Phase structure of balck branes

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Introduction: Partition function and Euclidean path integral

- Partition function (canonical ensemble):

$$Z = \mathrm{Tr}\exp\{-\beta H\} = \sum_n \langle n|\exp\{-\beta H\}|n\rangle \qquad W = -\ln Z$$

- In Path integral language:
  Minkowski:

$$\langle j|\exp\{-itH\}|i\rangle = \int_{\phi_0=\phi_i}^{\phi_t=\phi_j} [d\phi]\exp\{i\int \mathrm{d}\tau\mathcal{L}\}$$

  Euclideanize: $t \to -it$, and periodic boundary condition $i = j = n$ with period β, trace:

$$Z = \sum_n \langle n|\exp\{-\beta H\}|n\rangle = \int_{\phi_0=\phi_\beta} [d\phi]\exp\{\int \mathrm{d}\tau\mathcal{L}_E\}$$

- Define Euclidean action: $I_E = -\int \mathrm{d}\tau \mathcal{L}_E$
- To the leading order, we use the classical solution to calculate the classical action: $Z \approx e^{-I_E}$ and $W = -\ln Z \approx I_E$.
- For canonical ensembles, $W = \beta F$, F is Helmholz free energy. $E = \langle H \rangle = \frac{\partial W}{\partial \beta}|_{V,N}$.
- For grand canonical ensemble:

$$\begin{aligned} Z &= \mathrm{Tr}\exp\{-\beta(H-\mu N\} = \sum_n \langle n| \exp\{-\beta(H-\mu N)\}|n\rangle \\ W &= -\ln Z = \beta(F-\mu N) = \beta G \approx I_E \end{aligned}$$

  G is the Gibbs free energy.
- These can be used to discuss the thermodynamics of the spacetime geometry such as black hole. We will use this formalism to discuss the thermodynamics of the black branes: Euclideanize the metric and compactify the time direction. Use the classical action to find the leading order approximation of the partition function.

Black p-brane without dilaton

Solution to D-dimensional spacetime gravity with a $p+1$ form gauge field:

$$I = \frac{1}{2\kappa^2}\int d^D x\sqrt{-g}\left[R - \frac{1}{2(d+1)!}F_{d+1}^2\right],$$

Black p-brane without dilaton

- Solution:

$$
\begin{aligned}
ds^2 &= \triangle_+ \triangle_-^{-\frac{d}{D-2}} dt^2 + \triangle_-^{\frac{\tilde{d}}{D-2}} dx^i dx^i + (\triangle_+ \triangle_-)^{-1} d\rho^2 + \rho^2 d\Omega_{\tilde{d}+1}^2, \\
A_{t1\cdots p} &= \left[\left(\frac{r_-}{r_+} \right)^{\tilde{d}/2} - \left(\frac{r_- r_+}{\rho^2} \right)^{\tilde{d}/2} \right], \\
F_{\rho t1\cdots p} &\equiv \partial_\rho A_{t1\cdots p} = \tilde{d}\, \frac{(r_- r_+)^{\tilde{d}/2}}{\rho^{\tilde{d}+1}},
\end{aligned}
$$

  where $\triangle_\pm = 1 - r_\pm^{\tilde{d}}/\rho^{\tilde{d}}$ and $\frac{d\tilde{d}}{D-2} = 2$.
- Asymptotic flat.
- isometry : Before compactification: $R \times E(d-1) \times SO(\tilde{d}+2)$, $d = 1+p, D = d + \tilde{d} + 2$
- horizon: $\rho = r_\pm$; Curvature singularity at $\rho = 0$
- $r_\pm = 0$, flat. $r_+ = r_- \neq 0$, extremal.

Black p-brane without dilaton

- Charge :

$$Q_d = \frac{\Omega_{\tilde{d}+1}}{\sqrt{2}\kappa}\tilde{d}(r_+ r_-)^{\tilde{d}/2},$$

If we define $Q_d^* = \left(\frac{\sqrt{2}\kappa Q_d}{\Omega_{\tilde{d}+1}\tilde{d}}\right)^{1/\tilde{d}}$, $r_- = \frac{{Q_d^*}^2}{r_+}$ not an independent variable. $r_\pm = 0 \Rightarrow Q = 0$, "hot flat space", no charge.

