Supersymmetric stabilisation of scale invariant quantum theories

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Scale anomalies in low energy effective theories

- Two simple examples

$$\hat{h}_D = \gamma^0 \gamma^j \hat{p}_j - \frac{\lambda}{r} , \qquad \hat{h}_S = \frac{\hat{p}^2}{2m} - \frac{\lambda}{r^2} . \tag{1}$$

- More general Hamiltonians have anisotropic scale invariance.[1]
- Boundary conditions break scale invariance and lead to bound states.
- $\lambda > \lambda_c$:

$$E = E_0 e^{-\frac{2\pi n}{\nu}} , \qquad n \in \mathbb{Z} . \tag{2}$$

- Residual discrete scale invariance (DSI): $r \mapsto e^{-\frac{2\pi}{\nu}} r$.

[1] Hornreich, Luban, and Shtrikman 1975; Grinstein 1981; Fradkin et al. 2004; Vishwanath, Balents, and Senthil 2004; Ardonne, Fendley, and Fradkin 2004.

Dynamical instabilities

- **Problem:** Consider

$$\hat{h}_S = -\left(d_r^2 + \frac{\lambda}{r^2}\right) , \qquad r \in [0, \infty) . \tag{3}$$

System is dynamically unstable:

$$E_n = E_0 e^{-\frac{2\pi n}{\nu}} , \quad E_0 < 0 , n \in \mathbb{Z} \text{ and } \nu \in \mathbb{R} . \tag{4}$$

- **Partial solution:**

$$\hat{h}_2 = \left(\hat{h}_S\right)^2 = \left(d_r^2 + \frac{\lambda}{r^2}\right)^2 , \qquad E_n = E_0^2 e^{-\frac{4\pi n}{\nu}} . \tag{5}$$

Note $\hat{h}_S$ is a conserved charge.

Everything you need to know if you switch off now...

1. Every scale covariant, spherically symmetric Hamiltonian without negative energy DSI can be embedded in a $\mathcal{N} = 2$ SUSY quantum mechanics.
2. SUSY systems have positive energy spectra therefore no dynamical instability.
3. Both components of the super-Hamiltonian can have zero modes.[2]
4. The landscape of scale covariant Hamiltonians can be decomposed into equivalence classes through the $r = 0$ power laws.

[2] Andrianov and Ioffe 2012.

Definition of $\hat{q}_N$

- Generic scale covariant, spherically symmetric Hamiltonian:

$$\hat{h}_N^{(\mathrm{b})} = \hat{p}^{2N} + \sum_{i=1}^{2N} \frac{\lambda_i}{r^i} d_r^{2N-i} \,, \qquad r \in [0, \infty) \,. \tag{6}$$

- Want to write $\hat{h}_N^{(\mathrm{b})} = \hat{q}_N^\dagger \hat{q}_N$.
- Ability to do this encoded in zero modes of $\hat{h}_N^{(\mathrm{b})}$:

$$\hat{h}_N^{(\mathrm{b})} \psi(r) = 0 \qquad \Rightarrow \qquad \psi(r) \sim r^\Delta \,. \tag{7}$$

- Constraints:
  1. Δ_i and Δ_i^* are both power laws.
  2. $2N - 1 - \Delta_i^*$ is a power law if Δ_i is.
  3. $\mathrm{Re}[\Delta_i] \neq N - 1/2$.

A second Hamiltonian (the superpartner)

- Define a new superpartner Hamiltonian: $\hat{h}_N^{(\mathrm{f})} = \hat{q}_N \hat{q}_N^\dagger$.
- Power laws of $\hat{h}_N^{(\mathrm{f})}$ are

$$\Delta_i - N \qquad \text{if } \hat{q}_N r^\Delta = 0 \, , \tag{8}$$

$$\Delta_i + N \qquad \text{if } \hat{q}_N r^\Delta \neq 0 \, . \tag{9}$$

- Intertwining relations:

$$\hat{q}_N \hat{h}_N^{(\mathrm{b})} = \hat{h}_N^{(\mathrm{f})} \hat{q}_N \, , \qquad \hat{h}_N^{(\mathrm{b})} \hat{q}_N^\dagger = \hat{q}_N^\dagger \hat{h}_N^{(\mathrm{f})} \, . \tag{10}$$

- Problems:
  1. Are these Hamiltonians self-adjoint?
  2. On what spaces of wavefunctions do $\hat{h}_N^{(\mathrm{b})}$ and $\hat{h}_N^{(\mathrm{f})}$ operate?

$\mathcal{N} = 2$ formalism

- Define formally self-adjoint supercharges:

$$\hat{Q}_N^+ = \begin{pmatrix} 0 & \hat{q}_N^\dagger \\ \hat{q}_N & 0 \end{pmatrix}, \quad \hat{Q}_N^- = \begin{pmatrix} 0 & -i\hat{q}_N^\dagger \\ i\hat{q}_N & 0 \end{pmatrix}.$$

- These give the super-Hamiltonian:

$$\hat{H}_N = \left(\hat{Q}_N^+\right)^2 = \left(\hat{Q}_N^-\right)^2 = \begin{pmatrix} \hat{h}_N^{(\mathrm{b})} & 0 \\ 0 & \hat{h}_N^{(\mathrm{f})} \end{pmatrix}. \quad (11)$$

- Now make sure $\hat{Q}_N^\pm$ is self-adjoint:

$$\int_{r=L}^{\infty} dr \left[\vec{\Phi}^\dagger(r)\hat{Q}_N^+\vec{\Psi}(r) - \left(\hat{Q}_N^+\vec{\Phi}(r)\right)^\dagger \vec{\Psi}(r)\right]. \quad (12)$$

