西北大学 NORTHWEST UNIVERSITY

公誠勤樸

Eclectic flavor group $\Delta(27) \rtimes S_3$ and lepton model building

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Based on **arXiv: 2308.16901**

彭桓武高能基础理论研究中心

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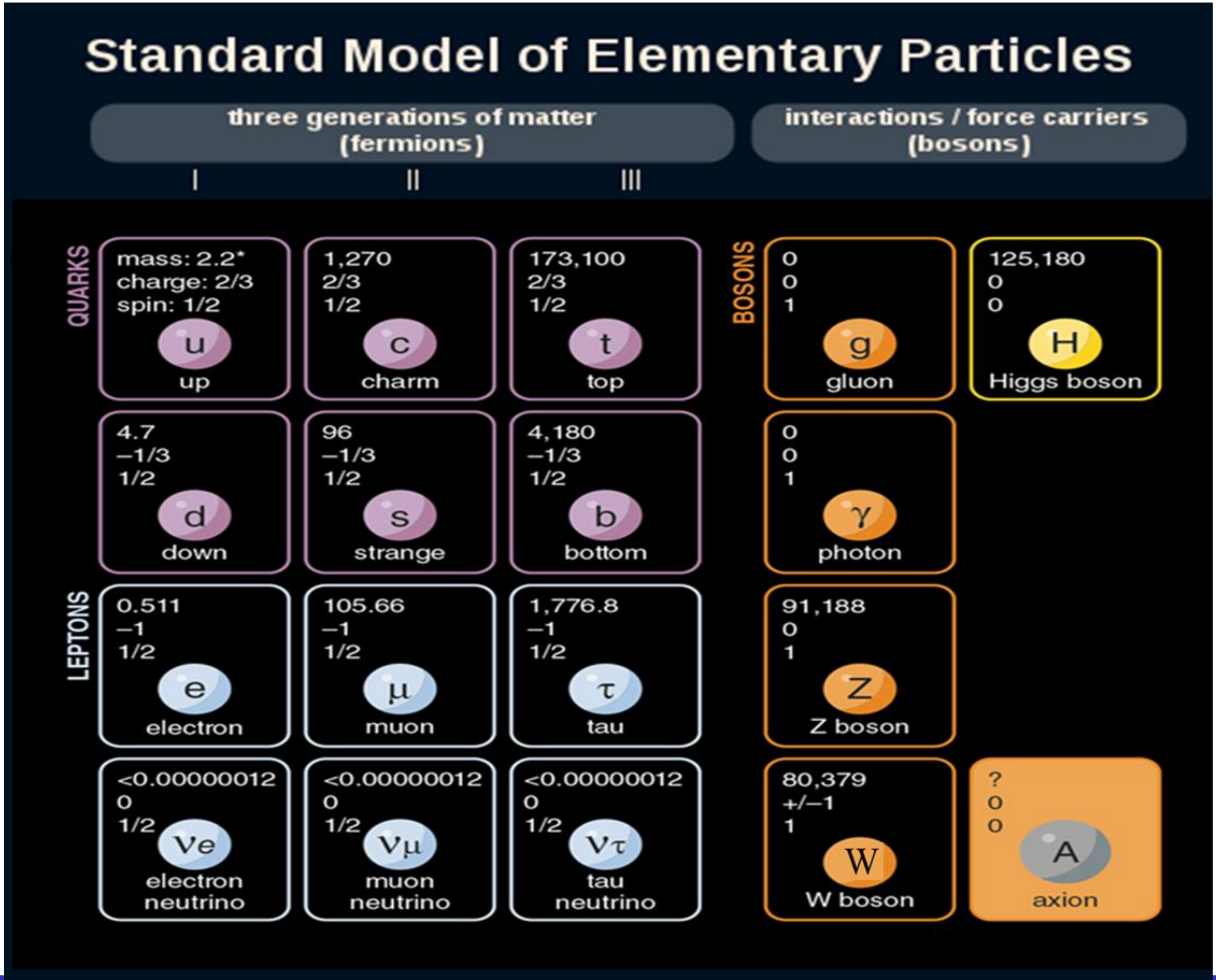
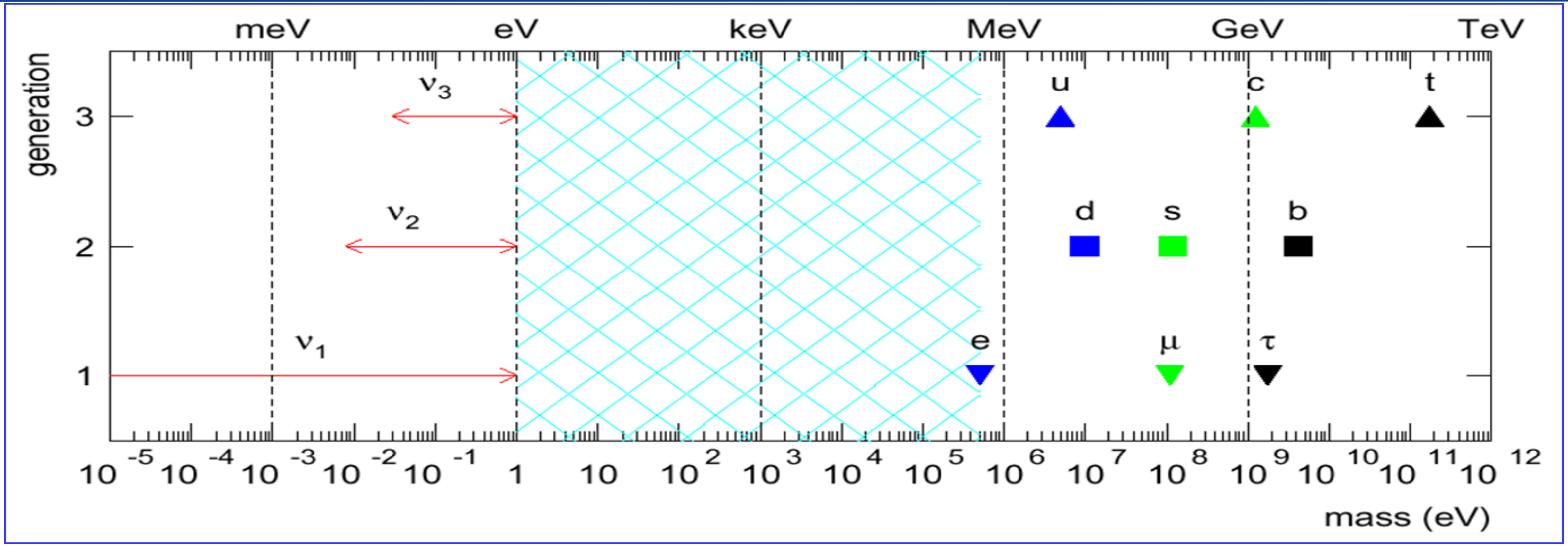
Outline

1. Motivation
2. Flavor symmetry
3. Eclectic Flavor Groups
4. Effective action invariant under $\Delta(27) \rtimes S_3$
5. $\Delta(27) \rtimes S_3$ eclectic lepton model
6. Conclusions

1. Motivation Flavor puzzle 1

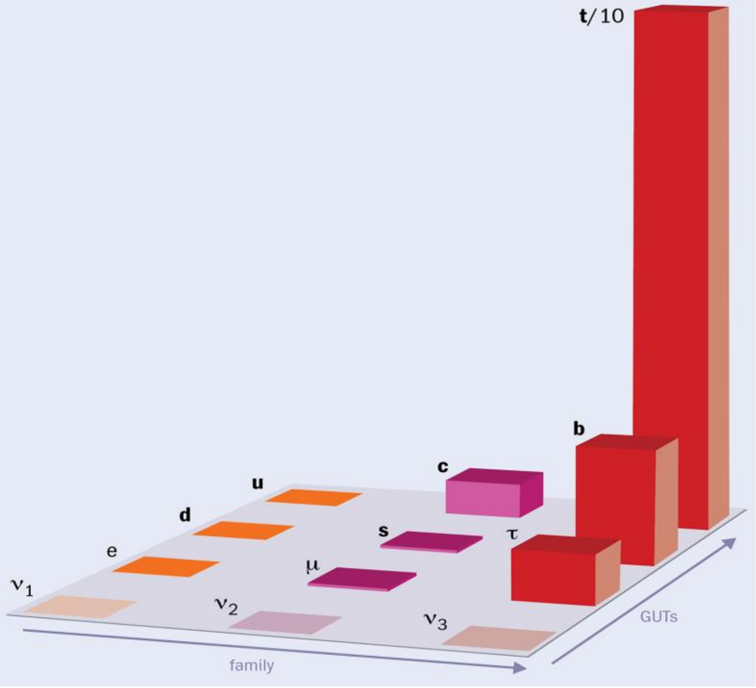


Flavor puzzle 2: Fermion mass hierarchies



[Z.Z.Xing, Phys.Rept. 854 (2020) 1-147]

- Why fermion mass hierarchies?
- Why are neutrino masses so small?



Flavor puzzle 3: Why different quark and lepton mixing

- **Cabibbo (1963)-Kobayashi-Maskawa (1973) Matrix:**

$$U_{CKM} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{CP}} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta_{CP}} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\theta_{12} \sim 13^\circ \rightarrow \theta_{23} \sim 2^\circ \rightarrow \theta_{13} \sim 0.2^\circ \rightarrow \delta \sim 65^\circ$

1963 | 1983 | 1990 | 2001

- **Pontecorvo (1957) -Maki-Nakawaga-Sakata (1962) Matrix:**

$$U_{PMNS} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{CP}} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta_{CP}} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha_{21}/2} & 0 \\ 0 & 0 & e^{i\alpha_{31}/2} \end{pmatrix}$$

$\theta_{23} \sim 42.2^\circ$ or $49^\circ \rightarrow \theta_{12} \sim 33.4 \rightarrow \theta_{13} \sim 8.58^\circ \rightarrow \delta/\alpha_{21}/\alpha_{31} \sim ???$

1998 | 2001 | 2012 | 20xy

Flavor puzzle 3: Why different quark and lepton mixing

- **Cabibbo (1963)-Kobayashi-Maskawa (1973) Matrix:**

$$U_{CKM}=\begin{pmatrix}1&0&0\\0&c_{23}&s_{23}\\0&-s_{23}&c_{23}\end{pmatrix}\begin{pmatrix}c_{13}&0&s_{13}e^{-i\delta_{CP}}\\0&1&0\\-s_{13}e^{i\delta_{CP}}&0&c_{13}\end{pmatrix}\begin{pmatrix}c_{12}&s_{12}&0\\-s_{12}&c_{12}&0\\0&0&1\end{pmatrix}$$

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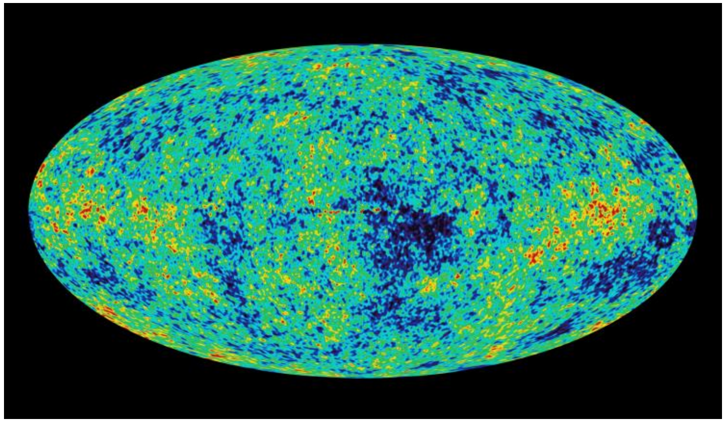
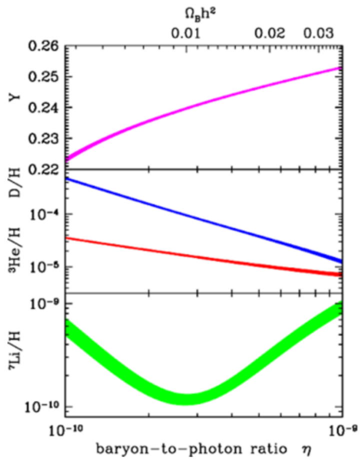
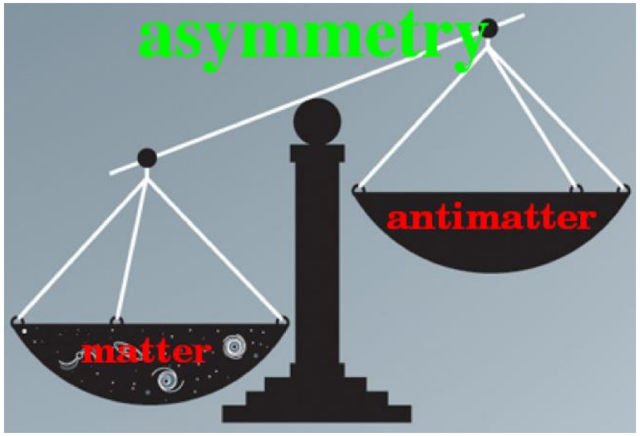
1998 2001 2012 20xy

