Geometries for Possible Kinematics

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Outline:

I. Introduction

II. 22 Possible Kinematics

III. Geometries for All Possible Kinematics

IV. Relations and Classification

V. Concluding Remarks

Outline:

Universe vs Multiverse

- Universe:

  The totality of everything that exists.

- Multiverse:

  The set of multiple possible universes that together comprise everything that exists.

Tegmark's classification (astro-ph/0302131, in Science and Ultimate Reality)

- Level I: Beyond our cosmological horizon

  Chaotic inflation $\Rightarrow$ infinite ergodic universes.

  (Physical laws & constants are the same in each U.)

- Level II: Universes with different physical constants

- Level III: Many-worlds interpretation of QM (Everett)

- Level IV: Ultimate Ensemble (Tegmark hypothesis)

  All universes can be defined by mathematical structures.

  They may even have different physical laws from our observable universe.

A possible math. structure for the multiverse.

Foundation of Physics

Law of Inertia:

A body will preserve its velocity and direction unless it is acted upon by an external unbalanced force.

The body moves uniformly along a straight line, *i.e.*

$$x^i = x_0^i + v^i(t - t_0) \quad \text{with } v^i = \frac{dx^i}{dt} = \text{consts.} \tag{1}$$

Q: What is the most general transformation which preserves a uniform motion along a straight line?

Umov (Physik. Zeit. **11** 905 (1910)), Weyl (Mathemathische Analyse des Raumproblems (1923)), Fock (The Theory of Space-Time & Gravitation (1964)), Hua, (Starting with the Unit Circle (1982)), Guo, Huang, Tian, Xu, Zhou, (Class Quant Grav **24** 4009 (2007))

The most general transfs. to preserve Eq.(1) are the linear fractional transformations:

$$T: \quad l^{-1}x'^{\mu} = \frac{A^{\mu}{}_{\nu} l^{-1} x^{\nu} + a^{\mu}}{b^t l^{-1} x + d} \in PGL(5, R) \qquad (2)$$

where $A = \{A^{\mu}{}_{\nu}\}$ *a* 4×4 *matrix,* a, b 4×1 *matrixes,* $d \in R$ *and* $b^t = \eta b$ *with* $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

Generators of $IM(1,3) \approx PGL(5,R)$ group

Guo, Huang, Wu, Zhou (Sci China: PMA 53 (2010), 591)

There are 24 independent generators.

They are, in terms of Beltrami coordinates,

3 rotation generators

$$J_i = \frac{1}{2}\epsilon_i{}^{jk}(x_j\partial_k - x_k\partial_j)$$

2 indep. time-translation generators among 4 kinds

$$H := \partial_t, \qquad H' = -\nu^2 t x^\mu \partial_\mu, \qquad H^\pm = \partial_t \mp \nu^2 t x^\mu \partial_\mu; \ (\nu = \frac{c}{l})$$

2 indep. sets of space-transl. generators among 4 sets

$$P_i := \partial_i, \qquad P'_i = -l^{-2} x_i x^\mu \partial_\mu, \qquad P_i^\pm = \partial_i \mp l^{-2} x_i x^\mu \partial_\mu;$$

2 indep. sets of boost generators among 4 sets

$$K_i := t\partial_i - c^{-2}x_i\partial_t, \qquad K_i^g = t\partial_i, \qquad K_i^c = -c^{-2}x_i\partial_t,$$
$$N_i = t\partial_i + c^{-2}x_i\partial_t;$$

and

$$R_{ij} = R_{ji} = x_i\partial_j + x_j\partial_i, \qquad (i < j)$$
$$M_0 = t\partial_t, \quad M_1 = x^1\partial_1, \quad M_2 = x^2\partial_2, \quad M_3 = x^3\partial_3.$$

$IM(1,3)$ contains different kinematics & thus physics, e.g.

- $\{H^\pm,\ P_i^\pm,\ K_i,\ J_i\}$ span $(\mathfrak{d}^\pm)$;
- $\{H = \frac{1}{2}(H^+ + H^-), P_i = \frac{1}{2}(P_i^+ + P_i^-),\ K_i,\ J_i\}$ span $\mathfrak{p}$;
- $\{H, P_i, K_i^g = \frac{1}{2}(K_i + N_i), J_i\}$ span $\mathfrak{g}$.

Are there any more kinematics?

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Inönü-Wigner Contraction (PNAS **39** (1953), 510.)

"Classical mechanics is a limiting case of relativistic mechanics. Hence the group of the former, the Galilei group, must be in some sense a limiting case of the relativistic mechanics' group, the representations of the former must be limiting cases of the latter's representations. ⋯ the inhomogeneous Lorentz group must be, in the same sense, a limiting case of the de Sitter groups."

$$\mathfrak{g}' = U(\varepsilon)\mathfrak{g}$$

with

$$\mathfrak{g} = V_R \oplus V_N$$

and

$$U(\varepsilon) = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & \varepsilon I \end{array}\right) \qquad \begin{array}{c} V_R \\ V_N \end{array}$$

When $\varepsilon \neq 0$, $\det |U(\varepsilon)| \neq 0$, $\mathfrak{g}' \approx \mathfrak{g}$
When $\varepsilon = 0$, $\det |U(\varepsilon)| = 0$, $\mathfrak{g} \to \mathfrak{g}'$.

$\uparrow$
a new algebra

Possible Kinematics (Bacry & Lévy-Leblond, JMP**9** 1605 (1968))

Theorem: Under the assumptions that:

(1) space is isotropic (rotation invariance);

(2) parity and time reversal are automorphisms of the kinematical group;

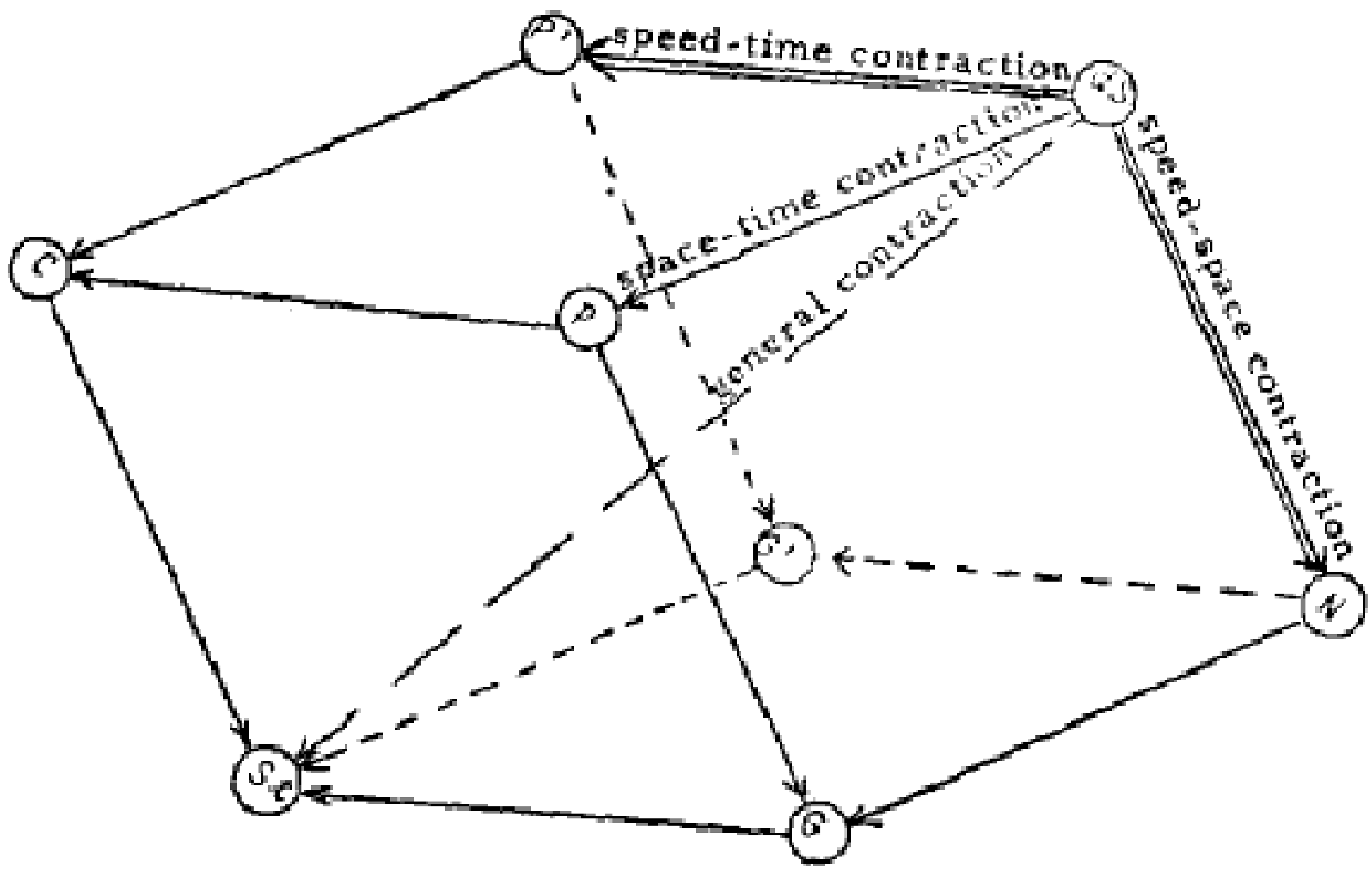
$$\begin{pmatrix} \Pi : H \to H, \boldsymbol{P} \to -\boldsymbol{P}, \boldsymbol{K} \to -\boldsymbol{K}, \boldsymbol{J} \to \boldsymbol{J} \\ \Theta : H \to -H, \boldsymbol{P} \to \boldsymbol{P}, \boldsymbol{K} \to -\boldsymbol{K}, \boldsymbol{J} \to \boldsymbol{J} \end{pmatrix}$$

(3) inertial transformations in any given direction form a noncompact subgroup, ↙— $\mathfrak{d}$, $\mathfrak{p}$, $\mathfrak{p}'$, $\mathfrak{n}$, $\mathfrak{g}$, $\mathfrak{g}'$, $\mathfrak{c}$ & $\mathfrak{s}$

then there are ***eight*** *types of Lie algebras for kinematical groups corresponding to* ***eleven*** *possible kinematics.*

$$\mathfrak{d}_\pm,\ \mathfrak{p},\ \mathfrak{h}_\pm(\mathfrak{p}'_\pm),\ \mathfrak{n}_\pm,\ \mathfrak{g},\ \mathfrak{g}',\ \mathfrak{c}\ \ \&\ \ \mathfrak{s}$$

They form a cube:



Bacry-Lévy-Leblond Contractions:

Speed-space contraction

$$\boldsymbol{P} \longrightarrow \epsilon \boldsymbol{P}, \quad \boldsymbol{K} \longrightarrow \epsilon \boldsymbol{K}, \quad \epsilon \longrightarrow 0; \tag{3}$$

Speed-time contraction

$$H \longrightarrow \epsilon H, \quad \boldsymbol{K} \longrightarrow \epsilon \boldsymbol{K}, \quad \epsilon \longrightarrow 0; \tag{4}$$

Space-time contraction

$$H \longrightarrow \epsilon H, \quad \boldsymbol{P} \longrightarrow \epsilon \boldsymbol{P}, \quad \epsilon \longrightarrow 0; \tag{5}$$

General contraction

$$H \longrightarrow \epsilon H, \quad \boldsymbol{P} \longrightarrow \epsilon \boldsymbol{P}, \quad \boldsymbol{K} \longrightarrow \epsilon \boldsymbol{K}, \quad \epsilon \longrightarrow 0. \tag{6}$$

In B coordinates, IW contraction realized by limit process.

