdots \langle n-1, n\rangle\langle n\, 1\rangle},$$

and

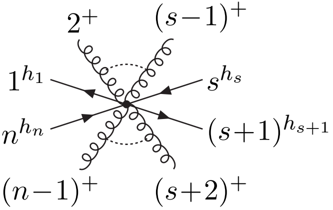


$$= A(1_{\mathrm{q}}^+, 2^+, \ldots, s^-, \ldots, (n-1)^+, n_{\bar{\mathrm{q}}}^-)$$

$$= ig^{n-2} \frac{\langle 1\, s\rangle \langle n\, s\rangle^3}{\langle 1\, 2\rangle\langle 2\, 3\rangle \cdots \langle n-1, n\rangle\langle n\, 1\rangle}.$$

CSW rules from MHV Lagrangian for QCD

Vertices with two pair of fermions:



$$= A(1_{\mathrm{q}}^{h_1} 2^+ \cdots s_{\bar{\mathrm{q}}}^{h_s} (s+1)_{\mathrm{q}}^{h_{s+1}} \cdots n_{\bar{\mathrm{q}}}^{h_n}),$$

$$A(1_{\mathrm{q}}^+ 2^+ \cdots s_{\bar{\mathrm{q}}}^- s+1_{\mathrm{q}}^+ \cdots n_{\bar{\mathrm{q}}}^-) = i g^{n-2} \frac{\langle s\, n\rangle^2}{\langle 1\, 2\rangle \langle 2\, 3\rangle \cdots \langle n\, 1\rangle} \times \left(\langle 1\, s\rangle \langle s+1, n\rangle + \frac{1}{N_c} \langle s, s+1\rangle \langle n\, 1\rangle \right)$$

$\cdots$

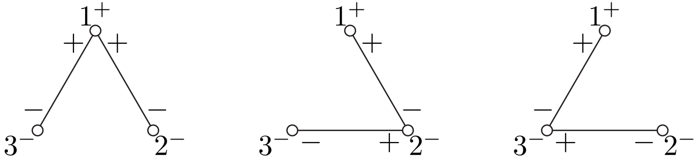
CSW rules from MHV Lagrangian for QCD

- The propagators for fermions is just like bosonic scalar propagators.

$$\langle \alpha^{-}(-p)\bar{\alpha}^{+}(p)\rangle = i\sqrt{2}\frac{\hat{p}}{p^2} \rightarrow \frac{i}{p^2}$$

Tree level 3-point $(++-)$ amplitude from completion vertices

1) Normal MHV vertices (with two "-" helicity) can not produce $(-++)$ amplitude in $(--++)$ signature or complex momentum.
2) The original amplitude is defined by A fields. But the MHV Lagrangian is in B fields. They are different by canonical transformation (translation kernel or completion vertex).
3) LSZ:Amplitude $A(-++) \sim \lim_{p^2\to 0} E_1^- E_2^+ E_3^+ p_1^2 p_2^2 p_3^2 \langle A_1 \bar{A}_2 \bar{A}_3 \rangle$



Tree level 3-point $(++-)$ amplitude from completion vertices

$$
\begin{aligned}
&A(1^-, 2^+, 3^+) \\
&= -\frac{g}{\sqrt{2}} p_1^2 p_2^2 p_3^2 \left\{ \frac{1}{p_2^2} \frac{1}{p_3^2} \Upsilon(123) - \frac{1}{p_3^2} \frac{1}{p_1^2} \frac{\hat{1}}{\hat{2}} \Xi^2(231) - \frac{1}{p_1^2} \frac{1}{p_2^2} \frac{\hat{1}}{\hat{3}} \Xi^1(312) \right\} \\
&= \frac{ig}{\sqrt{2}} \frac{\hat{1}^2}{(2\,3)} \left(\frac{p_1^2}{\hat{1}} + \frac{p_2^2}{\hat{2}} + \frac{p_3^2}{\hat{3}} \right) \\
&= ig\sqrt{2} \frac{\hat{1}}{\hat{2}\hat{3}} \{2\,3\} = ig \frac{[2\,3]^3}{[3\,1][1\,2]},
\end{aligned}
$$

The translation kernel has on-shell sigularity by itself.

Summary and Discussion

- By explicitly solving the canonical transformation, we obtain MHV lagrangian and prove the CSW rules directly from QFT.
- It is straightforward to extend this calculation to more than one quark fields.
- We show that the 3-point $(- + +)$ comes from the completion vertices.

THANK YOU!