Matter-antimatter asymmetry

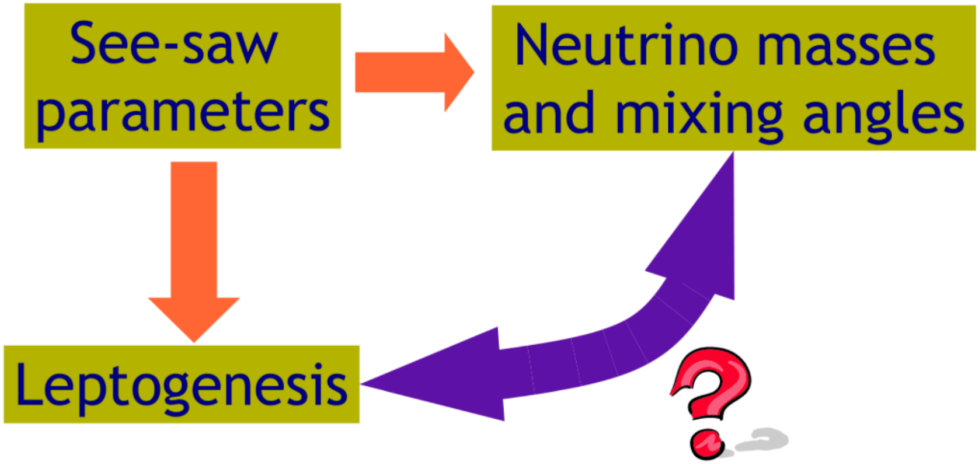






Leptogenesis

Fukugita, **Yanagida** **86**



2. Flavor symmetry

- TB mixing： [P.F. Harrison,W.G. Scott, Phys. Lett. B 535 (2002) 163-169]

$$U_{TB}=\begin{pmatrix}\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0\\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}\end{pmatrix}$$

$$\theta_{23}^{TB}=45^{\circ},\quad \theta_{12}^{TB}=35.26^{\circ},\quad \theta_{13}^{TB}=0^{\circ}$$

Exp data(NuFIT5.2)

$$39.7^{\circ}\le\theta_{23}^{\exp}\le 51.0^{\circ}$$
$$31.31^{\circ}\le\theta_{12}^{\exp}\le 35.74^{\circ}$$
$$8.23^{\circ}\le\theta_{23}^{\exp}\le 8.91^{\circ}$$

- In the basis of diagonal charged leptons, the neutrino mass matrix is：

$$M_{\nu}=U_{TB}\mathrm{diag}(m_1,m_2,m_3)U_{TB}^{T}$$

$$M_{\nu}=\frac{m_1}{6}\begin{pmatrix}4 & -2 & -2\\ -2 & 1 & 1\\ -2 & 1 & 1\end{pmatrix}+\frac{m_2}{3}\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}+\frac{m_3}{2}\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & -1\\ 0 & -1 & 1\end{pmatrix}$$

- Symmetry of TB neutrino mass matrix:

$$SM_\nu S^T = M_\nu, \qquad P_{23} M_\nu P_{23}^T = M_\nu,$$

with

$$S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \qquad P_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Rightarrow \quad S^2 = P_{23}^2 = 1.$$

- Diagonal charged lepton mass matrix is defined by invariance under T:

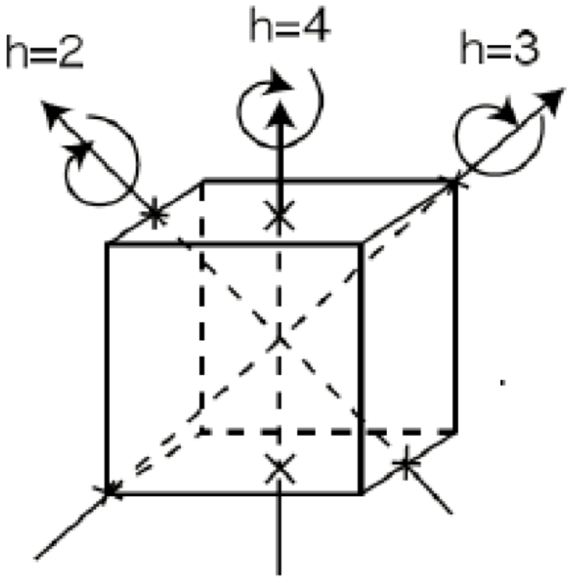
$$M_l = \mathrm{diag}(m_e, m_\mu, m_\tau), \qquad T^\dagger M_l^\dagger M_l T = M_l^\dagger M_l$$

minimal choice

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \qquad \omega = e^{2\pi i/3}. \quad \Rightarrow \quad T^3 = 1$$

- The group generated by S, T and P_{23}

$$S^2 = T^3 = P_{23}^2 = (ST)^3 = (SP_{23})^2 = (TP_{23})^2 = (STP_{23})^4 = 1 \quad \Rightarrow \quad S_4$$



- Breaking pattern: [Lam, 0708.3665, 0804.2622]

S_4

$\langle \varphi_l \rangle$ → Z_3^T (charged lepton)

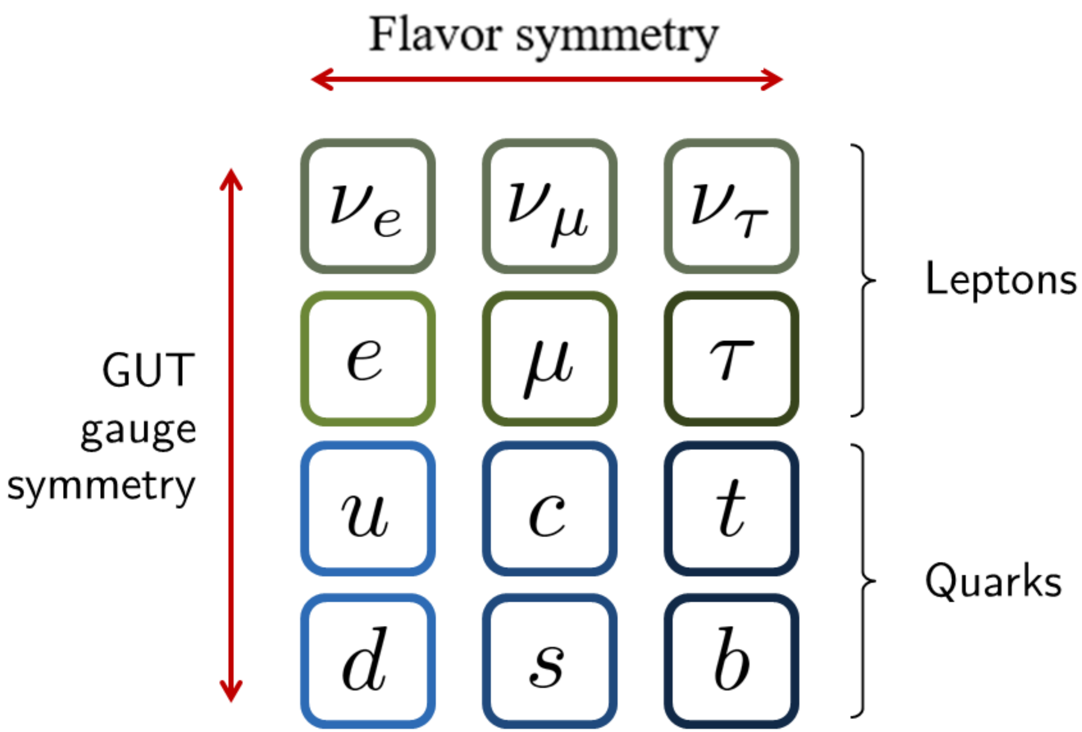
$\langle \varphi_\nu \rangle$ → $K_4^{(S,P_{23})}$ (neutrino)

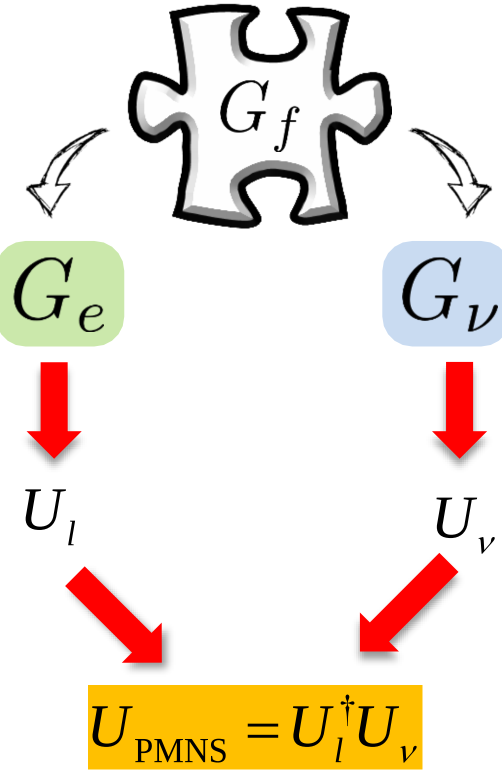
U_{PMNS} ⟷

- Mixing matrix: $U_{PMNS} = U_{TB}$

Flavor symmetries

Non Abelian discrete flavor symmetry

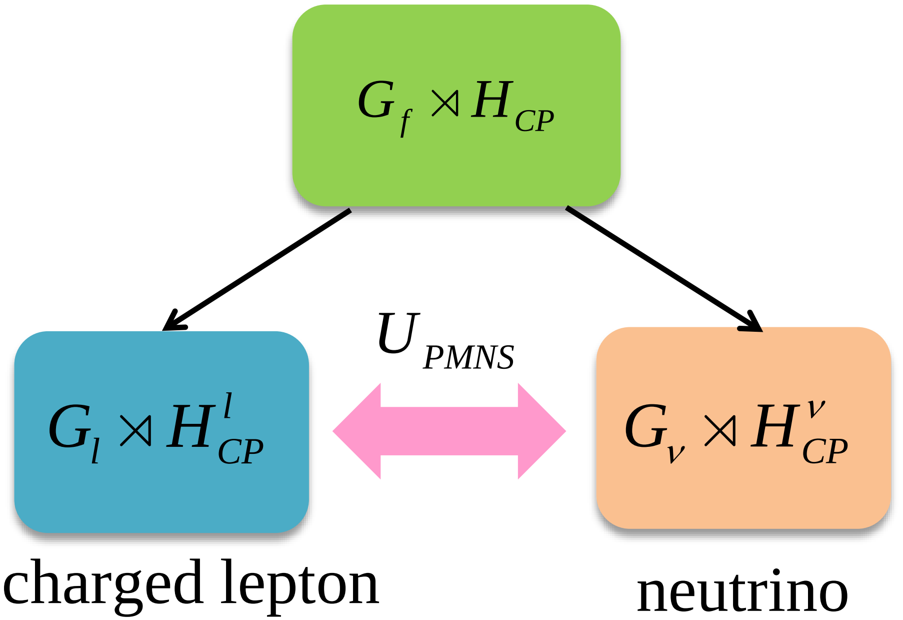




constrain mixing and Dirac phase

[Altarelli and Feruglio, Nucl.Phys.B 741 (2006) 215-235;
C.S. Lam, Phys.Lett.B 656 (2007) 193-198;
Feruglio, Romanino, Rev.Mod.Phys.93(2021);
Altarelli and Feruglio, Rev.Mod.Phys. 82, 2701 (2010)…]

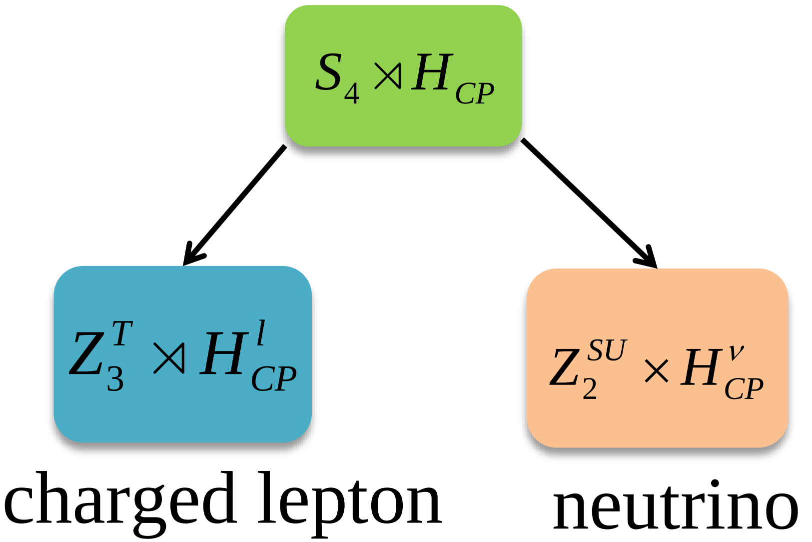
Flavor symmetries + gCP



Constrain mixing angles, Dirac phase and Majorana phase

[Feruglio, Hagedorn, Ziegler , JHEP 07 (2013) 027;
Holthausen, Lindner, Schmidt, JHEP 04 (2013) 122;
Ding, King, Luhn, Stuart, JHEP 05 (2013) 084;
Chen, Fallbacher, Mahanthappa, Ratz, Trautner, Nucl.Phys.B 883 (2014) 267-305…]