By the contraction of the generators of $\mathfrak{d}_\pm$, on one end

$$\mathfrak{p}: \qquad H = \lim_{l_r\to\infty} H_r^\pm, \qquad \boldsymbol{P} = \lim_{l_r\to\infty} \boldsymbol{P}_r^\pm, \quad \boldsymbol{K}, \quad \boldsymbol{J}, \tag{7}$$

and on the other

$$\begin{array}{ccc} & \partial_t \mp \nu_r^2 t x^\mu \partial_\mu & \partial_i \mp l_r^{-2} x_i x^\mu \partial_\mu \\ & \| & \| \end{array}$$

$$\mathfrak{p}_2: \qquad H' = \pm \lim_{l_r\to 0} \frac{l_r^2}{l^2} H_r^\pm, \;\; \boldsymbol{P}' = \pm \lim_{l_r\to 0} \frac{l_r^2}{l^2} \boldsymbol{P}_r^\pm, \;\; \boldsymbol{K}, \;\; \boldsymbol{J}. \tag{8}$$

$$\begin{array}{cc} \| & \| \\ -\nu^2 t x^\mu \partial_\mu & -\, l^{-2} x_i x^\mu \partial_\mu \end{array}$$

$\mathfrak{p}_2$: — the 2nd Poincaré algebra

$$
\begin{aligned}
&[P'_i, P'_j] = 0, && [K_i, K_j] = -c^{-2}\epsilon_{ijk}J_k, && [P'_i, K_j] = c^{-2}H', \\
&[H', P'_i] = 0, && [H', K_i] = P', && [J_i, H'] = 0, \\
&[J_i, P'_j] = \epsilon_{ijk}P'_k, && [J_i, K_j] = \epsilon_{ijk}K_k, && [J_i, J_j] = \epsilon_{ijk}J_k
\end{aligned}
$$

in comparison with $\mathfrak{p}$

$$
\begin{aligned}
&[P_i, P_j] = 0, && [K_i, K_j] = -c^{-2}\epsilon_{ijk}J_k, && [P_i, K_j] = c^{-2}H, \\
&[H, P_i] = 0, && [H, K_i] = P, && [J_i, H] = 0, \\
&[J_i, P_j] = \epsilon_{ijk}P_k, && [J_i, K_j] = \epsilon_{ijk}K_k, && [J_i, J_j] = \epsilon_{ijk}J_k.
\end{aligned}
$$

Note: H' and P'_i do not generate translations!

$\mathfrak{p}$, $\mathfrak{p}_2$: same algebraic relations, different physics & geometry!

More Results:

$$\mathfrak{n}_\pm : \quad H^\pm = \lim_{\substack{c_r,\, l_r \to \infty \\ \nu \text{ fixed}}} H_r^\pm, \ \boldsymbol{P} = \lim_{\substack{c_r,\, l_r \to \infty \\ \nu \text{ fixed}}} \boldsymbol{P}_r^\pm,$$

$$\boldsymbol{K}^g = \lim_{\substack{c_r,\, l_r \to \infty \\ \nu \text{ fixed}}} \boldsymbol{K}_r (\text{or } \boldsymbol{N}_r), \ \boldsymbol{J}, \tag{9}$$

$$\mathfrak{n}_{\pm 2} : \quad \pm H^\pm = \pm \lim_{\substack{c_r,\, l_r \to 0 \\ \nu \text{ fixed}}} H_r^\pm, \ \boldsymbol{P}' = \pm \lim_{\substack{c_r,\, l_r \to 0 \\ \nu \text{ fixed}}} \frac{l_r^2}{l^2} \boldsymbol{P}_r^\pm,$$

$$\boldsymbol{K}^c = \lim_{\substack{c_r,\, l_r \to 0 \\ \nu \text{ fixed}}} \frac{c_r^2}{c^2} \boldsymbol{K}_r (\text{or } - \frac{c_r^2}{c^2} \boldsymbol{N}_r), \ \boldsymbol{J}, \tag{10}$$

c_r : a running parameter of dimension LT^{-1}.

$$\mathfrak{e}: \qquad H = \lim_{l_r \to \infty} H_r^{\mp},\ \boldsymbol{P} = \lim_{l_r \to \infty} \boldsymbol{P}_r^{\pm},\ \boldsymbol{N},\ \boldsymbol{J}, \tag{11}$$

$$\mathfrak{e}_2: \qquad -H' = \pm \lim_{l_r \to 0} \frac{l_r^2}{l^2} H_r^{\mp},\ \boldsymbol{P}' = \pm \lim_{l_r \to 0} \frac{l_r^2}{l^2} \boldsymbol{P}^{\pm},\ \boldsymbol{N},\ \boldsymbol{J}. \tag{12}$$

$$\mathfrak{h}_{\pm}: \qquad H = \lim_{c_r \to 0} H_r^{\pm},\ \boldsymbol{P}^{\pm}, \boldsymbol{K}^{c} = \lim_{c_r \to 0} \frac{c_r^2}{c^2} \boldsymbol{K}_r,\ \boldsymbol{J}, \tag{13}$$

$$\mathfrak{e}', \mathfrak{p}': \qquad \pm H' = \lim_{c_r \to \infty} \frac{c^2}{c_r^2} H_r^{\pm},\ \boldsymbol{P}^{\pm}, \boldsymbol{K}^{g} = \lim_{c_r \to \infty} \boldsymbol{K}_r,\ \boldsymbol{J}, \tag{14}$$

$$\mathfrak{g}: \quad H = \lim_{\substack{c_r,\, l_r \to \infty \\ \nu_r \to 0}} H_r^{\pm}, \ \ \boldsymbol{P} = \lim_{\substack{c_r,\, l_r \to \infty \\ \nu_r \to 0}} \boldsymbol{P}^{\pm},$$

$$\boldsymbol{K}^g = \lim_{\substack{c_r,\, l_r \to \infty \\ \nu_r \to 0}} \boldsymbol{K}_r, \ \ \boldsymbol{J}, \tag{15}$$

$$\mathfrak{c}: \quad H = \lim_{\substack{l_r \to \infty \\ c_r \to 0}} H_r^{\pm}, \ \ \boldsymbol{P} = \lim_{\substack{l_r \to \infty \\ c_r \to 0}} \boldsymbol{P}^{\pm},$$

$$\boldsymbol{K}^c = \lim_{\substack{l_r \to \infty \\ c_r \to 0}} \frac{c_r^2}{c^2} \boldsymbol{K}_r, \ \ \boldsymbol{J}, \tag{16}$$

$$\mathfrak{g}': \quad H' = \lim_{\substack{l_r,\, c_r \to \infty \\ \nu_r \to \infty}} \frac{\nu^2}{\nu_r^2} H_r^{+}, \ \ \boldsymbol{P} = \lim_{\substack{l_r,\, c_r \to \infty \\ \nu_r \to \infty}} \boldsymbol{P}_r^{\pm},$$

$$\boldsymbol{K}^g = \lim_{\substack{l_r,\, c_r \to \infty \\ \nu_r \to \infty}} \boldsymbol{K}_r, \ \ \boldsymbol{J}, \tag{17}$$

$$\mathfrak{g}_2: \qquad H' = \pm \lim_{\substack{c_r,\, l_r \to 0 \\ \nu_r \to \infty}} \frac{\nu^2}{\nu_r^2} H_r^{\pm}, \;\; \boldsymbol{P}' = \pm \lim_{\substack{c_r,\, l_r \to 0 \\ \nu_r \to \infty}} \frac{l_r^2}{l^2} \boldsymbol{P}^{\pm},$$

$$\boldsymbol{K}^c = \lim_{\substack{c_r,\, l_r \to 0 \\ \nu_r \to \infty}} \frac{c_r^2}{c^2} \boldsymbol{K}_r, \;\; \boldsymbol{J}, \tag{18}$$

$$\mathfrak{c}_2: \qquad H' = \pm \lim_{\substack{l_r \to 0 \\ c_r \to \infty}} \frac{\nu^2}{\nu_r^2} H_r^{\pm}, \;\; \boldsymbol{P}' = \pm \lim_{\substack{l_r \to 0 \\ c_r \to \infty}} \frac{l_r^2}{l^2} \boldsymbol{P}^{\pm},$$

$$\boldsymbol{K}^g = \lim_{\substack{l_r \to 0 \\ c_r \to \infty}} \boldsymbol{K}_r, \;\; \boldsymbol{J}, \tag{19}$$

$$\mathfrak{g}_2': \qquad H = \lim_{\substack{l_r,\, c_r \to 0 \\ \nu_r \to 0}} H_r^{+}, \quad \boldsymbol{P}' = \pm \lim_{\substack{l_r,\, c_r \to 0 \\ \nu_r \to 0}} \frac{l_r^2}{l^2} \boldsymbol{P}_r^{\pm},$$

$$\boldsymbol{K}^c = \lim_{\substack{l_r,\, c_r \to 0 \\ \nu_r \to 0}} \frac{c_r^2}{c^2} \boldsymbol{K}_r \,(\text{or } -\lim_{\substack{l_r,\, c_r \to 0 \\ \nu_r \to 0}} \frac{\mathrm{c}_\mathrm{r}^2}{\mathrm{c}^2} \boldsymbol{N}_\mathrm{r}), \;\; \boldsymbol{J}. \tag{20}$$

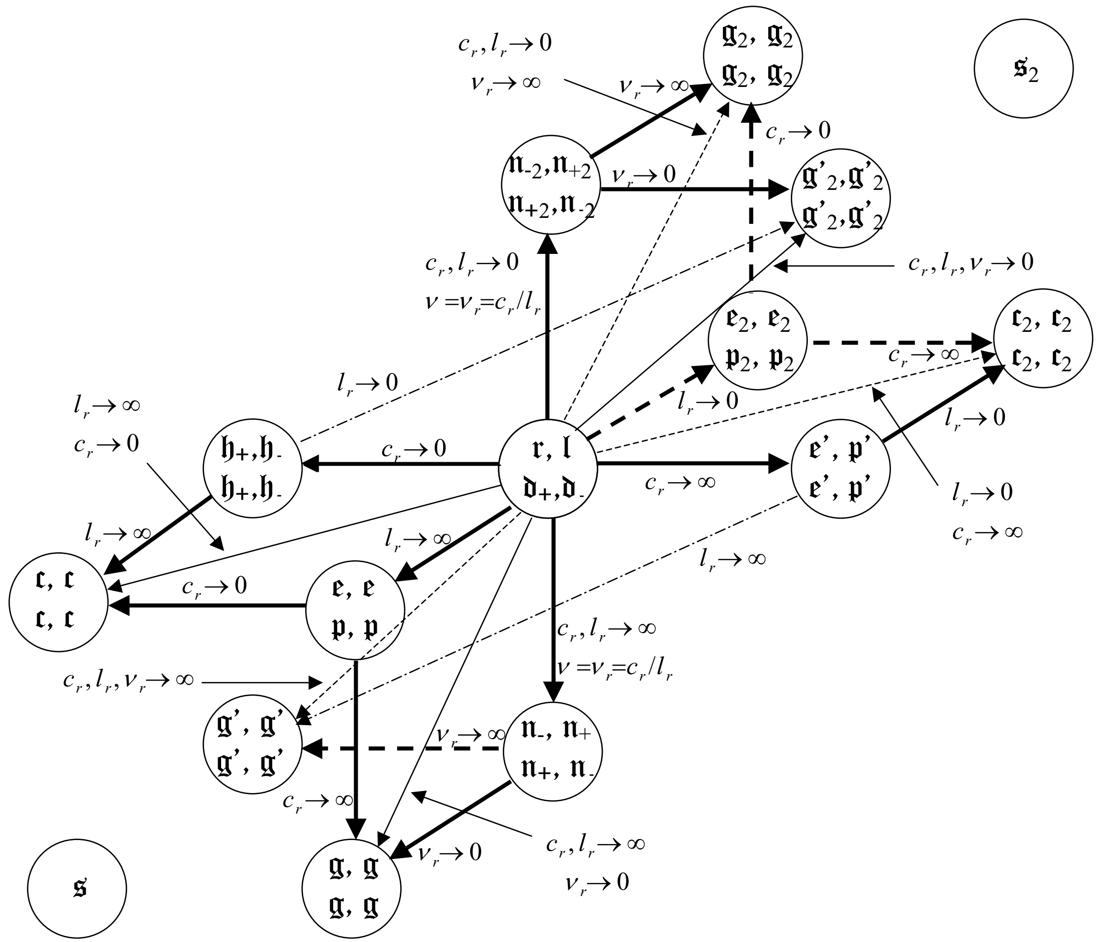
In summary, except static ones, there are 22 10-parameter kinematical (including geometrical) algebras with respect to the same $\mathfrak{so}(3)$ generated by $\boldsymbol{J}$. 9 more than the possible algebras in BLL paper (11 - 1 static +3 geometrical)!