- Temperature seen from infinity: $T^* = 1/\beta^*$

$$\beta^* = \frac{4\pi r_+}{\tilde{d}}\left(1 - \frac{r_-^{\tilde{d}}}{r_+^{\tilde{d}}}\right)^{-\frac{1}{2}+\frac{1}{\tilde{d}}},$$

Local temperature at ρ: $T = 1/\beta$,

$$\beta = \triangle_+^{1/2}\triangle_-^{-\frac{d}{2(D-2)}}\beta^* = \triangle_+^{1/2}\triangle_-^{-\frac{1}{\tilde{d}}}\frac{4\pi r_+}{\tilde{d}}\left(1 - \frac{r_-^{\tilde{d}}}{r_+^{\tilde{d}}}\right)^{-\frac{1}{2}+\frac{1}{\tilde{d}}},$$

General strategy

- We put the black p-brane inside a spherical cavity: outside the boundary is flat space in hot equilibrium (hot flat space). The black brane and the heat bath are in a thermal equilibrium. Two kinds of system:
  - Canonical ensemble: fix temperature at the boundary and Q charge.
  - Grand canonical ensemble: fix temperature and A potential at the boundary.
- Path integral of Euclidean action with Boundary terms = Partition function. At leading order: $e^{-I_E} \approx Z$
  The period of the Euclidean time $=\beta^* = 1/$(Temperature seen from infinity).

Canonical ensemble:boundary conditions

Boundary terms:

- Gravity sector:

$$I_E(g) = -\frac{1}{2\kappa^2}\int_M d^D x\sqrt{g_E}\,R_E + \frac{1}{\kappa^2}\int_{\partial M} d^{D-1}x\sqrt{\gamma}\,(K - K_0),$$

  K is the trace of the extrinsic curvature at the boundary. K_0 is the one for the flat metric.

- Form field sector:

$$\begin{aligned} I_E^C(F) &= \frac{1}{2\kappa^2}\frac{1}{2(d+1)!}\int_M d^D x\sqrt{g_E}F_{d+1}^2 \\ &\quad -\frac{1}{2\kappa^2}\frac{1}{d!}\int_{\partial M} d^{D-1}x\sqrt{\gamma}\,n_\mu\,F^{\mu\mu_1\mu_2\cdots\mu_d}A_{\mu_1\mu_2\cdots\mu_d}, \end{aligned}$$

Canonical ensemble:boundary conditions

- Matching with the flat metric at the boundary:

$$ds^2 = d\tau^2 + d\bar{x}^i d\bar{x}^i + \rho^2 d\Omega^2_{\tilde{d}+1}, \qquad (\rho \geq 0)$$

  - τ has the period of $\beta \Leftrightarrow$ the same temperature as the black brane
  - the same volume: $\bar{x}^i = \triangle_{-}^{\frac{\tilde{d}}{2(D-2)}} x^i$, $\Rightarrow$ $\int d^p \bar{x}^i = \int d^p x^i \sqrt{\det g_{ij}}$,
  - the radii for the $(\tilde{d}+1)$-sphere in both cases is the same

Canonical ensemble: Black p-brane partition function

Substitute the solution into the action we find the partition function:

$$
\begin{aligned}
-\ln Z = I_E \quad &= \quad -\frac{\beta V_p \Omega_{\tilde{d}+1}}{2\kappa^2}\rho^{\tilde{d}}\left[2\left(\frac{\triangle_+}{\triangle_-}\right)^{1/2}\right. \\
&\qquad \left. +\tilde{d}\left(\frac{\triangle_-}{\triangle_+}\right)^{1/2} + \tilde{d}(\triangle_+\triangle_-)^{1/2} - 2(\tilde{d}+1)\right].
\end{aligned}
$$

- From the definition of the partition function for canonical ensemble: free energy $F = -\ln Z/\beta = I_E/\beta$

Canonical ensemble: Black p-brane partition function

- Since $Z = \mathrm{Tr} e^{-\beta H}$, $E = -\frac{\mathrm{Tr} H e^{-\beta H}}{\mathrm{Tr} e^{-\beta H}} = -\left(\frac{\partial \ln Z}{\partial \beta}\right)_{V_p, Q}$ We find

$$E(\rho) = -\frac{V_p \Omega_{\tilde{d}+1}}{2\kappa^2} \rho^{\tilde{d}} \left[(\tilde{d}+2) \left(\frac{\triangle_+}{\triangle_-}\right)^{1/2} + \tilde{d}(\triangle_+ \triangle_-)^{1/2} - 2(\tilde{d}+1) \right],$$

which approaches the ADM mass at $\rho \to \infty$. From $S = \beta E - F$ we have entropy:

$$S = \frac{4\pi \, V_p \triangle_-^{-1/2 - \frac{d}{2(D-2)}} \Omega_{\tilde{d}+1}}{2\kappa^2} \, r_+^{\tilde{d}+1} \left(1 - \frac{r_-^{\tilde{d}}}{r_+^{\tilde{d}}}\right)^{1 - \frac{\tilde{d}}{2(D-2)}}.$$

- Another way: Geometry determines entropy $S = A/4$ then one finds E. We have the same results.

