THE EFFECT OF GAUGE CHOICE ON THE GROUP OF DRESSING TRANSFORMATIONS

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The Effect of Gauge Choice on the Group of Dressing Transformations (*Planning*)

- ***Symplectic and Poisson-Lie actions***
- ***Dressing transformations***
- ***Examples***
- ***Potential application***
- ***Conclusion***

1. Symplectic and Poisson-Lie actions

1.1. Symplectic actions

M – Poisson manifold equipped with Poisson bracket $\{,\}_M$;
H – Lie group with Lie algebra $\mathcal{H}$.
A symplectic action of H on M is such that for any $h^t \in H^t \subset H$ and any two functions f_1, f_2 on M,

$$\{f_1(h^t.x), f_2(h^t.x)\}_M = \{f_1, f_2\}_M(h^t.x), \tag{1}$$

where H^t is a one parameter subgroup of H.
The infinitesimal action is defined by introducing the vector field X: $X.f(x) \equiv \frac{d}{dt}f(h^t.x)\Big|_{t=0}$, which satisfies

$$\{X.f_1, f_2\}_M + \{f_1, X.f_2\}_M = X.\{f_1, f_2\}_M. \tag{2}$$

The action of any one parameter subgroup is locally Hamiltonian, i.e. one can introduce a local *Hamiltonian function* H_X for any infinitesimal action X,

$$X.f(x) = \{H_X(x), f(x)\}_M. \tag{3}$$

In standard physics texts, H_X is referred to as the canonical generator of the transformation corresponding to the Lie group action X.
Assume that H_X are globally defined on M and depend linearly on X with the additional property

$$\{H_X(x), H_Y(x)\}_M = H_{[X,Y]}(x). \tag{4}$$

Then there is a momentum map P:

$$\begin{aligned} P : M \quad &\longrightarrow \quad \mathcal{H}^* \\ P(x)(X) \quad &= \quad H_X(x) \qquad ; \qquad \forall\, X \in \mathcal{H} \end{aligned}$$

which maps the symplectic action of H on M into the coadjoint action of H on $\mathcal{H}^*$

$$H_X(h.x) \;=\; H_{(Ad_H h).X}(x), \qquad \forall\, x \in M\,;\, \forall\, X \in \mathcal{H}\,;\, \forall\, h \in H$$

1.2. Poisson-Lie groups

A Poisson-Lie group is a Lie group H equipped with a Poisson bracket $\{,\}_H$ such that the Lie group multiplication $H \times H \to H$ is a Poisson mapping.

Construction of a Poisson-Lie group structure: Let ℓ_1, ℓ_2 be any two functions on H. Then

$$\{\ell_1, \ell_2\}_H(h) = \sum_{a,b} \eta^{ab}(h)(\nabla_a^R \ell_1)(h)(\nabla_b^R \ell_2)(h) \tag{5}$$

is a Poisson bracket on H, where $\eta(h) = \sum_{a,b} \eta^{ab}(h) e_a \otimes e_b$, $\{e_a\}$ is a basis of $\mathcal{H}$, and ∇_a^R is the right-invariant vector field

$$\nabla_a^R \ell(h) = \frac{d}{dt} \ell(e^{te_a} h)|_{t=0}$$

In particular, if we take for ℓ the matrix elements of a representation of H, we get

$$\{h \otimes, h\}_H = \eta(h) \cdot \; h \otimes h \qquad ; \qquad h \in H \tag{6}$$

The Lie-Poisson property of the Poisson bracket $\{,\}_H$ is such that for any group multiplication $(h,g) \to hg$ viewed as a map $\rho^g : h \to hg$, the following equality hold:

$$\{\rho^g(h) \otimes, \rho^g(h)\}_{H \times H} = \rho^g(\{h \otimes, h\}_H), \tag{7}$$

where, by definition,

$$\begin{aligned} \{\rho^g(h) \otimes, \rho^g(h)\}_{H \times H} &= \{h \otimes, h\}_H g \otimes g + h \otimes h \{g \otimes, g\}_H \\ \rho^g(\{h \otimes, h\}_H) &= \eta(hg) \cdot hg \otimes hg. \end{aligned}$$

Inserting (6) into (7) one finds

$$\eta(hg) = \eta(h) + Ad\, h \cdot \eta(g), \tag{8}$$

a cocycle condition for $\eta(h)$.
The Poisson bracket $\{,\}_H$ can be used to define a Lie algebra structure on $\mathcal{H}^*$:

$$[d_e \ell_1, d_e \ell_2]_{\mathcal{H}^*} = d_e \{\ell_1, \ell_2\}_H, \tag{9}$$

where $d_e \ell$ is the differential of the function ℓ on H evaluated at the identity element e of H,

$$d_e \ell = \sum_a e^a \frac{d}{dt} \ell(e^{t e_a}) \Big|_{t=0},$$

here e^a is the dual of e_a.