They may also be obtained by the combinatory approach. (See: Guo, Huang, Wu, Zhou (Sci China: PMA 53 (2010), 591))

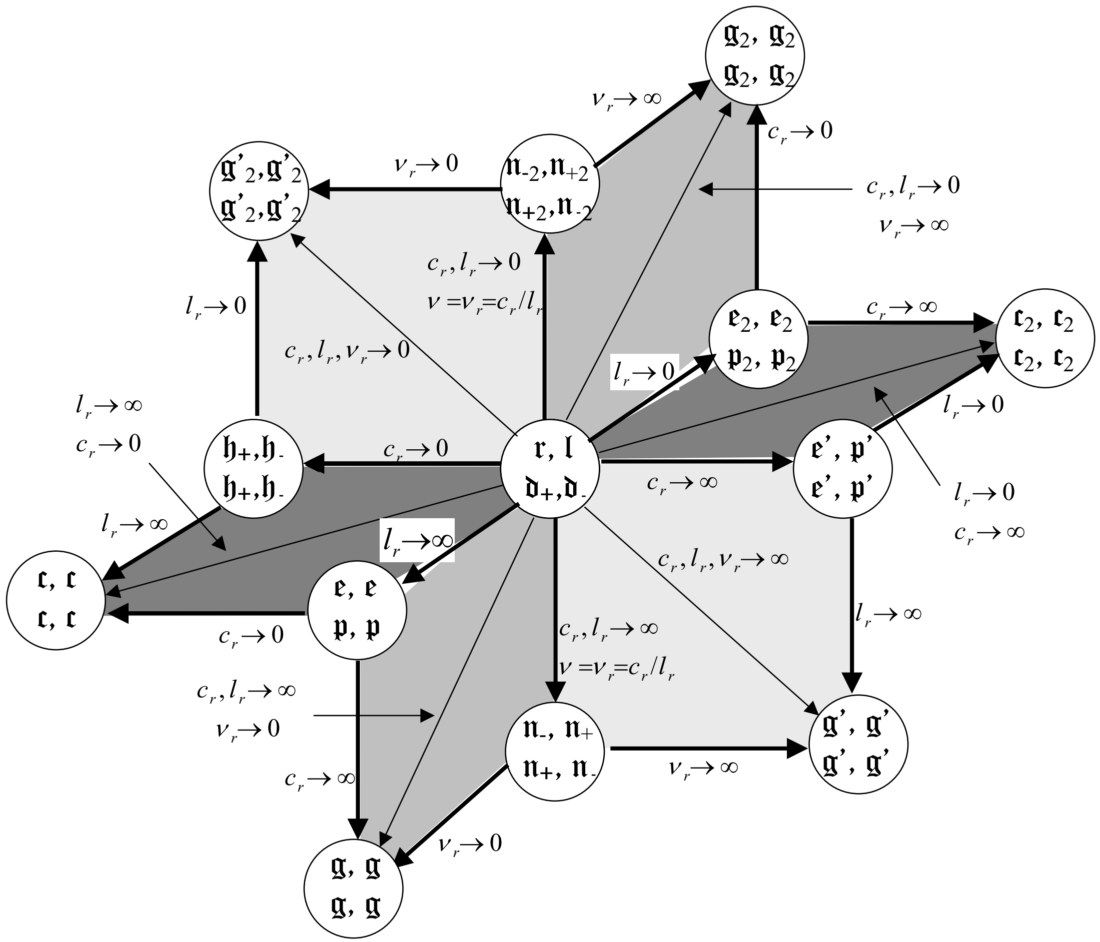
They are listed as follows.

Algebra	Symbol	Generator set	$[\mathcal{H},\mathbf{P}]$	$[\mathcal{H},\mathbf{K}]$	$[\mathbf{P},\mathbf{P}]$	$[\mathbf{K},\mathbf{K}]$	$[\mathbf{P},\mathbf{K}]$
dS	$\mathfrak{d}_+$	$(H^+,\mathbf{P}_i^+,\mathbf{K}_i,\mathbf{J}_i)$	$\nu^2\mathbf{K}$	$\mathbf{P}$	$l^{-2}\mathbf{J}$	$-c^{-2}\mathbf{J}$	$c^{-2}\mathcal{H}$
AdS	$\mathfrak{d}_-$	$(H^-,\mathbf{P}_i^-,\mathbf{K}_i,\mathbf{J}_i)$	$-\nu^2\mathbf{K}$	$\mathbf{P}$	$-l^{-2}\mathbf{J}$	$-c^{-2}\mathbf{J}$	$c^{-2}\mathcal{H}$
Poincaré	$\mathfrak{p}$ $\mathfrak{p}_2$	$(H,\mathbf{P}_i,\mathbf{K}_i,\mathbf{J}_i)$ $\{H',\mathbf{P}_i',\mathbf{K}_i,\mathbf{J}_i\}$	0	$\mathbf{P}$	0	$-c^{-2}\mathbf{J}$	$c^{-2}\mathcal{H}$
Riemann	$\mathfrak{r}$	$(H^-,\mathbf{P}_i^+,\mathbf{N}_i,\mathbf{J}_i)$	$-\nu^2\mathbf{K}$	$\mathbf{P}$	$l^{-2}\mathbf{J}$	$c^{-2}\mathbf{J}$	$-c^{-2}\mathcal{H}$
Lobachevsky	$\mathfrak{l}$	$(H^+,\mathbf{P}_i^-,\mathbf{N}_i,\mathbf{J}_i)$	$\nu^2\mathbf{K}$	$\mathbf{P}$	$-l^{-2}\mathbf{J}$	$c^{-2}\mathbf{J}$	$-c^{-2}\mathcal{H}$
Euclid	$\mathfrak{e}$ $\mathfrak{e}_2$	$(H,\mathbf{P}_i,\mathbf{N}_i,\mathbf{J}_i)$ $(H',\mathbf{P}_i',\mathbf{N}_i,\mathbf{J}_i)$	0	$\mathbf{P}$	0	$c^{-2}\mathbf{J}$	$-c^{-2}\mathcal{H}$
Galilei	$\mathfrak{g}$ $\mathfrak{g}_2$	$(H,\mathbf{P}_i,\mathbf{K}_i^{\mathfrak{g}},\mathbf{J}_i)$ $(H',\mathbf{P}_i',\mathbf{K}_i^{\mathfrak{c}},\mathbf{J}_i)$	0	$\mathbf{P}$	0	0	0
Carroll	$\mathfrak{c}$ $\mathfrak{c}_2$	$(H,\mathbf{P}_i,\mathbf{K}_i^{\mathfrak{c}},\mathbf{J}_i)$ $(H',\mathbf{P}_i',\mathbf{K}_i^{\mathfrak{g}},\mathbf{J}_i)$	0	0	0	0	$c^{-2}\mathcal{H}$
NH$_+$	$\mathfrak{n}_+$ $\mathfrak{n}_{+2}$	$(H^+,\mathbf{P}_i,\mathbf{K}_i^{\mathfrak{g}},\mathbf{J}_i)$ $(H^+,\mathbf{P}_i',\mathbf{K}_i^{\mathfrak{c}},\mathbf{J}_i)$	$\nu^2\mathbf{K}$	$\mathbf{P}$	0	0	0
NH$_-$	$\mathfrak{n}_-$ $\mathfrak{n}_{-2}$	$(H^-,\mathbf{P}_i,\mathbf{K}_i^{\mathfrak{g}},\mathbf{J}_i)$ $(-H^-,\mathbf{P}_i',\mathbf{K}_i^{\mathfrak{c}},\mathbf{J}_i)$	$-\nu^2\mathbf{K}$	$\mathbf{P}$	0	0	0
para-Galilei	$\mathfrak{g}'$ $\mathfrak{g}_2'$	$(H',\mathbf{P},\mathbf{K}^{\mathfrak{g}},\mathbf{J}_i)$ $(H,\mathbf{P}_i',\mathbf{K}_i^{\mathfrak{c}},\mathbf{J}_i)$	$\nu^2\mathbf{K}$	0	0	0	0
HN$_+$	$\mathfrak{h}_+$ $\mathfrak{h}_{+2}$	$(H,\mathbf{P}_i^+,\mathbf{K}_i^{\mathfrak{c}},\mathbf{J}_i)$ $(H',\mathbf{P}_i^+,\mathbf{K}_i^{\mathfrak{g}},\mathbf{J}_i)$	$\nu^2\mathbf{K}$	0	$l^{-2}\mathbf{J}$	0	$c^{-2}\mathcal{H}$
HN$_-$	$\mathfrak{h}_-$ $\mathfrak{h}_{-2}$	$(H,\mathbf{P}_i^-,\mathbf{K}_i^{\mathfrak{c}},\mathbf{J}_i)$ $(-H',\mathbf{P}_i^-,\mathbf{K}_i^{\mathfrak{g}},\mathbf{J}_i)$	$-\nu^2\mathbf{K}$	0	$-l^{-2}\mathbf{J}$	0	$c^{-2}\mathcal{H}$
Static	$\mathfrak{s}$ $\mathfrak{s}_2$	$(H^{\mathfrak{s}},\mathbf{P}_i',\mathbf{K}_i^{\mathfrak{c}},\mathbf{J}_i)$[1] $(H^{\mathfrak{s}},\mathbf{P}_i,\mathbf{K}_i^{\mathfrak{g}},\mathbf{J}_i)$	0	0	0	0	0

They fail to be put in 2 cubes with a common apex. (Huang, Tian, Wu, Xu, Zhou, arXiv: 1007.3618)



Instead, they may be put in the figure.



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Contraction of Geometries

Well-known Examples:

$l \to \infty$	dS, AdS	$\to$	Min
$l \to \infty$	$Riem, Lob$	$\to$	Euc
$c \to \infty$	Min	$\to$	G
$c \to 0$	Min	$\to$	C
$l \to \infty, c \to \infty, c/l$ fixed	dS	$\to$	NH_+
$l \to \infty, c \to \infty, c/l$ fixed	AdS	$\to$	NH_-

What about else?