Canonical ensemble: Black p-brane partition function

$$
\begin{aligned}
I_E &= \beta E - S \\
&= -\frac{\beta V_p \Omega_{\tilde{d}+1}}{2\kappa^2}\rho^{\tilde{d}}\left[(\tilde{d}+2)\left(\frac{\triangle_+}{\triangle_-}\right)^{1/2} + \tilde{d}(\triangle_+\triangle_-)^{1/2} - 2(\tilde{d}+1)\right] \\
&\quad - \frac{4\pi V_p \triangle_-^{-1/2-\frac{d}{2(D-2)}}\Omega_{\tilde{d}+1}}{2\kappa^2} r_+^{\tilde{d}+1}\left(1-\frac{r_-^{\tilde{d}}}{r_+^{\tilde{d}}}\right)^{1-\frac{\tilde{d}}{2(D-2)}} .
\end{aligned}
$$

- Generalize to nonequilibrium: We fix the T (β), V_p on the boundary and Q, and choose arbituary geometric parameters r_+ not satisfying the equilibrum temperature equation $\beta(r_+)$. These determine the E and entropy S and hence free energy F. Notice $r_- = r_-(r_+, Q)$is not an independent variable.
- The equilibrium should be at the point: $\left(\frac{\partial F}{\partial r_+}\right)_{V_p,Q,\beta} = 0$, we can solve $\beta(r_+)$. The result should be the same β we used in equilibrium.

Canonical ensemble: Black p-brane partition function

$$
\begin{aligned}
&\left(\frac{\partial F}{\partial r_+}\right)_{V_p,Q,\beta} \\
\sim\quad &\left[\tilde{d}+2+\left(\frac{\tilde{d}}{2}-\frac{\tilde{d}+2}{2\triangle_-}\right)\left(1-\frac{r_-^{\tilde{d}}}{r_+^{\tilde{d}}}\right)\right] \\
&\times\left[\beta\tilde{d}-4\pi r_+\triangle_+^{1/2}\triangle_-^{-\frac{d}{2(D-2)}}\left(1-\frac{r_-^{\tilde{d}}}{r_+^{\tilde{d}}}\right)^{-\frac{\tilde{d}}{2(D-2)}}\right]=0,
\end{aligned}
$$

$$
\Rightarrow \beta=\frac{4\pi r_+\triangle_+^{1/2}\triangle_-^{-\frac{d}{2(D-2)}}}{\tilde{d}\left(1-\frac{r_-^{\tilde{d}}}{r_+^{\tilde{d}}}\right)^{\frac{\tilde{d}}{2(D-2)}}}=\triangle_+^{1/2}\triangle_-^{-\frac{d}{2(D-2)}}\beta^*,
$$

The same as before.

Canonical ensemble: stability condition

We define

$$x = \left(\frac{r_+}{\rho}\right)^{\tilde{d}} \leq 1, \quad \bar{b} = \frac{\beta}{4\pi\rho}, \quad q = \left(\frac{Q_d^*}{\rho}\right)^{\tilde{d}},$$

We have $r_-/r_+ = Q_d^{*2}/r_+^2 < 1 \Rightarrow x > |q|$

- $$\frac{\partial \tilde{I}_E}{\partial x} = f(x,q)\left[\bar{b} - b(x,q)\right]$$

  where $f(x,q) > 0$.
- At the equilibrium

$$\bar{b} = b(x,q) = \frac{1}{\tilde{d}} \frac{x^{1/\tilde{d}}(1-x)^{1/2}}{\left(1 - \frac{q^2}{x^2}\right)^{\frac{\tilde{d}-2}{2\tilde{d}}} \left(1 - \frac{q^2}{x}\right)^{\frac{1}{\tilde{d}}}}$$

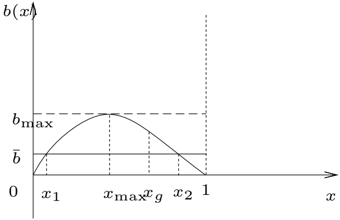
- $\frac{\partial^2 \tilde{I}_E}{\partial x^2} \sim -\frac{\partial b}{\partial x}$

$$\frac{\partial b}{\partial x} > 0, \qquad \frac{\partial^2 \tilde{I}_E}{\partial x^2} < 0, \qquad \text{local maximum, unstable}$$

$$\frac{\partial b}{\partial x} < 0, \qquad \frac{\partial^2 \tilde{I}_E}{\partial x^2} > 0, \qquad \text{local minimum, locally stable}$$

Chargeless Case

$$b(x) = \frac{1}{\tilde{d}} x^{1/\tilde{d}} (1-x)^{1/2},$$

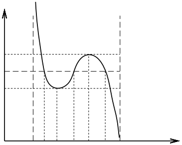


- $b_{\rm max} = \frac{1}{\sqrt{2\tilde{d}}} \left(\frac{2}{\tilde{d}+2}\right)^{1/2+1/\tilde{d}}$, at $x_{\rm max} = \frac{2}{\tilde{d}+2}$.
- $\bar{b} > b_{\rm max}$, no black brane, there is only "the hot empty space".
- $0 < \bar{b} < b_{\rm max}$ there are two black brane solutions. The larger one has $\partial b / \partial x < 0$, locally stable; the smaller one unstable.
- the "hot empty space" has $I_E = 0$, globaly stable black brane $I_E < 0 \Rightarrow \bar{x} > x_g = \frac{4(\tilde{d}+1)}{(\tilde{d}+2)^2} > x_{\rm max}$. $\bar{b} < b_g$
- For $b_g < \bar{b} < b_{\rm max}$, the black brane will tunnel to "hot flat space".

