Exact Solutions to Einstein's Field Equations

H. Lü

ICTS - USTC - June - 2004

Einstein { Special Relativity 1905
General Relativity 1915 }

Almost 100 years!

An encyclopaedic book of the same title by H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, E. Herlt (CUP 2003) $D=4$

This talk is inspired by the book.

A review of what's been achieved

$$\mathcal{L} = \sqrt{-g}\,R$$

GR is a beautiful theory; it seems to admit no elegant modification.

Fundamental theory of nature:

Principle of general coordinates invariance

Principle of Quantum mechanics

QM: God lacks basic concept of Estheticism

Exact solutions to a theory in D=4

* Local solutions

* Global structure

The problem in general relativity is that you don't know where you are, and you don't know ~~where~~ what time it is

—— Sidney Coleman

Let us consider a Ricci-flat

metric:

$$ds^2 = -dt^2 + \frac{r^2 + a^2 \cos^2\theta}{r^2 + a^2} dr^2$$

$$+ (r^2 + a^2 \cos^2\theta) [d\theta^2 + \sin^2\theta \, d\phi^2]$$

but this is not a new metric.

make a coordinate transformation:

$$r^2 \cos^2\theta = y^2 \cos^2\zeta$$

$$(r^2 + a^2) \sin^2\theta = y^2 \sin^2\zeta$$

we get:

$$ds^2 = -dt^2 + dy^2 + y^2 (d\zeta^2 + \sin^2\zeta \, d\phi^2)$$

BTZ is anly example of claiming fame by doing a Local transformation

We shall be concerned with a $\boxed{D=4}$ gravity coupled to a maxwell theory:

$$\mathcal{L} = \sqrt{-g}\left(R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \Lambda\right)$$

$$F_{\mu\nu} = \partial_{[\mu} A_{\nu]}$$

Equations of motion:

$$\nabla_\mu F^{\mu\nu} = 0$$

$$R_{\mu\nu} = \frac{1}{2}\left(F_{\mu\rho} F_\nu{}^\rho - \frac{1}{4} F_{\rho\sigma} F^{\rho\sigma} g_{\mu\nu}\right) + \frac{1}{2}\Lambda g_{\mu\nu}$$

SA

Schwarzschild black hole

$$ds^2 = -f\,dt^2 + \frac{dr^2}{f} + r^2 d\Omega_2^2$$

$$f = 1 - \frac{2m}{r}$$

$m<0$ naked singularity

Coordinate singularity: $r = r_0 = 2m$

$f = 0$: $r = r_0 = 2m$ $\boxed{\text{horizon}}$

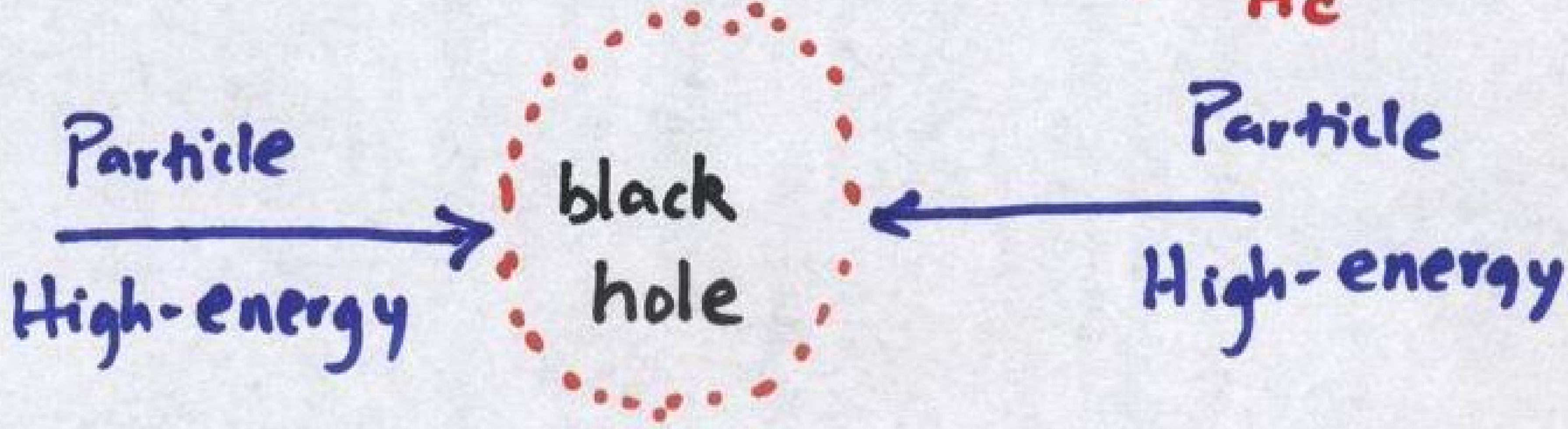
physical singularity : $r = 0$ $\frac{1}{2} m v^2 = \frac{GMm}{r_0^2}$