1.3. Poisson-Lie group action

The action of a Poisson-Lie group H on the Poisson manifold M is a Poisson-Lie action if for any $h \in H$ the following hold:

$$\{f_1(h.x), f_2(h.x)\}_{H\times M} = \{f_1, f_2\}_M(h.x), \tag{10}$$

where $\{,\}_{H\times M}$ is the product Poisson bracket.
The notion of Poisson-Lie group action is a direct generalization of the concept of the symplectic action.
One can introduce the infinitesimal Poisson-Lie group action also via the vector field X:

$$X.f(x) = \frac{d}{dt} f(e^{tX}.x)\Big|_{t=0}.$$

The LHS of the above can also be regarded as a linear function over $\mathcal{H}$, i.e.

$$X.f(x) = \langle \zeta_f(x), X\rangle,$$

with $\zeta_f(x)$ being an element of $\mathcal{H}^*$.

The infinitesimal form of (10) reads

$$\{X.f_1, f_2\}_M + \{f_1, X.f_2\}_M + \langle [\zeta_{f_1}, \zeta_{f_2}]_{\mathcal{H}^*}, X\rangle = X.\{f_1, f_2\}_M \qquad (11)$$

or equivalently,

$$\{\zeta_{f_1}, f_2\}_M + \{f_1, \zeta_{f_2}\}_M + [\zeta_{f_1}, \zeta_{f_2}]_{\mathcal{H}^*} = \zeta_{\{f_1,f_2\}_M}.$$

It follows that the Poisson-Lie action cannot be symplectic action unless $\mathcal{H}^*$ is abelian.

The notion of local Hamiltonian functions is generalized into the following definition for *nonabelian Hamiltonian*:

$$X.f = \langle \zeta_f, X\rangle \equiv \langle \Gamma^{-1}\{f, \Gamma\}, X\rangle. \qquad (12)$$

That (12) is consistent with (11) can be verified in not more than 10 lines of calculation.

The momentum map in this case is defined by

$$\begin{aligned} \mathcal{P} : M &\longrightarrow H^* \\ x &\longrightarrow \Gamma(x). \end{aligned} \qquad (13)$$

1.4. Specialization: $M = G$, $H = G^*$

Take for M a connected Lie group G with Lie algebra $\mathcal{G}$. The Poisson bracket on $M = G$ is defined via the classical r-matrix taking values on $\mathcal{G} \times \mathcal{G}$:

$$\{x \otimes, x\}_G = [r^{\pm}, x \otimes x] \qquad x \in G, \tag{14}$$

where $r^{\pm}$ are solutions of the classical Yang-Baxter equation

$$[r^{\pm}_{12}, r^{\pm}_{13}] + [r^{\pm}_{12}, r^{\pm}_{23}] + [r^{\pm}_{13}, r^{\pm}_{23}] = 0, \tag{15}$$

with $r^{+}_{12} = -r^{-}_{21}$ and their difference is equal to the second tensor Casimir $\Pi \equiv \sum_a t^a \otimes t^a$,

$$r^{+} - r^{-} = \Pi.$$

The Poisson bracket on G induces a Lie algebra structure on $\mathcal{G}^*$ as follows. First, one identifies $\mathcal{G}^*$ with $\mathcal{G}$ as vector spaces by use of the standard Cartan-Killing form $\langle , \rangle$ defined on $\mathcal{G}$. Then the Lie bracket induced by the Poisson bracket $\{ , \}_G$ can be written as

$$[X, Y]_R = \frac{1}{2}\left([R(X), Y] + [X, R(Y)]\right), \tag{16}$$

where $R = R^{+} + R^{-}$ with $R^{\pm}(X) \equiv \langle r^{\pm}, 1 \otimes X\rangle_2$.