Charged case

Analyse $\partial b/\partial x = 0$,

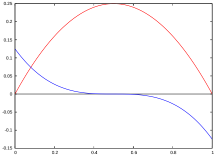
$$
\begin{aligned}
\frac{\partial b}{\partial x} \quad &\sim \quad -(1+\frac{\tilde{d}}{2})x^4 - [1+(2+\frac{\tilde{d}}{2})q^2]x^3 - 3q^2(\frac{\tilde{d}}{2}-1)x^2 \\
&\quad +q^2[\tilde{d}-1+\frac{3\tilde{d}}{2}q^2]x - \tilde{d}q^4 \\
&= \quad 0
\end{aligned}
$$



The discriminant:

$$
\triangle(q,\tilde{d}) = \frac{(1-q^2)^3 q^6}{16}\Big[\Big(-3\tilde{d}(4+\tilde{d})q^2 + 4(\tilde{d}-1)\Big)^3 - 108\tilde{d}^2(2+\tilde{d}-\tilde{d}^2)^2 q^2(1-q^2)\Big]
$$

- $q = q_c$, $\Delta = 0$, two solutions merge, critial point.
- $q < q_c$, $\Delta > 0$, two stationary points.
- $q > q_c$, $\Delta < 0$, no stationary point.



Charged case: $\tilde{d} > 2$

there is a critical point q_c

- $q > q_c$, there is no solution for $\partial b/\partial x = 0$, $\partial b/\partial x < 0$ for all $|q| < x < 1$

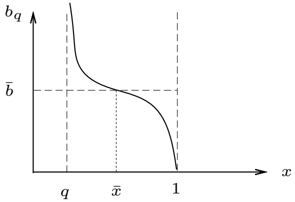
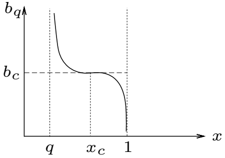


Figure: b decreases monotonically with x when $|q| < x < 1$.

There is only one stable black brane solution at all temperature.

- $q = q_c$ there is only one point where $\partial b/\partial x = 0$ or the two extrema merge. $x_{\min} = x_{\max} = x_c$ This is a critical point with $\partial^2 b/\partial x^2 = 0$.



- At the critial point, the reduced specific heat

$$\begin{aligned}\tilde{C}_v &= T\frac{\partial \tilde{S}}{\partial T} = \tau\frac{\partial \tilde{S}}{\partial \tau} \\ &= \frac{1}{3}\left.\frac{\partial \tilde{S}}{\partial x}\right|_{x=x_c}\left[\frac{1}{-\frac{1}{3!}\left.\frac{\partial^3 b}{\partial x^3}\right|_{x=x_c} b_c}\right]^{1/3}(\tau-\tau_c)^{-2/3}+\cdots.\end{aligned}$$

Therefore, the critical exponent α is $-2/3$.

Charged case: $\tilde{d} > 2$

- when $q < q_c$

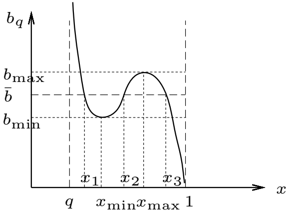


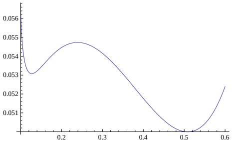
Figure: The typical behavior of b vs x when there is a phase transition.

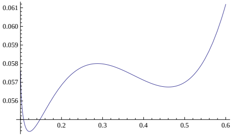
- $0 < \bar{b} < b_{\min}$ there is one stable solution
- $b_{\min} < \bar{b} < b_{\max}$ there are three solutions, the largest and smallest are locally stable solution and the middle one is unstable
- $b_{\max} < \bar{b}$ there is one stable solution

CHARGED CASE: $\tilde{d} > 2$

$q < q_c$

- $b_{\min} < \bar{b} < b_{\max}$, compare the free energies of the two local minima: There is a phase transation point b_t,
  - for $b < b_t$, $I_1 > I_3$, the larger one is globally stable.
  - for $b > b_t$, $I_1 < I_3$, the smaller one is globally stable.
  - at $b = b_t$, $I_1 = I_3$, the larger one and the smaller one could coexist. This is a first order phase transation.