Example. Geometries for the second Poincaré algebra
Metrics, metric-inverses, connections of (A)dS$_4$ in B coord.:

$$ds_{\pm}^2 = \frac{1}{\sigma_r^{\pm}}\left(\eta_{\mu\nu} \pm \frac{\eta_{\mu\kappa}\eta_{\nu\lambda}x^\kappa x^\lambda}{l_r^2\sigma^{\pm}}\right) dx^\mu dx^\nu, \tag{21}$$

$$\left(\frac{\partial}{\partial s}\right)^2 = \sigma_r^{\pm}(\eta^{\mu\nu} \mp l_r^{-2}x^\mu x^\nu)\frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu} \tag{22}$$

$$\Gamma^\lambda_{\mu\nu} = \pm\frac{(\delta^\lambda_\mu \eta_{\nu\kappa} + \delta^\lambda_\nu \eta_{\mu\kappa})x^\kappa}{l_r^2\sigma_r^{\pm}}, \tag{23}$$

where

$$\sigma_r^{\pm} = 1 \mp l_r^{-2}\eta_{\kappa\lambda}x^\kappa x^\lambda > 0. \tag{24}$$

In the limit of $l_r \to 0$,

$$0 < l_r^2 \sigma_r^{\pm} = \mp \eta_{\kappa\lambda} x^\kappa x^\lambda =: \mp x \cdot x. \tag{25}$$

Metrics, metric-inverses, connection of (A)dS$_4$ reduce to

$$\left.\begin{aligned} \boldsymbol{g}^{P_{2\pm}} &= \pm l^2 \frac{(x \cdot dx)^2 - (x \cdot x)(dx \cdot dx)}{(x \cdot x)^2} \\ \boldsymbol{h}_{P_{2\pm}} &= l^{-4} (x \cdot x)(x^\mu \partial_\mu)^2 \\ \Gamma_{P_{2\pm}}{}^{\lambda}_{\mu\nu} &= -\frac{(\delta^\lambda_\mu \eta_{\nu\kappa} + \delta^\lambda_\nu \eta_{\mu\kappa}) x^\kappa}{x \cdot x} \end{aligned}\right. \qquad \text{for } x \cdot x \begin{cases} < 0 \\ > 0 \end{cases}$$

They satisfy

a) $|g| = 0, \qquad \text{and} \qquad |h| = 0;$

b) when and only when $\forall \boldsymbol{\xi} \in \mathfrak{p}_2 \subset TM$,

$$\begin{cases} \mathcal{L}_{\boldsymbol{\xi}} g^{\pm}_{\mu\nu} = g^{\pm}_{\mu\nu,\lambda}\boldsymbol{\xi}^{\lambda} + g^{\pm}_{\mu\lambda}\partial_{\nu}\boldsymbol{\xi}^{\lambda} + g^{\pm}_{\lambda\nu}\partial_{\mu}\boldsymbol{\xi}^{\lambda} = 0, \\ \mathcal{L}_{\boldsymbol{\xi}} h^{\pm\mu\nu} = h^{\pm\mu\nu}{}_{,\lambda}\boldsymbol{\xi}^{\lambda} - h^{\pm\mu\lambda}\partial_{\lambda}\boldsymbol{\xi}^{\nu} - h^{\pm\lambda\nu}\partial_{\lambda}\boldsymbol{\xi}^{\mu} = 0, \\ [\mathcal{L}_{\boldsymbol{\xi}}, \nabla] = 0 \end{cases}$$

are valid simultaneously.

$$R^{\mu}{}_{\nu\lambda\sigma} = \pm l^{-2}(g^{\pm}_{\nu\lambda}\delta^{\mu}_{\sigma} - g^{\pm}_{\nu\sigma}\delta^{\mu}_{\lambda}), \qquad R_{\mu\nu} = \mp 3 l^{-2} g^{\pm}_{\mu\nu}. \qquad (26)$$

When $x \cdot x < 0$,

$$M^{+} = ISO(1,3)/ISO(1,2) = R \times dS_3$$

On dS_3, $\exists$ Beltrami translations:

$$\begin{aligned} {}^3\mathbf{H}^{+} &= c\partial_{z^0} - cl^{-2} z_0 z^{\beta} \partial_{z^{\beta}} = \frac{c^2}{l}\mathbf{K}_3, \\ {}^3\mathbf{P}_2^{+} &= \partial_{z^2} - l^{-2} z_2 z^{\beta} \partial_{z^{\beta}} = -l^{-1}\mathbf{J}_2, \\ {}^3\mathbf{P}_3^{+} &= \partial_{z^3} - l^{-2} z_3 z^{\beta} \partial_{z^{\beta}} = l^{-1}\mathbf{J}_1, \end{aligned}$$

where $z^0 = l\frac{x^0}{x^3},\ z^2 = l\frac{x^1}{x^3}, z^3 = l\frac{x^2}{x^3}$ are 3d B coordinates.

$\nexists$ spatial SO(3) isotropy at each point,

though $\exists$ algebraic SO(3) isotropy.

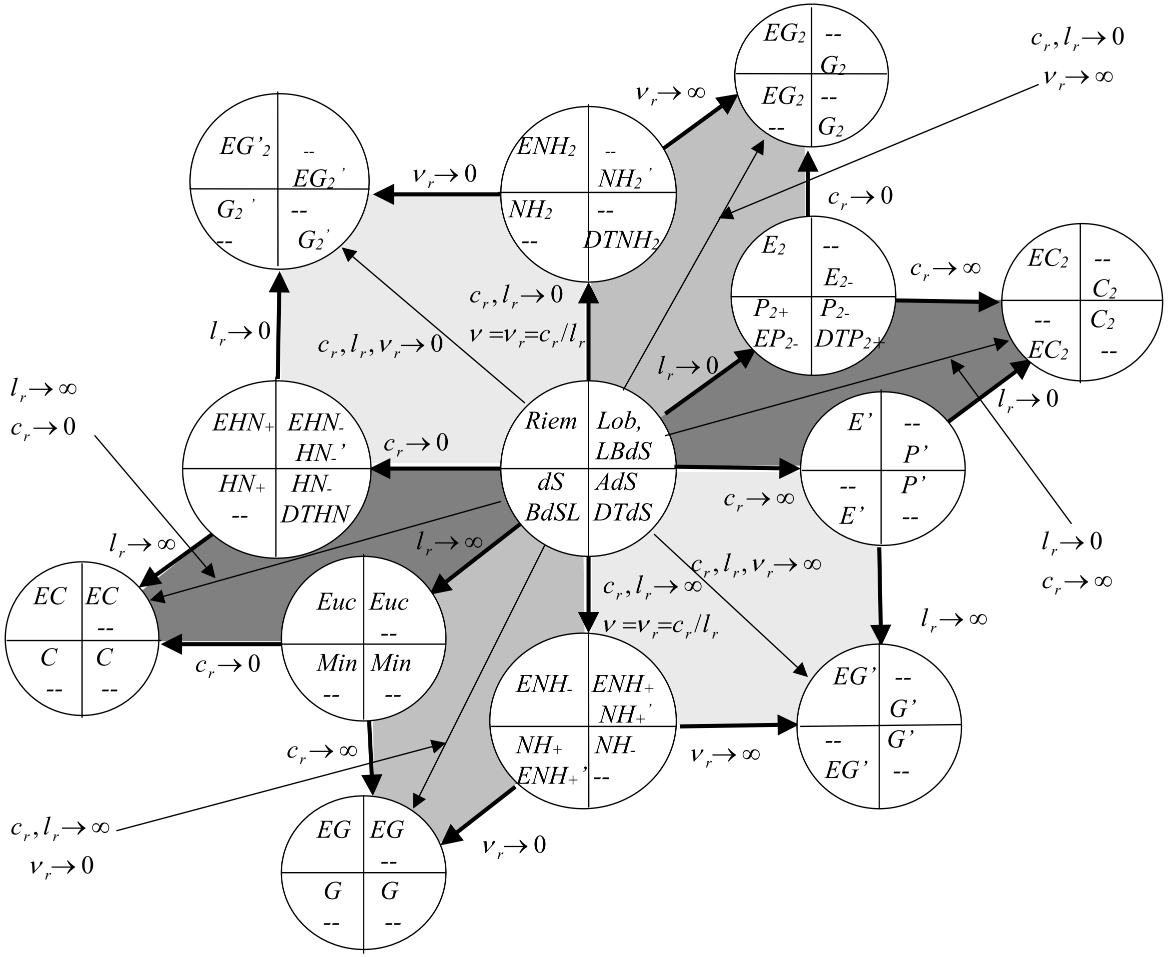
When $x \cdot x > 0$,

$$M^{-} = ISO(1,3)/ISO(3) = R \times H_3.$$

On H_3, $\exists$ Beltrami translations:

$${}^3\mathbf{P}_i^{-} = \partial_{z^i} + l^{-2} z_i z^j \partial_{z^j} = \frac{c}{l}\mathbf{K}_i$$

$\exists$ spatial SO(3) isotropy at each point.


Riem
Lob, LBdS
dS
BdSL
AdS
DTdS
EG'2
EG2'
G2'
ENH2
NH2'
NH2
DTNH2
EG2
G2
E2
E2-
P2+
EP2-
P2-
DTP2+
EC2
C2
EHN+
EHN-
HN-'
HN+
HN-
DTHN
E'
P'
EC
C
Euc
Min
ENH-
ENH+
NH+'
NH+
NH-
ENH+'
EG'
G'
EG
G
νr→0
νr→∞
cr, lr→0
ν=νr=cr/lr
cr→0
cr→∞
lr→0
lr→∞
cr, lr, νr→0
cr, lr, νr→∞
cr, lr→∞
cr, lr→0
νr→∞
lr→∞
cr→0
lr→0
cr→∞
cr, lr→∞
νr→0

Alg	Geom. name	Geometrical variables	$(\boldsymbol{g},\ \boldsymbol{h})$ Ranks	Signature	Contraction
$\mathfrak{r}$	*Riem*	$\boldsymbol{g}^{Riem} = \frac{1}{\sigma_E^+}\left((dx\cdot dx)_E - \frac{(x\cdot dx)_E^2}{l^2\sigma_E^+}\right)$	(4, 4)	$(+,+,+,+)$	No
$\mathfrak{l}$	*Lob*	$\boldsymbol{g}^{Lob} = \frac{1}{\sigma_E^-}\left((dx\cdot dx)_E + \frac{(x\cdot dx)_E^2}{l^2\sigma_E^-}\right)$	(4, 4)	$(+,+,+,+)$	No
$\mathfrak{l}$	*LBdS*	$\boldsymbol{g}^{LBdS} = -\frac{1}{\sigma_E^-}\left((dx\cdot dx)_E + \frac{(x\cdot dx)_E^2}{l^2\sigma_E^-}\right)$	(4, 4)	$(-,+,+,+)$	No
$\mathfrak{e}$	*Euc*	$\boldsymbol{g}^{Euc} = (dx\cdot dx)_E$	(4, 4)	$(+,+,+,+)$	$l_r \to \infty$
$\mathfrak{e}_2$	E_2	$\boldsymbol{g}^{E_2} = l^2\frac{(x\cdot x)_E(dx\cdot dx)_E - (x\cdot dx)_E^2}{(x\cdot x)_E^2}$ $\boldsymbol{h}_{E_2} = l^{-4}(x\cdot x)_E\ (x^\mu\partial_\mu)^2$ $\Gamma_{E_2\mu\nu}^{\lambda} = -\frac{(\delta_\mu^\lambda\delta_{\nu\kappa} + \delta_\nu^\lambda\delta_{\mu\kappa})x^\kappa}{(x\cdot x)_E}$	(3, 1)	$(+,+,+;+)$	$l_r \to 0$
$\mathfrak{e}_2$	E_{2-}	$\boldsymbol{g}^{E_{2-}} = l^2\frac{(x\cdot x)_E(dx\cdot dx)_E - (x\cdot dx)_E^2}{(x\cdot x)_E^2}$ $\boldsymbol{h}_{E_{2-}} = -l^{-4}(x\cdot x)_E\ (x^\mu\partial_\mu)^2$ $\Gamma_{E_{2-}}{}^{\lambda}_{\mu\nu} = -\frac{(\delta_\mu^\lambda\delta_{\nu\kappa} + \delta_\nu^\lambda\delta_{\mu\kappa})x^\kappa}{(x\cdot x)_E}$	(3, 1)	$(+,+,+;-)$	$l_r \to 0$

