Wave dynamics in the asymptotically flat spacetime

Schematic Picture of the wave evolution:

- Shape of the wave front (Initial Pulse)
- Quasi-normal ringing

  Unique fingerprint of the BH existence

  Detection is expected through GW observe

- Relaxation

K.D. Kokkotas & B.G. Schmidt. gr-qc/9909058

Perturbations in curved spacetime

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Outline:

I. Perturbations around black holes
1. Introduction
2. Perturbation equation
3. QNMs in AdS spacetimes
4. Support of AdS/CFT, dS/CFT correspondence

II. Inflation and cosmological perturbations
1. Introduction
2. Quantum fluctuations of a generic scalar field
3. Transplanckian Window
4. Transdimensional Window

I. Perturbation around black holes

1. Introduction:

Do BHs have a characteristic "Sound"?

Yes. During a certain time interval the evolution of initial perturbation is dominated by damped single-frequency oscillation

$\omega = \omega_R + i\omega_I$ $\propto$ black hole parameters

$\not\propto$ initial perturbation

Why it is called QNM?

1) They are not truly stationary, damped quite rapidly

2) They seem to appear only over a limited time interval, NMs extending from arbitrary early to late times.

What's the difference between QNM of BHs & QNM of stars?

1) Star: fluid making up star carry oscillations
Perturbations exist in metric and matter quantities over all space of star

2) BH: No matter could sustain such oscillation
Oscillations essentially involve the spacetime metric outside the horizon

2. The perturbation equation:

1) How to derive Eqs. governing the perturbation of BH?

2) How these Eqs. can be reduced to one-dimensional wave equation with a potential barrier?

2.1 Linear perturbations of BHs:

$$ds^2 = \mathring{g}_{\mu\nu} dx^\mu dx^\nu = -(1-\frac{2M}{r}) dt^2 + (1-\frac{2M}{r})^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \quad (1)$$

$\mathring{g}_{\mu\nu}$: metric of unperturbed static background spacetime

$$\mathring{R}_{\mu\nu} = 0 \quad (2)$$

Introducing small perturbation $h_{\mu\nu}$,

$$g_{\mu\nu} = \mathring{g}_{\mu\nu} + h_{\mu\nu} \quad (3)$$

where $|h_{\mu\nu}| << |\mathring{g}_{\mu\nu}|$ is assumed.

For static background, the behavior of the perturbed spacetime will be

$$R_{\mu\nu} = 0 \quad (4)$$

At first order of perturbation,

$$R_{\mu\nu} = \mathring{R}_{\mu\nu} + \delta R_{\mu\nu} = 0 \quad (5)$$

$\delta R_{\mu\nu} = R_{\mu\nu}(h_{\mu\nu})$

$$\delta R_{\mu\nu} = 0 \quad (6)$$

Eq.(6) can also be written in terms of the Christoffel symbols

$$\delta R_{\mu\nu} = -\delta\Gamma^{\beta}{}_{\mu\nu;\beta} + \delta\Gamma^{\beta}{}_{\mu\beta;\nu} = 0 \qquad (7)$$

where $\delta\Gamma^{\beta}{}_{\mu\nu} = \frac{1}{2}\overset{\circ}{g}{}^{\alpha\beta}(h_{\alpha\mu;\nu} + h_{\nu\alpha;\mu} - h_{\mu\nu;\alpha})$

These equations are linear in h, but they still form a system of ten coupled partial differential equations.

Can we simplify them?

Birkhoff's theorem: The Schwarzschild solution is the only spherically symmetric, asymptotically flat solution of Einstein equations in vacuum even if the spacetime is not static.

Thus: Nonrotating BHs can only be perturbed by nonradial perturbations and this forces to consider perturbations with complete angular dependence

$$h_{\mu\nu} = h_{\mu\nu}(t,r,\theta,\varphi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\sum_{n=1}^{10} C^{n}_{\ell m}(t,r)\,(Y^{n}_{\ell m})_{\mu\nu}(\theta,\varphi)$$

Different parts of h transform differently under rotations

$$h = \begin{pmatrix} h_{tt}^{S} & h_{tr}^{S} & h_{t\theta}^{V} & h_{t\varphi}^{V} \\ h_{rt}^{S} & h_{rr}^{S} & h_{r\theta}^{V} & h_{r\varphi}^{V} \\ h_{\theta t}^{V} & h_{\theta r}^{V} & h_{\theta\theta}^{T} & h_{\theta\varphi}^{T} \\ h_{\varphi t}^{V} & h_{\varphi r}^{V} & h_{\varphi\theta}^{T} & h_{\varphi\varphi}^{T} \end{pmatrix}$$

The scalar components of h can be represented directly by the scalar spherical harmonics $Y_{\ell m}(\theta,\varphi)$.

From the scalar function $S_{\ell m}(\theta,\varphi) = Y_{\ell m}(\theta,\varphi)$, vectors and tensors can be constructed as:

$$(\overset{1}{V}_{\ell m})_a = (S_{\ell m})_{;a} = \frac{\partial}{\partial x^a} Y_{\ell m}(\theta,\varphi)$$

$$(\overset{2}{V}_{\ell m})_a = \epsilon_a{}^{b} (S_{\ell m})_{;b} = \gamma^{bc} \epsilon_{ac} \frac{\partial}{\partial x^b} Y_{\ell m}(\theta,\varphi)$$

$$(\overset{1}{T}_{\ell m})_{ab} = (S_{\ell m})_{;ab}$$

$$(\overset{2}{T}_{\ell m})_{ab} = S_{\ell m} \gamma_{ab}$$

$$(\overset{3}{T}_{\ell m})_{ab} = \frac{1}{2}\left[\epsilon_a{}^{c} (S_{\ell m})_{;cb} + \epsilon_b{}^{c} (S_{\ell m})_{;ca}\right]$$

Indices a, b, c run from θ to φ; γ is the metric on 2-sphere of radius 1, ϵ is the totally antisymmetric tensor in 2-D, i.e. $\epsilon = \sin\theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$; the covariant derivatives are for 2-sphere.

[K. Thorne, Rev. Mod. Phys. 52, 299] (1980)

The scalar components of h can be expressed directly by the scalar spherical harmonics

$$S(r,\theta,\varphi) = \sum_{\ell m} a_{\ell m}(r)\, Y_{\ell m}(\theta,\varphi) \qquad (9)$$

The vector is achieved through an expansion in a series of vector spherical harmonics

$$V^{\alpha}(r,\theta,\varphi) = \sum_{\ell m} a_{\ell m}(r)\left[Y^{B}_{\ell m}(\theta,\varphi)\right]^{\alpha} + \sum_{\ell m} b_{\ell m}(r)\left[Y^{E}_{\ell m}(\theta,\varphi)\right]^{\alpha} \qquad (10)$$

(magnetic type) (electric type)

For tensor

$$T_{\mu\nu}(t,r,\theta,\varphi) = \sum_{\ell m} a_{\ell m}(t,r)\left[A^{ax}_{\ell m}(\theta,\varphi)\right]_{\mu\nu} + \sum_{\ell m} b_{\ell m}(t,r)\left[B^{pol}_{\ell m}(\theta,\varphi)\right]_{\mu\nu}$$

(ax, pol indicate the parity transformations) (11)

If $\mathcal{P}$ is the parity operator to produce a parity transformation on a rank 2 symmetric tensor $F_{\mu\nu}$

$$\mathcal{P}\left([F_{\ell m}(\theta,\varphi)]_{\mu\nu}\right) \longrightarrow \left[\tilde{F}_{\ell m}(\pi-\theta,\pi+\varphi)\right]_{\mu\nu}$$

The tensor spherical harmonics can be classified according to their behavior "under Parity change".

$$P(F_{\mu\nu}) = \tilde{F}_{\mu\nu} = (-1)^{\ell+1} F_{\mu\nu} \qquad \text{axial tensor harmonics}$$

$$P(F_{\mu\nu}) = \tilde{F}_{\mu\nu} = (-1)^{\ell} F_{\mu\nu} \qquad \text{polar}$$

checking the behavior under space inversions, we find

$$S_{\ell m} = Y_{\ell m} \qquad \text{polar} \qquad (-1)^{\ell}$$

$$\overset{1}{V}_{\ell m} = \frac{\partial}{\partial x^a} Y_{\ell m}(\theta,\varphi) \qquad \text{polar} \qquad (-1)^{\ell}$$

$$\overset{2}{V}_{\ell m} = \gamma^{bc} \epsilon_{ac} \frac{\partial}{\partial x^b} Y_{\ell m}(\theta,\varphi) \qquad \text{axial} \qquad (-1)^{\ell+1}$$

$$\overset{1}{T}_{\ell m} = (S_{\ell m})_{;ab} \qquad \text{polar} \qquad (-1)^{\ell}$$

$$\overset{2}{T}_{\ell m} = S_{\ell m} \gamma_{ab} \qquad \text{polar} \qquad (-1)^{\ell}$$

$$\overset{3}{T}_{\ell m} = \frac{1}{2}\left[\epsilon_a{}^c (S_{\ell m})_{;cb} + \epsilon_b{}^c (S_{\ell m})_{;ca}\right] \qquad \text{axial} \qquad (-1)^{\ell+1}$$

This classification of tensor spherical harmonics is reflected also on the metric perturbations, as a result are classified as "axial" & "polar" respectively.