$$(R^{\mu}{}_{\nu\rho\sigma})^2 = \frac{48m^2}{r^6} \longrightarrow \infty \quad \text{at } r=0$$

$$r_0 = \frac{2GM}{c^2}$$

$$\shortparallel$$

$$r_0 = \frac{\hbar}{Mc}$$



Minimum Scale: Plank Scale $= \sqrt{\frac{G\hbar}{c^3}} \sim 10^{-33}$ cm

$M_{PL} = \sqrt{\frac{c\hbar}{G}} \sim 10^{-5}$ gram

Reissner - Nordström BH

(charged black hole)

$$ds^2 = -\frac{\Delta_r}{r^2}dt^2 + \frac{r^2 dr^2}{\Delta_r} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

$$\Delta_r = r^2 - 2mr + p^2 + q^2$$

$$= (r-m)^2 + (p^2 + q^2 - m^2)$$

$$A_{(1)} = \frac{q}{r}dt + p\cos\theta\, d\phi$$

electric (q)

magnetic (p)

$r=0 \rightarrow$ curvature singularity

$$\begin{cases} m^2 > p^2+q^2 & \text{singularity inside (two) horizons} \\ m^2 < p^2+q^2 & \text{naked singularity} \end{cases}$$

$m^2 = p^2 + q^2$?

$m^2 = p^2 + q^2$ $\qquad$ $\Delta_r = (r-m)^2$

$r \to \infty$, $\qquad ds^2 \to$ Minkowski

$r \to m$, $\qquad$ define $r - m = \rho \to 0$

$$ds^2 = -\frac{\rho^2}{m^2}\,dt^2 + \frac{m^2\,d\rho^2}{\rho^2} + m^2\,d\Omega_2^2$$

$$adS_2 \quad \times \quad S^2$$

Interpolates

$AdS_2 \times S^2$ $\qquad$ $(\text{Minkowski})_4$

$\boxed{\text{BPS}}$

(A)dS Charge Black Hole

$$ds^2 = -\frac{\Delta_r}{r^2}dt^2 + \frac{r^2 dr^2}{\Delta_r} + r^2 d\Omega_2^2$$

$$\Delta_r = r^2(1+g^2r^2) - 2mr + p^2 + q^2$$

$$A_{(1)} = \frac{q}{r}dt + p\cos\theta\, d\phi$$

Cosmological constant $\Lambda = -6g^2$

if $p = q = 0 \Rightarrow$ AdS Schwarzschild

Interesting feature

Large AdS black hole can be stable

$r \to \infty$ asymptotic

$$ds^2 \xrightarrow[r\to\infty]{} -(1+g^2r^2)dt^2 + \frac{dr^2}{1+g^2r^2} + r^2 d\Omega_2^2$$ AdS_4 boundary

NUT solution

Newman Unti Tambourino

$$ds^2 = -\frac{r^2-N^2-2Mr}{r^2+N^2}\left(dt - 2N\cos\theta\, d\phi\right)^2$$

$$+\frac{r^2+N^2}{r^2-N^2-2Mr}\, dr^2 + (r^2+N^2)\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

more natural in Euclidean signature

$N \to iN, \quad t \to i\psi$

$$ds^2 = +\frac{r^2+N^2-2Mr}{r^2-N^2}\left(d\psi - 2N\cos\theta\, d\phi\right)^2$$

$$+\frac{r^2-N^2}{r^2+N^2-2Mr}\, dr^2 + (r^2-N^2)\, d\Omega_2^2$$

BPS limit : $M = N$

$$ds^2_{NUT} = \frac{r-N}{r+N}(d\psi - 2N\cos\theta\, d\phi)^2 + \frac{r+N}{r-N}dr^2 + (r^2-N^2)(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$r \to N \Rightarrow R^4 \qquad r \to \infty \quad R^3 \times S^1$

