## 11D, SOGRA: $g,\ C,\quad \psi_\mu$

$G = dC$

$$S = \frac{1}{2\kappa^2}\int d^{11}x \sqrt{-g}\, R - \frac{1}{2} G\wedge * G - \frac{1}{6} C\wedge G\wedge G .$$

**EOM:**

$$\begin{cases} R_{\mu\nu} = \frac{1}{12}\left( G^2_{\mu\nu} - \frac{1}{12} g_{\mu\nu} G^2 \right) \\ d * G + \frac{1}{2} G\wedge G = 0 \\ dG = 0 \end{cases}$$

$$G^2_{\mu\nu} = G_{\mu\sigma_1\sigma_2\sigma_3} G_\nu{}^{\sigma_1\sigma_2\sigma_3}, \quad G^2 = G_{\sigma_1\sigma_2\sigma_3\sigma_4} G^{\sigma_1\sigma_2\sigma_3\sigma_4}$$

**SUSY:**

$$\begin{cases} \delta g \sim \epsilon\psi \\ \delta C \sim \epsilon\psi \\ \delta\psi \sim \hat{\nabla}\epsilon + \epsilon\psi\psi \end{cases}$$

**SUSY solution:** $\hat{\nabla}\epsilon = 0$ — Killing Spinors

**Cliff(10,1):** $\{\Gamma^\alpha\}, \quad \{\Gamma^\alpha, \Gamma^\beta\} = 2\eta^{\alpha\beta}$

$$\frac{1}{4}\Gamma^{\alpha\beta} = \frac{1}{8}\left(\Gamma^\alpha\Gamma^\beta - \Gamma^\beta\Gamma^\alpha\right)$$

$$\begin{cases} \hat{\nabla}_\mu \epsilon = \nabla_\mu \epsilon + \frac{1}{288}\left(\Gamma_\mu^{\nu_1\nu_2\nu_3\nu_4} - 8\delta_\mu^{\nu_1}\Gamma^{\nu_2\nu_3\nu_4}\right) G_{\nu_1\nu_2\nu_3\nu_4}\epsilon = 0 \\ \nabla_\mu \epsilon = \left(\partial_\mu + \frac{1}{4}\omega_{\mu\alpha\beta}\Gamma^{\alpha\beta}\right)\epsilon . \end{cases}$$

**Flux G:** $\frac{1}{(2\pi l_p)^3} G - \frac{\lambda}{2} \in H^4(Y, \mathbb{Z})$,

$\lambda(Y) = p_1(Y)/2$, $\quad Y$: 11D spin manifold.

$G = 0$: Ricci flat

$$\begin{cases} R_{\mu\nu} = 0 \\ \nabla_\mu \epsilon = 0 \end{cases}$$ & covariant spinor

Special holonomy: $[\nabla_\mu, \nabla_\nu]\epsilon = \underbrace{\frac{1}{4} R_{\mu\nu\alpha\beta}\Gamma^{\alpha\beta}}_{\text{generate a subgroup of } Spin(10,1)}\epsilon = 0$

$$[\nabla_\mu, \nabla_\nu]\,\varepsilon = \frac{1}{4} R_{\mu\nu\alpha\beta}\Gamma^{\alpha\beta}\varepsilon = 0$$

$\{R_{\mu\nu\alpha\beta}\Gamma^{\alpha\beta}\}$ generates a subgroup of $Spin(10,1)$

---

<u>$d=8$:</u> $Spin(7)$ - holonomy

$\varepsilon^{1234} = \varepsilon_1\wedge\varepsilon_2\wedge\varepsilon_3\wedge\varepsilon_4$

<u>4-form:</u>

$$\Psi = \varepsilon^{1234} + \varepsilon^{1256} + \varepsilon^{1278} + \varepsilon^{3456} + \varepsilon^{3478} + \varepsilon^{5678} + \varepsilon^{1357} - \varepsilon^{1368} - \varepsilon^{1458} - \varepsilon^{1467} - \varepsilon^{2358} - \varepsilon^{2367} - \varepsilon^{2457} + \varepsilon^{2468}$$