2.2 Axial perturbations: the Regge-Wheeler Eq.

The general form of axial perturbation with given ℓ and m is

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & -h_0(t,r)\frac{1}{\sin\theta}\frac{\partial Y_{\ell m}}{\partial\varphi} & h_0(t,r)\sin\theta\frac{\partial Y_{\ell m}}{\partial\theta} \\ 0 & 0 & -h_1(t,r)\frac{1}{\sin\theta}\frac{\partial Y_{\ell m}}{\partial\varphi} & h_1(t,r)\sin\theta\frac{\partial Y_{\ell m}}{\partial\theta} \\ * & * & \frac{1}{2}h_2(t,r)\frac{1}{\sin\theta}X_{\ell m} & -\frac{1}{2}h_2(t,r)\sin\theta\, W_{\ell m} \\ * & * & * & -\frac{1}{2}h_2(t,r)\sin\theta\, X_{\ell m} \end{pmatrix}$$

*: a component fixed by the symmetry of h

$$X_{\ell m}(\theta,\varphi) = 2\left(\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi}Y_{\ell m} - \cot\theta\frac{\partial}{\partial\varphi}Y_{\ell m}\right)$$

$$W_{\ell m}(\theta,\varphi) = \left(\frac{\partial^2}{\partial\theta^2}Y_{\ell m} - \cot\theta\frac{\partial}{\partial\theta}Y_{\ell m} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}Y_{\ell m}\right)$$

where $h_0(t,r)$, $h_1(t,r)$ and $h_2(t,r)$ are unknown functions.

The Einstein equations with the metric perturbations $h_{\mu\nu}$ can be simplified if suitable gauge conditions are chosen.

In the linearized approach, infinitesimal coordinate transformation

$$x^{\alpha'} = x^{\alpha} + \eta^{\alpha}$$

will lead to new metric perturbation

$$g'_{\mu\nu}(x') = \overset{\circ}{g}_{\mu\nu}(x') + h'_{\mu\nu}$$

where $h'_{\mu\nu} = h_{\mu\nu} + \eta_{\mu;\nu} + \eta_{\nu;\mu}$

Taking the gauge vector

$$\eta^{\alpha} = \Lambda(t,r)\left[0,0,\overset{2}{V}(\theta,\varphi)\right] = \Lambda(t,r)\left[0,0,-\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}Y_{lm},\ \sin\theta\frac{\partial}{\partial\theta}Y_{lm}\right]$$

computing the changes in the metric perturbation, the new tensor $h'_{\mu\nu}$ has the correct general form.

The changes to the coefficients h_0, h_1, h_2 are

$$\delta h_0 = \frac{\partial}{\partial t}\Lambda(t,r)$$

$$\delta h_1 = \frac{\partial}{\partial r}\Lambda(t,r) - 2\frac{\Lambda(t,r)}{r}$$

$$\delta h_2 = -2\Lambda(t,r)$$

Taking $\Lambda(t,r) = -\frac{1}{2}h_2(t,r) \Rightarrow h_2(t,r) = 0$

— Regge-Wheeler gauge.

Inserting $h_{\mu\nu}$ in Regge-Wheeler form into

$$\delta R_{\mu\nu} = 0$$

Nontrivial radial equations are

δR_{23}: $\quad 0 = R_1(h_0, h_1, t, r) = \frac{1}{B(r)}\frac{\partial}{\partial t}h_0 - \frac{\partial}{\partial r}(B(r)h_1)$

δR_{13}: $\quad 0 = R_2(h_0, h_1, t, r)$

$$= \frac{1}{B(r)}\left(\frac{\partial^2 h_1}{\partial t^2} - \frac{\partial^2 h_0}{\partial t \partial r} + \frac{2}{r}\frac{\partial h_0}{\partial t}\right) + \frac{1}{r^2}(\ell(\ell+1)-2)h_1$$

δR_{03}: $\quad 0 = R_3(h_0, h_1, t, r)$

$$= \frac{1}{2}B(r)\left(\frac{\partial^2 h_0}{\partial r^2} - \frac{\partial^2 h_1}{\partial t \partial r} - \frac{2}{r}\frac{\partial h_1}{\partial t}\right) + \frac{1}{r^2}\left(r\frac{\partial}{\partial r}B(r) - \frac{\ell(\ell+1)}{2}\right)h_0$$

where $B(r) = 1 - 2M/r$

Defining $\Psi_\ell(t,r) = \frac{1}{r}B(r)h_1(t,r)$, $\Psi_\ell(t,r)$

satisfies the differential equation

$$\frac{\partial^2\Psi}{\partial t^2} - B(r)\frac{\partial}{\partial r}B(r)\frac{\partial}{\partial r}(\Psi/r) + \frac{2}{r}B^2(r)\frac{\partial}{\partial r}(r\Psi) + \frac{1}{r}((\ell+1)\ell - 2)B(r)\Psi = 0$$

Introducing tortoise coordinate

$$r_* = r + 2M\ln\left(\frac{r}{2M} - 1\right)$$

$r \to \infty$, $r_* \to r$

$r \to 2M$, $r_* \to -\infty$

The tortoise coordinate is suited to study the perturbation propagation near the BH horizon, $r^* \to -\infty$, does not suffer from coordinate singularities

2.3. Polar perturbations: the Zerilli Equation

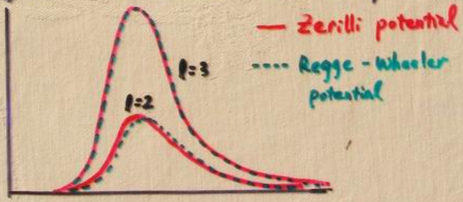
The analysis for polar perturbations proceeds along similar lines. However it is considerably more complicated and larger numbers of functions involved.

$$\frac{\partial^2 Z}{\partial t^2} - \frac{\partial^2 Z}{\partial r_*^2} + \tilde{V} Z = 0$$

$$\tilde{V} = \left(1-\frac{2M}{r}\right)\left[\frac{2\lambda(\lambda+1)r^3+6\lambda^2 M r^2+18\lambda M^2 r+18M^3}{r^3(\lambda r+3M)^2}\right]$$

where $\lambda = (l-1)(l+2)/2$

The Regge-Wheeler and Zerilli potentials look rather different, but actual values are quite close



After all complicated analysis, we are left with just two one-dimensional wave equations which completely determine the behavior of any perturbations of the black hole.

2.4 QNMs of Black Holes

If a harmonic time dependence is introduced for the perturbations,

$$\Psi, Z \sim \exp(i\omega_n t)$$

ω_n is the oscillation frequency of the n-th mode and is a complex number of the type

$$\omega_n = \omega_{r,n} + i\omega_{i,n} \qquad n = 0, 1, 2 \ldots$$

it is then possible to define the QNMs of the black hole as the solutions of equations

$$\partial^2_{r_*}\Psi + [\omega^2 - V]\Psi = 0$$

$$\partial^2_{r_*}Z + [\omega^2 - \tilde{V}]Z = 0$$

Boundary conditions:

$r_* \to \infty$, $\Psi, Z \sim \exp(+i\omega r_*)$ pure outgoing wave

$r_* \to -\infty$, $\Psi, Z \sim \exp(-i\omega r_*)$ pure ingoing wave

2.5. Summary of main results on QNMs of Schwarzschild BHs

1) All QNMs of Schwarzschild BH

ω_I positive $\longrightarrow$ damped modes

Schwarzschild BH is linearly stable against perturbations.