Alg	Geom. name	Geometrical variables	$(\boldsymbol{g},\ \boldsymbol{h})$ Ranks	Signature	Contraction
$\mathfrak{d}_+$	dS	$\boldsymbol{g}^{dS} = \frac{1}{\sigma^+}\left(dx\cdot dx + \frac{(x\cdot dx)^2}{l^2\sigma^+}\right)$	(4, 4)	$(+,-,-,-)$	No
$\mathfrak{d}_+$	$BdSL$	$\boldsymbol{g}^{BdSL} = \frac{1}{\sigma^+}\left(dx\cdot dx + \frac{(x\cdot dx)^2}{l^2\sigma^+}\right)$	(4, 4)	$(+,+,+,+)$	No
$\mathfrak{d}_-$	AdS	$\boldsymbol{g}^{AdS} = \frac{1}{\sigma^-}\left(dx\cdot dx - \frac{(x\cdot dx)^2}{l^2\sigma^-}\right)$	(4, 4)	$(+,-,-,-)$	No
$\mathfrak{d}_-$	$DTdS$	$\boldsymbol{g}^{DTdS} = -\frac{1}{\sigma^-}\left(dx\cdot dx - \frac{(x\cdot dx)^2}{l^2\sigma^-}\right)$	(4, 4)	$(+,+,-,-)$	No
$\mathfrak{p}$	Min	$\boldsymbol{g}^{Min} = dx\cdot dx$	(4, 4)	$(+,-,-,-)$	$l_r\to\infty$
$\mathfrak{p}_2$	$P_{2\pm}$	$\boldsymbol{g}^{P_{2\pm}} = \pm l^2\frac{(x\cdot dx)^2-(x\cdot x)(dx\cdot dx)}{(x\cdot x)^2}$ $\boldsymbol{h}_{P_{2\pm}} = l^{-4}(x\cdot x)(x^\mu\partial_\mu)^2$ $\Gamma_{P_{2\pm}}{}^{\lambda}_{\mu\nu} = -\frac{(\delta^\lambda_\mu\eta_{\nu\kappa}+\delta^\lambda_\nu\eta_{\mu\kappa})x^\kappa}{x\cdot x}$	(3, 1)	$(+,-,-;-)$ $(-,-,-;+)$	$l_r\to 0$
$\mathfrak{p}_2$	EP_{2-}	$\boldsymbol{g}^{EP_{2-}} = l^2\frac{(x\cdot dx)^2-(x\cdot x)(dx\cdot dx)}{(x\cdot x)^2}$ $\boldsymbol{h}_{EP_{2-}} = l^{-4}(x\cdot x)(x^\mu\partial_\mu)^2$ $\Gamma_{EP_{2-}}{}^{\lambda}_{\mu\nu} = -\frac{(\delta^\lambda_\mu\eta_{\nu\kappa}+\delta^\lambda_\nu\eta_{\mu\kappa})x^\kappa}{x\cdot x}$	(3, 1)	$(+,+,+;+)$	$l_r\to 0$
$\mathfrak{p}_2$	DTP_{2+}	$\boldsymbol{g}^{DTP_{2+}} = l^2\frac{(x\cdot dx)^2-(x\cdot x)(dx\cdot dx)}{(x\cdot x)^2}$ $\boldsymbol{h}_{DTP_{2+}} = -l^{-4}(x\cdot x)(x^\mu\partial_\mu)^2$ $\Gamma_{DTP_{2+}}{}^{\lambda}_{\mu\nu} = -\frac{(\delta^\lambda_\mu\eta_{\nu\kappa}+\delta^\lambda_\nu\eta_{\mu\kappa})x^\kappa}{x\cdot x}$	(3, 1)	$(+,-,-;+)$	$l_r\to 0$

Alg	Geom. name	Geometrical variables	$(\boldsymbol{g},\ \boldsymbol{h})$ Ranks	Signature	Contraction
$\mathfrak{n}_\pm$	$NH_\pm$	$\boldsymbol{g}^{NH_\pm} = (\sigma_{\mathfrak{n}}^\pm)^{-2} c^2 dt^2$ $\boldsymbol{h}_{NH_\pm} = -\sigma_{\mathfrak{n}}^\pm \delta^{ij} \partial_i \partial_j$ $\Gamma_{NH_\pm}{}^0_{00} = \pm \frac{2\nu^2 t}{c\sigma_{\mathfrak{n}}^\pm}$ $\Gamma_{NH_\pm}{}^i_{0j} = \Gamma_{NH_\pm}{}^i_{j0} = \pm \frac{\nu^2 t}{c\sigma_{\mathfrak{n}}^\pm} \delta^i_j$	(1, 3)	$(+;-,-,-)$	$l_r, c_r \to \infty$ $\nu = c_r/l_r$ finite
$\mathfrak{n}_\pm$	$ENH_\pm$	$\boldsymbol{g}^{ENH_\pm} = (\sigma_{\mathfrak{n}}^\pm)^{-2} c^2 dt^2$ $\boldsymbol{h}_{ENH_\pm} = \sigma_{\mathfrak{n}}^\pm \delta^{ij} \partial_i \partial_j$ $\Gamma_{ENH_\pm}{}^0_{00} = \pm \frac{2\nu^2 t}{c\sigma_{\mathfrak{n}}^\pm}$ $\Gamma_{NH_+}{}^i_{0j} = \Gamma_{NH_+}{}^i_{j0} = \pm \frac{\nu^2 t}{c\sigma_{\mathfrak{n}}^\pm} \delta^i_j$	(1, 3)	$(+;+,+,+)$	$l_r, c_r \to \infty$ $\nu = c_r/l_r$ finite
$\mathfrak{n}_+$	NH'_+	$\boldsymbol{g}^{NH'_+} = -(\sigma_{\mathfrak{n}}^+)^{-2} c^2 dt^2$ $\boldsymbol{h}_{NH'_+} = -\sigma_{\mathfrak{n}}^+ \delta^{ij} \partial_i \partial_j$ $\Gamma_{NH'_+}{}^0_{00} = \frac{2\nu^2 t}{c\sigma_{\mathfrak{n}}^+}$ $\Gamma_{NH'_+}{}^i_{0j} = \Gamma_{NH'_+}{}^i_{j0} = \frac{\nu^2 t}{c\sigma_{\mathfrak{n}}^+} \delta^i_j$	(1, 3)	$(-;+,+,+)$	$l_r, c_r \to \infty$ $\nu = c_r/l_r$ finite
$\mathfrak{n}_+$	ENH'_+	$\boldsymbol{g}^{ENH'_+} = (\sigma_{\mathfrak{n}}^+)^{-2} c^2 dt^2$ $\boldsymbol{h}_{ENH'_+} = -\sigma_{\mathfrak{n}}^+ \delta^{ij} \partial_i \partial_j$ $\Gamma_{ENH'_+}{}^0_{00} = \frac{2\nu^2 t}{c\sigma_{\mathfrak{n}}^+}$ $\Gamma_{ENH'_+}{}^i_{0j} = \Gamma_{ENH'_+}{}^i_{j0} = \frac{\nu^2 t}{c\sigma_{\mathfrak{n}}^+} \delta^i_j$	(1, 3)	$(+;+,+,+)$	$l_r, c_r \to \infty$ $\nu = c_r/l_r$ finite

Alg	Geom. name	Geometrical variables	$(\boldsymbol{g},\ \boldsymbol{h})$ Ranks	Signature	Contraction
$\mathfrak{n}_{+2}$	NH_2	$\boldsymbol{g}^{NH_2} = l^2 \frac{(\boldsymbol{x}\cdot\boldsymbol{dx})^2 - (\boldsymbol{x}\cdot\boldsymbol{x})(\boldsymbol{dx}\cdot\boldsymbol{dx})}{(\boldsymbol{x}\cdot\boldsymbol{x})^2}$ $\boldsymbol{h}_{NH_2} = l^{-4}\boldsymbol{x}\cdot\boldsymbol{x}\left[\nu^{-2}\partial_t^2 - (x^\mu\partial_\mu)^2\right]$ $\Gamma_{NH_2}{}^0_{0i} = \Gamma_{NH_2}{}^0_{i0} = -\frac{x^i}{\boldsymbol{x}\cdot\boldsymbol{x}}$ $\Gamma_{NH_2}{}^i_{jk} = -\frac{\delta^i_j x^k + \delta^i_k x^j}{\boldsymbol{x}\cdot\boldsymbol{x}}$	(2, 2)	$(-,-;+,-)$	$l_r, c_r \to 0$ $\nu = c_r/l_r$ finite
$\mathfrak{n}_{+2}$	NH'_2	$\boldsymbol{g}^{NH'_2} = -l^2 \frac{(\boldsymbol{x}\cdot\boldsymbol{dx})^2 - (\boldsymbol{x}\cdot\boldsymbol{x})(\boldsymbol{dx}\cdot\boldsymbol{dx})}{(\boldsymbol{x}\cdot\boldsymbol{x})^2}$ $\boldsymbol{h}_{NH'_2} = l^{-4}\boldsymbol{x}\cdot\boldsymbol{x}\left[\nu^{-2}\partial_t^2 - (x^\mu\partial_\mu)^2\right]$ $\Gamma_{NH'_2}{}^0_{0i} = \Gamma_{NH'_2}{}^0_{i0} = -\frac{x^i}{\boldsymbol{x}\cdot\boldsymbol{x}}$ $\Gamma_{NH'_2}{}^i_{jk} = -\frac{\delta^i_j x^k + \delta^i_k x^j}{\boldsymbol{x}\cdot\boldsymbol{x}}$	(2, 2)	$(+,+;+,-)$	$l_r, c_r \to 0$ $\nu = c_r/l_r$ finite
$\mathfrak{n}_{-2}$	ENH_2	$\boldsymbol{g}^{ENH_2} = l^2 \frac{(\boldsymbol{x}\cdot\boldsymbol{x})(\boldsymbol{dx}\cdot\boldsymbol{dx}) - (\boldsymbol{x}\cdot\boldsymbol{dx})^2}{(\boldsymbol{x}\cdot\boldsymbol{x})^2}$ $\boldsymbol{h}_{ENH_2} = l^{-4}\boldsymbol{x}\cdot\boldsymbol{x}\left[\nu^{-2}\partial_t^2 + (x^\mu\partial_\mu)^2\right]$ $\Gamma_{ENH_2}{}^0_{0i} = \Gamma_{ENH_2}{}^0_{i0} = -\frac{x^i}{\boldsymbol{x}\cdot\boldsymbol{x}}$ $\Gamma_{ENH_2}{}^i_{jk} = -\frac{\delta^i_j x^k + \delta^i_k x^j}{\boldsymbol{x}\cdot\boldsymbol{x}}$	(2, 2)	$(+,+;+,+)$	$l_r, c_r \to 0$ $\nu = c_r/l_r$ finite
$\mathfrak{n}_{-2}$	$DTNH_2$	$\boldsymbol{g}^{DTNH} = l^2 \frac{(\boldsymbol{x}\cdot\boldsymbol{dx})^2 - (\boldsymbol{x}\cdot\boldsymbol{x})(\boldsymbol{dx}\cdot\boldsymbol{dx})}{(\boldsymbol{x}\cdot\boldsymbol{x})^2}$ $\boldsymbol{h}_{DTNH} = l^{-4}\boldsymbol{x}\cdot\boldsymbol{x}\left[\nu^{-2}\partial_t^2 + (x^\mu\partial_\mu)^2\right]$ $\Gamma_{DTNH_2}{}^0_{0i} = \Gamma_{DTNH_2}{}^0_{i0} = -\frac{x^i}{\boldsymbol{x}\cdot\boldsymbol{x}}$ $\Gamma_{DTNH_2}{}^i_{jk} = -\frac{\delta^i_j x^k + \delta^i_k x^j}{\boldsymbol{x}\cdot\boldsymbol{x}}$	(2, 2)	$(-,-;+,+)$	$l_r, c_r \to 0$ $\nu = c_r/l_r$ finite