covariant const $\Rightarrow$ $d\Psi = 0$

$$\Psi_{mnpq} = -\bar{\rho}\,\gamma_{mnpq}\,\rho, \quad m,n,p,q = 1,\cdots,8.$$

---

<u>$G_2$ - holonomy:</u> $d=7$,

<u>3-form:</u>

$$\phi = e^{246} - e^{235} - e^{145} - e^{136} + e^{127} + e^{347} + e^{567}$$

covariant const $\Rightarrow$

$$d\phi = d*\phi = 0$$

$$\phi_{mnp} = -i\bar{\rho}\,\gamma_{mnp}\,\rho.$$

---

$SU(n)$ - holonomy. $d=2n$

$$\begin{cases} J = e^{12} + e^{34} + \cdots + e^{(2n-1)(2n)} \\ \Omega = (e^1 + ie^2)(e^3 + ie^4)\cdots(e^{2n-1} + ie^{2n}) \end{cases}$$

$$dJ = d\Omega = 0$$

$$J_{mn} = i\rho^\dagger\gamma_{mn}\rho$$

$$\Omega_{m_1\cdots m_{2n}} = \rho^T\gamma_{m_1\cdots m_{2n}}\rho.$$

~~$C$ — 3-form~~

~~$\delta C = d\Lambda$, $\Lambda$ — 2-form~~

~~$X \sim \mathbb{R} \times Y$ cone~~ ~~$K = H^2(Y; U(1))$~~

## ~~Membranes and~~ Fivebranes Geometry

$G \neq 0$

$$ds^2 = H^{-\frac{1}{3}}\left[d\xi^i d\xi^j \eta_{ij}\right] + H^{\frac{2}{3}}\left[dx^I dx^I\right].$$

$$G_{I_1 I_2 I_3 I_4} = -c\, \varepsilon_{I_1 I_2 I_3 I_4 J}\, \partial_J H, \qquad c = \pm 1.$$

$$\varepsilon = H^{-\frac{1}{12}} \varepsilon_0, \qquad \underline{\Gamma^{012345} \varepsilon = c\, \varepsilon} \Rightarrow \begin{array}{l} \left(\Gamma^{012345}\right)^2 = 1 \\ \mathrm{Tr}\, \Gamma^{012345} = 0 \end{array}$$

$\Rightarrow$ 16 independent Killing spinors

$dG = 0 \Rightarrow H$ harmonic

e.g. $H = 1 + \dfrac{a_5 N}{r^3}$, $\quad r^2 = x^I x^I$

$$\frac{1}{(2\pi l_p)^3} \int_{S^4} G = cN.$$

Near horizon geometry: $AdS_7 \times S^4$

$N$ units of flux on the four sphere

Membrane geometry:

$$ds^2 = H^{-\frac{2}{3}}\left[d\xi^i d\xi^j \eta_{ij}\right] + H^{\frac{1}{3}}\left(dx^I dx^I\right)$$

$$C = c\, H^{-1} d\xi^0 \wedge d\xi^1 \wedge d\xi^2,$$

$$H = 1 + \frac{a_2 N}{r^6}, \quad r^2 = x^I x^I$$

$$\frac{1}{(2\pi l_p)^6} \int_{S^7} *G = cN.$$

orthonormal frame:

$\{H^{-\frac{1}{3}} d\xi^i,\ H^{\frac{1}{6}} dx^I\}$

$\varepsilon = H^{-\frac{1}{6}} \varepsilon_0$

$\Gamma^{012} \varepsilon = c\, \varepsilon$

World volume theory $\Rightarrow$ calibrated geometry

Calibration. $M$ — Riemannian manifold,

$\varphi$ — $p$-form

$$d\varphi = 0, \qquad \varphi|_{\xi^p} \le \underbrace{\mathrm{vol}}|_{\xi^p}, \quad \forall \xi^p \text{ — } p\text{-plane}$$

Volume form induced from the metric on $M$.

- $\Sigma_p$ — $p$-cycle is called calibrated by $\varphi$ if

$$\varphi|_{\Sigma_p} = \mathrm{vol}|_{\Sigma_p}.$$

- minimal surface: $\Sigma, \Sigma'$, $\quad \Sigma - \Sigma' = \partial \Xi$

$$\mathrm{vol}(\Sigma) = \int_\Sigma \varphi = \underbrace{\int_\Xi d\varphi}_{=0} + \int_{\Sigma'} \varphi \le \mathrm{vol}(\Sigma').$$

---

Spin(7) $\quad d^* \Psi = 0$, Cayley 4-cycles

$G_2$: $\quad \phi, *\phi$ — associative, co-associative

$SU(n)$: $\quad J^n$ — Kahler calibration, cycles holomorphic

$e^{i\theta}\Omega$ — calibrate special Lagrangian $n$-cycles.