2) The QNMs in BH are isospectral,

axial / polar perturbations $\Rightarrow$ same ω

This is due to the uniqueness in which BH react to a perturbation. Not true for relativistic stars

3) Damping time $\tau \sim M$ (i.e. $\omega_{I,n} \sim 1/M$),

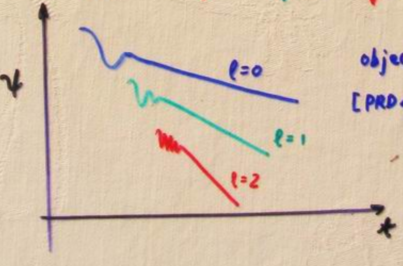
shorter for higher-order modes ($\omega_{I,n+1} > \omega_{I,n}$)

Detection of GW emitted from a perturbed BH

$\longrightarrow$ direct measure of BH mass

4)

n	$\ell=2$		$\ell=3$		$\ell=4$	
0	0.37367	−0.08896 i	0.59944	−0.09270 i	0.8092	−0.09416 i
1	0.34671	−0.27391 i	0.58264	−0.28130 i	0.79663	−0.28403 i
2	0.3011	−0.4783 i	0.5517	−0.4791 i	0.7727	−0.4799 i
3	0.2515	−0.7051 i	0.5119	−0.6903 i	0.7398	−0.6839 i



objective picture [PRD 49, 883 (1994)]

3. QNMs of AdS black holes

The quasinormal frequencies of AdS black holes have a direct interpretation in terms of the dual CFT.

AdS/CFT: a large static black hole in asymptotically AdS spacetime corresponds to an (approximately) thermal state in CFT.

Perturbing the black hole corresponds to perturbing this thermal state, and the decay of the perturbation describes the return to thermal equilibrium.

3.1 Schwarzschild - AdS black holes.

D-dimensional SAdS metric

$$ds^2 = -f(r)\,dt^2 + f(r)^{-1}\,dr^2 + r^2\,d\Omega_{d-2}^2$$

where $f(r) = \frac{r^2}{R^2} + 1 - \left(\frac{r_0}{r}\right)^{d-3}$

R: AdS radius, r_0 related to the black hole mass

$$M = \frac{(d-2)A_{d-2}\,r_0^{d-2}}{16\pi G_d}$$

$$A_{d-2} = 2\pi^{(d-1)/2}/\Gamma\left(\frac{d-1}{2}\right)$$

the area of a unit $(d-2)$-sphere

The black hole horizon is at $r = r_+$, the largest zero of f.
The Hawking temperature is

$$T = \frac{f'(r_+)}{4\pi} = \frac{(d-1)r_+^2 + (d-3)R^2}{4\pi r_+ R^2}$$

The minimally coupled scalar wave equation

$$\nabla^2 \phi = 0.$$

If we consider modes

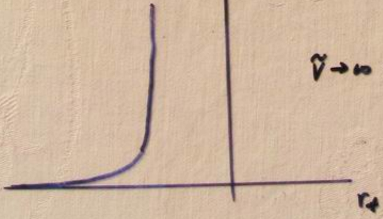
$$\phi(t, r, \text{angles}) = r^{(2-d)/2} \psi(r) Y(\text{angles}) e^{-i\omega t}$$

where Y denotes the spherical harmonics on S^{d-2}, and introduce the tortoise coordinate $dr_* = dr/f(r)$, the wave equation reduces to the form

$$[\partial_{r_*}^2 + \omega^2 - \tilde{V}(r_*)]\psi = 0$$

$\tilde{V}$: positive $r_* \to -\infty$ (Location of the horizon) $\tilde{V} = 0$

r_* $(r \to \infty)$ $\tilde{V}$ diverge



$\tilde{V} \to \infty \Rightarrow \Phi \to 0$

In the absence of a black hole, r_* has only a finite range and solutions exist for only a discrete set of real ω.

Once the BH is added, ω may have any values.

Outgoing wave coming from the (past) horizon, scattering off the potential and becoming an ingoing wave entering the (future) horizon.

Definition of QNM in AdS BHs:

QNMs are defined to be modes with only ingoing waves near the horizon.

Exists for only a discrete set of complex ω.

We want modes which behave like $e^{-i\omega(t+r_*)}$ near the horizon, it is convenient to set $v = t + r_*$ and work with ingoing Eddington coordinates.

The metric of SAdS BH in d-dimensions in ingoing Eddington coordinates is

$$ds^2 = -f(r)\,dv^2 + 2\,dv\,dr + r^2 d\Omega_{d-2}^2$$

where $f = \frac{r^2}{R^2} + 1 - \left(\frac{r_0}{r}\right)^{d-3}$

By separation of variables

$$\Phi(v, r, \text{angles}) = r^{(2-d)/2}\,\psi(r)\,Y(\text{angles})\,e^{-i\omega v}$$

the minimally coupled scalar wave equation

$$\nabla^2 \Phi = 0$$

reduce to the radial equation for $\psi(r)$

$$f\frac{d^2}{dr^2}\psi + [f'(r) - 2i\omega]\frac{d}{dr}\psi(r) - V(r)\psi(r) = 0$$

where $V = \frac{(d-2)(d-4)}{4r^2}f(r) + \frac{d-2}{2r}f'(r) + \frac{c}{r^2}$

and $c = l(l+d-3)$

Boundary:

Near the (future) horizon, ingoing modes $e^{-i\omega v}$

Near the (past) horizon, outgoing modes $e^{-i\omega(t-r_*)} = e^{-i\omega v} * e^{2i\omega r_*}$

$$r_* = \int \frac{dr}{f(r)} \approx \frac{1}{f'(r_h)}\ln(r-r_h)$$

near the horizon $r=r_h$, the outgoing modes behave like $e^{-i\omega(t-r_*)} = e^{-i\omega v}e^{2i\omega r_*} \approx e^{-i\omega v}(r-r_h)^{2i\omega/f'(r_h)}$

We wish to find the complex values of ω such that Eq. has a solution with only ingoing modes near the horizon and vanishing at infinity.

Numerical approach

To compute the QNM, we will expand the solution in a power series about the horizon and impose the boundary condition that the solution vanish at infinity.

In order to map the entire region of interest, $r_+ < r < \infty$, into a finite parameter range, we change variables to $x = 1/r$. Then the wave equation

$$f(r)\frac{d^2}{dr^2}\psi(r) + [f'(r) - 2i\omega]\frac{d}{dr}\psi(r) - V(r)\psi(r) = 0$$

becomes

$$s(x)\frac{d^2}{dx^2}\psi(x) + \frac{t(x)}{x - x_+}\frac{d}{dx}\psi(x) + \frac{u(x)}{(x - x_+)^2}\psi(x) = 0 \qquad (*)$$

where

$$s(x) = \frac{r_0^{d-3}x^{d+1} - x^4 - x^2}{x - x_+} = \frac{x_+^2 + 1}{x_+^{d-1}}x^d + \cdots + \frac{x_+^2 + 1}{x_+^3}x^4 + \frac{x^3}{x_+^2} + \frac{x^2}{x_+}$$

$$t(x) = (d-1)r_0^{d-3}x^d - 2x^3 - 2x^2 i\omega$$

$$u(x) = (x - x_+)V(x)$$

where $r_0^{d-3} = (x_+^2 + 1)/x_+^{d+1}$

s, t, u can be expanded about the horizon $x = x_+$

e.g. $s(x) = \sum_{n=0}^{d} s_n (x - x_+)^n$

To determine the behavior of the solutions near the horizon, we look for a solution of the form

$$\psi(x) = \sum_{n=0}^{\infty} a_n (x - x_+)^n$$

Substitute into (*) and equating coefficients of $(x-x_+)^n$ for each n, we obtain the following recursion relations for the a_n:

$$a_n = -\frac{1}{P_n} \sum_{k=0}^{n-1} \left[k(k-1) s_{n-k} + k t_{n-k} + u_{n-k} \right] a_k$$

where

$$P_n = n(n-1) s_0 + n t_0 = 2x_+^2 \, n(n\kappa - i\omega)$$

Boundary condition:

$r \to \infty \; (x \to 0)$, $\psi \to 0$

⇓

$x = 0$, $\psi(0) = \sum_{n=0}^{\infty} a_n (x - x_+)^n = 0$ — Satisfied only for special (discrete) values of ω

In order to find QNM, we need zeros of

$\sum_{n=0}^{\infty} a_n(\omega)(-x_+)^n$ in the complex ω plane.

This is done by truncating the series after a large number of terms and computing the partial sum as a function ω. One can find zeros of this partial sum and check the accuracy by seeing how much the location of zeros changes going to H.