It follows from the fact $R^+ - R^- = 1$ that every $X \in \mathcal{G}$ admits a unique decomposition

$$X = X^+ - X^-, \qquad X^\pm \equiv R^\pm(X). \tag{17}$$

Then the Lie bracket $[,]_R$ can be rewritten as

$$[X, Y]_R = [X^+, Y^+] - [X^-, Y^-], \tag{18}$$

the RHS being defined in terms of the Lie bracket of $\mathcal{G}$. One immediately recognizes that any pair of elements X^+, Y^- commutes in $\mathcal{G}_R \equiv \mathcal{G}^*$. In other words, $\mathcal{G}_\pm \equiv \Im\mathrm{m} R^\pm$ are two commuting subalgebras in $\mathcal{G}^*$. Correspondingly, the Lie group $G_R \equiv G^*$ generated by $\mathcal{G}^*$ has the composition law

$$(g_-, g_+).(h_-, h_+) = (g_- h_-, g_+ h_+). \tag{19}$$

Just as $\mathcal{G}$ and $\mathcal{G}^*$ are identified as vector spaces, G and G^* can be identified as manifolds:

$$(g_-, g_+) \in G^* \longrightarrow g = g_-^{-1} g_+ \in G. \tag{20}$$

The last factorization of $g \in G$ is unique for every g which can be regarded as a purely algebraic version of Riemann-Hilbert problem.

The group G^* itself becomes a Poisson-Lie group if we introduce on it the Semenov-Tian-Shansky Poisson bracket:

$$\begin{aligned} \{g_+\otimes, g_+\}_{G^*} &= -[r^{\pm}, g_+ \otimes g_+] \\ \{g_-\otimes, g_-\}_{G^*} &= -[r^{\mp}, g_- \otimes g_-] \\ \{g_-\otimes, g_+\}_{G^*} &= -[r^{-}, g_- \otimes g_+] \\ \{g_+\otimes, g_-\}_{G^*} &= -[r^{+}, g_+ \otimes g_-] \end{aligned} \tag{21}$$

or, for the factorized element $g = g_-^{-1} g_+$:

$$\begin{aligned} \{g\otimes, g\}_{G^*} &= -(g\otimes 1)r^{+}(1\otimes g) - (1\otimes g)r^{-}(g\otimes 1) \\ &\quad +(g\otimes g)r^{\pm} + r^{\mp}(g\otimes g). \end{aligned} \tag{22}$$

The action of G^* on G:

For any $(g_-, g_+) \in G^*$, The action is given as follows:

$$(g_-, g_+) \in G^*,\ x \in G \ \rightarrow x^g \ = \ (xgx^{-1})_\pm\, x\, g_\pm^{-1} \ \in G \quad \text{with} \quad g = g_-^{-1} g_+ \qquad (23)$$

That the two signs give the same result for x^g is required in order that the action (23) indeed satisfies the group composition law for G^*.

The infinitesimal form of (23) is,

$$\delta_X\, x \ = \ Y_\pm\, x - x\, X_\pm \quad \text{with} \quad Y_\pm = (xXx^{-1})_\pm. \qquad (24)$$

The corresponding nonabelian Hamiltonian is the group element itself, i.e.:

$$\delta_X\, x = \langle (1 \otimes Xx^{-1}) \{x \otimes, x\}_G \rangle_2 \quad \forall X \in \mathcal{G},\ \forall x \in G. \qquad (25)$$

The proof is simply done by inserting the Poisson bracket (14) into (25) and remembering the definition of $R^\pm$.

2. Dressing transformations

– A number of different historical names (Riemann-Hilbert transformations, hidden symmetry transforms, non-local transformations, ..., and dressing transformations);

– Some subtleties: Riemann-Hilbert or hidden symmetries were originally treated as Hamiltonian symmetries for certain (1+1)-dimensional soliton equations and, as such, they (usually) lead to infinite dimensional Kac-Moody algebras. However, they are actually not Hamiltonian symmetries but rather symmetries only on the space of solutions to the soliton equations.

The key difference lies in that dressing symmetries do not commute with the Hamiltonian of the soliton equation. Their charges generate a non-local nonabelian symmetry algebra and do not belong to the commutative algebra generated by the set of commuting integrals of motion.

– Traditional Riemann-Hilbert transformations depend crucially on the solution of a certain Riemann-Hilbert problem like $U(\lambda) = U_-^{-1}(\lambda)U_+(\lambda)$, where $U_\pm(\lambda)$ are required to be analytic respectively on two regions separated by a contour C on the λ plane. The choice of the contour C seems to have no dynamical or algebraic reasoning.