if there is a nut, there has to be a $\boxed{\text{bolt}}$:

$$ds^2_{Bolt} = \frac{(r-2N)(r-\frac{N}{2})}{r^2-N^2}(d\psi - \underline{\underline{2N\cos\theta\, d\phi}})^2 + \frac{dr^2\,(r^2-N^2)}{(r-2N)(r-\frac{N}{2})} + (r^2-N^2)(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$r \to 2N \Rightarrow R^2 \times S^2 \qquad r \to \infty \quad R^3 \times S^1$

AdS-Charged NUT black hole

$g^2 \qquad p^2+q^2 \qquad N \qquad M$

$$ds_4^2 = -\frac{P}{r^2+N^2}(dt - 2N\cos\theta\, d\phi)^2 + \frac{r^2+N^2}{P}dr^2 + (r^2+N^2)(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$A_{(1)} = \frac{qr - Np}{r^2+N^2}dt - \frac{p(r^2-N^2)+2Nqr}{r^2+N^2}\cos\theta\, d\phi$$

$$P(r) = g^2(r^2+N^2) + (1+4g^2N^2)(r^2-N^2) - 2Mr + q^2 + p^2$$

Properties of solutions so far

1. One is forced to make a symmetry assumption.

2. Cohomogeneity one

3. EOM reduces to non-linear ordinary differential equations

More Challenging task

Cohomogeneity two

have to deal with

Non-linear partial differential equations.

techniques

* Kerr-schild form

* Separation of variables [Carter]

* Harmonic superposition

Kerr-Schild form

Example: Schwarzschild

$$ds^2 = -f\,dt^2 + \frac{dr^2}{f} + r^2 d\Omega_2^2 \qquad f = 1 - \frac{2m}{r}$$

$$= -f\left[dt - \frac{dr}{f}\right]\left[dt + \frac{dr}{f}\right] + r^2 d\Omega_2^2$$

define $du = dt - \frac{dr}{f}$

$$ds^2 = -du^2 + 2\,dr\,du + r^2 d\Omega_2^2 + \frac{2m}{r} du$$

$$du = d\tilde{t} - dr$$

$$ds^2 = \underbrace{-d\tilde{t}^2 + dr^2 + r^2 d\Omega_n^2}_{\text{Minkowski}} + \underbrace{\frac{m}{r^{n-1}}(d\tilde{t} - dr)^2}_{\text{linear fructuation}}$$

Kerr solution (rotating black hole)

$$ds^2 = ds_0^2 + f\,(k_\mu dx^\mu)^2$$

K_μ a null vector under ds_0^2

f : linear fructuation

$$\overset{\leftarrow \text{Minkowski}}{ds_0^2} = -dt + F\,dr^2 + (r^2 + a^2\cos^2\theta)[d\theta^2 + \sin^2\theta\, d\phi^2]$$

$$K_\mu dx^\mu = dt + F\,dr - \underset{\uparrow \text{angular momentum}}{a}\sin^2\theta\, d\phi$$

rotating black hole $\quad F = \frac{r^2 + a^2\cos^2\theta}{r^2 + a^2}$

$$ds^2 = ds_0^2 + \frac{2\overset{\swarrow \text{mass}}{m}r}{r^2 + a^2\cos^2\theta}(K_\mu dx^\mu)^2$$

Cohomogeneity two : $\boxed{r, \ \theta}$

AdS - Kerr - Newman solution

$\uparrow$ rotating (Kerr) $\uparrow$ Charged (Newman)

Boyer-Lindquist form.

$$ds^2 = \rho^2\left(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta}\right)$$

$$+ \frac{\sin^2\theta\,\Delta_\theta}{\rho^2}\left(a\,dt - (r^2+a^2)\,d\phi\right)^2$$