Results: (set $R=1$)

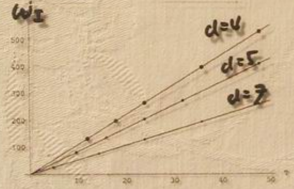
1. For large BH ($r_+ \gg R$)

r_+	4D BH modes ω_I	4D BH modes ω_R	5D BH modes ω_I	5D BH modes ω_R	7D BH modes ω_I	7D BH modes ω_R
100	266.3856	184.9534	274.6655	311.9627	261.2	500.8
50	133.1933	92.4937	137.3296	156.0077	130.7	250.4
10	26.6418	18.6070	27.4457	31.3699	26.07	50.35
5	13.3255	9.4711	13.6914	15.9454	12.96	25.57
1	2.6712	2.7982	2.5547	4.5788	2.16	7.27
0.8	2.1304	2.5878	1.9676	4.1951		
0.6	1.5797	2.4316	1.3656	3.8914		
0.4	1.0064	2.3629	0.7462	3.7174		

Both the ω_R, $\omega_I \propto r_+$

Temperature of a large BH $T = \frac{(d-1)r_+}{4\pi}$,

ω_R, ω_I are linear functions of T.



$\omega_I = 11.16\,T \quad d=4$

$\omega_I = 8.63\,T \quad d=5$

$\omega_I = 5.47\,T \quad d=7$

($\tau = 1/\omega_I$: time scale approach to thermal equilibrium

In table, as a function of r_+, ω_I is almost independent of dimension. The difference in these slopes is almost entirely due to the dimension dependence of the relation between r_+ and T.

ω_R does depend on the dimension

$\omega_R = 10.5T \quad d=7$

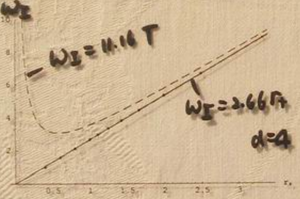
$\omega_R = 9.8T \quad d=5$

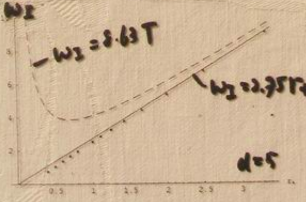
$\omega_R = 7.75T \quad d=4$

2. For intermediate size BH ($r_+ \sim R$)

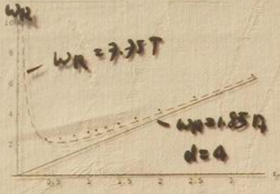
The quasinormal frequencies do not scale with the temperature.

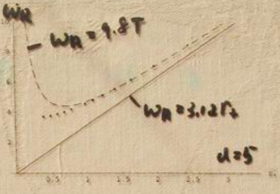
Imaginary part





The real part





ω_R approximates the temperature more closely than the BH size.

3. Dependence of the quasinormal frequency on ℓ

$\ell \uparrow$, $\omega_I \downarrow$ $\omega_R \uparrow$

cf. SBH case

4. For small BH ($r_+ < R$)

Speculation: $r_+ \to 0$, $\omega_I \to 0$, $\omega_R \to$ const

Decay of the field is due to absorption by the BH, BH becomes arbitrarily small, the field will no longer decay.

Summary of the QNM in SAdS BH:

- QNM is determined by two dimensionful parameters

  the AdS radius R & BH radius r_+

- For large BHs ($r_+ \gg R$), there is an additional symmetry which ensures that ω can depend only on the BH temperature $T \sim r_+/R^2$

- For smaller BHs, $T \uparrow$ as $r_+ < R$, $\omega_I \propto r_+$

cf. Asymptotically flat BHs

An ordinary SBH has only one dimensionful parameter T

ω must be multiples of this temperature.

Difference between AdS SBH & SBH:

1. Small AdS BH do not behave like BHs in asymptotically flat spacetime.

   reason: The boundary conditions at infinity are changed. Physically, the late time behavior of the field is affected by waves bouncing off the potential at large r.

2. Decay at very late times

   SBH: power law tail

   SAdS BH: exponential decay

More discussion on ω for small SAdS BH:

$\omega_I \downarrow$ as $r_+ \downarrow$, $\omega_R \to$ const. $(r_+ \ll R)$

This result was challenged by the superpotential approach

(T.R. Govindarajan, V. Suneeta, CQG 18, 265 (2001))

The mode is still proportional to the surface gravity

Object Picture: (J.M. Zhu, B. Wang, E. Abdalla PRD63, 124004 (2001))

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega_{d-2}^2$$

where $f(r) = \frac{r^2}{R^2} + 1 - \left(\frac{r_0}{r}\right)^{d-3}$

Using the seperation

$$\Phi(t, r, \text{angles}) = r^{(2-d)/2}\,\psi(t, r)\,Y(\text{angles})$$

The radial wave function ψ satisfies the equation

$$-\frac{\partial^2 \psi}{\partial t^2} + f\frac{\partial}{\partial r}\left(f\frac{\partial \psi}{\partial r}\right) = V\psi$$

where

$$V = f\left[\frac{l(l+d-3)}{r^2} - \frac{(2-d)(d-4)}{4r^2}f - \frac{2-d}{2r}\frac{\partial f}{\partial r}\right]$$

Using tortoise coordinate $r^* = \int dr/f$, the wave eq. become

$$-\frac{\partial^2 \psi}{\partial t^2} + \frac{\partial^2 \psi}{\partial r^{*2}} = V\psi$$

$$-4\frac{\partial^2}{\partial u \partial v}\psi = V(r)\psi \qquad (u = t - r^*,\ v = t + r^*)$$

The two-dimensional wave equation can be integrated numerically, using for example the finite difference method suggested by Price, Pullin.

Using Taylor's theorem, it is discretized as

$$\psi_N = \psi_E + \psi_W - \psi_S - \delta u \delta v\, V\left(\frac{v_N + v_W - u_N - u_E}{4}\right)\frac{\psi_W + \psi_E}{8} + O(\epsilon^4)$$

where the points N, S, E, W form a null rectangle with relative positions as:

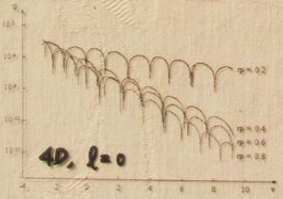
N: $(u+\delta u,\ v+\delta v)$, W: $(u+\delta u,\ v)$

E: $(u,\ v+\delta v)$ S: $(u,\ v)$

Initial condition $\psi(u, v=v_0)=0$

Initial perturbation $\psi(u=u_0, v) = \exp\left[-\frac{(v-v_c)^2}{2\sigma^2}\right]$.

Results for small BH ($r_h < R$) ($R=1$)



1. ringing: oscillatory exponential fall off
2. $r_h \downarrow$, damping time $\uparrow$, $\omega_I \downarrow$
3. oscillation time scale do not differ much for different r_h. $\omega_R \to$ const.

Relation between ω and spacetime dimensions for small SAdS BH

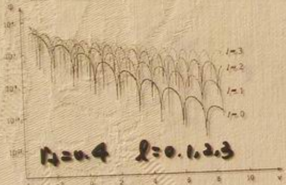
$\omega_R \uparrow$ with $d \uparrow$

$\omega_I \downarrow$ with $d \uparrow$

Different from the behavior of ω for big SAdS BH

ω_I almost independent of dimension

ω_R does depend on the dimension (Consistent with ω_R for small BH)



Wave dynamics behavior for different multipole index l

$l \uparrow$. $\omega_I \downarrow$. $\omega_R \uparrow$

The dependence of quasinormal frequencies on the multipole index is universal for big. intermediate and small AdS BHs.

QNMs of RN AdS BHs

B. Wang, C. Y. Lin, E. Abdalla, PLB
B. Wang, C. Molina, E. Abdalla, PRD

Besides r_+, R, the RN AdS BH has another parameter, the charge Q.