[C. C. Li and G. J. Ding, Nucl.Phys.B 881 (2014) 206-232]



- Mixing matrix:

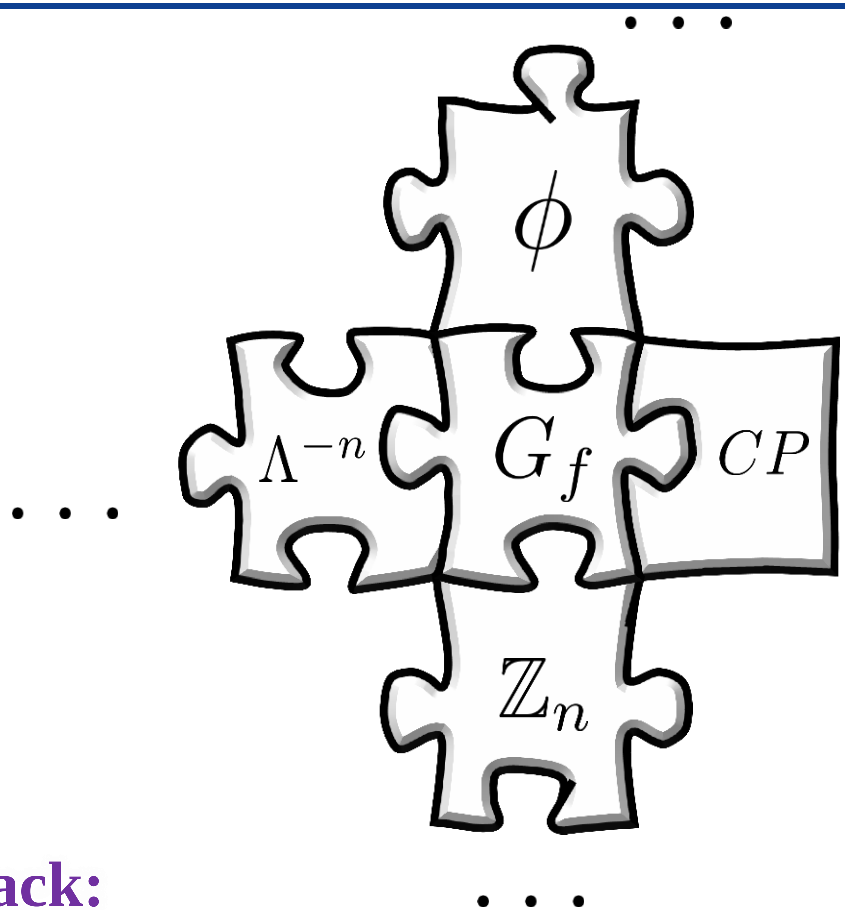
$$U_{PMNS} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{\cos\theta}{\sqrt{3}} & \frac{\sin\theta}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{\cos\theta}{\sqrt{3}} + \frac{i\sin\theta}{\sqrt{2}} & -\frac{i\cos\theta}{\sqrt{2}} + \frac{\sin\theta}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{\cos\theta}{\sqrt{3}} - \frac{i\sin\theta}{\sqrt{2}} & \frac{i\cos\theta}{\sqrt{2}} + \frac{\sin\theta}{\sqrt{3}} \end{pmatrix}$$

- Mixing parameters:

$$\left|\sin\delta_{CP}\right| = 1, \quad \sin\alpha_{21} = \sin\alpha_{31} = 0, \quad \sin^2\theta_{23} = \frac{1}{2}$$

$$\sin^2\theta_{13} = \frac{1}{3}\sin^2\theta, \quad \sin^2\theta_{12} = \frac{\cos^2\theta}{2+\cos^2\theta} = \frac{1}{3} - \frac{2}{3}\tan^2\theta_{13}$$

Problems with the traditional approach



❑Drawback:

- Introduced many gauge singlet flavons.
- Higher dimensional operators.
- High number of free parameters in effective Lagrangian.
- Spontaneous breaking in flavon sector.

Modular symmetry: a new approach to flavor puzzle

- The homogeneous modular group

$$\Gamma = SL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, ad - bc = 1, \quad a,b,c,d \in \mathbb{Z} \right\}.$$

$$\tau \mapsto \gamma\tau \equiv \frac{a\tau + b}{c\tau + d},$$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

- Generators S and T

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

$$S : \tau \mapsto -\frac{1}{\tau}, \qquad T : \tau \mapsto \tau + 1$$

$$S^4 = (ST)^3 = 1, \; S^2 T = T S^2 .$$

- Principal congruence subgroups of $SL(2,\mathbb{Z}),\quad N = 2,3,4...$

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$$

$\Rightarrow$ $T^N \in \Gamma(N)$

- Inhomogeneous finite modular groups

$$\Gamma_N \equiv \Gamma / \pm\Gamma(N) \quad \rightarrow \quad S^2 = (ST)^3 = T^N = 1, \quad \text{for} \quad N \leq 5$$

[Feruglio, 1706.08749]

$$\Gamma_2 \cong S_3, \qquad \Gamma_3 \cong A_4, \qquad \Gamma_4 \cong S_4, \qquad \Gamma_5 \cong A_5$$

- Homogeneous finite modular groups:

$$\Gamma'_N \equiv \Gamma / \Gamma(N) \quad \rightarrow \quad S^4 = (ST)^3 = T^N = 1, \; S^2T = TS^2 \; \text{for} \quad N \leq 5$$

[X.G.Liu and G.J.Ding,
JHEP 08 (2019) 134]

$$\Gamma'_2 \cong S_3, \qquad \Gamma'_3 \cong T', \qquad \Gamma'_4 \cong S'_4, \qquad \Gamma'_5 \cong A'_5$$

The Kahler potential is not under control and can not obtain analytical results !

The Kahler potential under control

- In traditional flavor symmetry the Kahler potential is under control

[Chen M.C. et al, 1208.2947,…]

$$\mathcal{K} \supset (\varphi L\bar{L})_{1_0} / \Lambda \quad \text{and/or} \quad \left(\varphi\bar{\varphi}L\bar{L}\right)_{1_0} / \Lambda^2 \,.$$

- Combine the advantages of both approaches traditional flavor symmetry and modular symmetry

$$G_{\text{flavor}} = G_{\text{traditional}} \times G_{\text{modular}}$$

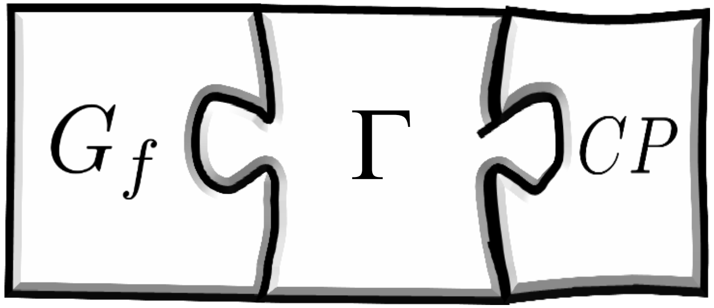
[Chen M.C. et al, 2108.02240]

$$G_{\text{eclectic}} = G_{\text{traditional}} \cup G_{\text{modular}}$$

[A. Baur et al, 1901.03251;
A. Baur et al, 1908.00805]

Eclectic flavor groups

How does this work?



[Hans Peter Nilles et al, 2001.01736;
Hans Peter Nilles et al, 2004.05200]

3. Eclectic Flavor Groups

- Transformation properties

TFS:

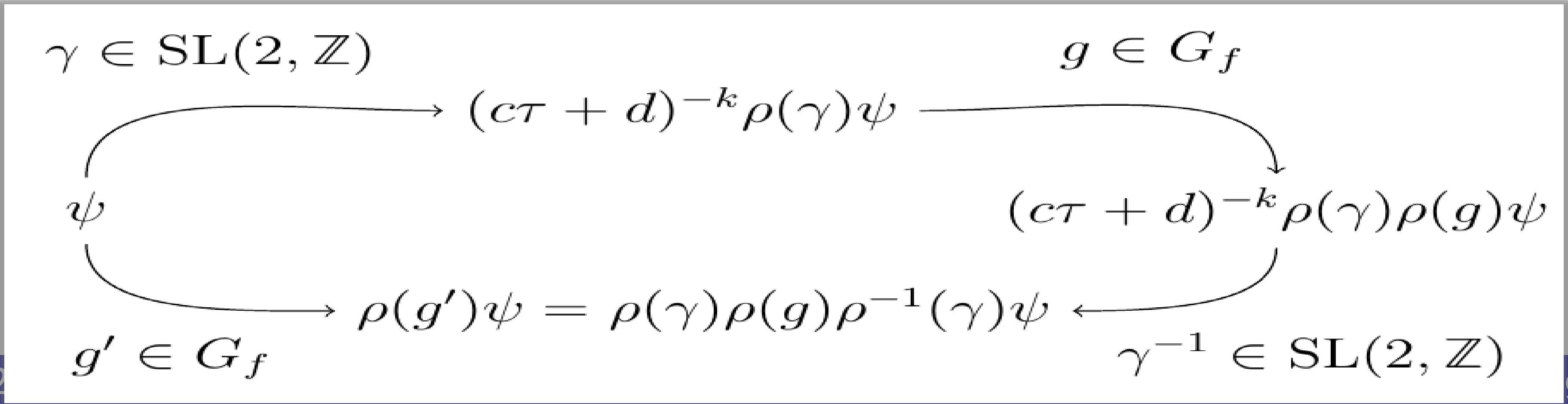
$$\tau \overset{g}{\longmapsto} \tau, \qquad \psi \overset{g}{\longmapsto} \rho(g)\,\psi, \qquad g \in G_f\,,$$

MS:

$$\tau \xrightarrow{\gamma} \gamma\tau \equiv \frac{a\tau+b}{c\tau+d}, \quad Y(\tau) \xrightarrow{\gamma} Y(\gamma\tau) = (c\tau+d)^k \rho(\gamma)\, Y(\tau)\,,$$

$$\psi \xrightarrow{\gamma} (c\tau+d)^{-k_\psi}\rho(\gamma)\psi, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})\,,$$

Combining MS with TFS



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Combining MS with TFS

$\gamma \in \mathrm{SL}(2,\mathbb{Z})$ $\qquad$ $g \in G_f$

$$\psi \longrightarrow (c\tau+d)^{-k}\rho(\gamma)\psi \longrightarrow (c\tau+d)^{-k}\rho(\gamma)\rho(g)\psi$$

$$\rho(\gamma)\rho(g)\rho^{-1}(\gamma) = \rho(g'), \qquad g, g' \in G_f, \quad \gamma \in \Gamma.$$

$$\psi \longrightarrow \rho(g')\psi = \rho(\gamma)\rho(g)\rho^{-1}(\gamma)\psi \longleftarrow (c\tau+d)^{-k}\rho(\gamma)\rho(g)\psi$$

$g' \in G_f$ $\qquad$ $\gamma^{-1} \in \mathrm{SL}(2,\mathbb{Z})$

- The consistency condition

$$\rho(\gamma)\rho(g)\rho^{-1}(\gamma) = \rho(u_\gamma(g)), \qquad \forall g \in G_f,$$

$\Rightarrow$ $u_\gamma : G_f \to G_f$ $\Rightarrow$ $G_{ecl} \cong G_f \rtimes \Gamma'_N \, (G_f \rtimes \Gamma_N)$.

traditional flavor transfromation

$$W(\psi,\phi,Y(\tau)) \xrightarrow{g} W(\rho_{r_1}(g)\psi, \rho_{r_2}(g)\phi, Y(\tau)) = W(\psi,\phi,Y(\tau))$$

- If γ corresponds to an inner automorphism

$$W(\psi,\phi,Y(\tau)) \xrightarrow{\gamma\sim g} W(\rho_{r_1}(g)\psi, \rho_{r_2}(g)\phi, \rho_{r_3}(g)Y(\tau)) = W(\psi,\phi,Y(\tau))$$

$\Rightarrow$ $Y(\tau) \xrightarrow{\gamma\sim g} \rho_{r_3}(\gamma)Y(\tau) = \rho_{r_3}(g)Y(\tau) = Y(\tau)$

Finite modular group must be subgroup of the ***outer*** automorphism group of G_f.

$\Rightarrow$ $G_f = Z_3 \times Z_3, \Delta(27), \Delta(54), \ldots$

- The consistency condition

$$\rho(\gamma)\rho(g)\rho^{-1}(\gamma) = \rho(u_\gamma(g)), \qquad \forall g \in G_f,$$

→ $u_\gamma : G_f \to G_f$ → $G_{ecl} \cong G_f \rtimes \Gamma'_N \, (G_f \rtimes \Gamma_N)$.