---

Normal bundle $\quad T(M)|_\Sigma = T(\Sigma) \oplus N(\Sigma)$

$J$ — Kahler calibration

$\beta^1(\Sigma) = \dim (H^1(\Sigma, \mathbb{R}))$

Turning on Neveu-Schwarz fluxes

$\Rightarrow$ deforming Calabi-Yau manifold into a manifold with a non-integrable complex structure

manifold with SU(3) structure or half-flat six manifold

$M^n$ — almost complex manifold $(M, g, J)$

$J^2 = -Id$, $g$ - almost Hermitian metric

There exist a real two-form $J$ & a complex three form $\Omega$, s.t.

$$\begin{cases} J\wedge J\wedge J = \frac{3i}{4}\,\Omega\wedge\bar{\Omega} \\ J\wedge\Omega = 0 \end{cases}$$

$dJ$, $d\Omega$ may not be zero.

Half-flat condition:
$$\begin{cases} \Omega = \mathrm{Re}\,\Omega + i\,\mathrm{Im}\,\Omega = \Omega_+ + i\Omega_- \\ d\Omega_- = 0, \quad J\wedge dJ = 0 \end{cases}$$

Generalized Mirror Symmetry Conjecture:

Type IIA(B) on a Calabi-Yau with RR & NS fluxes

$\overset{\text{mirror}}{\approx}$ Type IIB(A) on a half-flat six manifold.

$$\boxed{\begin{aligned}\Omega &= \mathrm{Re}\,\Omega + i\,\mathrm{Im}\,\Omega \\ &= \Omega_+ + i\Omega_-\end{aligned}}$$

<u>half-flat condition,</u> $\begin{cases} \hat{d}\Omega_- = 0 \\ J\wedge \hat{d}J = 0 \end{cases}$

$$d = \hat{d} + d\tilde{y}\,\partial_{\tilde{y}}$$

$$\begin{cases} \varphi = J\wedge d\tilde{y} + \Omega_- \\ *\varphi = -\Omega_+\wedge d\tilde{y} + \frac{1}{2}J\wedge J \end{cases}$$

6)

$$\begin{aligned} d\varphi &= (\hat{d} + d\tilde{y}\,\partial_{\tilde{y}})(J\wedge d\tilde{y}) \\ &\quad + (\hat{d} + d\tilde{y}\,\partial_{\tilde{y}})\Omega_- \\ &= \hat{d}J\wedge d\tilde{y} + \cancel{\partial_{\tilde{y}}J\,d\tilde{y}} \\ &\quad + \hat{d}\Omega_- + \partial_{\tilde{y}}\Omega_-\,d\tilde{y} \\ &= \hat{d}J\wedge d\tilde{y} + \partial_{\tilde{y}}\Omega\wedge d\tilde{y} \\ &= (\hat{d}J - \partial_{\tilde{y}}\Omega_-)\wedge d\tilde{y} \end{aligned}$$

$\hat{d}\Omega_-$

$$\begin{aligned} d(*\varphi) &= (\hat{d} + d\tilde{y}\,\partial_{\tilde{y}})(-\Omega_+\wedge d\tilde{y} + \tfrac{1}{2}J\wedge J) \\ &= -\hat{d}\Omega_+\wedge d\tilde{y} + \underline{\hat{d}J\wedge J} \\ &\quad + \tfrac{1}{2}d\tilde{y}\;\partial_{\tilde{y}}(J\wedge J) \\ &= -\hat{d}\Omega_+\wedge d\tilde{y} + \tfrac{1}{2}d\tilde{y}\;\partial_{\tilde{y}}(J\wedge J) \\ &= \left(-\hat{d}\Omega_+ + \tfrac{1}{2}\partial_{\tilde{y}}(J\wedge J)\right)\wedge d\tilde{y} \end{aligned}$$

<u>SU(3) structure,</u>

$$\begin{cases} J\wedge J\wedge J = \frac{3i}{4}\Omega\wedge\bar{\Omega} \\ J\wedge\Omega = 0 \end{cases}$$

<u>Hitchin's theorem;</u> (*) preserves SU(3) structure.

$$\boxed{\begin{aligned} \hat{d}J - \partial_{\tilde{y}}\Omega_- &= 0 \\ d\hat{\Omega}_+ &= \tfrac{1}{2}\partial_{\tilde{y}}(J\wedge J) \end{aligned}} \quad (*)$$

Hitchin's flow equation

---

$\mathbb{R}^{1,3}\times X_6$, $X_6$ half-flat six manifold

It has BPS domain wall solutions with 4 Killing spinors.