It possesses richer physics to be explored.

$$ds^2 = -h\,dt^2 + h^{-1}dr^2 + r^2 d\Omega^2, \quad A = Q/r\,dt$$

where
$$h = 1 - \frac{r_+}{r} - \frac{r_+^3}{R^2 r} - \frac{Q^2}{r_+ r} + \frac{Q^2}{r^2} + \frac{r^2}{R^2}$$

The mass of the BH is
$$M = \frac{1}{2}\left(r_+ + \frac{r_+^3}{R^2} + \frac{Q^2}{r_+}\right)$$

The Hawking temperature is given by
$$T_H = \frac{1 - Q^2/r_+^2 + 3r_+^2/R^2}{4\pi r_+}$$

and the potential by $\phi = \dfrac{Q}{r_+}$

In the extreme case r_+, Q satisfy the relation
$$1 - \frac{Q^2}{r_+^2} + \frac{3r_+^2}{R^2} = 0$$

Consider a massless scalar field Φ in the RN AdS spacetime, obeying the wave equation

$$\Box\Phi = 0$$

where $\Box = g^{\alpha\beta}\nabla_\alpha\nabla_\beta$

If we decompose the scalar field

$$\Phi = \sum_{lm} \frac{1}{r} \psi_l(t,r) Y_{lm}(\theta,\phi)$$

then each wave function $\psi_l(r)$ satisfies

$$-\frac{\partial^2 \psi_l}{\partial t^2} + \frac{\partial^2 \psi_l}{\partial r^{*2}} = V_l \psi_l$$

where

$$V_l = h\left[\frac{l(l+1)}{r^2} + \frac{1}{r}\frac{dh}{dr}\right]$$

$$= h\left[\frac{l(l+1)}{r^2} + \frac{r_+ + r_+^3/R^2 + Q^2/r_+}{r^3} - \frac{2Q^2}{r^4} + \frac{2}{R^2}\right]$$

r^* is the tortoise coordinate $r^* = \int dr/h$.

V_l is positive and vanishes at the horizon, however it diverges at $r=\infty$, which requires that Φ vanishes at infinity.

QNMs of AdS space are defined to be modes with only ingoing wave near horizon. Using the ingoing Eddington coordinates $v = t + r^*$, the metric changes to

$$ds^2 = -h\,dv^2 + 2\,dv\,dr + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

Adopt the seperation

$$\Phi = \frac{1}{r}\psi(r)Y(\theta,\varphi)e^{-i\omega v}$$

the radial wave eq:

$$h(r)\frac{d^2\psi}{dr^2} + (h'(r) - 2i\omega)\frac{d\psi}{dr} - V(r)\psi(r) = 0$$

where the potential is given by

$$V(r) = \frac{h(r)}{r} + \frac{\ell(\ell+1)}{r^2}$$

$$= \frac{1}{r}\left(\frac{r_+}{r^2} + \frac{r_+^3}{R^2 r^3} + \frac{Q^2}{r_+ r^2} - \frac{2Q^2}{r^3} + \frac{2r}{R^2} \right) + \frac{\ell(\ell+1)}{r^2}$$

To find the complex values of ω such that

ψ finite at the horizon $r=r_+$, and vanishing at infinity, we use numerical method suggested by Horowitz. We will expand the solution in power series about the horizon and impose the boundary condition that the solution vanishes at infinity.

Adopting $x = 1/r$, the radial wave eq. becomes

$$S(x)\frac{d^2}{dx^2}\psi(x) + \frac{t(x)}{x-x_+}\frac{d}{dx}\psi(x) + \frac{u(x)}{(x-x_+)^2}\psi(x) = 0$$

where

$$S(x) = \frac{r_0 x^5 - x^4 - x^2 - Q^2 x^6}{x - x_+}$$

$$t(x) = 3r_0 x^4 - 2x^3 - 4Q^2 x^5 - 2i\omega x^2$$

$$u(x) = (x - x_+) V(x)$$

and

$$r_0 = \frac{1 + x_+^2 + Q^2 x_+^4}{x_+^3}$$

Expanding s, t, u about the horizon $x = x_+$ in the form

$$s(x) = \sum S_n (x - x_+)^n$$

The first terms are

$s_0 = 2x_+^2\kappa$, $t_0 = 2x_+^2(\kappa - i\omega)$ $u_0 = 0$

where $\kappa = (x_+ + 3/x_+ - Q^2 x_+^3)/2$ is the surface gravity.

The solution of the wave eq. can be expressed as a power series

$$\psi(x) = \sum_{n=0}^{\infty} a_n (x - x_+)^n$$

Substituting it into wave eq. and equating coefficients of $(x-x_+)^n$ for each n, we have the recursion relations for a_n

$$a_n = -\frac{1}{P_n} \sum_{k=0}^{n-1} [k(k-1)s_{n-k} + k t_{n-k} + u_{n-k}] a_k$$

where

$$P_n = n(n-1)s_0 + n t_0 = 2x_+^2 n(n\kappa - i\omega)$$

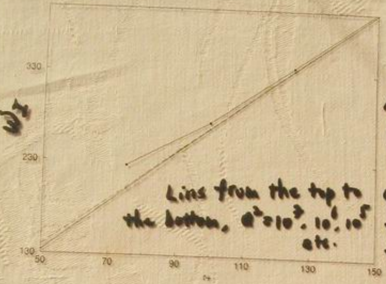
The boundary condition

$$r \to \infty \; (x \to 0) \; , \; \psi = \sum_{n=0}^{\infty} a_n(\omega)(-x_+)^n = 0$$

The algorithm to find ω:

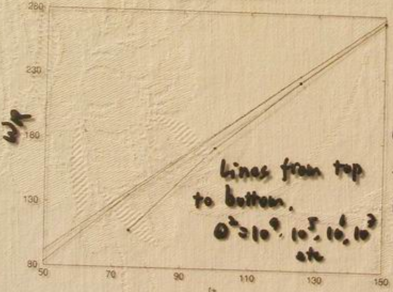
1. truncate at a number N of terms
$$\sum_{n=0}^{N} a_n(\omega)(-x_+)^n = 0$$
2. find roots of interest of this function
3. increase N until these roots become constant within the desired precision

Results: 1. ω depend on Q



$Q\uparrow, \omega_I \uparrow$

AdS/CFT:
For big Q, it is quicker for the quasinormal ringing to settle down to thermal equilibrium



$Q\uparrow, \omega_R \downarrow$

Frequency of the oscillation becomes small as Q increases

With an additional parameter, charge Q, neither ω_R nor ω_I is a linear function of r_+ as found is SAdS BH. The bigger the charge Q is, the larger is the deviation from the linear relation we observe.

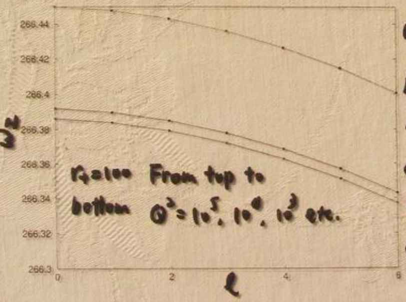
$Q\uparrow$, $\omega_I \uparrow$, $\omega_R \downarrow$ tells us.

If we perturb a RN AdS BH with high charge, the surrounding geometry will not "ring" as much and long as that of the BH with small Q.

It is easy for the perturbation on the highly charged AdS BH background to return to thermal equilibrium.

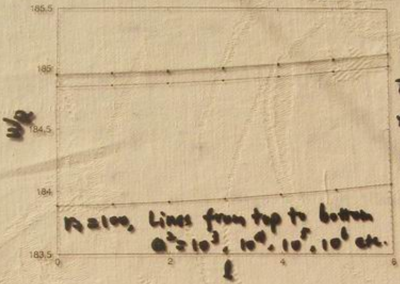
<u>This is the new physics brought by Q.</u>

2. ω depend on ℓ.



$\omega_I \downarrow$ with $\ell \uparrow$.

Different values of Q do not change the qualitative characteristic



$l\uparrow$. $\omega_R\uparrow$
This behavior is the same for different Q.

When Q increases to nearly the extreme value satisfying $1-\frac{Q^2}{R^2}+\frac{3R^2}{R^2}=0$, this numerical method breaks down.

Object picture of the quasinormal ringing?