Alg	Geom. name	Geometrical variables	$(\boldsymbol{g},\ \boldsymbol{h})$ Ranks	Signature	Contraction
$\mathfrak{h}_{\pm}$	$HN_{\pm}$	$\boldsymbol{g}^{HN_{\pm}} = -\frac{1}{\sigma^{\pm}_{E,\,3}}\left[d\boldsymbol{x}\cdot d\boldsymbol{x} \mp \frac{(\boldsymbol{x}\cdot d\boldsymbol{x})^2}{l^2\sigma^{\pm}_{E,\,3}}\right]$ $\boldsymbol{h}_{HN_{\pm}} = \sigma^{\pm}_{E,\,3}\partial^2_{ct}$ $\Gamma_{HN_{\pm}}{}^{0}_{0i} = \Gamma_{HN_{\pm}}{}^{0}_{i0} = \mp\frac{x^i}{l^2\sigma^{+}_{E,\,3}}$ $\Gamma_{HN_{\pm}}{}^{i}_{jk} = \mp\frac{\delta^i_j x^k + \delta^i_k x^j}{l^2\sigma^{+}_{E,\,3}}$	$(3,\ 1)$	$(-,-,-;+)$	$c_r \to 0$
$\mathfrak{h}_{\pm}$	$EHN_{\pm}$	$\boldsymbol{g}^{EHN_{\pm}} = \frac{1}{\sigma^{\pm}_{E,\,3}}\left[d\boldsymbol{x}\cdot d\boldsymbol{x} \mp \frac{(\boldsymbol{x}\cdot d\boldsymbol{x})^2}{l^2\sigma^{\pm}_{E,\,3}}\right]$ $\boldsymbol{h}_{EHN_{\pm}} = \sigma^{\pm}_{E,\,3}\partial^2_{ct}$ $\Gamma_{EHN_{\pm}}{}^{0}_{0i} = \Gamma_{EHN_{\pm}}{}^{0}_{i0} = \mp\frac{x^i}{l^2\sigma^{\pm}_{E,\,3}}$ $\Gamma_{EHN_{\pm}}{}^{i}_{jk} = \mp\frac{\delta^i_j x^k + \delta^i_k x^j}{l^2\sigma^{\pm}_{E,\,3}}$	$(3,\ 1)$	$(+,+,+;+)$	$c_r \to 0$

Alg	Geom. name	Geometrical variables	$(\boldsymbol{g},\ \boldsymbol{h})$ Ranks	Signature	Contraction
$\mathfrak{h}_-$	HN'_-	$\boldsymbol{g}^{HN'_-} = -\frac{1}{\sigma^-_{E,\,3}}\left[d\boldsymbol{x}\cdot d\boldsymbol{x} + \frac{(\boldsymbol{x}\cdot d\boldsymbol{x})^2}{l^2\sigma^-_{E,\,3}}\right]$ $\boldsymbol{h}_{HN'_-} = -\sigma^-_{E,\,3}\partial^2_{ct}$ $\Gamma_{HN'_-}{}^0_{0i} = \Gamma_{HN'_-}{}^0_{i0} = \frac{x^i}{l^2\sigma^-_{E,\,3}}$ $\Gamma_{HN'_-}{}^i_{jk} = \frac{\delta^i_j x^k + \delta^i_k x^j}{l^2\sigma^-_{E,\,3}}$	(3, 1)	$(-,+,+;+)$	$c_r \to 0$
$\mathfrak{h}_-$	$DTHN$	$\boldsymbol{g}^{DTHN} = \frac{1}{\sigma^-_{E,3}}\left[d\boldsymbol{x}\cdot d\boldsymbol{x} + \frac{(\boldsymbol{x}\cdot d\boldsymbol{x})^2}{l^2\sigma^-_{E,3}}\right]$ $\boldsymbol{h}_{DTHN} = -\sigma^-_{E,3}\partial^2_{ct}$ $\Gamma_{DTHN}{}^0_{0i} = \Gamma_{DTHN}{}^0_{i0} = \frac{x^i}{l^2\sigma^-_3}$ $\Gamma_{DTHN}{}^i_{jk} = \frac{\delta^i_j x^k + \delta^i_k x^j}{l^2\sigma^-_{E,\,3}}$	(3, 1)	$(+,-,-;+)$	$c_r \to 0$
$\mathfrak{e}'$	E'	$\boldsymbol{g}^{E'} = \frac{1}{\nu^2 t^2}\left[\frac{l^2\sigma^+_3}{t^2}dt^2 + d\boldsymbol{x}\cdot d\boldsymbol{x} - \frac{2}{t}\boldsymbol{x}\cdot d\boldsymbol{x}\,dt\right]$	(4, 4)	$(+,+,+,+)$	$c_r \to \infty$
$\mathfrak{p}'$	P'	$\boldsymbol{g}^{P'} = \frac{1}{\nu^2 t^2}\left[\frac{l^2\sigma^-_3}{t^2}dt^2 - d\boldsymbol{x}\cdot d\boldsymbol{x} + \frac{2}{t}\boldsymbol{x}\cdot d\boldsymbol{x}\,dt\right]$	(4, 4)	$(+,-,-,-)$	$c_r \to \infty$

Alg	Geom. name	Geometrical variables	$(\boldsymbol{g},\ \boldsymbol{h})$ Ranks	Signature	Contraction
$\mathfrak{g}$	G	$\boldsymbol{g}^{G} = c^2 dt^2$ $\boldsymbol{h}_{G} = -\delta^{ij}\partial_i\partial_j$ $\Gamma^{\lambda}_{G\mu\nu} = 0$	(1, 3)	$(+;-,-,-)$	$l_r, c_r \to \infty$ $\nu_r \to 0$
$\mathfrak{g}$	EG	$\boldsymbol{g}^{EG} = c^2 dt^2$ $\boldsymbol{h}_{EG} = \delta^{ij}\partial_i\partial_j$ $\Gamma^{\lambda}_{G\mu\nu} = 0$	(1, 3)	$(+;+,+,+)$	$l_r, c_r \to \infty$ $\nu_r \to 0$
$\mathfrak{c}$	C	$\boldsymbol{g}^{C} = -d\boldsymbol{x}\cdot d\boldsymbol{x}$ $\boldsymbol{h}_{C} = \partial^2_{ct}$ $\Gamma^{\lambda}_{C\mu\nu} = 0$	(3, 1)	$(-,-,-;+)$	$l_r \to \infty$ $c_r \to 0$
$\mathfrak{c}$	EC	$\boldsymbol{g}^{EC} = d\boldsymbol{x}\cdot d\boldsymbol{x}$ $\boldsymbol{h}_{EC} = \partial^2_{ct}$ $\Gamma^{\lambda}_{C\mu\nu} = 0$	(3, 1)	$(+,+,+;+)$	$l_r \to \infty$ $c_r \to 0$

Alg	Geom. name	Geometrical variables	$(\boldsymbol{g},\ \boldsymbol{h})$ Ranks	Signature	Contraction
$\mathfrak{c}_2$	C_2	$\boldsymbol{g}^{C_2} = -d(\boldsymbol{x}/\nu t)\cdot d(\boldsymbol{x}/\nu t)^{\,2}$ $\boldsymbol{h}_{C_2} = l^{-2}\nu^2 t^2 x^\mu x^\nu \partial_\mu \partial_\nu$ ${\Gamma_{C_2}}^0_{00} = -\frac{2}{ct}$ ${\Gamma_{C_2}}^i_{j0} = {\Gamma_{C_2}}^i_{0j} = -\frac{1}{ct}\delta^i_j$	(3, 1)	$(-,-,-;+)$	$l_r \to 0$ $c_r \to \infty$
$\mathfrak{c}_2$	EC_2	$\boldsymbol{g}^{EC_2} = d(\boldsymbol{x}/(\nu t))\cdot d(\boldsymbol{x}/(\nu t))$ $\boldsymbol{h}_{EC_2} = l^{-4}c^2 t^2 x^\mu x^\nu \partial_\mu \partial_\nu$ ${\Gamma_{EC_2}}^0_{00} = -\frac{2}{ct}$ ${\Gamma_{EC_2}}^i_{j0} = {\Gamma_{EC_2}}^i_{0j} = -\frac{1}{ct}\delta^i_j$	(3, 1)	$(+,+,+;+)$	$l_r \to 0$ $c_r \to \infty$
$\mathfrak{g}_2$	EG_2	$\boldsymbol{g}^{EG_2} = l^2\,\frac{(\boldsymbol{x}\cdot\boldsymbol{dx})^2 - (\boldsymbol{x}\cdot\boldsymbol{x})(\boldsymbol{dx}\cdot\boldsymbol{dx})}{(\boldsymbol{x}\cdot\boldsymbol{x})^2}$ $\boldsymbol{h}_{EG_2} = -l^{-4}\boldsymbol{x}\cdot\boldsymbol{x}(x^\mu\partial_\mu)^2$ Free parameter: $lx^0/\sqrt{\boldsymbol{x}\cdot\boldsymbol{x}}$ ${\Gamma_{G_2}}^0_{0i} = {\Gamma_{G_2}}^0_{i0} = -\frac{x^i}{\boldsymbol{x}\cdot\boldsymbol{x}}$ ${\Gamma_{G_2}}^l_{ij} = -\frac{\delta^l_i x^j + \delta^l_j x^i}{\boldsymbol{x}\cdot\boldsymbol{x}}$	(2, 1)	$(-,-;-)$	$l_r, c_r \to 0$ $\nu_r \to \infty$
$\mathfrak{g}_2$	G_2	$\boldsymbol{g}^{G_2} = l^2\,\frac{(\boldsymbol{x}\cdot\boldsymbol{x})(\boldsymbol{dx}\cdot\boldsymbol{dx}) - (\boldsymbol{x}\cdot\boldsymbol{dx})^2}{(\boldsymbol{x}\cdot\boldsymbol{x})^2}$ $\boldsymbol{h}_{G_2} = -l^{-4}\boldsymbol{x}\cdot\boldsymbol{x}(x^\mu\partial_\mu)^2$ Free parameter: $lx^0/\sqrt{\boldsymbol{x}\cdot\boldsymbol{x}}$ ${\Gamma_{G_2}}^0_{0i} = {\Gamma_{G_2}}^0_{i0} = -\frac{x^i}{\boldsymbol{x}\cdot\boldsymbol{x}}$ ${\Gamma_{G_2}}^l_{ij} = -\frac{\delta^l_i x^j + \delta^l_j x^i}{\boldsymbol{x}\cdot\boldsymbol{x}}$	(2, 1)	$(+,+;-)$	$l_r, c_r \to 0$ $\nu_r \to \infty$