Calibration: $\phi$: $k$-form, $M$ — oriented manifold

$\phi(\xi)\big|_p \le 1$, $\forall$ oriented $k$-plane $\xi$ in $T_pM$.

Generalized calibration: $\phi$ may not be closed

$N$ — cycle, $\partial N = 0$

$K \subset N \subset M$, $\partial L = K$, $L \subset M$

$(\alpha, \beta)$ trivial class in $H_{dR}(M, N)$, $\beta$ calibration form in $N$

$\alpha = d\beta$

$(K, L)$ calibrated by $(\alpha, \beta)$ if $\beta(\xi)\big|_p \le 1$, $\xi$ — $k$-plane in $T_pM$.

It minimizes:

$$\mathcal{E}(K, L) = vol(K) - \int_L \alpha$$

$U(n)$ holonomy: $(M, g, J)$ $g$: almost Hermitian metric, $J$: almost complex structure

$J^2 = -Id$

$$g(JX, JY) = g(X, Y)$$

Define: (Chern Connection)

$$\nabla_X Y = \nabla^g_X Y - \tfrac{1}{2} J(\nabla^g_X J) Y.$$

We have: $\nabla J = 0$, $\nabla g = 0$

Let $\omega(X, Y) = g(X, JY)$, we also have $\nabla \omega = 0$

But: $dJ \neq 0$, $d\omega \neq 0$.

Type IIB supergravity

$$S_{IIB} = \frac{1}{2k^2}\{\int d^{10}x\sqrt{-\det G}R - \frac{1}{2}\int(d\Phi\wedge *d\Phi + e^{\Phi}H_3\wedge *H_3$$

$$+e^{2\Phi}F_1\wedge *F_1 + e^{\Phi}F_3\wedge *F_3 + \frac{1}{2}F_5\wedge *F_5 + C_4\wedge H_3\wedge F_3)\}$$

$$H_3 = dB_2, F_1 = dC_0, F_3 = dC_2, F_5 = dC_4, \tilde{F}_3 = F_3 - C_0\wedge H_3, \tilde{F}_5 = F_5 - C_2\wedge H_3.$$

Solution Anzatz:

$$ds^2 = H^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + H^{1/2}(d\rho^2 + \rho^2 d\theta^2 + \delta_{mn}dx^m dx^n)$$

$$\tilde{F}_5 = d(H^{-1}dx^0\wedge\ldots\wedge dx^3) + *d(H^{-1}dx_0\wedge\ldots\wedge dx^3)$$

H is a harmonic function of the transverse space.

Supergravity solution of N D3-branes in flat 10D space

$$ds^2 = f(r)^{-1/2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + f(r)^{1/2}(dr^2 + r^2 d\Omega^2),$$

$$f(r) = \frac{1}{1 + \frac{R^4}{r^4}}$$

Let $r \to 0$, near-horizon geometry = $AdS_5$ geometry

Limits in parameter space

$$T = \frac{R^2}{2\pi\alpha} = \frac{1}{2\pi(g_{YM}^2 N)^{1/2}}, 4\pi g_s = g_{YM}^2$$

Let $g_s \to 0, N \to \infty$ Type IIB supergravity in $AdS_5 \times S^5$
== N=4 SYM at the strong 't Hooft coupling $\lambda = g_{YM}^2 N$