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$$W(\psi,\phi,Y(\tau)) \xrightarrow{\gamma\sim g} W(\rho_{r_1}(g_1)\psi, \rho_{r_2}(g_2)\phi, \rho_{r_3}(g_3)Y(\tau)) = W(\psi,\phi,Y(\tau))$$

→ $Y(\tau) \xrightarrow{\gamma\sim g} \rho_{r_3}(g_3)Y(\tau) \neq Y(\tau)$

Finite modular group must be subgroup of the automorphism group of G_f.

→

Traditional flavor group
Q_8
$A_4 \cong [12,3]$
$C_2 \times Q_8 \cong [16,12]$
$(C_4 \times C_2) \rtimes C_2 \cong [16,13]$
$(C_3 \times C_3) \rtimes C_2 \cong [18,4]$
$T' = [24,3]$
$C_3 \times Q8 \cong [24,11]$
$C_2 \times A_4 \cong [24,13]$
$C_2 \times C_2 \times S_3 \cong [24,14]$
$\Delta(27) \cong [27,3]$
$C_9 \rtimes C_3 \cong [27,4]$

- It is sufficient to only discuss:

$$\rho(S)\,\rho(g)\,\rho^{-1}(S) = \rho(u_S(g)),$$
$$\rho(T)\,\rho(g)\,\rho^{-1}(T) \;=\; \rho(u_T(g))\;.$$

- The finite modular group:

$$\Gamma_N \quad = \left\{S,T \middle| S^2 = (ST)^3 = T^N = 1\right\}\;,$$
$$\Gamma'_N \quad = \left\{S,T \middle| S^4 = (ST)^3 = T^N = 1,\; S^2T = TS^2\right\}, \qquad N \leq 5\,.$$

$$(u_S)^{N_s} = (u_T)^N = (u_S \circ u_T)^3 = 1, \qquad (u_S)^2 \circ u_T = u_T \circ (u_S)^2\;,$$

$$N_s = 2 \quad \Rightarrow \quad \Gamma_N, \qquad\qquad N_s = 4 \quad \Rightarrow \quad \Gamma'_N$$

[Hans Peter Nilles et al, 2001.01736]

EFG and GCP

- The consistency between the modular symmetry and gCP

$$\rho(K_*)\rho^*(S)\rho^{-1}(K_*) = \rho^{-1}(S), \qquad \rho(K_*)\rho^*(T)\rho^{-1}(K_*) = \rho^{-1}(T)\,.$$

[Novichkov et al, 1905.11970;
Ding et al, 2102.06716]

- The consistency between the traditional symmetry and gCP

$$\rho(K_*)\rho^*(g)\rho^{-1}(K_*) = \rho(u_{K_*}(g))\,, \qquad \forall g \in G_f\,,$$

[Feruglio, Hagedorn, Ziegler , JHEP 07 (2013) 027;
Holthausen, Lindner, Schmidt, JHEP 04 (2013) 122;
Ding, King, Luhn, Stuart, JHEP 05 (2013) 084;
Chen, Fallbacher, Mahanthappa, Ratz, Trautner, Nucl.Phys.B 883 (2014) 267-305…]

- The automorphisms of G_f satisfy

$$(u_S)^{N_s} = (u_T)^N = (u_S \circ u_T)^3 = 1, \qquad (u_S)^2 \circ u_T = u_T \circ (u_S)^2\,,$$
$$(u_{K_*})^2 = 1, \qquad u_{K_*} \circ u_S \circ u_{K_*} = u_S^{-1}, \qquad u_{K_*} \circ u_T \circ u_{K_*} = u_T^{-1}\,,$$

[Hans Peter Nilles et al, 2001.01736]

Examples of traditional flavor groups

flavor group $\mathcal{G}_{\text{fl}}$	GAP ID	$\text{Aut}(\mathcal{G}_{\text{fl}})$	finite modular groups		eclectic flavor group
Q_8	[8, 4]	S_4	without $\mathcal{CP}$	S_3	$\text{GL}(2,3)$
			with $\mathcal{CP}$	–	–
$\mathbb{Z}_3 \times \mathbb{Z}_3$	[9, 2]	$\text{GL}(2,3)$	without $\mathcal{CP}$	S_3	$\Delta(54)$
			with $\mathcal{CP}$	$S_3 \times \mathbb{Z}_2$	[108, 17]
A_4	[12, 3]	S_4	without $\mathcal{CP}$	S_3	S_4
				S_4	S_4
			with $\mathcal{CP}$	–	–
T'	[24, 3]	S_4	without $\mathcal{CP}$	S_3	$\text{GL}(2,3)$
			with $\mathcal{CP}$	–	–
$\Delta(27)$	[27, 3]	[432, 734]	without $\mathcal{CP}$	S_3	$\Delta(54)$
				T'	$\Omega(1)$
			with $\mathcal{CP}$	$S_3 \times \mathbb{Z}_2$	[108, 17]
				$\text{GL}(2,3)$	[1296, 2891]
$\Delta(54)$	[54, 8]	[432, 734]	without $\mathcal{CP}$	T'	$\Omega(1)$
			with $\mathcal{CP}$	$\text{GL}(2,3)$	[1296, 2891]

[Hans Peter Nilles et al, 2001.01736]

flavor group	finite modular group	EFG	Group id
Q_8	S_3	$GL(2,3) = [48,29]$	—
	A_4	$Q_8 \rtimes A_4 = [96,204]$	$[192,1494]$
	S_4	$Q_8 \rtimes S_4$	$[192,1494]$
$Z_3 \times Z_3$	S_3	$\Delta(54)$	$[54,8]$
	T'	$(Z_3 \times Z_3) \rtimes T'$	$[216,153]$
A_4	S_3	$A_4 \rtimes S_3$	$[72,43]$
	A_4	$A_4 \rtimes A_4$	$[144,184]$
	S_4	$A_4 \rtimes S_4$	$[288,1026]$
S_4	S_3	$S_4 \rtimes S_3$	$[144,183]$
	A_4	$S_4 \rtimes A_4$	$[288,1024]$
	S_4	$S_4 \rtimes S_4$	$[576,8653]$
T'	S_3	$T' \rtimes S_3$	$[144,125]$
	A_4	$T' \rtimes A_4$	$[288,860]$
	S_4	$T' \rtimes S_4$	$[576,8282]$
$\Delta(27)$	S_3	$\Delta(27) \rtimes S_3$	$[162,46]$
	T'	$\Delta(27) \rtimes T' \cong \Omega(1)$	$[648,533]$
$\Delta(54)$	S_3	$\Delta(54) \rtimes S_3$	$[324,122]$
	T'	$\Delta(54) \rtimes T'$	$[1296,2895]$

- The multiplication rules of $\Delta(27)$

$$A^3 = B^3 = (AB)^3 = (AB^2)^3 = 1\,.$$

our working basis

$\omega = e^{2\pi i/3}$

$$\mathbf{1}_{r,s} \quad : \quad \rho_{\mathbf{1}_{r,s}}(A) = \omega^r\,, \qquad \rho_{\mathbf{1}_{r,s}}(B) = \omega^s\,, \quad \text{with} \quad r,s = 0,1,2\,,$$

$$\mathbf{3} \quad : \quad \rho_{\mathbf{3}}(A) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \rho_{\mathbf{3}}(B) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix},$$

$$\bar{\mathbf{3}} \quad : \quad \rho_{\bar{\mathbf{3}}}(A) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \rho_{\bar{\mathbf{3}}}(B) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}.$$

- The outer automorphism group of $\Delta(27)$

$$\mathrm{Aut}\left(\Delta(27)\right) \cong [432, 734]$$

finite modular subgroups: $S_3, \quad Z_3 \times S_3, \quad T', \quad GL(2,3)$

- The three outer automorphisms u_S , u_T and u_{K^*}

$$\begin{aligned} u_S(A) &= A^2, & u_T(A) &= A^2\,, \\ u_S(B) &= B^2, & u_T(B) &= A^2B^2A\,, \\ u_{K_*}(A) &= A\,, & u_{K_*}(B) &= AB^2A\,. \end{aligned}$$

[Hans Peter Nilles et al, 2001.01736]

→ $u_S,\ u_T \quad \Rightarrow \quad \Gamma_2 \cong S_3$

no GCP → $\Delta(27) \rtimes S_3 \cong [162, 46]$

EFG+GCP → [324, 121]

- Solving the following consistency conditions

$$\begin{aligned} \rho_{\boldsymbol{r}}(S)\,\rho_{\boldsymbol{r}}(A)\,\rho_{\boldsymbol{r}}^{-1}(S) &= \rho_{\boldsymbol{r}}(A^2), & \rho_{\boldsymbol{r}}(T)\,\rho_{\boldsymbol{r}}(A)\,\rho_{\boldsymbol{r}}^{-1}(T) &= \rho_{\boldsymbol{r}}(A^2)\,, \\ \rho_{\boldsymbol{r}}(S)\,\rho_{\boldsymbol{r}}(B)\,\rho_{\boldsymbol{r}}^{-1}(S) &= \rho_{\boldsymbol{r}}(B^2), & \rho_{\boldsymbol{r}}(T)\,\rho_{\boldsymbol{r}}(B)\,\rho_{\boldsymbol{r}}^{-1}(T) &= \rho_{\boldsymbol{r}}(A^2B^2A)\,, \end{aligned}$$

- u_S and u_T act on irreducible representations as

$$u_S,\ u_T: \quad \mathbf{1}_{0,1} \leftrightarrow \mathbf{1}_{0,2}\,,\quad \mathbf{1}_{1,0} \leftrightarrow \mathbf{1}_{2,0}\,,\quad \mathbf{1}_{1,1} \leftrightarrow \mathbf{1}_{2,2}\,,\quad \mathbf{1}_{1,2} \leftrightarrow \mathbf{1}_{2,1}\,,$$
$$\mathbf{1}_{0,0} \to \mathbf{1}_{0,0}\,,\quad \mathbf{3} \to \mathbf{3},\quad \bar{\mathbf{3}} \to \bar{\mathbf{3}}\,.$$

- The representations of EFG must be

$$1_{0,0}, 2_i, 3 \ \mathit{and}\ \bar{3}$$

where 2_i (i=1,2,3,4) are defined as

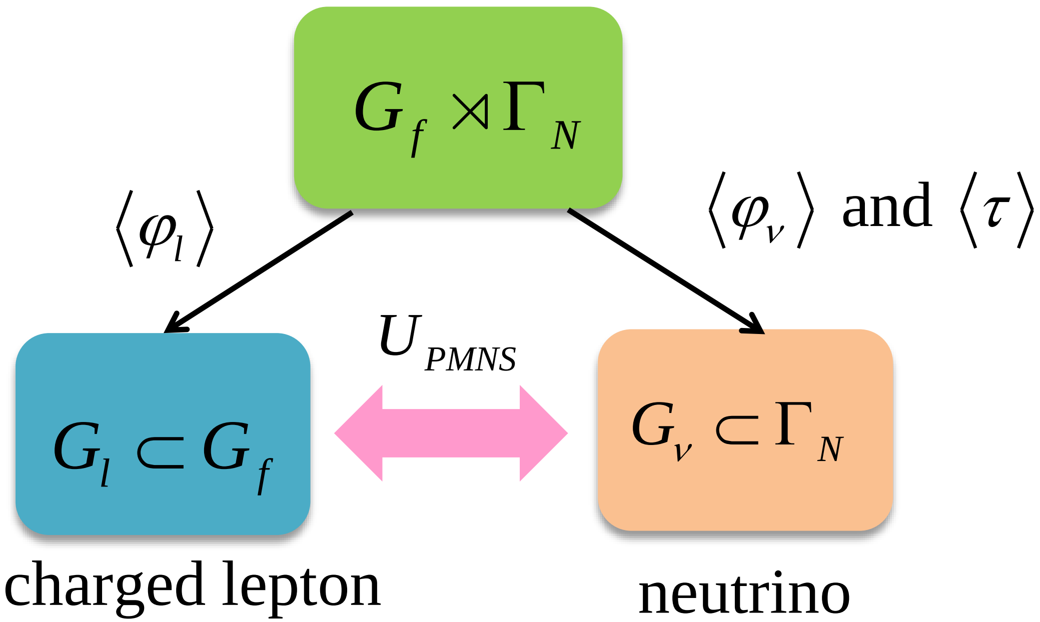
$$\begin{aligned} \mathbf{2}_1 &\equiv (\mathbf{1}_{0,1}, \mathbf{1}_{0,2})^T, & \mathbf{2}_2 &\equiv (\mathbf{1}_{1,1}, \mathbf{1}_{2,2})^T, \\ \mathbf{2}_3 &\equiv (\mathbf{1}_{1,0}, \mathbf{1}_{2,0})^T, & \mathbf{2}_4 &\equiv (\mathbf{1}_{1,2}, \mathbf{1}_{2,1})^T\,. \end{aligned}$$