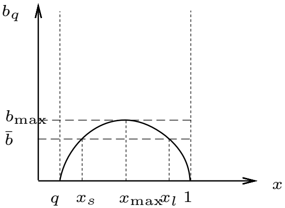
Charged case: $\tilde{d} = 1$

$$b = x(1-x)^{1/2}\left(1-\frac{q^2}{x^2}\right)^{\frac{1}{2}}\left(1-\frac{q^2}{x}\right)^{-1}.$$

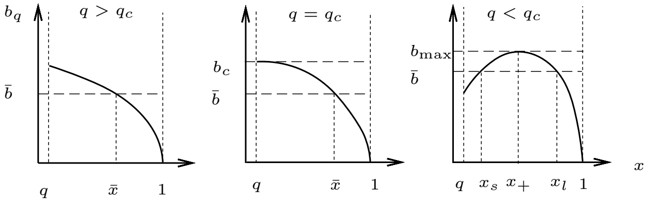
$$\frac{\partial b(x,q)}{\partial x} \sim \frac{3}{2}x^4 - \left(1+\frac{5}{2}\,q^2\right)x^3 + \frac{3}{2}\,q^2 x^2 + \frac{3}{2}\,q^4\,x - q^4 = 0.$$

$$\triangle(q,1) = \frac{(15)^3(q^2-1)^3 q^8}{16}\left[q^4 - \frac{16}{125}q^2 + \frac{16}{125}\right] < 0,$$

- $b(q) = b(1) = 0$, $\Delta < 0$, one maximum in $q < x < 1$.
- $0 < b < b(x_{\max})$, two black brane: smaller— unstable, larger—stable.



Charged case: $\tilde{d} = 2$



$$b(x) = \frac{1}{2} x^{1/2} (1-x)^{1/2} \left(1 - \frac{q^2}{x}\right)^{-1/2} .$$

solution to $\frac{\partial q(x)}{\partial x} = 0$, $x_\pm = \frac{1}{4}\left[1 + 3q^2 \pm \sqrt{B}\right]$, where $B = (1-q^2)(1-9q^2)$

- $q \geq q_c = 1/3$, no real solution. $\frac{\partial q(x)}{\partial x} \leq 0$, one stable black brane for $0 < \bar{b} < b(q)$.
- $q < q_c = 1/3$: (1) $0 < b < b(q)$, one stable black brane. (2)$b(q) < b < b(x_+)$, one small unstable , the other larger stable.

Black p-brane with dilaton

Solution to D-dimensional spacetime gravity with a dilaton field and a $p+1$ form gauge field:

$$I_E = I_E(g) + I_E(\phi) + I_E(F),$$

$$\begin{aligned}
I_E(g) &= -\frac{1}{2\kappa^2}\int_M d^D x\sqrt{g_E}\,R_E + \frac{1}{\kappa^2}\int_{\partial M} d^{D-1}x\sqrt{\gamma}\,(K-K_0)\,, \\
I_E(\phi) &= -\frac{1}{2\kappa^2}\int_M d^D x\sqrt{g_E}\left(-\frac{1}{2}(\partial\phi)^2\right), \\
I_E^C(F) &= \frac{1}{2\kappa^2}\frac{1}{2(d+1)!}\int_M d^D x\sqrt{g_E}\,e^{a(d)\phi}\,F_{d+1}^2 \\
&\quad -\frac{1}{2\kappa^2}\frac{1}{d!}\int_{\partial M} d^{D-1}x\sqrt{\gamma}\,n_\mu\,e^{a(d)\phi}\,F^{\mu\mu_1\mu_2\cdots\mu_d}A_{\mu_1\mu_2\cdots\mu_d},
\end{aligned}$$

Black brane solution

$$
\begin{aligned}
ds^2 &= \triangle_+ \triangle_-^{-\frac{d}{D-2}} dt^2 + \triangle_-^{\frac{\tilde{d}}{D-2}} dx^i dx^i + \triangle_+^{-1} \triangle_-^{\frac{a^2}{2\tilde{d}}-1} d\rho^2 + \rho^2 \triangle_-^{\frac{a^2}{2\tilde{d}}} \Omega_{\tilde{d}+1}^2, \\
A_{t1\cdots p} &= -ie^{-a\phi_0/2} \left[\left(\frac{r_-}{r_+} \right)^{\tilde{d}/2} - \left(\frac{r_- r_+}{\rho^2} \right)^{\tilde{d}/2} \right], \\
F_{\rho t1\cdots p} &\equiv \partial_\rho A_{t1\cdots p} = -ie^{-a\phi_0/2}\, \tilde{d}\, \frac{(r_- r_+)^{\tilde{d}/2}}{\rho^{\tilde{d}+1}}, \\
e^{-2(\phi-\phi_0)} &= \triangle_-^a ,
\end{aligned}
$$

where $\phi_0 = \phi(\infty)$, and $a^2 = 4 - \frac{2d\tilde{d}}{(d+\tilde{d})}$.

