# Blowup Equations for Refined Topological Strings

Xin Wang

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base on M. x. Huang, K. Sun and XW, arXiv:1711.XXXXX.

- Donaldson theory of 4-d manifold $X$. $F = *F$
- Topological quantum field theory in 4-d, partition function $Z_X$.
- Blowup equation: $Z_{\tilde{X}}$ on blowup geometry $\tilde{X} = X \# \mathbb{P}^2$.
- Supersymmetric Yang-Mills theory, special case of Donaldson theory on $R^4$

- Omega background Nekrasov,03' for SYM.

$$(z_1, z_2) \sim (e^{i\epsilon_1} z_1, e^{i\epsilon_2} z_2)$$

- Refined partition. For $SU(N)$ gauge group Nakajima & Yoshioka,03'

$$\begin{aligned}\widehat{Z}_{m,k,d}(\epsilon_1, \epsilon_2, \vec{a}; \mathfrak{q}) \sim \\ \sum_{\{\vec{k}\}=-k/N} Z_m \left(\epsilon_1, \epsilon_2 - \epsilon_1, \vec{a} + \epsilon_1 \vec{k}; \exp\left(\epsilon_1 (d + m(-\tfrac{1}{2} + \tfrac{k}{N}) - \tfrac{N}{2})\right) \mathfrak{q}\right) \\ \times Z_m \left(\epsilon_1 - \epsilon_2, \epsilon_2, \vec{a} + \epsilon_2 \vec{k}; \exp\left(\epsilon_2 (d + m(-\tfrac{1}{2} + \tfrac{k}{N}) - \tfrac{N}{2})\right) \mathfrak{q}\right) \ .\end{aligned} \tag{1}$$

- This turn out to be 0 or $Z_m(\epsilon_1, \epsilon_2, \vec{a}; \mathfrak{q})$

- Topological string $\sim$ M theory

$$Z^{\text{top.string}}(X) = Z^{\text{M theory}}_{BPS}(S^1 \times X \times R^4)$$

- Compactified on $X$ with $A_N$ type singularities fibered on $\mathbb{P}^1$, gives 4-d gauge theory.

BPS contents $\sim$ gravitational corrections of 4-d gauge theory

- Lift to 5d with extra $S^1$

$$Z^{\text{topological string}} = Z^{\text{5-d gauge}}$$

- Blowup equation for refined topological string should exist and even have more things.

$$\sum_{n\in\mathbb{Z}^g}\frac{(-1)^{|\mathbf{n}|}Z(\epsilon_1-\epsilon_2,\epsilon_2,\mathbf{t}+i\mathbf{R}\epsilon_2)Z(\epsilon_1,\epsilon_2-\epsilon_1,\mathbf{t}+i\mathbf{R}\epsilon_1)}{Z(\epsilon_1,\epsilon_2,\mathbf{t})}=\Lambda(\mathbf{t}_m,\epsilon_1,\epsilon_2),$$

where $\mathbf{R}=\mathbf{C}\cdot\mathbf{n}+\mathbf{r}$.

- Modular property of top.string $\Rightarrow\Lambda$ is modular invariant $\Rightarrow$Solve $\mathbf{r}$ completely.

# Property of blowup equations and summary

- Solve partition function recursively. Better than holomorphic anomaly equation.
- Modular property lead to blowup equation on general points of moduli space.
- Existence for compact CY3?

- The holomorphic anomaly equation BCOV 92'
- $W(t_i, \bar{t}_{\bar{i}}) = e^{\sum_{g=2}^{\infty} g_s^{2g-2} \mathcal{F}_g(t_i, \bar{t}_{\bar{i}})}$ is the wave function of the quantization of $x_i = t_i, p_i$ space.
- BCOV as a quantization condition.
- Wave function is invariant under symplectic transformations of phase space parameters $(x_i, p_i) \to$ Modular invariant.

- $|Z\rangle$ is invariant, but $\langle t_i|Z\rangle$ indeed have changed.
- $F_g(t_i) = \lim_{\bar{t}_{\bar{i}}\to\infty} \mathcal{F}_g(t_i, \bar{t}_{\bar{i}})$
- Different polarization $Z(t_i) = e^{\sum_{g=0}^{\infty} g_s^{2g-2} F_g(t_i)}$
- For $g >= 2$, $\mathcal{F}_g(t_i, \bar{t}_{\bar{i}})$ is a weight 0 unholomorphic form, and $F_g(t_i)$ is a weight 0 holomorphic quasi-modular form. ABK 06'
- Also true after refinement.

A modular form of weight k for the modular group

$$\mathrm{SL}(2,\mathbf{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a,b,c,d \in \mathbf{Z},\ ad-bc=1 \right\}$$

is a complex-valued function f on the upper half-plane $H = z \in C, \mathrm{Im}(z) > 0$, satisfying the following three conditions:

- $f$ is a holomorphic function on $H$.
- For any $z \in H$ and any matrix in $SL(2,Z)$ as above, we have:

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

- $f$ is required to be holomorphic as $z \to i\infty$.