2.1. Dressing transformations in the ultralocal gauge – Poisson-Lie symmetries

Every classical integrable soliton equation admits a zero curvature description which is the compatibility condition for the Lax equations

$$D_\mu T(x) = 0, \qquad D_\mu = \partial_\mu - A_\mu, \qquad \mu = (0,1), \tag{26}$$

where the vector A_μ is called the Lax connection which encodes all the dynamical information of the system. Usually A_μ takes value on some Lie algebra $\mathcal{G}$ which generates the Lie group G on which the transport matrix $T(x) \equiv T(x,0) = P\exp(\int_0^x A_1(x,t)dx)$ lives. It is clear from (26) that the transport matrix $T(x)$ is defined up to the right multiplication by a constant group element. This degree of freedom is removed by the standard normalization condition

$$T(0,0) = 1, \qquad T(x,z) = T(x,y)T(y,z). \tag{27}$$

For a large number of integrable models there exist the so-called fundamental Poisson relation (FPR)

$$\{L(x)\otimes, L(y)\} = [r^{\pm}_{12}, L(x)\otimes 1 + 1\otimes L(x)]\delta(x-y), \qquad L \equiv A_1, \tag{28}$$

where $r^{\pm}$ satisfy the classical Yang-Baxter equation. Such FPRs are called ultralocal because they integrate into the following Poisson bracket for the transport matrix:

$$\{T(x)\otimes, T(x)\} = [r^{\pm}, T(x)\otimes T(x)], \tag{29}$$

$$\{T(x,y)\otimes, T(z,w)\} = 0 \text{ if } (x,y)\cap(z,w)=\varnothing. \tag{30}$$

For Lax systems with an ultralocal FPR, the dressing transformations are defined as

$$T(x)\to T^g(x) = \Theta_{\pm}(x)T(x)g_{\pm}^{-1}, \tag{31}$$

where

$$\Theta(x) \equiv T(x)gT^{-1}(x), \tag{32}$$

and the suffices $\pm$ are in accordance with the unique factorizations

$$\Theta(x) = \Theta_-(x)^{-1}\Theta_+(x), \quad g = g_-^{-1}g_+ \tag{33}$$

determined by the classical r-matrices $r^{\pm}$.

We see that the action of dressing transformations are exactly the Poisson-Lie group action of the group G^* on G. In particular, the nonabelian Hamiltonian of the dressing group action is exactly the monodromy matrix $T(L)$ with L being the boundary value of x:

$$\delta_X T(x) = \langle 1 \otimes X T(L) \{T(L) \otimes, T(x)\} \rangle_2. \tag{34}$$

The induced transformation on the Lax connection takes the form of a gauge transformation, i.e.

$$A_\mu(x) \to A^g_\mu(x) = \partial_\mu \Theta_\pm(x) \Theta^{-1}_\pm(x) + \Theta_\pm(x) A_\mu(x) \Theta^{-1}_\pm(x). \tag{35}$$

However, these must be form-preserving gauge transformations which maps one solution to the soliton equation to another. In particular, they change the soliton number as well as the total energy of the system.

2.2. Non-ultralocal gauges

– Ultralocal gauge does not always exist for soliton systems, e.g. Riemann-Sigma models are not ultralocalizable.
– Ultralocal gauges are extremely difficult to find, even if they do exist.
– Needs a description of dressing transformations which is essentially independent of the ultralocality of the gauge choice for the Lax system.
The FPR written in a generic (non-ultralocal) gauge takes the form

$$\{L(x)\otimes, L(y)\} = ([r+s, L(x)\otimes 1] + [r-s, 1\otimes L(x)])\,\delta(x-y) + 2s\delta'(x-y), \tag{36}$$

where r is antisymmetric under the exchange of its two tensor product components, while s is symmetric under the same exchange. It should be reminded that any FPR can be casted into the above form, with r and s being possibly dynamical.
When $s = 0$, the FPR (36) degenerates into (28). In the presence of generic s, the r in (36) no longer satisfies the classical Yang-Baxter equation, and this renders the factorization problem (33) more complicated.

Way Out:

$$\{D(x)\otimes, D(y)\} = [\hat{r}+\hat{s}, D(x)\otimes 1] + [\hat{r}-\hat{s}, 1\otimes L(x)], \tag{37}$$

where $D(x) = D_1$, $\hat{r} = r\delta(x-y)$, $\hat{s} = s\delta(x-y)$. The condition that (37) obeys Jacobian identity implies the following equations,

$$[d_{12}, d_{13}+d_{23}] + [d_{32}, d_{13}] = 0, \tag{38}$$

$$[c_{12}, c_{13}+c_{23}] + [c_{32}, c_{13}] = 0, \tag{39}$$

where $d_{12} \equiv r_{12}+s_{12}$, $d_{21} \equiv Pd_{12}P = -(r_{12}-s_{12}) \equiv -c_{12}$. The equations (38,39) are known as modified classical Yang-Baxter equations.