$$- \frac{\Delta_r}{\rho^2}\left(dt - a\sin^2\theta\,d\phi\right)^2$$

$$A_{(1)} = \frac{qr}{\rho^2}\left(dt - a\sin^2\theta\,d\phi\right)$$

$$+ \frac{p\cos\theta}{\rho^2}\left(a\,dt - (r^2+a^2)\,d\phi\right)$$

$$\rho^2 = r^2 + a^2\cos^2\theta$$

$$\Delta_r = (r^2+a^2)(1+g^2r^2) - 2mr + p^2 + q^2$$

$$\Delta_\theta = 1 - g^2a^2\cos^2\theta$$

Can also be in Kerr-Schild form

Kerr Newman

Rotating AdS Charged NUT

$$ds^2 = \frac{p^2+q^2}{X}dp^2 + \frac{p^2+q^2}{Y}dq^2$$
$$+ \frac{X}{p^2+q^2}(d\tau + q^2 d\sigma)^2 - \frac{Y}{p^2+q^2}(d\tau - p^2 d\sigma)^2$$

$$X = \gamma - g^2 - \epsilon p^2 - \lambda p^4 + 2lp$$

$$Y = \gamma + e^2 + \epsilon q^2 - \lambda q^4 - 2mq$$

$(e, g) \leftarrow$ electric, magnetic charges

(γ, m, l) related to angular momentum mass and NUT charges.

λ : cosmological constant

ϵ : 0, 1, −1

Carter: Separation of variables

Plebañski metric

$$ds^2 = \frac{1}{(1-pq)^2}\left[\frac{p^2+q^2}{X}dp^2 + \frac{p^2+q^2}{Y}dq^2 + \frac{X}{p^2+q^2}\left(d\tau + q^2 d\sigma\right)^2 - \frac{Y}{p^2+q^2}\left(d\tau - p^2 d\sigma\right)^2\right]$$

$X(p)$

$Y(q)$

I have Not yet studied it in detail

Harmonic superposition

Reissner - Nordström BPS black hole

$$ds^2 = -H^{-2}dt^2 + H^2(dr^2 + r^2 d\Omega_2^2)$$

$$= -H^{-2}dt^2 + H^2 dy^i dy^i$$

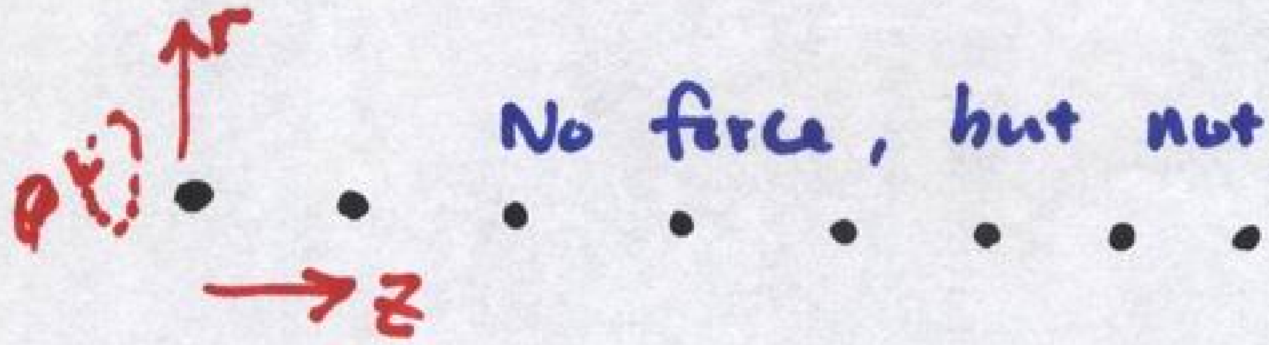
$$A_{(1)} = (H^{-1}-1)dt$$

EOM $\Rightarrow$ $\quad \Box H = \partial_i \partial_i H = 0$

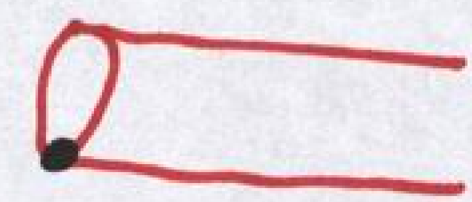
$$H = \sum \frac{q_i}{|\vec{y} - \vec{y}_i|} + 1$$

No-force condition

Array of Schwarzschild black hole



No force, but not stable



Can be stable.

$$ds^2 = -e^{2U} dt^2 + e^{2K-2U}(dr^2 + dz^2) + e^{-2U} r^2 d\theta^2$$

$U = U(r, z)$ $\qquad$ $K = K(r, z)$

$$\nabla^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \frac{d^2}{dz^2}$$

Laplacian of 3-D space in cylindrical Polar coordinates

$\nabla^2 U = 0$ harmonic

Superposition.