The object picture of the evolution of a massless scalar field in RN AdS BH

$$ds^2 = -h\,dt^2 + h^{-1}dr^2 + r^2 d\Omega^2$$

$$h = 1 - \frac{r_+}{r} - \frac{r_+^3}{R^2 r} - \frac{Q^2}{r_+ r} + \frac{Q^2}{r^2} + \frac{r^2}{R^2}$$

$$= \frac{1}{R^2 r^2}(r-r_+)(r-r_-)(r-r_1)(r-r_2)$$

where r_1, r_2 are two complex roots relating to r_+, r_- by

$$r_1 + r_2 = -(r_+ + r_-)$$

$$r_1 r_2 = R^2 + r_+ r_- + r_+^2 + r_-^2$$

Introducing the surface gravity κ_i associated with r_i by the relation $\kappa_i = \frac{1}{2}\left|dh/dr\right|_{r=r_i}$, we have

$$\kappa_{r_+} = \frac{1}{2R^2}(r_+ - r_-)(r_+ - r_1)(r_+ - r_2)/r_+^2$$

$$\kappa_{r_-} = \frac{1}{2R^2}(r_+ - r_-)(r_- - r_1)(r_- - r_2)/r_-^2$$

$$\kappa_{r_1} = \frac{1}{2R^2}(r_+ - r_1)(r_- - r_1)(r_1 - r_2)/r_1^2$$

$$\kappa_{r_2} = \frac{1}{2R^2}(r_+ - r_2)(r_- - r_2)(r_2 - r_1)/r_2^2$$

These quantities allow us to write

$$h^{-1} = \frac{1}{2\kappa_{r_+}(r-r_+)} - \frac{1}{2\kappa_{r_-}(r-r_-)} + \frac{1}{2\kappa_{r_1}(r-r_1)} - \frac{1}{2\kappa_{r_2}(r-r_2)}$$

then the tortoise coordinate r_* is in the form

$$r_* = \frac{1}{2\kappa_2}\ln(r-r_2) - \frac{1}{2\kappa_1}\ln(r-r_1)$$

$$+ \frac{R^2[(r_1 r_2)^2 - r_+ r_- r_1 r_2]}{[r_+^2 - r_+(r_1+r_2)+r_1 r_2][r_-^2 - r_-(r_1+r_2)+r_1 r_2]} \times \int \frac{dr}{r^2 - r(r_1+r_2)+r_1 r_2}$$

$$+ \frac{R^2[r_+ r_-(r_1+r_2) - r_+ r_1 r_2 - r_- r_1 r_2]}{[r_+^2 - r_+(r_1+r_2)+r_1 r_2][r_-^2 - r_-(r_1+r_2)+r_1 r_2]} \times \int \frac{dr}{r^2 - r(r_1+r_2)+r_1 r_2}$$

Using the separation

$$\Phi = \sum_{\ell m} \frac{1}{r}\psi_\ell(t,r)\,Y_{\ell m}(\theta,\phi)$$

the wave function $\psi_\ell(r)$ satisfies

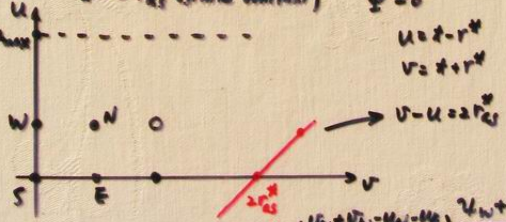
$$-\frac{\partial^2 \psi_\ell}{\partial t^2} + \frac{\partial^2 \psi_\ell}{\partial r_*^2} = V_\ell \psi_\ell$$

where $V_\ell = h\left[\frac{\ell(\ell+1)}{r^2} + \frac{1}{r}\frac{dh}{dr}\right]$

$r \to \infty$, $r_* \to r^*_{as}$ (finite constant) $\quad \Phi = 0$

$u = t - r^*$

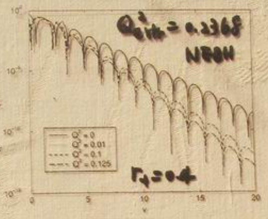
$v = t + r^*$



$$\psi_N = \psi_E + \psi_W - \psi_S - \delta u\,\delta v\,V\left(\frac{v_N + v_W - u_N - u_E}{4}\right)\frac{\psi_W + \psi_E}{8}$$

$$+ O(\epsilon^4)$$

Object picture of QNM:



$r_+ = 0.4$ ($Q^2_{crit} = 0.2368$)

Q^2	r_-	ω_I^1	ω_R^1
0	0	1.007	2.363
0.01	2.14E-2	1.034	2.327
0.1	0.196	1.42	2.05
0.125	0.238	1.53	2.04

$r_+ = 1$ ($Q^2_{crit} = 4$)

Q^2	r_-	ω_I^1	ω_R^1
0	0	2.67	2.79
0.01	4.9875E-003	2.68	2.78

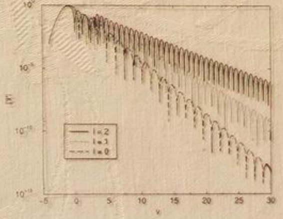
1. ω dependence on Q

$Q\uparrow$, $\omega_I \uparrow$, $\omega_R \downarrow$

AdS/CFT: For bigger Q, it is quicker for the quasinormal ringing to settle down to thermal equilibrium.

If we perturb a RN AdS BH with high charge, the surrounding geometry will not "ring" as much and as long as that of the BH with small Q.

2. Wave evolution with l

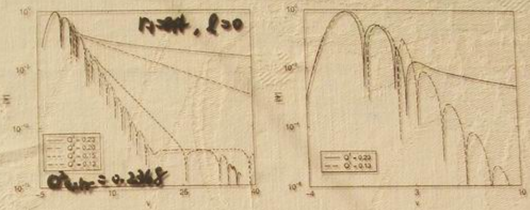


$l\uparrow$, $\omega_I \downarrow$, $\omega_R \uparrow$

Keeps the same as Q is introduced

c.f. asymptotically flat spacetime

3. QNM for highly charged AdS BH



$Q > Q_{ctr}$, $\omega_I \downarrow$ with $Q \uparrow$

$\omega_R \to 0$

Different properties of imaginary frequencies with the increase of the charge Q reflect different phase characteristics?

Second-order phase transition in the extreme limit of BH?

Consistent with the result found in Kerr BH (PRL 84, 4537 (2000))

Excellent agreements with our results have been found in [E. Berti and K.D. Kokkotas, PRD 67, 064020 (2003)]

QNMs in topological BH backgrounds

B. Wang, E. Abdalla, R.B. Mann, PRD 65, 084006 (2002)

Dependence of QNMs on the curvature-coupling constant, spacetime topology

$$ds^2 = -N_g(r)\,dt^2 + N_g(r)^{-1}dr^2 + r^2 d\Omega^2$$

$$N_g(r) = r^2/\ell^2 - \epsilon(g-1) - 2M/r$$

where

$$\epsilon(g-1) = [\Theta(g-1) - \Theta(1-g)] = \begin{cases} -1 & g=0 \\ 0 & g=1 \\ 1 & g>1 \end{cases}$$

$$d\Omega^2 = \begin{cases} d\theta^2 + \sin^2\theta\, d\phi^2 & g=0 \quad \text{SAdS BH} \\ d\theta^2 + d\phi^2 & g=1 \quad \text{toroidal spacetime} \\ d\theta^2 + \sinh^2\theta\, d\phi^2 & g>1 \quad \text{topological BH (hyperbolic space)} \end{cases}$$

The massless scalar field Φ obeys the wave equation

$$(\Box - \xi R)\Phi = 0$$

where $R = -4|\Lambda|$ the Ricci scalar $\quad (\ell = \sqrt{3/|\Lambda|})$

ξ a tunable curvature coupling constant

Decompose $\Phi = \sum \frac{1}{r}\psi(t,r)\,Y(\theta,\varphi)$

$\psi(r)$ satisfies

$$-\frac{\partial^2\psi}{\partial t^2} + \frac{\partial^2\psi}{\partial x_g^2} = V\psi \qquad x_g = \int N_g^{-1}\,dr \text{ tortoise coord.}$$

where $V = N_g\left(\frac{2}{\ell^2}(1-6\xi) + \frac{2M}{r^3}\right)$

To compare the wave behavior for BHs of different topologies, we perform a rescaling $r = r_+ z$ so that the event horizon is at unit dimensionless distance. Adopting this dimensionless variable z, we obtain

$$\tilde{N}_g = \ell^2 N_g / r_+^2 = \frac{z-1}{z}\left(z^2+z+1-\frac{\ell^2}{r_+^2}\,6(g-1)\right)$$

After rescaling the scalar wave equation becomes

$$-\frac{\partial^2 \tilde{\psi}}{\partial \tilde{t}^2} + \frac{\partial^2 \tilde{\psi}}{\partial \tilde{x}_g^2} = \tilde{V}\tilde{\psi}$$

where $\tilde{t} = \frac{r_+}{\ell^2} t$, $\tilde{x}_g = \frac{r_+}{\ell^2} x_g$

$$\tilde{V} = \left[1-3\beta^2 6(g-1)\right]\frac{z-1}{z^4}\left[z^2+z+1-3\beta^2 6(g-1)\right]$$
$$+ 2\chi\frac{z-1}{z}\left[z^2+z+1-3\beta^2 6(g-1)\right]$$

and $\chi = 1-6\xi$. $\beta = \frac{\ell}{\sqrt{3}\,r_+} > 0$

For different BHs with different topologies, $\tilde{x}_g$ are

$$\tilde{x}_{g=0} = \frac{1}{3(1+\beta^2)}\left[\ln\frac{z-1}{\sqrt{z^2+z+1+3\beta^2}} + \frac{\sqrt{3}(1+2\beta^2)}{\sqrt{1+4\beta^2}}\arctan\left(\frac{2z+1}{\sqrt{3+12\beta^2}}\right)\right]$$

$$\tilde{x}_{g=1} = \frac{1}{3}\ln\frac{z-1}{\sqrt{z^2+z+1}} + \frac{1}{3}\arctan\left(\frac{2z+1}{\sqrt{3}}\right)$$