Introduce two linear maps R_d and R_c on $\mathcal{G}$,

$$R_d : \mathcal{G} \to \mathcal{G}, \qquad R_d(A) \equiv \langle d_{12}, 1\otimes A\rangle_2,$$

$$R_c : \mathcal{G} \to \mathcal{G}, \qquad R_c(A) \equiv \langle c_{12}, A\otimes 1\rangle_1, \quad \forall A \in G.$$

$$r_{12} = r^{ab} X_a \otimes X_b, \qquad s_{12} = s^{ab} X_a \otimes X_b,$$

$$\begin{aligned} R_d(A) &= (r^{ab} + s^{ab}) X_a \langle X_b, A \rangle \\ &= -(r^{ab} - s^{ab}) \langle X_a, A \rangle X_b = -R_c(A). \end{aligned}$$

One can simply write

$$R(A) \equiv R_d(A) = -R_c(A).$$

Applying the operation $\langle A \otimes B, \cdot \rangle_{12}$ to the FPR (37), one gets

$$\{D(A), D(B)\} = \langle D, [R(A), B] + [A, R(B)] \rangle \equiv D([A, B]_R), \qquad (40)$$

where $[A, B]_R \equiv [R(A), B] + [A, R(B)]$ is called the Baxter-Lie bracket. Notice that in writing (40), we have taken ∂ as the generator of a one dimensional Lie algebra independent of $\mathcal{G}$, with $\langle \partial, \mathcal{G} \rangle = 0$. The symbol $D(A)$ is defined as

$$D(A) = \langle D, A \rangle.$$

The condition for the left hand side of (40) to satisfy Jacobi identities now becomes the same condition for $[A, B]_R$ to satisfy Jacobi identity, i.e. the linearized form of the modified classical Yang-Baxter equation,

$$[RA, RB] - R([RA, B] + [A, RB]) = -[A, B]. \tag{41}$$

It follows immediately that the following classical Yang-Baxter equations hold,

$$[R_\pm A, R_\pm B] - R_\pm([R_\pm A, B] + [A, R_\pm B]) = 0, \tag{42}$$

where

$$R_\pm \equiv \frac{1}{2}(R \pm 1)\,.$$

The images of $R_\pm$ in $\mathcal{G}$,

$$\mathcal{G}_\pm \equiv \Im m R_\pm \subset \mathcal{G} \tag{43}$$

are respectively Lie algebras which commutes with each other, thanks to (42). Moreover, $R_\pm$ are also homomorphisms from $\mathcal{G}_R$ (i.e. the set $\mathcal{G}$ equipped with the Lie bracket $[,]_R$) to $\mathcal{G}$,

$$(R \pm 1)[X, Y]_R = [(R \pm 1)X, (R \pm 1)Y]. \tag{44}$$

For $X \in \mathcal{G}$, we need to make the unique factorization

$$X = X_+ - X_-.$$

This can be achieved as follows,

$$X = X_+ - X_- \equiv \frac{1}{2}(R+1)X - \frac{1}{2}(R-1)X. \tag{45}$$

Since the operator R is uniquely determined by the r and s matrices as shown above, this factorization is also unique, and correspondingly the group element factorization $g = g_-^{-1} g_+$ is also uniquely determined. With this solution of the factorization problem, the dressing transformation in the non-ultralocal case can be performed in exactly the same way as in the ultralocal case.

– *Everything works smoothly and it seems that there isn't much differences between ultralocal and non-ultralocal cases.*

——Anonymous

No! The structure of the dressing group is significantly modified by the gauge choice! We shall illustrate the differences below using simple examples.

3. Examples

3.1. Toda fields

Let us start by recalling the Lax pair of Toda field theory in the ultralocal gauge:

$$\begin{aligned} \partial_\pm T(x) &= A_\pm T(x), \\ A_\pm &= \pm\left[\frac{1}{2}\partial_\pm\Phi + \exp\left(\mp\frac{1}{2}ad\Phi\right)E_\pm\right], \end{aligned} \tag{46}$$

$$\Phi = \sum \varphi^i H_i, \quad E_\pm = \sum E_{\pm i},$$

– $\{H_i, E_{\pm i}\}$: the Chevalley basis for the Lie algebra $\mathcal{G}$. FPR:

$$\{L(x) \otimes, L(y)\} = [r^\pm, L(x) \otimes 1 + 1 \otimes L(y)]\delta(x-y),$$

$$r^\pm = \pm\frac{1}{2}[\sum_{ij}(K^{-1})^{ij}H_i \otimes H_j + 2\sum_{\alpha\in\Delta_+} E_{\pm\alpha} \otimes E_{\mp\alpha}],$$

K is the symmetrized Cartan matrix of $\mathcal{G}$.