- Physical radius: $\bar{\rho} = \rho \triangle_-^{\frac{\alpha^2}{4\tilde{d}}}$,
- Define $\bar{r}_\pm = r_\pm \triangle_-^{\frac{\alpha^2}{4\tilde{d}}}$, $\triangle_\pm = 1 - \frac{r_\pm^{\tilde{d}}}{\rho^{\tilde{d}}} = 1 - \frac{\bar{r}_\pm^{\tilde{d}}}{\bar{\rho}^{\tilde{d}}}$.
- Temperature: $\beta^* = \frac{4\pi r_+}{\tilde{d}} \left(1 - \frac{r_-^{\tilde{d}}}{r_+^{\tilde{d}}} \right)^{\frac{1}{\tilde{d}} - \frac{1}{2}}$,

$$
\beta = \triangle_+^{1/2} \triangle_-^{-\frac{d}{2(d+\tilde{d})}} \beta^* = \triangle_+^{1/2} \triangle_-^{-1/\tilde{d}} \frac{4\pi \bar{r}_+}{\tilde{d}} \left(1 - \frac{\bar{r}_-^{\tilde{d}}}{\bar{r}_+^{\tilde{d}}} \right)^{\frac{1}{\tilde{d}} - \frac{1}{2}}
$$

The same as before except $r_\pm \to \bar{r}_\pm$, $\rho \to \bar{\rho}$.

- We fix ϕ at the boundary ϕ_ρ.
- Charge:

$$Q = \frac{i}{\sqrt{2}\kappa}\int e^{a(d)\phi}F^{r01\cdots(d-1)}\sqrt{g_E}dx^{\tilde{d}-1}$$
$$= \frac{\tilde{d}}{\sqrt{2}\kappa}e^{a\phi_\rho/2}\triangle_-^{\frac{a^2}{4}}(r_+r_-)^{\tilde{d}/2}\Omega_{\tilde{d}+1}$$

We define $(Q^*)^{\tilde{d}} = e^{-a\phi_\rho/2}\frac{\sqrt{2}\kappa}{\tilde{d}\,\Omega_{\tilde{d}}}Q = (\bar{r}_+\bar{r}_-)^{\tilde{d}/2}$

The same form as before except $r_\pm \to \bar{r}_\pm$.

- Euclidean action:

$$I_E = -\frac{\beta V_p\Omega_{\tilde{d}+1}}{2\kappa^2}\bar{\rho}^{\tilde{d}}\left[2\left(\frac{\triangle_+}{\triangle_-}\right)^{1/2} + \tilde{d}\left(\frac{\triangle_-}{\triangle_+}\right)^{1/2} + \tilde{d}(\triangle_+\triangle_-)^{1/2} - 2(\tilde{d}+1)\right]$$

The same form as the nondilaton case except $r_\pm \to \bar{r}_\pm$, $\rho \to \bar{\rho}$.

- We define: $x = \left(\frac{\bar{r}_+}{\bar{\rho}}\right)^{\tilde{d}} \le 1, \quad b = \frac{\beta}{4\pi\bar{\rho}}, \quad q = \left(\frac{Q_d^*}{\bar{\rho}}\right)^{\tilde{d}}$, then

$$b = \frac{1}{\tilde{d}}\, x^{1/\tilde{d}}(1-x)^{1/2}\left(1-\frac{q^2}{x^2}\right)^{\frac{1}{\tilde{d}}-\frac{1}{2}}\left(1-\frac{q^2}{x}\right)^{-\frac{1}{d}}.$$

They are exactly the same equations as in the nondilaton case. All the discussions in the nondilaton case can be applied here with no change only with new definition of variables.

Summary of canonical ensemble

- $q = 0$,
  - $T < T_{\min}, (b > b_{\max})$ there is no black brane solution.
  - $T_{\min} < T < T_g$ there are two solution: smaller—unstable, larger —locally stable but not globally stable. Tunnel to "hot flat space".
  - $T_g < T$, there are two solutions: smaller one —unstable, larger one —globally stable. T larger$\rightarrow r_+$ larger.
- $q > 0$, $\tilde{d} > 2$
  - $q \geq q_c$, one globally stable black brane. $q = q_c$, there is a critical point.
  - $q < q_c$, $T < T_{\min}$ or $T > T_{\max}$, there is one globally stable black brane. T greater $\rightarrow r_+$ larger.
  - $q < q_c$, $T_{\min} < T < T_{\max}$: three solutions, the largest and the smallest ones—locally stable, the middle one unstable. At b_t: A first order transation between the largest one and the smallest one.
- $q > 0$, $\tilde{d} = 1$, $T > T_{\min}$, there are two solutions: smaller one —unstable, larger one — stable.
- $\tilde{d} = 2$,
  - $q > q_c$, $T > T_c$: one stable solution. T increase $\rightarrow r_+$ larger.
  - $0 < q < q_c$, $T_{\min} < T < T(q)$: two solutions: smaller—unstable, larger — stable; $T(q) < T$: one stable solution.