(independent of parameter β)

$$\tilde{x}_{g\geq 2} = \frac{1}{3(1-\beta^2)}\left[\ln\frac{z-1}{\sqrt{z^2+z+1-3\beta^2}} + \frac{\sqrt{3}(1-2\beta^2)}{\sqrt{1-4\beta^2}}\tan^{-1}\left(\frac{2z+1}{\sqrt{3-12\beta^2}}\right)\right]$$

$(0<\beta<1/2)$

$$\tilde{x}_{g\geq 2} = \frac{4}{9}\ln\frac{z-1}{z+1/2} - \frac{2}{3(2z+1)} \qquad (\beta = 1/2)$$

$$\tilde{x}_{g\geq 2} = \frac{1}{3(1-\beta^2)}\left(\ln\frac{z-1}{\sqrt{z^2+z+1-3\beta^2}} + \frac{\sqrt{3}(1-2\beta^2)}{\sqrt{4\beta^2-1}}\ln\frac{2z+1-\sqrt{3}\sqrt{4\beta^2-1}}{2z+1+\sqrt{3}\sqrt{4\beta^2-1}}\right)$$

$(1/2<\beta<1)$

$$\tilde{x}_{g\geq 2} = \frac{2}{9}\ln\frac{z-1}{z+2} - \frac{1}{3(z-1)} \qquad (\beta=1)$$

For the higher genus cases, $0<\beta\leq 1$, $\beta>1$ naked singularity.

For $g\geq 2$, BH mass

$\sqrt{3}(M/2) = 1/\beta^3 - 3/\beta$

$0<\beta<1/\sqrt{3}$	positive M
$\beta = 1/\sqrt{3}$	zero M
$1/\sqrt{3}<\beta<1$	negative M

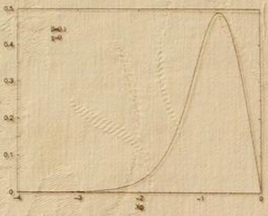
Using the null coordinates $u = \tilde{t} - \tilde{x}_g$, $v = \tilde{t} + \tilde{x}_g$, the wave equation becomes

$$-4\frac{\partial^2}{\partial u\partial v}\tilde{\psi}(u,v) = \tilde{V}(z)\tilde{\psi}(u,v)$$

Numerical Results

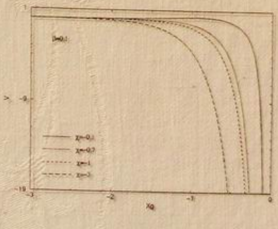
1. SAdS BH background $(g=0)$

The curvature coupling constant $\xi = (1-\chi)/6$ plays an important role in the behavior of $\tilde{V}$.

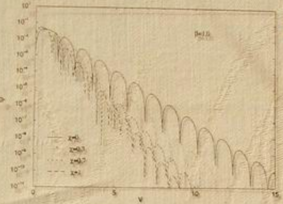


$\xi = 1/6 \ (\chi = 0)$

$\xi < 1/6 \ (\chi > 0)$



$\xi > 1/6 \ (\chi < 0)$



a. $\xi \le 1/6$ ($\chi \geq 0$)

$\chi \uparrow$, ω_R, $\omega_I \uparrow$

This behavior remains the same for different β

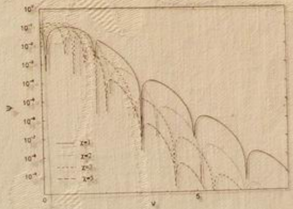
AdS/CFT: The more weakly the scalar field is non-minimally coupled to the curvature, the faster the thermal state in the CFT settles down to thermal equilibrium.

b. $\xi > 1/6$

The wave propagation amplifies instead of decays outside the black hole.

This behavior can be attributed to the negative infinite potential, which implies that the wave outside the BH gains energy from the spacetime.

2. Toroidal BH ($g=1$) background.



a. $\xi \le 1/6$ $\chi \geq 0$

$\xi \downarrow \Rightarrow \chi \uparrow \Rightarrow \omega_R, \omega_I \uparrow$

b. $\chi < 0$. ($\xi > 1/6$)

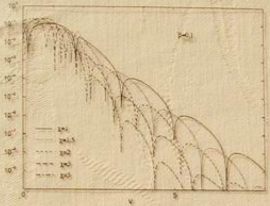
amplification mode

Similar behavior ($g=1$, $g=0$) is due to their similar V

3. Higher genus topological black holes ($g \geq 2$)

a. Positive mass ($0 < \beta < 1/\sqrt{3}$)

Potential has a shape similar to the $g=0$ case.



i) $\xi \leq 1/6$.

$\xi \downarrow, \to \chi \uparrow \Rightarrow \omega_I, \omega_R \uparrow$

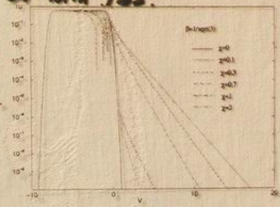
Outside the topological BH with positive mass, if the field is more weakly non-minimally coupled to spacetime curvature, the thermal perturbation will settle down faster.

(same as $g=0$, $g=1$ case)

ii) $\xi > 1/6$, amplification of the mode again

b. Zero mass ($\beta = 1/\sqrt{3}$)

$\beta = 1/\sqrt{3}$. V vanishes for a scalar field conformally coupled to curvature ($\xi = 1/6$) for the topological BH with $g \geq 2$.



i) $\chi = 0$, the "blip" due to $V = 0$

ii) $\xi < 1/6$ ($\chi > 0$)

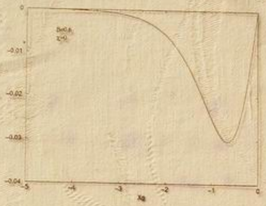
$\chi \uparrow$, $\omega_I \uparrow$, $\omega_R = 0$

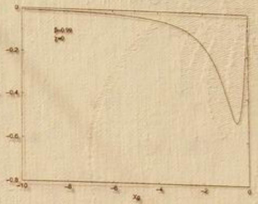
iii) $\xi > 1/6$ ($\chi < 0$)

potential tends to $-\infty$, amplification appears again

c. Negative mass ($1/\sqrt{3} < \beta \leq 1$)

i) For $\xi = 1/6$ ($\chi = 0$), the potential for $\beta > 1/\sqrt{3}$ outside the BH becomes everywhere negative

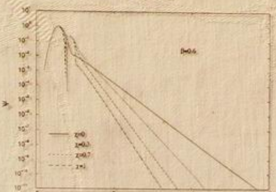




This is quite different from the $\xi = 0.1$ cases and also different from the positive mass higher genus BH case.

Amplification Again

ii) $\xi < 1/6$, the potential becomes positive and tends to infinite at infinity



The scalar fields decay faster when the non-minimal coupling to the spacetime curvature is weaker

iii) $\xi > 1/6$ ($\alpha < 0$).

The potential goes to negative infinity for all ρ.

The wave outside the BH experiences amplification instead of decay.

Summary:

1. For $\xi < 1/6$, the field decays monotonically with a decay constant that increases with decreasing ξ.

This behavior holds for all topological BHs with different genus. (V similar)

cf. dS case (PRD. 60, 064003 (1999))

Decay constant increases with increasing ξ for $\xi < 1/6$.

However in dS, V falls off exponentially at both r_+, r_c.

$\omega_R \neq 0$ for all positive BH with different topologies.

($\omega_R \uparrow$ as $\xi \downarrow$ when $\xi < 1/6$)

$\omega_R = 0$ for zero and negative mass higher genus BHs.

2. For $\xi = 1/6$, conformally coupled to curvature (PRD 59, 064025 (1999))

3. For $\xi > 1/6$

Potentials tend to negative infinity for large distance outside topological BHs.

Waves amplify instead of decay outside BHs of any genus.

4. Support of AdS/CFT, dS/CFT correspondences from perturbations of spacetime

4.1 AdS/CFT correspondence (Birmingham et al., PRL 88, 151301, 2002)

For a small perturbation, the relaxation process is completely determined by the poles, in the momentum representation, of the retarded correlation function of the perturbation.