The factorization problem is now solved with the aid of $r^{\pm}$:

$$R_{\pm}(H_i) = \pm\frac{1}{2}H_i, \quad R_{\pm}(E_{\pm\alpha}) = \pm E_{\pm\alpha}, \quad R_{\pm}(E_{\mp\alpha}) = 0. \tag{47}$$

This factorization is completely symmetric in the $+$ and the $-$ parts. In particular, the Cartan subalgebra generators appear in both the $+$ and the $-$ subalgebras of the dressing algebra. This $+-$ symmetry will be lost if we were working in a non-ultralocal gauge as will be shown below in (49).

Change the working gauge by setting

$$\Psi(x) \equiv \exp\left(\frac{1}{2}\Phi\right) T(x).$$

In the gauge with $\Psi(x)$ playing the role of transport matrix, one has the Lax pair

$$\partial_+\Psi(x) = (\partial_+\Phi + E_+)\,\Psi(x), \quad \partial_-\Psi(x) = -\left[\exp\left(ad\Phi\right) E_-\right]\Psi(x). \tag{48}$$

One has

$$\{L(x)\otimes, L(y)\} = ([r+s, L(x)\otimes 1] + [r-s, 1\otimes L(x)])\,\delta(x-y) + 2s\delta'(x-y),$$

with

$$\begin{aligned} s &= \sum_{ij} (K^{-1})^{ij} H_i \otimes H_j, \\ r &= \sum_{\alpha\in\Delta_+} (E_\alpha \otimes E_{-\alpha} - E_{-\alpha}\otimes E_\alpha) \quad \text{mod} \quad \mathbb{C}\Pi, \\ \Pi &= \sum_{ij} (K^{-1})^{ij} H_i \otimes H_j + \sum_{\alpha\in\Delta_+} (E_\alpha \otimes E_{-\alpha} + E_{-\alpha}\otimes E_\alpha). \end{aligned}$$

One can write

$$d_{12} = r_{12} + s_{12} = \sum_{ij} (K^{-1})^{ij} H_i \otimes H_j + \sum_{\alpha \in \Delta_+} \left(E_\alpha \otimes E_{-\alpha} - E_{-\alpha} \otimes E_\alpha \right).$$

This leads to the linear map R, with

$$R(H_i) = H_i, \quad R(E_{\pm\alpha}) = \pm E_{\pm\alpha}.$$

Consequently, one has

$$\begin{aligned} R_+(E_\alpha) &= E_\alpha, \quad R_+(E_{-\alpha}) = 0, \quad R_+(H_i) = H_i, \\ R_-(E_\alpha) &= 0, \quad R_-(E_{-\alpha}) = -E_{-\alpha}, \quad R_-(H_i) = 0. \end{aligned} \tag{49}$$

Thus for $\forall X \in \mathcal{G}$, a unique factorization $X = X_+ - X_-$ is given explicitly.
–Notice the sharp contrast between the solutions (47) (for ultralocal case) and (49) (for non-ultralocal case) to the factorization problem!

Dressing of the Toda fields:

According to the form of the Lax equation (48), the following vectors

$$\xi^{(\rho)} \equiv \langle \lambda^{(\rho)}_{\max} | \Psi(x), \quad \bar{\xi}^{(\rho)} \equiv \Psi^{-1}(x) \exp(\Phi) | \lambda^{(\rho)}_{\max} \rangle$$

are respectively chiral and antichiral,

$$\partial_- \xi^{(\rho)} = \partial_+ \bar{\xi}^{(\rho)} = 0.$$

The dressing transformation of $\Psi(x)$ naturally induces the transformation for $\xi^{(\rho)}$ and $\bar{\xi}^{(\rho)}$:

$$\xi^{(\rho)} \to \xi^{(\rho)} g_-^{-1}, \quad \bar{\xi}^{(\rho)} \to g_+ \bar{\xi}^{(\rho)}. \tag{50}$$

Therefore, one has

$$\exp(\lambda^{(\rho)}_{\max} \cdot \Phi) = \xi^{(\rho)} \cdot \bar{\xi}^{(\rho)} \to \xi^{(\rho)} g_-^{-1} g_+ \bar{\xi}^{(\rho)} = \xi^{(\rho)} g \bar{\xi}^{(\rho)}. \tag{51}$$

More concretely, writing

$$\Theta_\pm = K_\pm M_\pm$$

where $M_\pm$ are group elements generated by $E_{\pm\alpha}$, $K_\pm$ belong to the Cartan subgroup and, in particular, $K_- = 1$. One finds that Φ transforms according to

$$\Phi \to \Phi + \ln K_+.$$

3.2. Principal chiral model

The model is defined via the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}\langle J_\mu J^\mu \rangle, \qquad J_\mu = \partial_\mu g g^{-1}, \qquad g \in G.$$