The representations of the EFG $\Delta(27)\rtimes S_3 \cong [162,46]$				
	A	B	S	T
$\mathbf{1}^{a}_{\mathbf{0},\mathbf{0}}$	1	1	$(-1)^a$	$(-1)^a$
$\mathbf{2_0}$	$\mathbb{1}_2$	$\mathbb{1}_2$	$\rho_{\mathbf{2}}(S)$	$\rho_{\mathbf{2}}(T)$
$\mathbf{2}_{i,m}$	$\rho_{\mathbf{2}_i}(A)$	$\rho_{\mathbf{2}_i}(B)$	$\rho_{\mathbf{2}_{i,m}}(S)$	$\rho_{\mathbf{2}_{i,m}}(T)$
$\mathbf{3}^a$	$\rho_{\mathbf{3}}(A)$	$\rho_{\mathbf{3}}(B)$	$\rho_{\mathbf{3}^a}(S)$	$\rho_{\mathbf{3}^a}(T)$
$\bar{\mathbf{3}}^a$	$\rho_{\bar{\mathbf{3}}}(A)$	$\rho_{\bar{\mathbf{3}}}(B)$	$\rho^*_{\mathbf{3}^a}(S)$	$\rho^*_{\mathbf{3}^a}(T)$
$\mathbf{6}=\mathbf{3}^0\otimes\mathbf{2}_0$	$\rho_{\mathbf{3}}(A)\otimes\mathbb{1}_2$	$\rho_{\mathbf{3}}(B)\otimes\mathbb{1}_2$	$\rho_{\mathbf{3}^0}(S)\otimes\rho_{\mathbf{2}}(S)$	$\rho_{\mathbf{3}^0}(T)\otimes\rho_{\mathbf{2}}(T)$
$\bar{\mathbf{6}}=\bar{\mathbf{3}}^0\otimes\mathbf{2}_0$	$\rho_{\bar{\mathbf{3}}}(A)\otimes\mathbb{1}_2$	$\rho_{\bar{\mathbf{3}}}(B)\otimes\mathbb{1}_2$	$\rho_{\bar{\mathbf{3}}^0}(S)\otimes\rho_{\mathbf{2}}(S)$	$\rho_{\bar{\mathbf{3}}^0}(T)\otimes\rho_{\mathbf{2}}(T)$

$$\rho_{\mathbf{2}_{i,m}}(S)=\begin{pmatrix}0&1\\1&0\end{pmatrix},\qquad \rho_{\mathbf{2}_{i,m}}(T)=\begin{pmatrix}0&\omega^m\\\omega^{-m}&0\end{pmatrix}.$$

$$\rho_{\mathbf{3}^a}(S)=(-1)^a\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix},\quad \rho_{\mathbf{3}^a}(T)=(-1)^a\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix},$$

Analytical results ?



$$\langle \varphi_l \rangle = (1,1,1)^T, \quad \langle \varphi_\nu \rangle = (0,1,-1)^T, \quad \langle \tau \rangle = i.$$

$$\Delta(27) \rtimes S_3$$

$$\Rightarrow \quad G_l = Z_3^A, \; G_\nu = Z_2^S.$$

$$\Rightarrow \quad U_{PMNS} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta e^{-i\delta} & 0 \\ -\sin\theta e^{i\delta} & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4. Effective action invariant under $\Delta(27) \rtimes S_3$

Superpotential

- Level 2 modular form multiplets

$$Y_{\mathbf{1}}^{(k_Y)}(\tau) = Y_1\,, \qquad Y_{\mathbf{1}'}^{(k_Y)}(\tau) = Y_2\,, \qquad Y_{\mathbf{2}}^{(k_Y)}(\tau) = \begin{pmatrix} Y_3 \\ Y_4 \end{pmatrix}.$$

- The assignment

$$\psi \sim \mathbf{3}^0, \psi^c \sim \mathbf{3}^0, \qquad \Phi \equiv (\phi_1, \phi_2, \phi_3)^T \sim \mathbf{3}^0.$$

- The mass matrix of the fermion ψ

$$M_\psi = \frac{v_{u,d}}{\Lambda}\left[\alpha_1 Y_1 \begin{pmatrix} \phi_1 & 0 & 0 \\ 0 & \phi_2 & 0 \\ 0 & 0 & \phi_3 \end{pmatrix} + \alpha_2 Y_1 \begin{pmatrix} 0 & \phi_3 & \phi_2 \\ \phi_3 & 0 & \phi_1 \\ \phi_2 & \phi_1 & 0 \end{pmatrix} + \alpha_3 Y_2 \begin{pmatrix} 0 & -\phi_3 & \phi_2 \\ \phi_3 & 0 & -\phi_1 \\ -\phi_2 & \phi_1 & 0 \end{pmatrix}\right], \tag{3.18}$$

- The assignment

$$\psi \sim \mathbf{3}^0, \psi^c \sim \overline{\mathbf{3}}^0, \qquad \Phi \equiv (\phi_1, \phi_2)^T \sim 2_{1,1}.$$

- The mass matrix of the fermion ψ

$$M''_\psi = \frac{\alpha v_{u,d}}{\Lambda} \begin{pmatrix} 0 & \omega\phi_2(Y_3 + iY_4) & \phi_1(Y_3 - iY_4) \\ \phi_1(Y_3 - iY_4) & 0 & \omega\phi_2(Y_3 + iY_4) \\ \omega\phi_2(Y_3 + iY_4) & \phi_1(Y_3 - iY_4) & 0 \end{pmatrix}. \qquad (3.34)$$

$\mathcal{W}$	$\Delta(27)\rtimes S_3$	$Y_{\boldsymbol{r}}^{(k_Y)}$	M_ψ	Constraints of gCP
$\mathcal{W}_{\psi 1}$	$\psi,\,\psi^c\sim\mathbf{3^0},\,\Phi\sim\mathbf{3^0}$	$Y_{\mathbf{1}}^{(k_Y)},\,Y_{\mathbf{1'}}^{(k_Y)}$	M_ψ in Eq. (3.18)	Eq. (3.19)
$\mathcal{W}_{\psi 2}$	$\psi,\,\psi^c\sim\mathbf{3^0},\,\Phi\sim\mathbf{3^1}$	$Y_{\mathbf{1}}^{(k_Y)},\,Y_{\mathbf{1'}}^{(k_Y)}$	$M_\psi\,(Y_1\to Y_2, Y_2\to Y_1)$	Eq. (3.19)
$\mathcal{W}_{\psi 3}$	$\psi,\,\psi^c\sim\mathbf{\bar{3}^0},\,\Phi\sim\mathbf{\bar{3}^0}$	$Y_{\mathbf{1}}^{(k_Y)},\,Y_{\mathbf{1'}}^{(k_Y)}$	M_ψ	Eq. (3.22)
$\mathcal{W}_{\psi 4}$	$\psi,\,\psi^c\sim\mathbf{\bar{3}^0},\,\Phi\sim\mathbf{\bar{3}^1}$	$Y_{\mathbf{1}}^{(k_Y)},\,Y_{\mathbf{1'}}^{(k_Y)}$	$M_\psi\,(Y_1\to Y_2, Y_2\to Y_1)$	Eq. (3.22)
$\mathcal{W}_{\psi 5}$	$\psi\sim\mathbf{3^0},\,\psi^c\sim\mathbf{\bar{3}^0},$ $\Phi\sim\mathbf{2_{1,0}}$	$Y_{\mathbf{1}}^{(k_Y)},\,Y_{\mathbf{1'}}^{(k_Y)}$	M'_ψ in Eq. (3.28)	Eq. (3.30)
$\mathcal{W}_{\psi 6}$	$\psi\sim\mathbf{3^0},\,\psi^c\sim\mathbf{\bar{3}^0},$ $\Phi\sim\mathbf{2_{2,2}}$	$Y_{\mathbf{1}}^{(k_Y)},\,Y_{\mathbf{1'}}^{(k_Y)}$	$D_L M'_\psi\left(Y_1\to\omega^2 Y_1, Y_2\to i\omega Y_2\right)D_R$	Eq. (3.30)
$\mathcal{W}_{\psi 7}$	$\psi\sim\mathbf{3^0},\,\psi^c\sim\mathbf{\bar{3}^0},$ $\Phi\sim\mathbf{2_{3,2}}$	$Y_{\mathbf{1}}^{(k_Y)},\,Y_{\mathbf{1'}}^{(k_Y)}$	$\phi_1(\alpha_1 Y_1+\alpha_2 Y_2)\text{diag}(\omega,\omega^2,1)$ $+\phi_2(\alpha_1 Y_1-\alpha_2 Y_2)\text{diag}(\omega,1,\omega^2)$	—
$\mathcal{W}_{\psi 8}$	$\psi\sim\mathbf{3^0},\,\psi^c\sim\mathbf{\bar{3}^0},$ $\Phi\sim\mathbf{2_{4,2}}$	$Y_{\mathbf{1}}^{(k_Y)},\,Y_{\mathbf{1'}}^{(k_Y)}$	$P_{23}D_R M'_\psi(P_{23}D_R)^\dagger$	—
$\mathcal{W}_{\psi 9}$	$\psi\sim\mathbf{3^0},\,\psi^c\sim\mathbf{\bar{3}^0},$ $\Phi\sim\mathbf{2_{1,1}}$	$Y_{\mathbf{2}}^{(k_Y)}$	M''_ψ in Eq. (3.34)	—
$\mathcal{W}_{\psi 10}$	$\psi\sim\mathbf{3^0},\,\psi^c\sim\mathbf{\bar{3}^0},$ $\Phi\sim\mathbf{2_{1,2}}$	$Y_{\mathbf{2}}^{(k_Y)}$	$M''_\phi\left(\phi_1\to\omega^2\phi_1, Y_4\to -Y_4\right)$	—
$\mathcal{W}_{\psi 11}$	$\psi\sim\mathbf{3^0},\,\psi^c\sim\mathbf{\bar{3}^0},$ $\Phi\sim\mathbf{2_{2,0}}$	$Y_{\mathbf{2}}^{(k_Y)}$	$\omega^2 D_L M''_\psi D_R$	$\Im\alpha=0$
$\mathcal{W}_{\psi 12}$	$\psi\sim\mathbf{3^0},\,\psi^c\sim\mathbf{\bar{3}^0},$ $\Phi\sim\mathbf{2_{2,1}}$	$Y_{\mathbf{2}}^{(k_Y)}$	$D_L M''_\phi\left(\phi_1\to\phi_1,\phi_2\to\omega\phi_2, Y_4\to -Y_4\right)D_R$	$\Im\alpha=0$
$\mathcal{W}_{\psi 13}$	$\psi\sim\mathbf{3^0},\,\psi^c\sim\mathbf{\bar{3}^0},$ $\Phi\sim\mathbf{2_{3,0}}$	$Y_{\mathbf{2}}^{(k_Y)}$	$\alpha[\phi_1(Y_3-iY_4)\text{diag}(\omega,\omega^2,1)$ $+\phi_2(Y_3+iY_4)\text{diag}(\omega^2,\omega,1)]$	—
$\mathcal{W}_{\psi 14}$	$\psi\sim\mathbf{3^0},\,\psi^c\sim\mathbf{\bar{3}^0},$ $\Phi\sim\mathbf{2_{3,1}}$	$Y_{\mathbf{2}}^{(k_Y)}$	$\alpha[\phi_1(Y_3+iY_4)\text{diag}(1,\omega,\omega^2)$ $+\phi_2(Y_3-iY_4)\text{diag}(\omega^2,\omega,1)]$	—
$\mathcal{W}_{\psi 15}$	$\psi\sim\mathbf{3^0},\,\psi^c\sim\mathbf{\bar{3}^0},$ $\Phi\sim\mathbf{2_{4,0}}$	$Y_{\mathbf{2}}^{(k_Y)}$	$P_{23}D_L^\dagger M''_\psi D_R^\dagger P_{23}$	—
$\mathcal{W}_{\psi 16}$	$\psi\sim\mathbf{3^0},\,\psi^c\sim\mathbf{\bar{3}^0},$ $\Phi\sim\mathbf{2_{4,1}}$	$Y_{\mathbf{2}}^{(k_Y)}$	$P_{23}D_L^\dagger M''_\psi\left(\phi_1\to\omega^2\phi_1, Y_4\to -Y_4\right)D_R^\dagger P_{23}$	—

Kahler potential

- The Kahler potential of the fermion ψ:

$$K = K_{LO} + K_{NLO} + K_{NNLO} + \ldots$$

where

$$\mathcal{K}_{\rm LO} = g(\tau,\bar{\tau})\left(\psi_1^\dagger\psi_1 + \psi_2^\dagger\psi_2 + \psi_3^\dagger\psi_3\right)$$

without off-diagonal element of the Kahler metric

K_{NLO} can be absorbed into the K_{LO}

K_{NNLO} which is suppressed by $\frac{\langle\Phi\rangle^2}{\Lambda^2}$ will give rise to off-diagonal elements.