**D-brane action**

$$S_{DBI} = -T_p \int d^{p+1}\xi e^{-\Phi}(C \det(g_{ab} + B_{ab} + 2\pi\alpha F_{ab})^{1/2}$$

$$1/g_s = e^{-\Phi}$$

**DBI brane action reduction**
Let $g_{ab} = \eta_{ab}, B_{ab} = 0, \alpha \to 0,$

$$S_{DBI} \to S_{YM} = -\frac{T_p(2\pi\alpha)^2}{4g_s} \int d^{p+1}\xi(F_{ab}F^{ab})$$

$$g_{YM}^2 = \frac{g_s}{T_p}(2\pi\alpha)^{-2}.$$

**Klebanov-Strassler Supergravity solution, deformed conifold**

To get N=1 Super Yang-Mills of gauge group SU(N) Klebanov and Strassler consider adding M D5-branes wrapped on $S^2$ in addition to N D3-branes at the conifold singularity, then the gauge group is changed to SU(N+M) x SU(N).

$$ds^2 = h^{-\frac{1}{2}}\eta_{\mu\nu}dx^\mu dx^\nu + h^{\frac{1}{2}}(dr^2 + r^2 ds^2_{T^{1,1}})$$

$$h(r) = \frac{27\pi}{4r^4}\alpha^2 g_s M(N + g_s M(\frac{3}{8\pi} + \frac{3}{2\pi}\log\frac{r}{r_{max}}))$$

$$\frac{1}{(2\pi)^2\alpha}\int_A F = M, \frac{1}{(2\pi)^2\alpha}\int_B H = N$$

$$ds^2_{T^{1,1}} = \frac{1}{g}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 + \frac{1}{6}\Sigma_{i=1}^2(d\theta_i^2 \sin^2\theta_i d\phi_i^2),$$

$\psi \in [0, 4\pi), \phi_1, \phi_2 \in [0, 2\pi), \theta_1, \theta_2 \in [0, \pi)$.

KS solution gives:

world volume theory of D3-branes with M fluxes.

Maldacena and Nunez considers N D5 branes wrapped on a two sphere inside a Calabi-Yau manifold. Bound states of N D5-branes wrapped on a two-sphere inside a CY threefold == N=1 SU(N) SYM

Twisting of normal bundle

$$D\epsilon = 0$$

There is usually no solution to the equation $(\partial + \omega)\epsilon = 0$

One is adding a gauge field then the equation becomes $\partial + \omega - A = 0$. One then let $\omega = A$, then one needs to solve $\partial\epsilon = 0$.

$AdS_7 \times S^4$: Identify $SO(3) \subset SO(5)$ with spin connection of the normal bundle of a SLAG 3-cycle.

D=7 SO(5) gauged supergravity uplift to 11D, or

Truncation of the KK reduction on a four sphere of D=11 SUGRA.

**Maldacena-Nunez Solution, resolved conifold**
Wrapping on a 2 sphere = 7D supergravity and uplift to ten dimension
5D effective action:

$$S = \alpha'^{-2} \int d^4x d\rho e^{2k} (4\partial_i k \partial^i k - 2\partial_i g \partial^i g - 1/2 \partial_i a \partial^i a e^{-2g} + V)$$

$g_{ij} = \eta_{ij}, V = 4 + 2e^{-2g} - \frac{(1-a^2)^2}{4} e^{-4g}$. By assuming functions $k, g, a$ depend on $\rho$ only we get a SUGRA solution.

$$ds^2 = e^{\Phi}(\underbrace{dx_{1,3}^2}_{\mathbb{R}^{1,3}} + \frac{e^{2g}}{\lambda^2} \underbrace{(d\theta^2 + \sin^2\theta d\phi^2)}_{S^2}) + \frac{1}{\lambda^2} \underbrace{d\rho^2 + \frac{1}{4\lambda^2} \Sigma_{a=1}^3 (w^a - \frac{A^a}{\lambda})^2}),$$

$$= H^{-1/2}(dx_{1,3}^2 + zr_0^2 d\Omega_2^2) + H^{1/2}(d\rho^2 + \rho^2 d\tilde{\phi}_2^2) + H^{1/2}/z(d\sigma^2 + \sigma^2(\phi_1 + \cos\theta d\phi)^2).$$

$\lambda^{-2} = N g_s \alpha'$, $w^a$ parametrizez the 3-sphere, and the gauge field $A^a$ is written as: $A^1 = -\lambda a d\theta, A^2 = \lambda a \sin\theta d\phi, A^3 = -\lambda \cos\theta d\phi$.

Conjecture: KS solution $\overset{\text{dual}}{\approx}$ MN Solution

Type IIB N D3 branes with fluxes $\Longleftrightarrow$ M theory fivebrane geometry