The equation of motion and the Lax connection are respectively given by

$$\partial_\mu J^\mu = 0,$$
$$A_0 = \frac{\lambda J_1 + J_0}{\lambda^2 - 1}, \qquad A_1 = L = \frac{\lambda J_0 + J_1}{\lambda^2 - 1}.$$

The FPR can be directly evaluated, using the standard canonical Poisson brackets for the fundamental fields, yielding

$$\begin{aligned} \{L(x,\lambda) \otimes, L(y,\mu)\} &= [(r+s)(\lambda,\mu), L(x,\lambda)\otimes 1]\delta(x-y) \\ &\quad + [(r-s)(\lambda,\mu), 1 \otimes L(y,\mu)]\delta(x-y) \\ &\quad + 2s(\lambda,\mu)\delta'(x-y), \end{aligned} \tag{52}$$

where

$$s(\lambda,\mu) = \frac{\lambda+\mu}{(\lambda^2-1)(\mu^2-1)}\Pi,$$

Π is the symmetric tensor Casimir operator of G,

$$\Pi = g^{ab} X_a \otimes X_b.$$

Since the gauge group for the Lax connection is not G but rather $\tilde{G}$, the loop group based on G, the matrix $r(\lambda, \mu)$ is only determined modulo a constant multiple of the symmetric tensor Casimir $\tilde{\Pi}$ of $\tilde{G}$, i.e.

$$\begin{aligned} r(\lambda, \mu) &= \frac{1}{\lambda-\mu}\left(\frac{\lambda^2}{\lambda^2-1}+\frac{\mu^2}{\mu^2-1}\right)\Pi \quad \mathrm{mod} \quad \mathbb{C}\tilde{\Pi}, \\ \langle \tilde{\Pi}, X_2 \rangle &= X, \quad \forall X \in \tilde{G}. \end{aligned}$$

Following the last section, one can rewrite the FPR (52) into the form

$$\{D(x,\lambda)\otimes, D(y,\mu)\} = [\hat{d}_{12}(\lambda,\mu), D(x,\lambda)\otimes 1] + [\hat{c}_{12}(\lambda,\mu), 1\otimes D(y,\mu)],$$

where $\hat{d}_{12} = d_{12}\delta(x-y)$, $\hat{c}_{12} = c_{12}\delta(x-y)$, and

$$\begin{aligned} d_{12} &= \left\{\frac{1}{\lambda-\mu}\left(\frac{\lambda^2}{\lambda^2-1}+\frac{\mu^2}{\mu^2-1}\right)+\frac{\lambda+\mu}{(\lambda^2-1)(\mu^2-1)}\right\}\Pi \\ &= \left(\frac{2}{\lambda-\mu}\frac{\mu^2}{\mu^2-1}\right)\Pi, \\ c_{12} &= -d_{21}. \end{aligned}$$

Substituting the expressions for d_{12} and c_{12} back into (38,39), one sees that the factor $\frac{\mu^2}{\mu^2-1}$ appears in every term in the modified Yang-Baxter equation, which means that the modified Yang-Baxter equation holds true independent of this algebraically trivial factor. For this reason, one can simply drop this factor while defining the linear map R over the loop algebra $\tilde{\mathcal{G}}$, i.e. introduce the map R as

$$R[X(\lambda)] = \oint \frac{d\mu}{2\pi i} \frac{\mu^2 - 1}{\mu^2} tr_2 \left\{ d_{12}(\lambda, \mu) \left[1 \otimes X(\mu) \right] \right\}.$$

Consequently,

$$R_{\pm} \left[X(\lambda) \right] \equiv \frac{1}{2} \left(R \pm 1 \right) \left[X(\lambda) \right] = \frac{1}{2} \Big(R[X(\lambda)] \pm \langle \tilde{\Pi}, X_2(\lambda) \rangle \Big)$$

is well-defined. Explicitly, for $\lambda^n X_a \in \tilde{\mathcal{G}}$, one has

$$[\lambda^n X_a]_+ = \begin{cases} \lambda^n X_a, & (n \geq 0) \\ 0, & (n < 0) \end{cases},$$

$$[\lambda^n X_a]_- = \begin{cases} 0, & (n \geq 0) \\ -\lambda^n X_a, & (n < 0) \end{cases}.$$

Accordingly, for $\forall g(\lambda) \in \tilde{G}$, the factorization $g(\lambda) = g_-^{-1}(\lambda) g_+(\lambda)$ is uniquely determined, where $g_+(\lambda)$ is analytic inside the unit circle on the λ-plane, while $g_-(\lambda)$ is analytic outside the unit circle on the λ-plane. Such a factorization agrees exactly with the historical notion of Riemann-Hilbert transformations for the principal chiral model (which is the infinitesimal form of dressing transformations from the modern view point), which was invented before the relationship between dressing transformations and the Poisson-Lie structure was understood.