- $k$ is even
- Every modular form is product of Eisenstein series $E_4, E_6$.
- The second Eisenstein series $E_2$ transform like a modular form, but with extra terms appear.
- Quasi-modular forms are defined as product of $E_2, E_4, E_6$ and their sums.
- Division and root of (quasi-)modular forms may define a form with negative or rational weight, it is no more holomorphic, but with not bad behavior. We may abuse the name modular form include this kind of forms.

$$\sum_{n\in\mathbb{Z}^g} \exp\left(\frac{1}{2}R^2\partial_t^2 F^{(0,0)}(t) + F^{(0,1)}(t) - F^{(1,0)}(t) + in\pi\right) = 1, \quad (2)$$

For local Calabi-Yau

$$e^{F^{(1,0)}(t)-F^{(0,1)}(t)}$$

is always weight $1/2$.

Every modular form of weight $1/2$ is a linear combination of unary theta series.

weight match for all identities $\Rightarrow$ weight 0

# An example: Local $\mathbb{P}^2$

- Simplest local Calabi-Yau described by toric diagram with genus one mirror curve.
- Non-lagrangian gauge theory.

modular group of local $\mathbb{P}^2$ is $\Gamma(3) \in SL(2,\mathbb{Z})$. It has generators

$$a := \theta^3 \begin{bmatrix} \frac{1}{6} \\ \frac{1}{6} \end{bmatrix}, \quad b := \theta^3 \begin{bmatrix} \frac{1}{6} \\ \frac{1}{2} \end{bmatrix}, \quad c := \theta^3 \begin{bmatrix} \frac{1}{6} \\ \frac{5}{6} \end{bmatrix}, \quad d := \theta^3 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{6} \end{bmatrix},$$

with all weight $3/2$. We also introduce Dedekind $\eta$ function satisfy $\eta^{12} = \frac{i}{3^{3/2}} abcd$.

$$F^{(0,1)} = -\frac{1}{6}\log(d\eta^3), \quad F^{(1,0)} = \frac{1}{6}\log(\eta^3/d),$$

and

$$F^{(0,1)} - F^{(1,0)} = \log(\eta(\tau)).$$

Since $R$ for this model is $R = 3n + 1/2$. The the first equation (2) indicate

$$\sum_n (-1)^n e^{\frac{1}{2}(n+1/6)^2 3*2\pi i\tau} = \eta(\tau), \tag{3}$$

the Euler identity, or the Pentagonal number theorem.

$$\sum_n \frac{1}{2}(3n+1/2)(-1)^n e^{\frac{1}{2}(n+1/6)^2 3*2\pi i\tau} = \frac{b}{2i} + \frac{d}{2\sqrt{3}}.$$

$$\sum_n (n+1/2)(-1)^n e^{\frac{1}{2}(n+1/2)^2 *6\pi i\tau} = \frac{d}{3\sqrt{3}}.$$

- The blowup equation is quasi-modular of weight 0 for arbitrary choice of $\mathbf{r}$.
- For special choice of $\mathbf{r}$, if $\Lambda$ is a finite series $\Rightarrow$ weight 0 modular form.
- Product of quasi-modular objects become modular!
- Modular $\Rightarrow$ defined $\Lambda$ and fix $\mathbf{r}$ completely.

Finite series, weight $0 = \Lambda$ is a constant.

$$\Lambda = \Lambda(\mathbf{t}_m, \epsilon_1, \epsilon_2)$$

I will show how this condition is used to determine $\mathbf{r}$.

The free energy have the genus expansion

$$F(\mathbf{t},\epsilon_1,\epsilon_2)=\frac{1}{\epsilon_1\epsilon_2}(a_{ijk}t_it_jt_k)+c_it_i+\mathcal{O}(e^{t_i}).$$

$$\Lambda(\mathbf{t}_m,\epsilon_1,\epsilon_2,\mathbf{r})=\lambda(\epsilon_1,\epsilon_2,\mathbf{r})\sum_{\mathbf{n}\in\mathcal{I}}(-1)^{|\mathbf{n}|}e^{-\mathrm{i}(\epsilon_1+\epsilon_2)a_{ijk}R_iR_jR_k-\frac{1}{6}a_{ijk}R_iR_jt_k},$$

$$\mathcal{I}=\{\mathbf{n}\in\mathbb{Z}^g|\forall k,f^k(\mathbf{n})\equiv\sum_{i,j}a_{ijk}R_iR_j=f^k(\mathbf{0})\}. \tag{4}$$

$$\mathcal{I}^a=\{\mathbf{0},n_j=\delta_{aj}\},\ a=1,\cdots,g. \tag{5}$$

# Conclusion

- From modular properties, we fix the expression of the blowup equations.
- A compact expression for the blowup equation near conifold points.
- More test and generalizations?