Grand canonical ensemble

Fix temperature T, volume V_p, ρ_B, and potential A at the boundary

- Fix the potential in the local inertial frame:

$$A_{\hat{0}\hat{1}\cdots\hat{p}} = (\triangle_+\triangle_-)^{-1/2}A_{01\cdots p} = e^{-a\phi_\rho}\left(\frac{r_-}{r_+}\right)^{\tilde{d}/2}\left(\frac{\triangle_+}{\triangle_-}\right)^{1/2} \equiv \sqrt{2\kappa}\Phi,$$

- Subtract the boundary term of the canonical action⇔ Legendre transformation.

$$\begin{aligned} I_E^{GC} &= I_E^C + \frac{1}{2\kappa^2}\frac{1}{d!}\int_{\partial M} d^{D-1}x\sqrt{\gamma}\,n_\mu\,e^{-a(d)\phi}\,F^{\mu\mu_1\mu_2\cdots\mu_d}A_{\mu_1\mu_2\cdots\mu_d} \\ &= I_E^C - \beta V_p Q_d \Phi \end{aligned}$$

Equilibrium condition

Equilibrium: $\frac{\partial I_E}{\partial r_+} = 0$; $\frac{\partial I_E}{\partial Q} = 0 \Rightarrow \beta, \Phi$. Define

$$Q_d^* = \left(\frac{\sqrt{2}\kappa Q_d}{\Omega_{\tilde{d}+1}\tilde{d}} e^{a\bar{\phi}/2}\right)^{\frac{1}{\tilde{d}}}, x = \left(\frac{\bar{r}_+}{\bar{\rho}_B}\right)^{\tilde{d}}, \bar{b} = \frac{\beta}{4\pi\bar{\rho}_B}, q = \left(\frac{Q_d^*}{\bar{\rho}_B}\right)^{\tilde{d}}, \bar{\varphi} = \sqrt{2}\kappa e^{-a\bar{\phi}/2}\Phi$$

- At the equilibrium, temperature and Φ:
  $\bar{b} = b(x, q), \quad \bar{\varphi} = \varphi(x, q)$ where we have defined

$$b(x,q) = \frac{1}{\tilde{d}} \frac{x^{\frac{1}{\tilde{d}}}(1-x)^{\frac{1}{2}}}{\left(1 - \frac{q^2}{x^2}\right)^{\frac{1}{2}-\frac{1}{\tilde{d}}}\left(1 - \frac{q^2}{x}\right)^{\frac{1}{\tilde{d}}}}, \qquad \varphi(x,q) = \frac{q}{x}\left(\frac{1-x}{1-\frac{q^2}{x}}\right)^{\frac{1}{2}}.$$

- Solve $\bar{\varphi} = \varphi(x, q)$ and reformulate $b(x, q)$ in x and $\bar{\varphi}$:

$$\frac{q^2}{x^2} = \frac{\bar{\varphi}^2}{1 - (1-\bar{\varphi}^2)x}, \quad \bar{b} = b_{\bar{\varphi}}(x) \equiv b(x,q) = \frac{x^{\frac{1}{\tilde{d}}}\left[1 - \left(1 - \bar{\varphi}^2\right)x\right]^{\frac{1}{2}}}{\tilde{d}\,(1-\bar{\varphi}^2)^{\frac{1}{2}-\frac{1}{\tilde{d}}}},$$

- For a grand canonical ensembel, we fix $\bar{b}$, V_p, $\bar{\varphi}$: Given $\bar{b}$ and $\bar{\varphi}$ we can solve x from the second eq. and this determines q in the first eq. So we only need to consider when there is a solution of the second equation.

Stability analysis

- stability condition: Define $\tilde{I}_{ij} \equiv \frac{\partial^2 \tilde{I}_E(x,q)}{\partial z_i \partial z_j}$, $z_1 = q, z_2 = x$.

$$\text{positive eigenvalues} \Leftrightarrow \tilde{I}_{qq} > 0, \quad \frac{\tilde{I}_{qq}}{\det \tilde{I}_{ij}} > 0$$

Equivalent to

$$\frac{db_{\bar{\varphi}}(x)}{dx} = \frac{b_{\bar{\varphi}}(x)\left[2 - (\tilde{d}+2)x(1-\bar{\varphi}^2)\right]}{2\tilde{d}\,x\left[1 - x(1-\bar{\varphi}^2)\right]} < 0 \Rightarrow 2 - (2+\tilde{d})\bar{x}(1-\bar{\varphi}^2) < 0.$$

Phase structure

Equilibrium condition:

$$\bar{b} = \frac{x^{\frac{1}{\tilde{d}}}\left[1-\left(1-\bar{\varphi}^2\right)x\right]^{\frac{1}{2}}}{\tilde{d}(1-\bar{\varphi}^2)^{\frac{1}{2}-\frac{1}{\tilde{d}}}}, \quad b(0) = 0, b_{\bar{\varphi}}(1) = \frac{\bar{\varphi}}{\tilde{d}(1-\bar{\varphi}^2)^{\frac{1}{2}-\frac{1}{\tilde{d}}}},$$