Notice that the λ-independent subalgebra $\mathcal{G}$ lies completely inside the subalgebra $\tilde{\mathcal{G}}_+$. This is another example in which the $+$ and $-$ subalgebras are asymmetric. The asymmetry between the $+$ and the $-$ subalgebras in the dressing group algebra may be a universal property for the dressing transformations defined in the non-ultralocal gauges.

The dressing group of transformations is actually the semiclassical ancestor of quantum group symmetries (including finite quantum groups, quantum affine algebras and Yangian doubles, each corresponds to different types of classical r-matrices). Therefore, for classical integrable systems admitting dressing transformations, the problem of finding the quantum spectrum is fairly simple: one simply needs to find all the states in certain highest weight representations of the corresponding quantum groups (or quantum affine algebras or Yangian doubles etc).

4. Potential Application

– I'm pretty aware that this is a workshop on superstrings and related matters. So why bother to talk about these algebraic issues here?

The answer is related to the recent advances in the study of IIB Green-Schwarz superstring on $AdS_5 \times S^5$ background.

As is well known, IIB superstring on $AdS_5 \times S^5$ is dual to $D = 4$ $N = 4$ SYM in the framework of AdS/CFT correspondence. Dolan, Nappi and Witten made some important discovery on the hidden symmetry from the SYM side (hep-th/0308089, 0401243). They actually found a Yangian algebra symmetry for the weakly coupled SYM. The corresponding structure on the AdS side is just the algebra of nonlocal charges associated with the nonlocal conserved currents. The latter was found even earlier by Bena, Polchinski and Roiban (hep-th/0305116). The nonlocal Yangian symmetry for the IIB superstring on $AdS_5 \times S^5$ is enlarged into super Yangian symmetry by Hatsuda and Yoshida (hep-th/0407044).

Now the problem arises: The approaches used by Bena-Pochinski-Roiban and Hatsuda-Yoshida are in a sense like treating the symmetries generated by non-local conserved currents as Hamiltonian symmetries (that's why the symmetry algebras they found lack a classical double structure). The situation is very much similar to the early days in the study of dressing symmetries. Since we KNOW that the symmetries generated by the nonlocal charges are related to dressing symmetries, their actions on the superstring variables should NOT be Hamiltonian – there must be a Poisson-Lie group action and hence a classical double structure.

What do we need to justify the last statement?

– We need a proper worldsheet action for the IIB superstring on $AdS_5 \times S^5$, and this is provided by Roiban-Siegal (hep-th/0010104) or alternatively by Metsaev-Tseytlin (hep-th/9805028);

– We need a Lax representation for the equations of motion, and this is given by Hatsuda-Yoshida as

$$[\partial_\tau + L_\tau, \partial_\sigma + L_\sigma] = 0, \tag{53}$$

$$(L_\tau)_M{}^N = -\frac{2\lambda}{\lambda^2 - 1}\left[\lambda (J_\tau)_M{}^N + (J_\sigma)_M{}^N\right], \tag{54}$$

$$(L_\sigma)_M{}^N = -\frac{2\lambda}{\lambda^2 - 1}\left[\lambda (J_\sigma)_M{}^N + (J_\tau)_M{}^N\right] \tag{55}$$

where $J_{\tau,\sigma}$ are right invariant vector fields on the super coset $\frac{GL(4|4)}{(Sp(4)\times GL(1))^2}$. Another Lax pair was given earlier than Hatsuda-Yoshida by Hou et al (hep-th/0406239) but with more complicated notations which I omit here;

– We see that the Lax structure is very much like the one for principal chiral model and naturally bears a non-ultralocal character, so we need a method to do dressings right from the non-ultralocal gauge – this is outlined here;

– To make the dressing procedure work, we need the explicit form for the r and s matrices. This is still NOT presented anywhere yet! Still waiting for this KEY input ...

5. Conclusions

If all the necessary conditions just listed are met and all the speculations I just outlined turns out to be correct, then it is highly hopeful that the theory of Yangian doubles will play a role in the quantization of IIB Green-Schwarz superstring on $AdS_5 \times S^5$ background.

Thank you!

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