$' = \frac{\partial}{\partial r}$

$\dot{} = \frac{\partial}{\partial z}$

$K' = (U'^2 - \dot{U}^2) r$

$\dot{K} = \dot{U} U' r$

Schwarzschild in axially symmetric

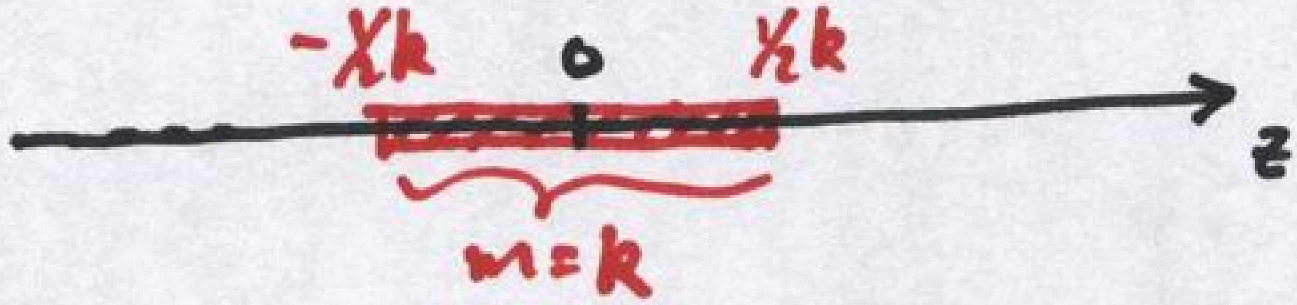
Coordinates:

$$U = \frac{1}{2} \log \frac{\sigma + \tilde{\sigma} - k}{\sigma + \tilde{\sigma} + k}$$

$$\sigma = \sqrt{r^2 + \left(z - \frac{k}{2}\right)^2} \qquad \tilde{\sigma} = \sqrt{r^2 + \left(z + \frac{k}{2}\right)^2}$$

Potential of a uniform thin rod of

(Newtonian) mass k and length $\frac{1}{2}k$



$$K = \frac{1}{2} \log \frac{(\sigma + \tilde{\sigma} - k)(\sigma + \tilde{\sigma} + k)}{4\sigma\tilde{\sigma}}$$

$$\boxed{M = 8k}$$

Periodic Array of black holes

$$U = \frac{1}{2}\sum_{n} \frac{\sigma_n + \tilde{\sigma}_n - k_n}{\sigma_n + \tilde{\sigma}_n + k_n}$$

$$\sigma_n = \sqrt{r^2 + (z - z_n - k_n/2)^2}$$

$$\tilde{\sigma}_n = \sqrt{r^2 + (z - z_n + k_n/2)^2}$$

$$K = \frac{1}{4}\sum_{m,n=1}^{N} \log \frac{\sigma_m \tilde{\sigma}_n + (z - z_m - \tfrac{1}{2}k_m)(z - z_n + \tfrac{1}{2}k_n) + r^2}{\sigma_m \sigma_n + (z - z_m - \tfrac{1}{2}k_m)(z - z_n - \tfrac{1}{2}k_n) + r^2}$$

$$+ \frac{1}{4}\sum_{m,n=1}^{N} \log \frac{\tilde{\sigma}_m \sigma_n + (z - z_m + \tfrac{1}{2}k_m)(z - z_n - \tfrac{1}{2}k_n) + r^2}{\tilde{\sigma}_m \sigma_n + (z - z_m + \tfrac{1}{2}k_m)(z - z_n + \tfrac{1}{2}k_n) + r^2}$$

whoa!

what if there are only two centers i.e. two black holes?