$\mathcal{N} = 1$

- With a lot of redefinitions we write the boundary term as:

$$i\left[\left(\vec{\phi}^{+}_{Q^+_N}(L)\right)^\dagger\left(\vec{\psi}^{+}_{Q^+_N}(L)\right)-\left(\vec{\phi}^{-}_{Q^+_N}(L)\right)^\dagger\left(\vec{\psi}^{-}_{Q^+_N}(L)\right)\right]\,.$$

- Therefore generic self-adjoint boundary conditions are:

$$\vec{\psi}^{+}_{Q^+_N}(L) = U_N\vec{\psi}^{-}_{Q^+_N}(L)\,. \tag{13}$$

- This ensures $\mathcal{N} = 1$. $\mathcal{N} = 2$ requires $\hat{Q}^-_N$ is self-adjoint and

$$\hat{Q}^+_N\hat{Q}^-_N\,, \qquad \hat{Q}^-_N\hat{Q}^+_N\,, \tag{14}$$

are well-defined.[3]

[3] Falomir and Pisani 2005.

$\mathcal{N} = 2$

- Use a trick:

$$\hat{Q}_N^- = e^{\frac{i\pi}{4}\sigma_3}\hat{Q}_N^+ e^{-\frac{i\pi}{4}\sigma_3}\,, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,.$$

- This leads to a restriction on the unitary parameter:

$$U_N^2 = \mathbb{1}_N\,. \tag{15}$$

- Now we have $\mathcal{N} = 2$ and boundary conditions can be separated:

$$(\mathbb{1}_N - U_N)\,\vec{\psi}_{(\mathrm{b})}(L) = 0\,, \tag{16}$$

$$(\mathbb{1}_N + U_N)\,P_{(\mathrm{f})}\vec{\psi}_{(\mathrm{f})}(L) = 0\,. \tag{17}$$

- $L = 0$ essentially the same.

The inverse square potential

- Bosonic hamiltonian

$$\hat{h}_S^{(\mathrm{b})} = -\left(d_r^2 + \frac{\lambda_2}{r^2}\right) . \tag{18}$$

- The superpartner coupling:

$$\tilde{\lambda}_2 \quad = \quad (\lambda_2 - 1) + 2\left(\frac{1}{4} - \lambda_2\right)^{1/2} . \tag{19}$$

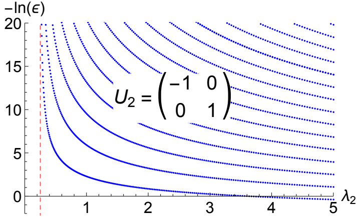
- $U_1 = \pm 1$ and

	$U_1 = +1$	$U_1 = -1$
$\lambda_2 < 1/4$	no zero modes	$\hat{h}_N^{(\mathrm{f})}$ has zero modes
$\lambda_2 < -3/4$	$\hat{h}_N^{(\mathrm{b})}$ has zero modes	$\hat{h}_N^{(\mathrm{f})}$ has zero modes

Square of the inverse square potential

- $\lambda_2 > 1/4$ then $\hat{h}_S$ has negative energy DSI and cannot be supersymmetrized.
- Can supersymmetrize the square:

$$\hat{H}_2 = \begin{pmatrix} \hat{h}_S^2 & 0 \\ 0 & \hat{h}_S^2 \end{pmatrix} .$$



$$\hat{h}_2^{(\mathrm{b})} = \hat{h}_2^{(\mathrm{f})} = \hat{h}_S^2 = \left(d_r^2 + \frac{\lambda_2}{r^2}\right)^2$$

- $U_2 = \pm\mathbb{1}_2$ no bound states.
- No zero modes.

The multiplicity of superpartner Hamiltonians

- Can permute roots of $\hat{h}_N^{(\mathrm{b})}$ without affecting operator:

$$\hat{h}_N^{(\mathrm{b})} = (-1)^N \hat{D}_{2N}^{\sigma} \ldots \hat{D}_1^{\sigma} \,, \qquad (20)$$

$$\hat{D}_k^{\sigma} = i\left(d_r - \frac{\Delta_{\sigma(k)} - k + 1}{r}\right) \,, \qquad (21)$$

where σ is a permutation.

- $\hat{q}_N^{\sigma}$ can be affected by permutation. Must maintain

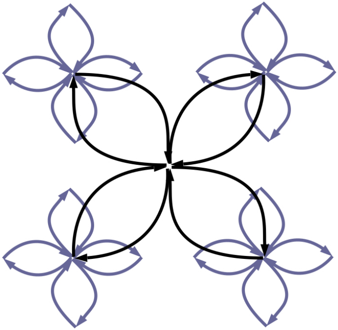
$$\Delta_{\sigma(2N-i+1)} = 2N - 1 - \Delta^*_{\sigma(i)} \,. \qquad (22)$$

- Many superpartners:

$$\hat{h}_N^{(\mathrm{f})} = \hat{q}_N^{\sigma} \left(\hat{q}_N^{\sigma}\right)^{\dagger} \,. \qquad (23)$$

Heads, petals and flowers

- Consider the following procedure:
  1. Head: $\hat{h}_N^{(\mathrm{b})} = \hat{q}_N^\dagger \hat{q}_N$.
  2. Petals: $\hat{h}_N^{(\mathrm{f})} = \hat{q}_N \hat{q}_N^\dagger$.
  3. Permute Δ_i in $\hat{q}_N$ and $\hat{q}_N^\dagger$.
  4. Redefine $\hat{h}_N^{(\mathrm{b})} := \hat{h}_N^{(\mathrm{f})}$. Repeat.
- Can arrange for N power laws $\Delta_i + m_i N$, $m_i \in \mathbb{Z}$ with other N given by $2N - 1 - \Delta_i^*$.



$N = 2$

Thanks for listening!

Further reading