5. $\Delta(27) \rtimes S_3$ eclectic lepton model

Fields	L	E^c	N^c	H_u	H_d	ϕ	φ	χ
$\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$	$(\mathbf{2}, -\frac{1}{2})$	$(\mathbf{1}, 1)$	$(\mathbf{1}, 0)$	$(\mathbf{2}, \frac{1}{2})$	$(\mathbf{2}, -\frac{1}{2})$	$(\mathbf{1}, 0)$	$(\mathbf{1}, 0)$	$(\mathbf{1}, 0)$
$\Delta(27) \rtimes S_3$	$\mathbf{\bar{3}^0}$	$\mathbf{\bar{3}^0}$	$\mathbf{3^0}$	$\mathbf{1^0_{0,0}}$	$\mathbf{1^0_{0,0}}$	$\mathbf{\bar{3}^0}$	$\mathbf{2_{2,0}}$	$\mathbf{3^0}$
Modular weight	0	0	0	0	0	6	4	0
Z_3	ω	ω	1	1	1	ω	ω^2	1

Kahler potential :

$$\propto 1$$

$$\mathcal{K} = \mathcal{K}_{\mathrm{LO}} + \mathcal{K}_{\mathrm{NLO}} + \mathcal{K}_{\mathrm{NNLO}} + \ldots$$

Correction $\sim \frac{\langle\Phi\rangle^2}{\Lambda^2}$

Ignore!

$$\sum_{m,n,\boldsymbol{r_1},\boldsymbol{r_2},s} \frac{1}{\Lambda^2}(-i\tau + i\bar{\tau})^{-k_\psi - k_\Theta + m}\left(Y^{(m)\dagger}_{\boldsymbol{r_1}} Y^{(n)}_{\boldsymbol{r_2}} \psi^\dagger \psi \Theta^\dagger \Phi\right)_{(\mathbf{1_{0,0}},\mathbf{1}),s} + \text{h.c.}$$

5. Δ(27) ⋊ S_3 eclectic lepton model

Fields	L	E^c	N^c	H_u	H_d	ϕ	φ	χ
$\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$	$(\mathbf{2}, -\frac{1}{2})$	$(\mathbf{1}, 1)$	$(\mathbf{1}, 0)$	$(\mathbf{2}, \frac{1}{2})$	$(\mathbf{2}, -\frac{1}{2})$	$(\mathbf{1}, 0)$	$(\mathbf{1}, 0)$	$(\mathbf{1}, 0)$
$\Delta(27) \rtimes S_3$	$\bar{\mathbf{3}}^{\mathbf{0}}$	$\bar{\mathbf{3}}^{\mathbf{0}}$	$\mathbf{3}^{\mathbf{0}}$	$\mathbf{1}^{\mathbf{0}}_{\mathbf{0,0}}$	$\mathbf{1}^{\mathbf{0}}_{\mathbf{0,0}}$	$\bar{\mathbf{3}}^{\mathbf{0}}$	$\mathbf{2}_{\mathbf{2,0}}$	$\mathbf{3}^{\mathbf{0}}$
Modular weight	0	0	0	0	0	6	4	0
Z_3	ω	ω	1	1	1	ω	ω^2	1

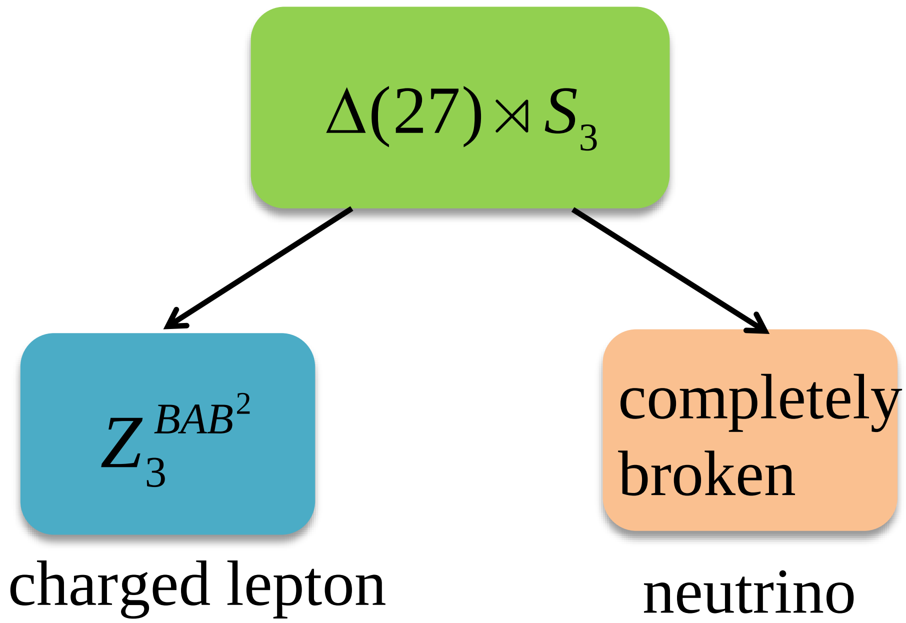
Superpotential :

$$\mathcal{W}_l = \frac{\alpha}{\Lambda}\left(E^c L \phi Y_{\mathbf{1}}^{(6)}\right)_{(\mathbf{1_{0,0,1}}),1} H_d + \frac{\beta}{\Lambda}\left(E^c L \phi Y_{\mathbf{1}}^{(6)}\right)_{(\mathbf{1_{0,0,1}}),2} H_d + \frac{\gamma}{\Lambda}\left(E^c L \phi Y_{\mathbf{1'}}^{(6)}\right)_{(\mathbf{1_{0,0,1}})} H_d ,$$

$$\mathcal{W}_\nu = \frac{h}{\Lambda}\left(N^c L \varphi Y_{\mathbf{2}}^{(2)}\right)_{(\mathbf{1_{0,0,1}})} H_u + \frac{g_1}{2}\left(N^c N^c \chi\right)_{(\mathbf{1_{0,0,1}}),1} + \frac{g_2}{2}\left(N^c N^c \chi\right)_{(\mathbf{1_{0,0,1}}),2} , \quad (4$$

$$\langle\phi\rangle = (1, \omega, \omega^2)^T v_\phi, \qquad \langle\varphi\rangle = (1, 1)^T v_\varphi, \qquad \langle\chi\rangle = (1, x, 1)^T v_\chi ,$$

Symmetry breaking



$$U_l = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & -\omega^2 & 1 \\ \omega^2 & -1 & 1 \\ \omega & -\omega & 1 \end{pmatrix},$$

$$m_e = \left|(\alpha + 2\beta) Y_{\mathbf{1}}^{(6)}\right| \frac{v_\phi v_d}{\Lambda},$$

$$m_\mu = \left|(\alpha - \beta) Y_{\mathbf{1}}^{(6)} + \sqrt{3} i \gamma Y_{\mathbf{1'}}^{(6)}\right| \frac{v_\phi v_d}{\Lambda},$$

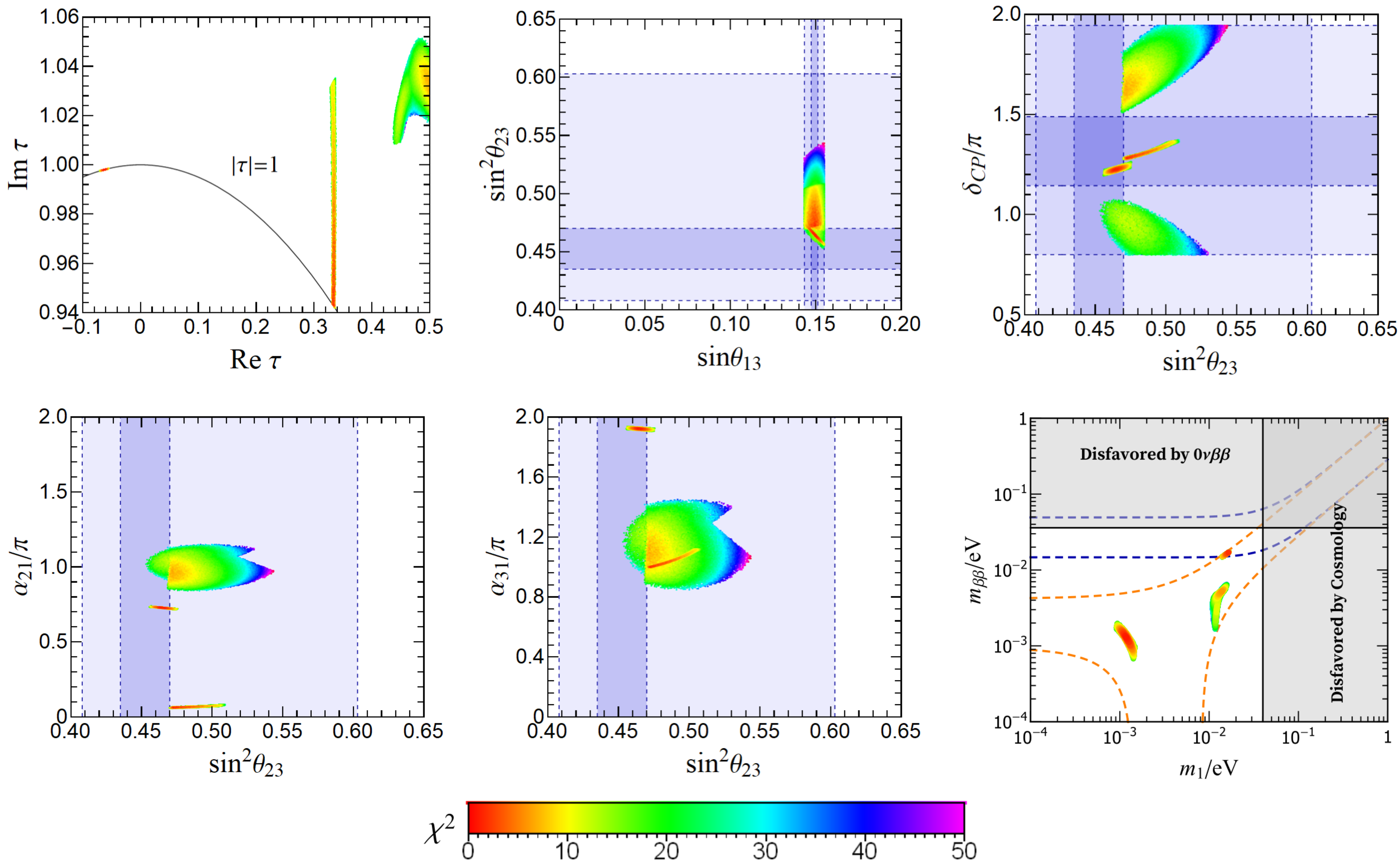
$$m_\tau = \left|(\alpha - \beta) Y_{\mathbf{1}}^{(6)} - \sqrt{3} i \gamma Y_{\mathbf{1'}}^{(6)}\right| \frac{v_\phi v_d}{\Lambda}.$$

■ Best fit values of the free parameters

$$\Re\langle\tau\rangle = -0.0621\,(0.496), \quad \Im\langle\tau\rangle = 0.998\,(0.965), \quad \arg(\alpha) = 0.100\pi\,(0.0833\pi),$$
$$\gamma/|\alpha| = 0.207\,(-0.778), \quad \arg(g_1) = 0.119\,(0.0111), \quad x = 1.882(0.987),$$
$$|\alpha|v_d v_\phi/\Lambda = 739.4\,\mathrm{GeV}\,(159.3\,\mathrm{GeV})\,, \quad h^2 v_u^2 v_\varphi^2/(|g_1|v_\chi \Lambda^2) = 0.815\,\mathrm{eV}\,(1.803\,\mathrm{eV})\,,$$

■ The predictions for various observable quantities

$$\sin^2\theta_{13} = 0.02233\,(0.02216), \quad \sin^2\theta_{12} = 0.3041\,(0.3025), \quad \sin^2\theta_{23} = 0.4645\,(0.5945),$$
$$\delta_{CP} = 1.226\pi\,(1.498\pi)\,, \quad \alpha_{21} = 0.729\pi\,(1.998\pi), \quad \alpha_{31} = 1.923\pi\,(0.994\pi),$$
$$m_1 = 1.173\,\mathrm{meV}\,(72.26\,\mathrm{meV}), \quad m_2 = 8.676\,\mathrm{meV}\,(72.77\,\mathrm{meV})\,, \quad m_3 = 50.13\,\mathrm{meV}\,(52.84\,\mathrm{meV})$$
$$\sum_{i=1}^{3} m_i = 59.98\,\mathrm{meV}\;(197.9\,\mathrm{meV})\,, \quad m_{\beta\beta} = 1.298\,\mathrm{meV}\,(71.98\,\mathrm{meV})\,,$$
$$m_e = 0.511\,\mathrm{MeV}\,(0.511\,\mathrm{MeV}), \quad m_\mu = 107.9\,\mathrm{MeV}\,(107.9\,\mathrm{MeV}), \quad m_\tau = 1.837\,\mathrm{GeV}\,(1.836\,\mathrm{GeV})\,.$$



The correlations among mixing parameters

5. Conclusions

- **We develop methods to determine the eclectic flavor groups**

- **A comprehensive analysis of the superpotential and Kahler potential of models based on the EFG $\Delta(27) \rtimes S_3$**

- **Kahler potential are suppressed by powers of** $\langle\Phi\rangle^2/\Lambda^2$

- **A concrete lepton model invariant under the EFG $\Delta(27) \rtimes \mathrm{S}_3$**

Thank you!