Cosmological solution (time dependent)

Exact & well behaved solutions are rare:

Example: Cosmological bounce

Start from Kerr solution:

$$ds^2 = -\frac{\Delta_r}{\rho^2}\left(dt - a\sin^2\theta\, d\phi\right)^2 + \frac{\rho^2}{\Delta_r}dr^2$$

$$+\rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left(a\,dt - (r^2+a^2)\,d\phi\right)^2$$

$$\rho^2 = r^2 + a^2\cos^2\theta$$

$$\Delta_r = r^2 + a^2 - 2Mr$$

Horizon: $\Delta_r = 0$

Singularity: $\rho^2 = 0 \Rightarrow \underbrace{r = 0 \quad \cos\theta = 0 \;\; (\theta = \pi/2)}_{\text{ring of singularity.}}$

Wick rotations

$t \to i\psi \qquad r \to it \qquad a \to ia$

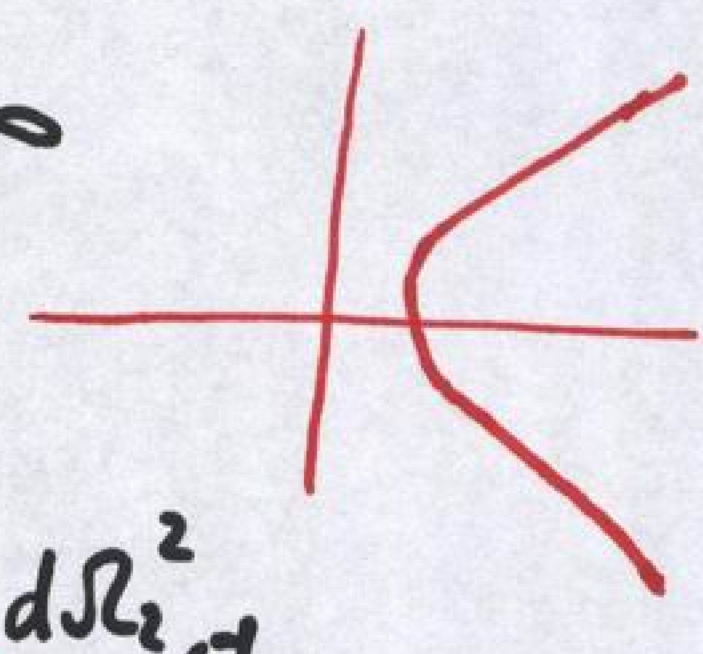
$$ds^2 = \frac{\Delta_t}{\rho^2}\left(d\psi + a\sinh^2\theta\, d\phi\right)^2 - \frac{\rho^2}{\Delta_t}dt^2 + \rho^2 d\theta^2 - \frac{\sinh^2\theta}{\rho^2}\left(a\,d\psi + (t^2+a^2)\,d\phi\right)^2$$

$\rho^2 = t^2 + a^2\cosh^2\theta > 0$ No singularity

$$\Delta_t = t^2 - 2Mt + a^2 = (t-M)^2 + a^2 - M^2$$

$a^2 > M^2 \qquad \Delta_t > 0$

Bounce : $-\infty < t < \infty$



$t \to \pm\infty$

$$ds^2 \to \underbrace{d\psi^2}_{} - \underbrace{dt^2 + t^2 d\Omega^2_{2,-1}}_{}$$

S^1 × Expanding M_3

Euclidean Signature

Euclidean space:

S^3: a group manifold

$$ds^2 = dr^2 + r^2 d\Omega_3^2$$

round

S^3:

$$\mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_4^2 = 1$$

$$ds^2 = d\mu_1^2 + d\mu_2^2 + d\mu_3^2 + d\mu_4^2$$

$$ds^2 = \frac{1}{4}\sigma_1^2 + \frac{1}{4}\sigma_2^2 + \frac{1}{4}\sigma_3^2$$

$$\sigma_1 = \cos\psi\, d\theta + \sin\psi \sin\theta\, d\phi$$

$$\sigma_2 = -\sin\psi\, d\theta + \cos\psi \sin\theta\, d\phi$$

$$\sigma_3 = d\psi + \cos\theta\, d\phi$$

θ, ϕ, ψ Euler angles

$$d\sigma_1 = -\sigma_2 \wedge \sigma_3$$

$$d\sigma_2 = -\sigma_3 \wedge \sigma_1$$

$$d\sigma_3 = -\sigma_1 \wedge \sigma_2$$

S^3: $ds^2 = \underbrace{\frac{1}{4}(d\psi + \cos\theta\, d\phi)^2}_{U(1)\text{ bundle}} + \underbrace{\frac{1}{4}(d\theta^2 + \sin^2\theta\, d\phi^2)}_{\text{over } S^2}$