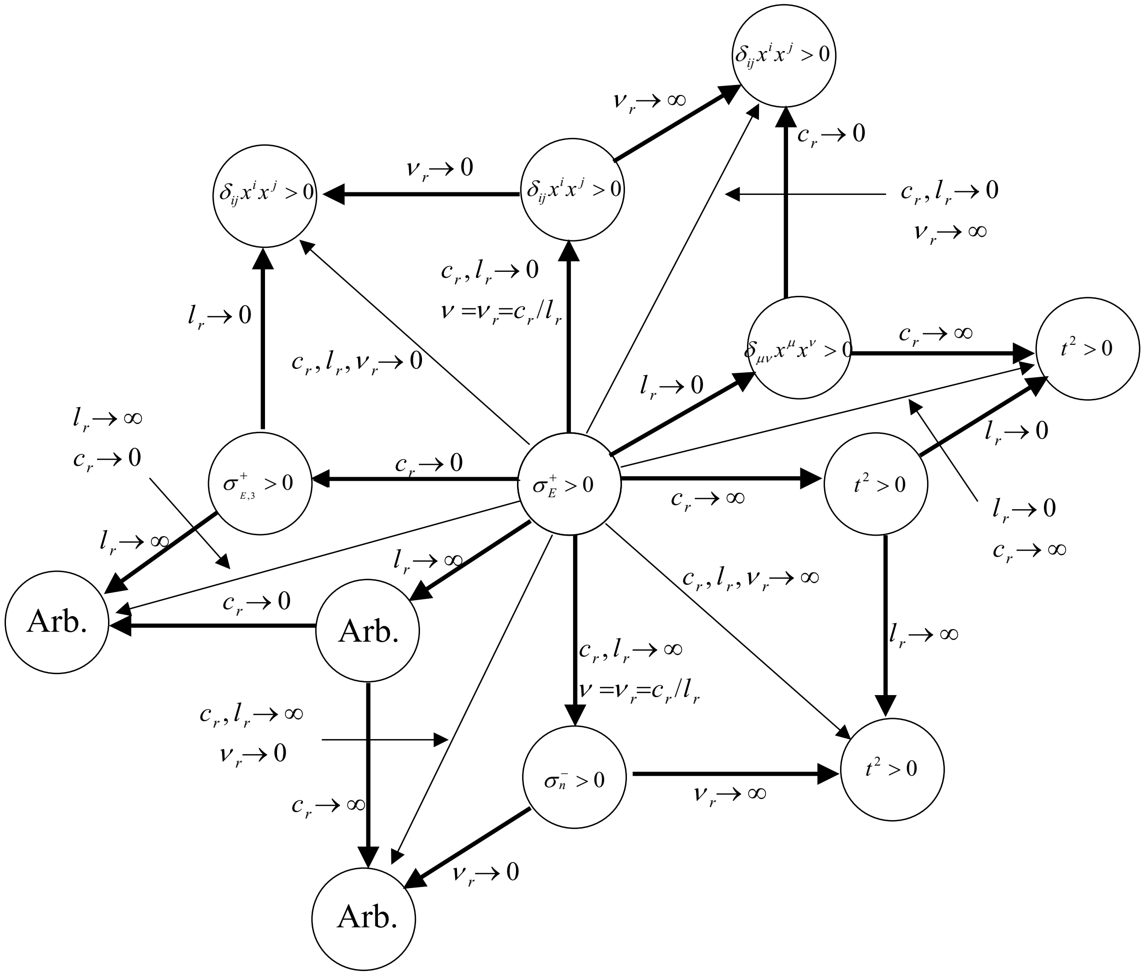
Alg	Geom. name	Geometrical variables	$(\boldsymbol{g},\ \boldsymbol{h})$ Ranks	Signature	Contraction
$\mathfrak{g}'$	G'	$\boldsymbol{g}^{G'} = -l^2(d\frac{1}{\nu t})^2$ $\boldsymbol{h}_{G'} = (\nu t)^{-2}\delta^{ij}\partial_i\partial_j$ $\Gamma_{\mathfrak{g}'}{}^{0}_{00} = -\frac{2}{ct}$ $\Gamma_{\mathfrak{g}'}{}^{i}_{j0} = \Gamma_{\mathfrak{g}'}{}^{i}_{0j} = -\frac{1}{ct}\delta^i_j$	(1, 3)	$(-;+,+,+)$	$c_r, l_r \to \infty$, $\nu_r \to \infty$
$\mathfrak{g}'$	EG'	$\boldsymbol{g}^{EG'} = l^2(d\frac{1}{\nu t})^2$ $\boldsymbol{h}_{EG'} = (\nu t)^{-2}\delta^{ij}\partial_i\partial_j$ $\Gamma_{\mathfrak{g}'}{}^{0}_{00} = -\frac{2}{ct}$ $\Gamma_{\mathfrak{g}'}{}^{i}_{j0} = \Gamma_{\mathfrak{g}'}{}^{i}_{0j} = -\frac{1}{ct}\delta^i_j$	(1, 3)	$(+;+,+,+)$	$c_r, l_r \to \infty$, $\nu_r \to \infty$
$\mathfrak{g}'_2$	G'_2	$\boldsymbol{g}^{G'_2} = l^2\frac{(\boldsymbol{x}\cdot\boldsymbol{dx})^2-(\boldsymbol{x}\cdot\boldsymbol{x})(\boldsymbol{dx}\cdot\boldsymbol{dx})}{(\boldsymbol{x}\cdot\boldsymbol{x})^2}$ $\boldsymbol{h}_{G'_2} = c^{-2}l^{-2}(\boldsymbol{x}\cdot\boldsymbol{x})(\partial_t)^2$ Free parameter: $l^2(\boldsymbol{x}\cdot\boldsymbol{x})^{-1/2}$ $\Gamma_{G'_2}{}^{0}_{0i} = \Gamma_{G'_2}{}^{0}_{i0} = -\frac{x^i}{\boldsymbol{x}\cdot\boldsymbol{x}}$ $\Gamma_{G'_2}{}^{l}_{ij} = -\frac{\delta^l_i x^j+\delta^l_j x^i}{\boldsymbol{x}\cdot\boldsymbol{x}}$	(2, 1)	$(-,-;+)$	$c_r, l_r \to 0$, $\nu_r \to 0$
$\mathfrak{g}'_2$	EG'_2	$\boldsymbol{g}^{EG'_2} = l^2\frac{(\boldsymbol{x}\cdot\boldsymbol{x})(\boldsymbol{dx}\cdot\boldsymbol{dx})-(\boldsymbol{x}\cdot\boldsymbol{dx})^2}{(\boldsymbol{x}\cdot\boldsymbol{x})^2}$ $\boldsymbol{h}_{EG'_2} = c^{-2}l^{-2}(\boldsymbol{x}\cdot\boldsymbol{x})(\partial_t)^2$ Free parameter: $l^2(\boldsymbol{x}\cdot\boldsymbol{x})^{-1/2}$ $\Gamma_{EG'_2}{}^{0}_{0i} = \Gamma_{EG'_2}{}^{0}_{i0} = -\frac{x^i}{\boldsymbol{x}\cdot\boldsymbol{x}}$ $\Gamma_{G'_2}{}^{l}_{ij} = -\frac{\delta^l_i x^j+\delta^l_j x^i}{\boldsymbol{x}\cdot\boldsymbol{x}}$	(2, 1)	$(+,+;+)$	$c_r, l_r \to 0$, $\nu_r \to 0$

Why so many?

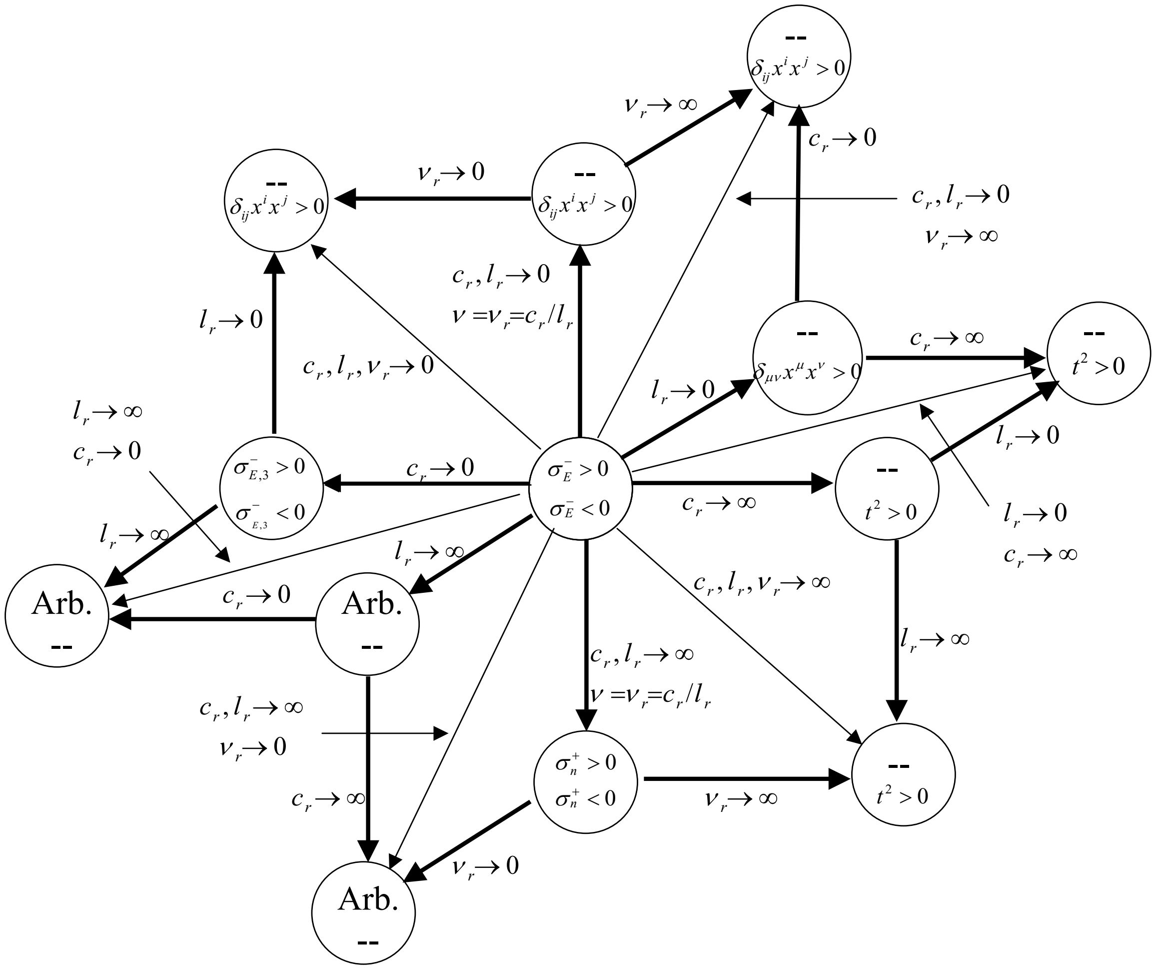
Domain conditions

$$
\begin{aligned}
\sigma_E^+ &:= 1 + l^{-2}\delta_{\mu\nu}x^\mu x^\nu > 0 \\
\sigma_E^- &:= 1 - l^{-2}\delta_{\mu\nu}x^\mu x^\nu \gtrless 0 \\
\sigma^+ &:= 1 - l^{-2}\eta_{\mu\nu}x^\mu x^\nu \gtrless 0 \\
\sigma^- &:= 1 + l^{-2}\eta_{\mu\nu}x^\mu x^\nu \gtrless 0
\end{aligned}
$$

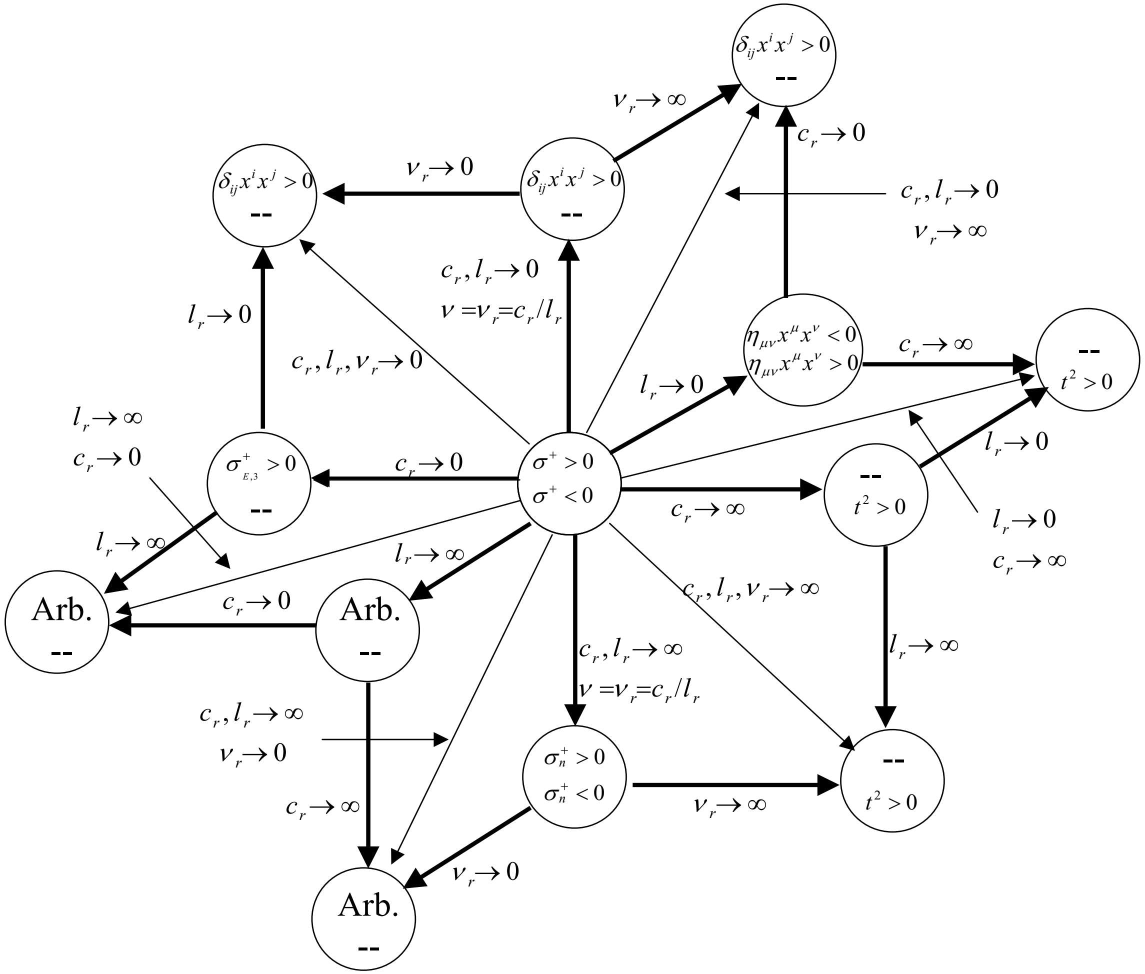
$$(\sigma_E^+ = 1 + l^{-2}\delta_{\mu\nu}x^\mu x^\nu)$$