The decay of small perturbations of a BH at equilibrium is described by the QNMs.

QNMs in AdS BH ⟷? Linear response theory in scale invariant finite temperature field theory

We consider the (2+1)-D AdS BH (B.T.Z. PRL 69, 1849 (1992))

$$ds^2 = -\sinh^2\mu\,(r_+ dt - r_- d\phi)^2 + d\mu^2 + \cosh^2\mu\,(-r_- dt + r_+ d\phi)^2$$

The angular coordinate ϕ has period 2π, the radii of inner and outer horizons are r_-, r_+. The conformal field theory splits into two independent sectors at thermal equilibrium with temperatures $T_L = (r_+ - r_-)/2\pi$, $T_R = (r_+ + r_-)/2\pi$.

Solving the wave equation in the bulk, the solutions are

$$\omega_R = -\frac{m}{\ell} - 2i\left(\frac{r_+ + r_-}{\ell^2}\right)(n+1)$$

$$\omega_L = \frac{m}{\ell} - 2i\left(\frac{r_+ - r_-}{\ell^2}\right)(n+1)$$

This result agrees exactly with location of the poles of the retarded correlation function of the corresponding perturbations in the dual CFT.

A quantitative test of the AdS/CFT correspondence

4.2 dS/CFT correspondence

(E. Abdalla, B. Wang et al PLB 2002
E. Abdalla et al. PRD 2002)

We live in a flat world with possibly a positive cosmological constant

Supernova observation, COBE satellite

Holographic duality: dS/CFT conjecture

A. Strominger, hep-th/0106113

Motivation: Quantitative test of the dS/CFT conjecture

3D dS toy model

$$ds^2 = -\left(M - \frac{r^2}{\ell^2} + \frac{J^2}{4r^2}\right)dt^2 + \left(M - \frac{r^2}{\ell^2} + \frac{J^2}{4r^2}\right)^{-1} dr^2$$
$$+ r^2\left(d\varphi - \frac{J}{2r^2}dt\right)^2$$

The horizon can be obtained from

$$M - \frac{r^2}{\ell^2} + \frac{J^2}{4r^2} = 0$$

The solutions r_+ corresponds to the cosmological horizon, another root r_- is imaginary

$$M = \frac{r_+^2 - r_-^2}{\ell^2} \quad , \; J = \frac{-2 r_+ r_-}{\ell}$$

Now consider the problem of scalar perturbations of such a spacetime.

Understand the problem as perturbations of a given spacetime.

Scalar perturbations are described by the equation

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\, g^{\mu\nu}\partial_\nu \phi) - \mu^2\phi = 0$$

where μ is the mass of the field.

Adopting the separation

$$\Phi(t, r, \varphi) = R(r)e^{-i\omega t}e^{im\varphi}$$

$$g_{rr} = \left(M - \frac{r^2}{\ell^2} + \frac{J^2}{4r^2}\right)^{-1}$$

The radial wave equation reads

49 $$\frac{1}{g_{rr}}\frac{d}{rdr}\left(\frac{r^2}{g_{rr}}\frac{dR}{rdr}\right) + \left[\omega^2 - \frac{m^2}{r^2}\left(M - \frac{r^2}{\ell^2}\right) - \frac{J}{r^2}m\omega\right]R = \frac{\mu^2}{g_{rr}}R$$

It can be simplified into

$$(1-z)\frac{d}{dz}\left(z\frac{dR}{dz}\right)+\left[\frac{1}{z}\left(\frac{\omega\ell^2 r_+ + m\ell r_-}{2(r_+^2+r_-^2)}\right)^2-\left(\frac{-\omega\ell^2 i r_- + i m\ell r_+}{2(r_+^2+r_-^2)}\right)^2+\frac{\mu^2\ell^2}{4(1-z)}\right]R=0$$

Set the ansatz

$$R(z)=z^\alpha(1-z)^\beta F(z)$$

we have

$$z(1-z)\frac{d^2F}{dz^2}+\left[1+2\alpha-(1+2\alpha+2\beta)z\right]\frac{dF}{dz}$$

$$+\left\{\left(\beta(\beta-1)+\frac{\mu^2\ell^2}{4}\right)\frac{1}{1-z}+\frac{1}{z}\left[\left(\frac{\omega\ell^2 r_+ + m\ell r_-}{2(r_+^2+r_-^2)}\right)^2+\alpha^2\right]\right.$$

$$\left.-\left[\left(\frac{-i\omega\ell^2 r_- + i m\ell r_+}{2(r_+^2+r_-^2)}\right)^2+\alpha^2+(1+2\alpha)\beta+\beta(\beta-1)\right]\right\}F=0$$

This equation can be reduced to the hypergeometric equation

$$z(1-z)\frac{d^2F}{dz^2}+\left[c-(1+a+b)z\right]\frac{dF}{dz}-abF=0$$

with solutions of QNM frequencies

$$\omega_R=\frac{im}{\ell}-2\left(\frac{ir_+ + r_-}{\ell^2}\right)(n+1)$$

$$\omega_L=-\frac{im}{\ell}-2\left(\frac{ir_+ - r_-}{\ell^2}\right)(n+1)$$

Now investigate QNM from the CFT side

Describing the coordinates in $SO(3,1)$, such that

$$X_1^2 + X_2^2 + X_3^2 - T^2 = \ell^2$$

The metric of 3D dS space can be reobtained by the change of variables

$$X_1 = \ell\sqrt{\alpha}\,\sin\left(\frac{r_+}{\ell}\varphi - \frac{r_-}{\ell^2}t\right),\quad X_2 = -\ell\sqrt{1-\alpha}\,\cosh\left(\frac{r_+}{\ell^2}t + \frac{r_-}{\ell}\varphi\right)$$

$$X_3 = \ell\sqrt{\alpha}\,\cos\left(\frac{r_+}{\ell}\varphi - \frac{r_-}{\ell^2}t\right),\quad T = -\ell\sqrt{1-\alpha}\,\sinh\left(\frac{r_+}{\ell^2}t + \frac{r_-}{\ell}\varphi\right)$$

where $\alpha = \frac{r^2 + r_-^2}{r_+^2 + r_-^2}$.

Define an invariant $P(X, X')$ associated to two points X and X' in dS space

$$P(X, X') = \eta_{AB} X^A X'^B$$

The Hadamard two-point function is defined as

$$G(X, X') = \text{const} \langle 0 | \Phi(X), \Phi(X') | 0 \rangle$$

which obeys

$$(\nabla_X^2 - m^2)\, G(X, X') = 0 \quad \Rightarrow G(P) = \text{Re}\, F\left(h_+, h_-, \tfrac{3}{2}, \tfrac{1+P}{2}\right)$$

Choosing boundary conditions for the fields

$$\lim_{r\to\infty} \phi(t, r, \varphi) \to r^{-h_-} \phi_-(t, \varphi)$$

$$h_\pm = 1 \pm \sqrt{1 - m^2\ell^2}$$

the two point correlator can be got (hep-th/0206113, NPB 648, 02)

$$\lim_{r\to\infty}\int dt\,d\varphi\,dt'\,d\varphi'\,\frac{(rr')^2}{\ell^2}\,\phi\overleftrightarrow{\partial}_{r^*}G\overleftrightarrow{\partial}_{r^*}\phi$$

using the seperation: $\phi(t,r,\varphi)=R(r)e^{-i\omega t}e^{im\varphi}$

the two-point function for QNM is

$$\int dt\,d\varphi\,dt'\,d\varphi'\,\frac{\exp(-im'\varphi'-i\omega't'+im\varphi+i\omega t)}{\left[2\sinh\frac{(ir_+ + r_-)(\ell\Delta\varphi-i\Delta t)}{2\ell^2}\sinh\frac{(ir_+ - r_-)(\ell\Delta\varphi+i\Delta t)}{2\ell^2}\right]^{h_+}}$$

$$\approx\delta_{mm'}\delta(\omega-\omega')\,\Gamma\left(h_+/2+\frac{im/2\ell+\omega/2}{2\pi T}\right)\Gamma\left(h_+/2-\frac{im/2\ell+\omega/2}{2\pi T}\right)$$

$$\times\,\Gamma\left(h_+/2+\frac{im/2\ell-\omega/2}{2\pi T}\right)\Gamma\left(h_+/2-\frac{im/2\ell-\omega/2}{2\pi T}\right)$$

The poles of such a correlator corresponds exactly to the QNM obtained from the wave equation in the bulk.

This work has been extended to 4D dS spacetimes (E. Abdalla et al PRD 2002)

These results provide a quantitative support of the dS/CFT correspondence