Cohomogeneity one:

$$ds_4^2 = dr^2 + a^2\sigma_1^2 + b^2\sigma_2^2 + c^2\sigma_3^2$$

$a(r)$ $b(r)$ $c(r)$

Ricci flat

$$R_{rr} = -\frac{\ddot{a}}{a} - \frac{\ddot{b}}{b} - \frac{\ddot{c}}{c} = 0$$

$$R_{11} = -\frac{\ddot{a}}{a} - \frac{\dot{a}\dot{b}}{ab} - \frac{\dot{a}\dot{c}}{ac} + \frac{a^4 - b^4 - c^4}{2a^2b^2c^2} + \frac{1}{a^2} = 0$$

R_{22}

R_{33} Cyclic order

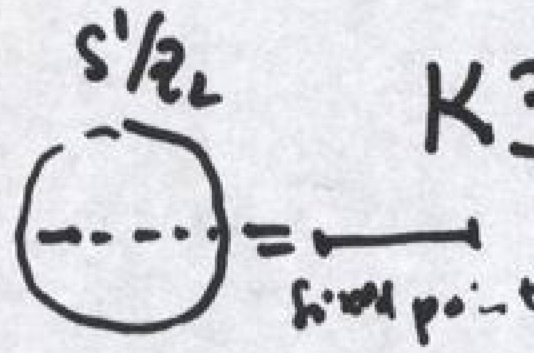
Two sets of first-order equations

(1)
$$\begin{cases} \dfrac{\dot a}{a} = \dfrac{b^2+c^2-a^2}{2abc} \\[2ex] \dfrac{\dot b}{b} = \dfrac{a^2+c^2-b^2}{2abc} \\[2ex] \dfrac{\dot c}{c} = \dfrac{a^2+b^2-c^2}{2abc} \end{cases}$$

regularity $\Rightarrow b=c$

Eguchi-Hanson metric

$R^2 \times S^2 \longleftrightarrow R^4/Z_2$



$K3 \rightarrow T^4/Z_2 \rightarrow 16$ fixed points

(2)
$$\begin{cases} \dfrac{\dot a}{a} = \dfrac{a^2-(b-c)^2}{abc} \\[2ex] \dfrac{\dot b}{b} = \dfrac{b^2-(c-a)^2}{abc} \\[2ex] \dfrac{\dot c}{c} = \dfrac{c^2-(a-b)^2}{abc} \end{cases}$$

$a \neq b \neq c$

Atiyah-Hitchin

$b=c$

Taub-NUT

a resolution of NUT with negative mass.

With a cosmological constant

$$R_{ij} = 3g_{ij}$$

For example: S^4, $S^2 \times S^2$, etc

Tri-axial first-order system

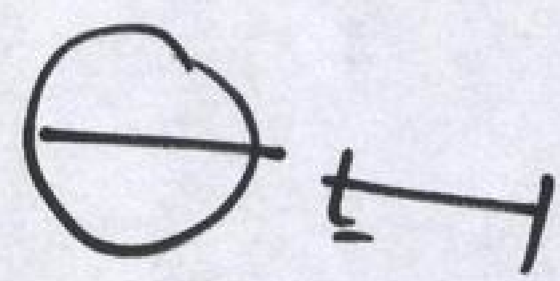
$$\frac{\dot{a}}{a} = -\frac{a^2-b^2-c^2}{2abc}$$

$$\frac{\dot{b}}{b} = -\frac{b^2-a^2-c^2}{2abc}$$

cosmological constant

$$\frac{\dot{c}}{c} = -\frac{c^2-a^2-b^2+2\Lambda a^2 b^2}{2abc}$$

$S^1/\mathbb{Z}_2$



$T^4/\mathbb{Z}_2 \to$

round S^4:

$$ds^2 = dr^2 + 4\sin^2\left(r+\tfrac{2}{3}\pi\right)\sigma_1^2 + 4\sin^2\left(r-\tfrac{2}{3}\pi\right)\sigma_2^2 + 4\sin^2 r\,\sigma_3^2$$

CP^2:

$$ds^2 = dr^2 + \sin^2 r\,\sigma_1^2 + \cos^2 r\,\sigma_2^2 + \cos^2 2r\,\sigma_3^2$$

$\vdots$

etc. an ~~ff~~ infinite number of almost regular solutions (with conical singularity)

Conclusions

* If you obtain a solution in $D=4$, it's probably known already

* Research direction $D \geq 5$

* numerical result – fit the experiment

General coordinate invariance

A principle likely survives in the "final" theory.

END