Backup

Modular invariant theory

[Ferrara et al, 1989; Feruglio, 1706.08749]

- $N = 1$ global supersymmetry
  - The action：

$$\mathcal{S} = \int d^4x d^2\theta d^2\bar{\theta} \mathcal{K}(\Phi_I, \bar{\Phi}_I; \tau, \bar{\tau}) + \int d^4x d^2\theta \mathcal{W}(\Phi_I, \tau) + \text{h.c.}$$

  - Kahler potential:

$$\mathcal{K} = -h \ln(-i\tau + i\bar{\tau}) + \sum_I (-i\tau + i\bar{\tau})^{-k_I} |\Phi_I|^2$$

  - The superpotential:

$$\mathcal{W} = \sum_n Y_{I_1 I_2 \ldots I_n}(\tau) \Phi_{I_1} \Phi_{I_2} \ldots \Phi_{I_n}$$

The Kahler potential is not under control

[Chen M.C. et al, 1909.06910,…]

- Transformation properties：

$$\tau \xrightarrow{\gamma} \gamma\tau = \frac{a\tau+b}{c\tau+d}\,,$$

$$Y^{(k)}(\tau) \rightarrow Y^{(k)}(\gamma\tau) = (c\tau+d)^k \rho(\gamma) Y^{(k)}(\tau),$$

$$\Phi \xrightarrow{\gamma} \gamma(c\tau+d)^{-k_\Phi} \rho(\gamma)\Phi\,, \quad \text{where } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$(-i\tau+i\bar{\tau})^k \xrightarrow{\gamma} \big((c\tau+d)(c\bar{\tau}+d)\big)^{-k} (-i\tau+i\bar{\tau})^k,$$

$$\left(Y^{(k)}\bar{Y}^{(k)}\right)_1 \xrightarrow{\gamma} (c\tau+d)^k (c\bar{\tau}+d)^k \left(Y^{(k)}\bar{Y}^{(k)}\right)_1,$$

$$\left(\bar{\Phi}\Phi\right)_1 \xrightarrow{\gamma} (c\tau+d)^{-k_\Phi} (c\bar{\tau}+d)^{-k_\Phi} \left(\bar{\Phi}\Phi\right)_1.$$

$$\mathbf{1}^a \otimes \mathbf{1}^b = \mathbf{1}^{[a+b]}, \qquad \mathbf{1}^a \otimes \mathbf{2}^b = \mathbf{2}^{[a+b]}, \qquad \mathbf{1}^a \otimes \mathbf{3} = \mathbf{3}, \qquad \mathbf{2}^a \otimes \mathbf{2}^b = \mathbf{1}^{[a+b]} \oplus \mathbf{3},$$
$$\mathbf{2}^a \otimes \mathbf{3} = \mathbf{2} \oplus \mathbf{2}' \oplus \mathbf{2}'', \qquad \mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{1}'' \oplus \mathbf{3}_S \oplus \mathbf{3}_A \,, \tag{B.3}$$

$$\mathbf{1}^a : \ \rho_{\mathbf{1}^a}(S) = 1, \qquad \rho_{\mathbf{1}^a}(T) = \omega^a \,,$$
$$\mathbf{2}^a : \ \rho_{\mathbf{2}^a}(S) = \frac{i}{\sqrt{3}} \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}, \qquad \rho_{\mathbf{2}^a}(T) = \omega^{a+1} \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}$$
$$\mathbf{3} : \quad \rho_{\mathbf{3}}(S) = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \qquad \rho_{\mathbf{3}}(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix},$$

- As a consequence

$$\mathcal{K} \supset \sum_{\Phi_n} \sum_{k\geq 0} (-i\tau + i\bar{\tau})^{-k+k_\Phi} \sum_a \kappa_a^{(k)} \left[Y^{(k)}(T) \otimes \bar{Y}^{(k)} \otimes \Phi(T) \otimes \bar{\Phi} \right]_{1,a}.$$

- The full Kahler potential includes additional terms, such as

$$\mathcal{K} = \alpha_0 (-i\tau + i\bar{\tau})^{-k_L} \left(\bar{L} L \right)_1 + \sum_{k=1} \alpha_k (-i\tau + i\bar{\tau})^{-k_L + k_Y} \left(Y \bar{Y} L \bar{L} \right)_{1,k} + \ldots$$

- The additional terms will modify the Kahler metric

$$K_L^{i\bar{j}} = \frac{\partial^2 K}{\partial L_i \, \partial \bar{L}_{\bar{j}}}$$

- This metric has to be diagonalized,

$$K_L = U_L^\dagger D^2 U_L$$

where U_L is unitary and D is diagonal and positive.

Therefore, the canonically normalized fields are

$$\hat{L} = DU_L\,L \quad \text{or equivalently} \quad L = U_L^{\dagger}\,D^{-1}\,\hat{L}\ .$$

After adding the $\alpha_{i>0}$ contributions and transforming the fields back to canonical normalization, we need to diagonalize

$$U_\nu^T m_\nu U_\nu = \text{diag}(m_1, m_2, m_3) \quad \text{and} \quad U_e^{\dagger} Y_e Y_e^{\dagger} U_e = \text{diag}(y_e^2, y_\mu^2, y_\tau^2).$$

$$\hat{U}_\nu^T\, D^{-1} U_L^{*}\, m_\nu\, U_L^{?}\, D^{-1} \hat{U}_\nu = \text{diag}(m_1, m_2, m_3)\ ,$$

$$\hat{U}_e^{\dagger} D^{-1} U_L^{*} Y_e Y_e^{?} U_L^T D^{-1} \hat{U}_e = \text{diag}(y_e^2, y_\mu^2, y_\tau^2).$$

We see that if D is proportional to the unit matrix, there would be no effect.
However, D is generically not proportional to the unit matrix.

$\mathcal{N}$=1 SUSY modular invariant theories

- known since late 1980s

S. Ferrara, D. Lust, A. D. Shapere and S. Theisen, Phys. Lett. B **225** (1989) 363.

S. Ferrara, .D. Lust and S. Theisen, Phys. Lett. B **233** (1989) 147.

focus on Yukawa interactions and $\mathcal{N}$=1 global SUSY

$$\mathcal{S} = \int d^4x d^2\theta d^2\bar{\theta}\ K(\Phi, \bar{\Phi}) + \int d^4x d^2\theta\ w(\Phi) + h.c.$$

$$\Phi = (\tau, \varphi)$$

Kahler potential, kinetic terms

superpotential, holomorphic function of Φ Yukawa interactions

S invariant if

$$\begin{cases} w(\Phi) \to w(\Phi) \\ K(\Phi, \bar{\Phi}) \to K(\Phi, \bar{\Phi}) + f(\Phi) + f(\bar{\Phi}) \end{cases}$$

- invariance of the Kahler potential easy to achieve. For example:

$$K(\Phi, \bar{\Phi}) = -h \log(-i\tau + i\bar{\tau}) + \sum_I (-i\tau + i\bar{\tau})^{-k_I} |\varphi^{(I)}|^2$$

minimal K

- extension to $\mathcal{N}$=1 SUGRA straightforward: ask invariance of $G=K+\log|w|^2$

Few facts about (level-N) Modular Forms

- transformation property under the modular group

$$f_i(\gamma\tau) = (c\tau + d)^k \rho(\gamma)_{ij} f_j(\tau)$$

unitary representation of the finite modular group $\Gamma_N \equiv \overline{\Gamma}/\overline{\Gamma}(N)$

- q-expansion

$$f(\tau + N) = f(\tau) \quad \Rightarrow \quad f(\tau) = \sum_{i=0}^{\infty} a_n q_N^n \qquad q_N = e^{\frac{i2\pi\tau}{N}}$$

- $k < 0 \Rightarrow f(\tau) = 0$

  $k = 0 \Rightarrow f(\tau) =$ constant

  $k > 0$ (even integer) $\Rightarrow f(\tau) \in \mathcal{M}_k(\Gamma(N))$ finite-dimensional linear space

- ring of modular forms generated by few elements

$$\mathcal{M}(\Gamma(N)) = \bigoplus_{k=0}^{\infty} \mathcal{M}_{2k}(\Gamma(N))$$

an explicit example in a moment

Kinetic Term

Kinetic term of the modulus τ $$\frac{|\partial_\mu \tau|^2}{\langle -i\tau + i\bar{\tau}\rangle^2}$$

Modular transformation $$\tau' = \frac{a\tau + b}{c\tau + d}, ad - bc = 1$$

- numerator

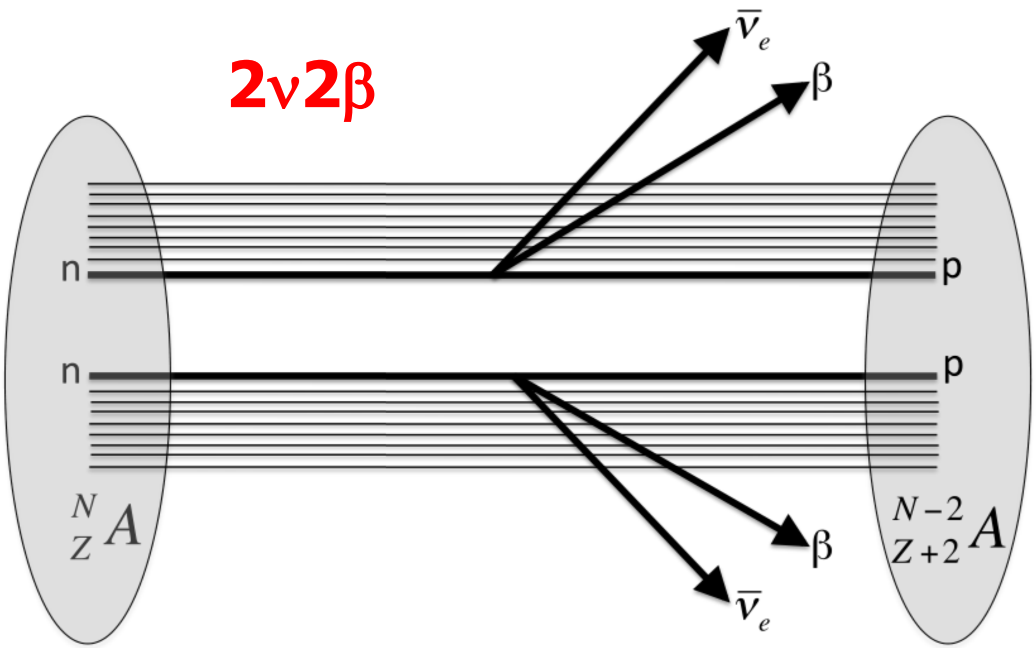
$$\partial_\mu \tau' = \frac{(a\partial_\mu \tau)(c\tau + d) - (a\tau + b)(c\partial_\mu \tau)}{(c\tau + d)^2} = \frac{(ad - bc)\partial_\mu \tau}{(c\tau + d)^2} = \frac{\partial_\mu \tau}{(c\tau + d)^2}$$

- denominator

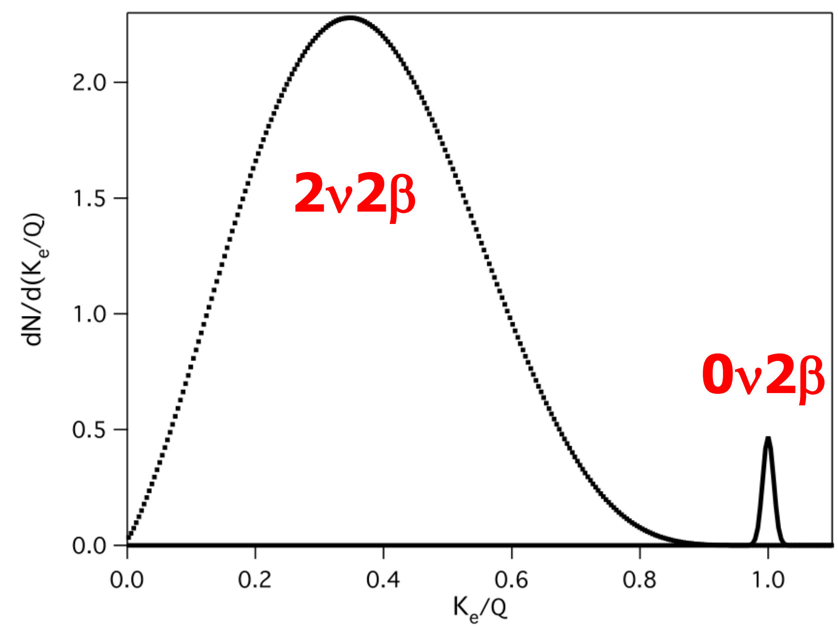
$$\tau' - \bar{\tau}' = \frac{(a\tau + b)(c\bar{\tau} + d) - (a\bar{\tau} + b)(c\tau + d)}{|c\tau + d|^2} = \frac{(ad - bc)(\tau - \bar{\tau})}{|c\tau + d|^2} = \frac{\tau - \bar{\tau}}{|c\tau + d|^2}$$