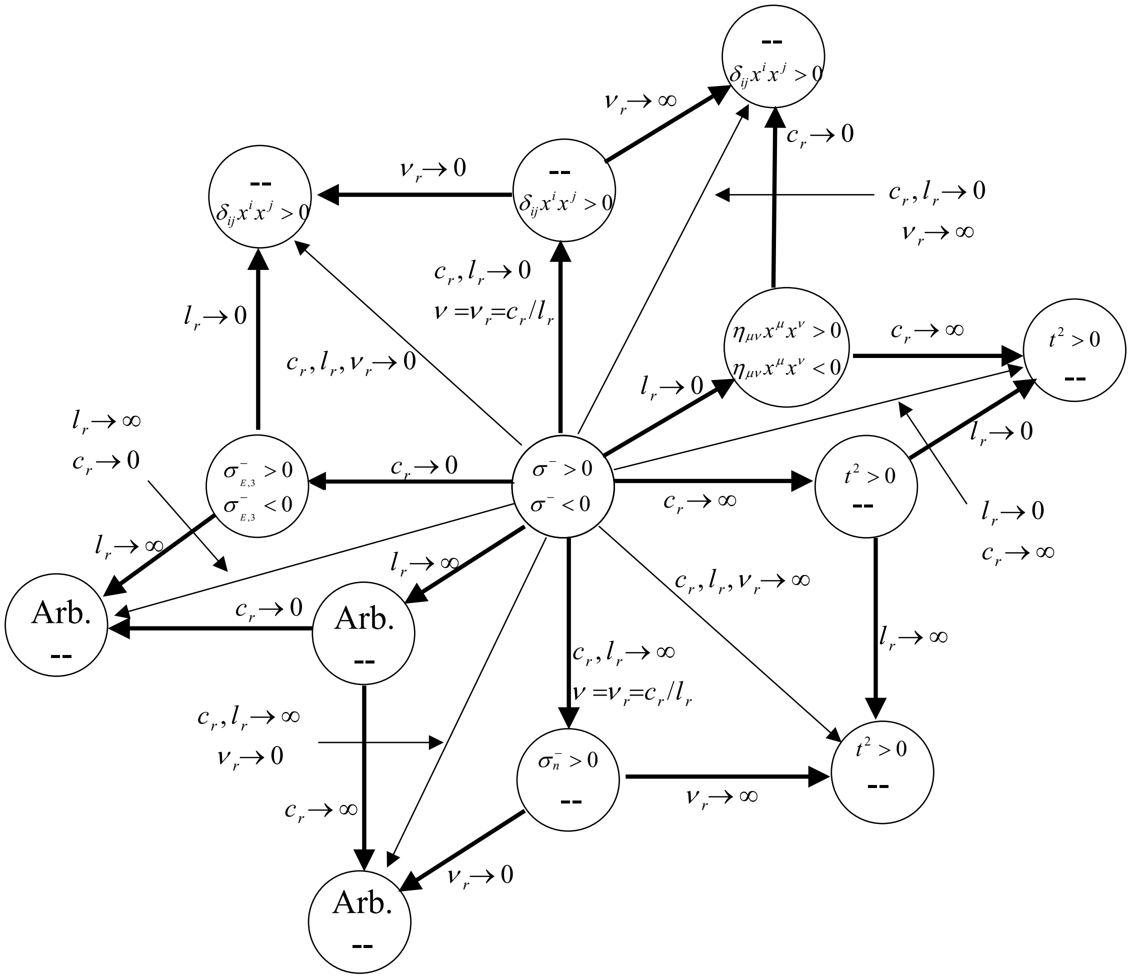
$$(\sigma_E^- = 1 - l^{-2}\delta_{\mu\nu}x^\mu x^\nu)$$



$$(\sigma^+ = 1 - l^{-2}\eta_{\mu\nu}x^\mu x^\nu)$$



$$(\sigma^- := 1 + l^{-2}\eta_{\mu\nu}x^\mu x^\nu)$$



Summary

There are 45 geometries in all.

All of them have the same vanishing Weyl projective curvature tensor

$$W^{\lambda}{}_{\mu\sigma\nu} := R^{\lambda}{}_{\mu\sigma\nu} + \frac{1}{3}(\delta^{\lambda}_{\sigma} R_{\mu\nu} - \delta^{\lambda}_{\nu} R_{\mu\sigma}). \tag{27}$$

From the viewpoint of DG, these geometries are not all independent. For example, dS and $LBdS$ describe the same space-time in different coordinate systems.

$$\left(dS : x^{\mu} := l\frac{\xi^{\mu}}{\xi^{4}}, \qquad LBdS : x^{\alpha} = l\frac{\xi^{\alpha}}{\xi^{0}} \right)$$

Therefore, the number of independent geometries will be less than 45.

Outline:

I. Introduction

II. 22 Possible Kinematics

III. Geometries for All Possible Kinematics

IV. Relations and Classification

V. Concluding Remarks

★ Duality btwn present & time infinity

The geometries for the possible kinematics almost appear in pairs.

Each pair are invariant under the transformation

$$\frac{1}{\nu^2 t} \Leftrightarrow t, \quad \frac{x^i}{\nu t} \Leftrightarrow x^i. \tag{28}$$

Table 1: Duality of the present and the time infinity

$t,\ x^i$	$\Leftrightarrow$	$1/\nu^2 t,\ x^i/\nu t$	$t,\ x^i$	$\Leftrightarrow$	$1/\nu^2 t,\ x^i/\nu t$
Min	$\Leftrightarrow$	P'	Euc	$\Leftrightarrow$	E'
G	$\Leftrightarrow$	G'	EG	$\Leftrightarrow$	EG'
C	$\Leftrightarrow$	C_2	EC	$\Leftrightarrow$	EC_2
G_2	$\Leftrightarrow$	G'_2	EG_2	$\Leftrightarrow$	EG'_2
HN_+	$\Leftrightarrow$	E_{2-}	EHN_+	$\Leftrightarrow$	E_2
HN_-	$\Leftrightarrow$	P_{2-}	EHN_-	$\Leftrightarrow$	EP_{2-}
HN'_-	$\Leftrightarrow$	P_{2+}	$DTHN_-$	$\Leftrightarrow$	DTP_{2+}
NH_+	$\Leftrightarrow$	NH'_+	ENH_+	$\Leftrightarrow$	ENH'_+
NH_2	$\Leftrightarrow$	NH'_2			
NH_-	$\Leftrightarrow$	NH_-	ENH_-	$\Leftrightarrow$	ENH_-
ENH_2	$\Leftrightarrow$	ENH_2	$DTNH_2$	$\Leftrightarrow$	$DTNH_2$

From the viewpoint of DG, the independent geometries are

$Riem$, Lob, dS, AdS, $DTdS$, Euc, Min, — 7 known ND

$ENH_\pm$, $NH_\pm$, EG, G, EC, C, — 8 known deg

E_2, EP_{2-}, $P_{2\pm}$, E_{2-}, DTP_{2+}, ENH_2, NH_2, $DTNH_2$, EG_2, G_2 — 11 new deg

There are 26 in all.

★ Spatial isotropy

Geom.	Ranks of $\boldsymbol{g}$&$\boldsymbol{h}$	Signature	topology
$DTdS$	(4,4)	$(+,+,-,-)$	
P_{2+}	(3,1)	$(+,-,-;-)$	$dS_3 \times \mathbb{R}_{\mathrm{sp}}$
DTP_{2+}	(3,1)	$(+,-,-;+)$	$dS_3 \times \mathbb{R}_{\mathrm{t}}$
ENH_2	(2,2)	$(+,+;+,+)$	$\mathbb{S}^2 \times \mathbb{R}^2$
NH_2	(2,2)	$(-,-;+,-)$	$\mathbb{S}^2 \times \mathbb{R}^2_{\mathrm{sp_t}}$
$DTNH_2$	(2,2)	$(-,-;+,+)$	$\mathbb{S}^2 \times \mathbb{R}^2_{DT}$
EG_2	(2,1)	$(+,+;+)$	$\mathbb{S}^2 \times \mathbb{R} \times \mathbb{R}_{\mathrm{f}}$
G_2	(2,1)	$(-,-;+)$	$\mathbb{S}^2 \times \mathbb{R}_{\mathrm{t}} \times \mathbb{R}_{\mathrm{f}}$

They've no spatial isotropy though they've $\mathfrak{so}(3)$ isotropy.

The remaining geometries are

$Riem$, Lob, dS, AdS, Euc, Min, — 6 known ND

$ENH_\pm$, $NH_\pm$, EG, G, EC, C, — 8 known deg

E_2, EP_{2-}, P_{2-}, E_{2-} — 4 new deg

There are 18 in all.

They appear in 9 pairs:

$$(Riem, dS) \quad (Lob, AdS) \quad (Euc, Min) \quad (ENH_\pm, NH_\pm)$$
$$(EG, G) \quad (EC, C) \quad (E_2, E_{2-}) \quad (EP_{2-}, P_{2-})$$

★ Geometries with Lorentz-like signature

The geometries for genuine possible kinematics should have right signature.

Then, only 9 geometries remains. They are

CC	>0	$=0$	<0
Relativistic	dS	Min	AdS
Absolute-time	NH_+	G	NH_-
Absolute-space	E_{2-}	C	P_{2-}

Outline:

I. Introduction

II. 22 Possible Kinematics

III. Geometries for All Possible Kinematics

IV. Relations and Classification

V. Concluding Remarks

The right requirements to pick up the genuine possible kinematics should be that

(1) *space is isotropic w.r.t. any point on the manifold;*

(2) *parity and time-reversal are automorphisms of the kinematical groups;*

(3) *the geometry has Lorentz-like signature.*

Then, the geometries for genuine possible kinematics are

3 relativistic geometries, dS, AdS, Min;

3 absolute-time geometries, $NH_{\pm}$, G;

3 absolute-space geometries E_{2-}, P_{2-}, C;

and their time dualities, P', NH'_{+}, G', $HN_{\pm}$ and C_2.

In DG, a geom & time-duality describe same geometry.

Hence, only 9 remains.

THANK YOU!