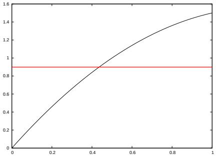
Stability condition: $\frac{db_{\bar{\varphi}}(x)}{dx} < 0 \Rightarrow 2 - (2+\tilde{d})\bar{x}(1-\bar{\varphi}^2) < 0.$

- Solve the $\frac{db_{\bar{\varphi}}(x_{\max})}{dx} = 0$: $x_{\max} = \frac{2}{(\tilde{d}+2)(1-\bar{\varphi}^2)}$

- If $\sqrt{\frac{\tilde{d}}{2+\tilde{d}}} < \bar{\varphi} < 1$, $x_{\max} > 1$:

  $0 < \bar{b} < b_{\bar{\varphi}}(1)$.

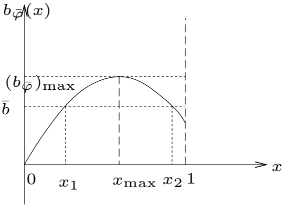
  There is an unstable black brane solution. No stable black brane solution. It will decay to "hot flat space".



Phase structure

If $\bar{\varphi} < \sqrt{\frac{\tilde{d}}{2+\tilde{d}}}$, $x_{\max} < 1$:

$$
\begin{aligned}
(b_{\bar{\varphi}})_{\max} &= \left(\frac{2}{2+\tilde{d}}\right)^{\frac{1}{\tilde{d}}} \left[\tilde{d}(\tilde{d}+2)\left(1-\bar{\varphi}^2\right)\right]^{-\frac{1}{2}} \\
&< \frac{1}{\sqrt{2\tilde{d}}} \left(\frac{2}{2+\tilde{d}}\right)^{\frac{1}{\tilde{d}}}
\end{aligned}
$$



- $0 < \bar{b} < b_{\varphi}(1)$, there is one unstable solution.
- $b_{\varphi}(1) < \bar{b} < b_{\max}$ there are two solutions. the smaller unstable, the larger locally stable.
- Since the Gibbs free energy of "hot flat space" is zero, for the larger one to be globally stable, we need $I_E < 0$.

  $\tilde{I}_E^{\bar{\varphi}} = -\frac{\bar{b}}{y}(\tilde{d}+2)(y-1)\left(y - \frac{\tilde{d}}{\tilde{d}+2}\right) < 0$ where $\frac{\tilde{d}}{\tilde{d}+2} < y \equiv \sqrt{1-\bar{x}(1-\bar{\varphi}^2)} < 1$

  $\Rightarrow$ when $0 < \bar{\varphi} < \frac{\tilde{d}}{\tilde{d}+2}$, $b_{\bar{\varphi}}(1) < \bar{b} < (b_{\bar{\varphi}})_g$, there is a global stable black brane solution at $\bar{x}_g = \frac{4(\tilde{d}+1)}{(\tilde{d}+2)^2\left(1-\bar{\varphi}^2\right)}$.

Summary of the Grand canonical ensemble

- $\sqrt{\frac{\tilde{d}}{2+\tilde{d}}} < \bar{\varphi} < 1$, $0 < \bar{b} < b_{\bar{\varphi}}(1)$: one unstable black brane solution, decay to "hot flat space".
- $\bar{\varphi} < \sqrt{\frac{\tilde{d}}{2+\tilde{d}}}$,
  - $0 < \bar{b} < b_{\varphi}(1)$: there is one unstable solution.
  - $\frac{\tilde{d}}{2+\tilde{d}} < \bar{\varphi} < \sqrt{\frac{\tilde{d}}{2+\tilde{d}}}$, $b_{\varphi}(1) < \bar{b} < b_{\max}$ there are two solutions. the smaller unstable, the larger locally stable, but globally unstable. They will finally tunnel to "hot flat space".
  - $\bar{\varphi} < \frac{\tilde{d}}{2+\tilde{d}}$,
    - $b_g < \bar{b} < b_{\max}$ two solutions: smaller—unstable, larger—locally stable, but globally unstable. They will finally tunnel to "hot flat space".
    - $b_{\varphi}(1) < \bar{b} < b_g$,two solutions: smaller—unstable, larger—globally stable.

Summary and Disscusion

- The phase structures of black branes in Canonical ensemble and Grand canonical ensemble are quite different because of the boundary condition.
- In canonical ensemble, there is a first order phase transation and a second order transation at the critical temperature. No such phase transation in GC. Similar to the phase structure of the van der Waals-Maxwell liquid-gas phase transation.
- The $q = 0$ case, $\tilde{d} = 1, 2$ case in canonical ensemble are quite different from $\tilde{d} > 2$ case but have some similarity with the GC cases.

Thank you!