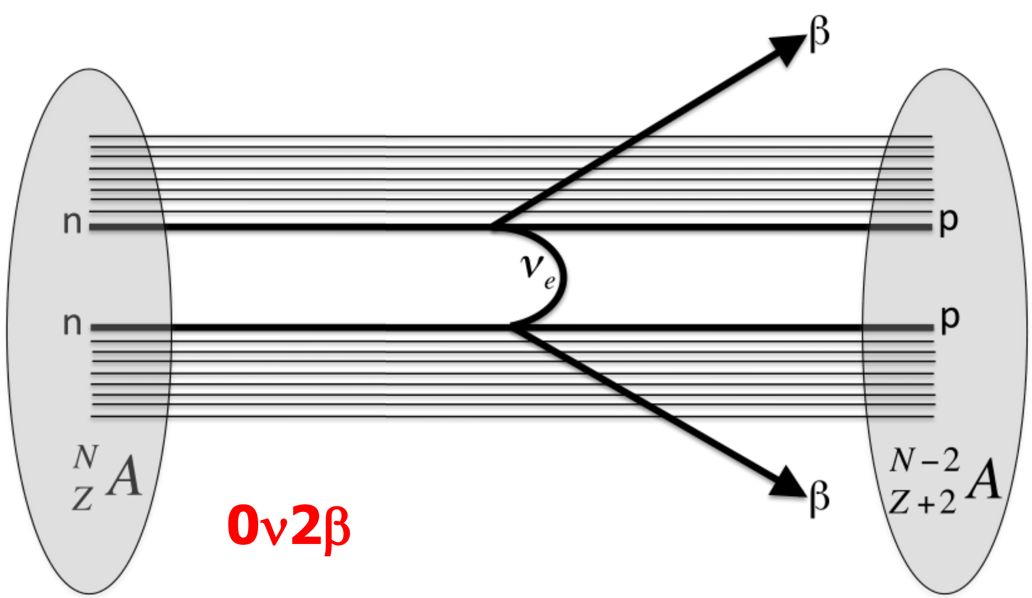
$\Rightarrow$ $$\frac{|\partial_\mu \tau'|^2}{\langle -i\tau' + i\bar{\tau}'\rangle^2} = \frac{|\partial_\mu \tau|^2}{\langle -i\tau + i\bar{\tau}\rangle^2}$$ Modular invariant

If this is the case, ...



$$ {}^{N}_{Z}A \rightarrow {}^{N-2}_{Z+2}A + 2\beta^{-} + 2\bar{\nu}_e $$



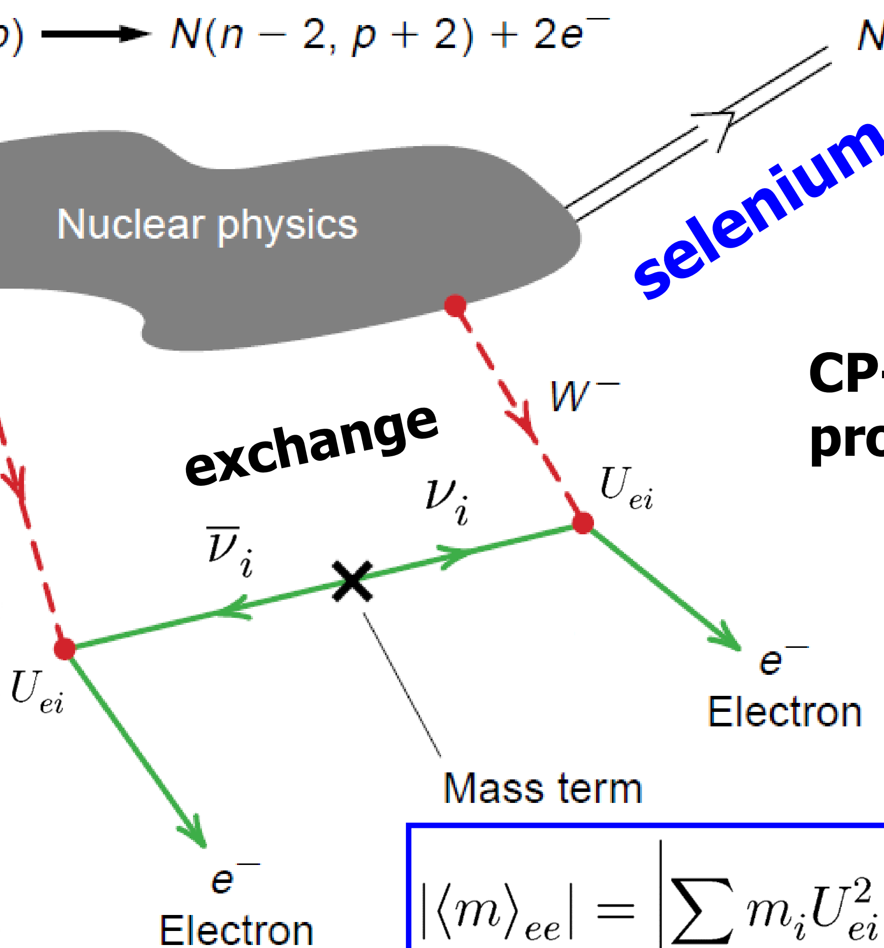


$$ {}^{N}_{Z}A \rightarrow {}^{N-2}_{Z+2}A + 2\beta^{-} $$

1939: 0ν2β decays

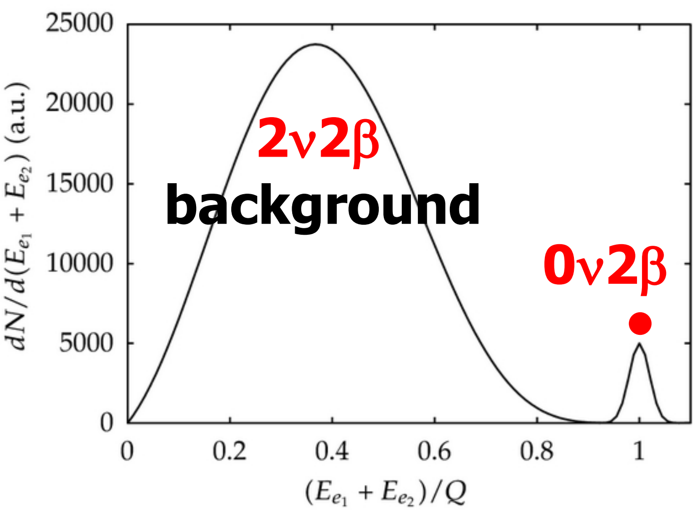
A 0ν2β decay can happen if massive ν's have the Majorana nature (Wendell Furry 1939)

$$T^{0\nu}_{1/2} = \left(G^{0\nu}\right)^{-1} \left|M^{0\nu}\right|^{-2} \left|\langle m \rangle_{ee}\right|^{-2}$$



Lepton number violation →

CP-conserving process ←



$$|\langle m \rangle_{ee}| = \left|\sum_i m_i U^2_{ei}\right|$$

Modular invariant theory [Ferrara et al, 1989; Feruglio, 1706.08749]

- $\mathcal{N}$=1 global supersymmetry theory with modular symmetry:

$$\mathcal{S} = \int d^4x d^2\theta d^2\bar{\theta} \mathcal{K}(\psi_I, \bar{\psi}_I; \tau, \bar{\tau}) + \int d^4x d^2\theta \mathcal{W}(\psi_I, \tau) + \text{h.c.}$$

- Minimal Kahler potential :

$$\mathcal{K} = -h \ln(-i\tau + i\bar{\tau}) + \sum_n (-i\tau + i\bar{\tau})^{-k_n} |\psi_n|^2$$

- Superpotential :

$$\mathcal{W} = \sum_n Y_{I_1 I_2 \ldots I_n}(\tau) \psi_{I_1} \psi_{I_2} \ldots \psi_{I_n}$$

- **Modular invariance requires Yukawa couplings are Modular Forms!**

$$\psi_{I_i} \xrightarrow{\gamma} (c\tau + d)^{-k_{\psi_{I_i}}} \rho_{I_i}(\gamma) \psi_{I_i}$$

$$Y_{I_1 I_2 \ldots I_n}(\tau) \to Y_{I_1 I_2 \ldots I_n}(\gamma\tau) = (c\tau + d)^{k_Y} \rho_Y(\gamma) Y_{I_1 I_2 \ldots I_n}(\tau)$$

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weight

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weight

Irreps of finite modular groups

Modular invariant theory [Ferrara et al, 1989; Feruglio, 1706.08749]

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$$\mathcal{S} = \int d^4x d^2\theta d^2\bar{\theta} \mathcal{K}(\psi_I, \bar{\psi}_I; \tau, \bar{\tau}) + \int d^4x d^2\theta \mathcal{W}(\psi_I, \tau) + \mathrm{h.c.}$$

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$$\psi_{I_i} \xrightarrow{\gamma} (c\tau + d)^{-k_{\psi_{I_i}}} \rho_{I_i}(\gamma)\psi_{I_i}$$

$$Y_{I_1 I_2 \ldots I_n}(\tau) \to Y_{I_1 I_2 \ldots I_n}(\gamma\tau) = (c\tau + d)^{k_Y} \rho_Y(\gamma) Y_{I_1 I_2 \ldots I_n}(\tau)$$

weight

Irreps of finite modular groups

$$\begin{cases} k_Y = k_{I_1} + k_{I_2} + \cdots + k_{I_n} \\ \rho_Y \otimes \rho_{I_1} \otimes \cdots \otimes \rho_{I_n} \supset 1 \end{cases}$$

❑ Remarks:

- Freedom of model building in this bottom-up approach: k_I, ρ_I
- For a given k_Y, ρ_Y , the modular forms space is finite-dimensional

➡ Only finite possible Yukawa couplings! Highly predictive !

N	$\dim\mathcal{M}_k(\Gamma(N))$	$\Gamma_N\ (\Gamma'_N)$	Modular forms multiplets			
			$k=1$	$k=2$	$k=3$	$k\geq 4$
2	$k/2+1\ (k\in\texttt{even})$	$S_3\ (S_3)$	$-$	$Y^{(2)}_{\mathbf{2}}$	$-$	...
3	$k+1$	$A_4\ (T')$	$Y^{(1)}_{\mathbf{2}}$	$Y^{(2)}_{\mathbf{3}}$	$Y^{(3)}_{\mathbf{2}},\ Y^{(3)}_{\mathbf{2}''}$	...
4	$2k+1$	$S_4\ (S'_4)$	$Y^{(1)}_{\hat{\mathbf{3}}'}$	$Y^{(2)}_{\mathbf{2}},\ Y^{(2)}_{\mathbf{3}}$	$Y^{(3)}_{\hat{\mathbf{1}}'},\ Y^{(3)}_{\hat{\mathbf{3}}},\ Y^{(3)}_{\hat{\mathbf{3}}'}$	...
5	$5k+1$	$A_5\ (A'_5)$	$Y^{(1)}_{\mathbf{6}}$	$Y^{(2)}_{\mathbf{3}},\ Y^{(2)}_{\mathbf{3}'},\ Y^{(2)}_{\mathbf{5}}$	$Y^{(3)}_{\mathbf{4}'},\ Y^{(3)}_{\mathbf{6}I},\ Y^{(3)}_{\mathbf{6}II}$	...

[Kobayashi, Tanaka, and Tatsuishi 2018; Feruglio 2017; Penedo and Petcov 2019; Novichkov et al. 2019; Ding, King, and Liu 2019b; Liu and Ding 2019; Liu, Yao, and Ding 2021; Novichkov, Penedo, and Petcov 2021; Wang, Yu, and Zhou 2021; Yao, Liu, and Ding 2021]

❑ Drawback: The Kahler potential is not under control!

[Chen, Ramos-Sanchez, Ratz 1909.06910]

$$\mathcal{K} \supset \sum_{\Phi_n}\sum_{k\geq 1}(-i\tau+i\bar{\tau})^{-k+k_n}\sum_{a}\kappa_a^{(k)}\left[Y^{(k)}(\tau)\otimes\bar{Y}^{(k)}(\tau)\otimes\psi_n\otimes\bar{\psi}_n\right]_{1,a}$$

❑ Advantages of EFG:

- There are very few candidates for self-consistently EFG groups G_{ecl} when u_γ nontrival: $G_f = \mathbb{Z}_3 \times \mathbb{Z}_3,\ \Delta(27),\ \Delta(54)\ \ldots$

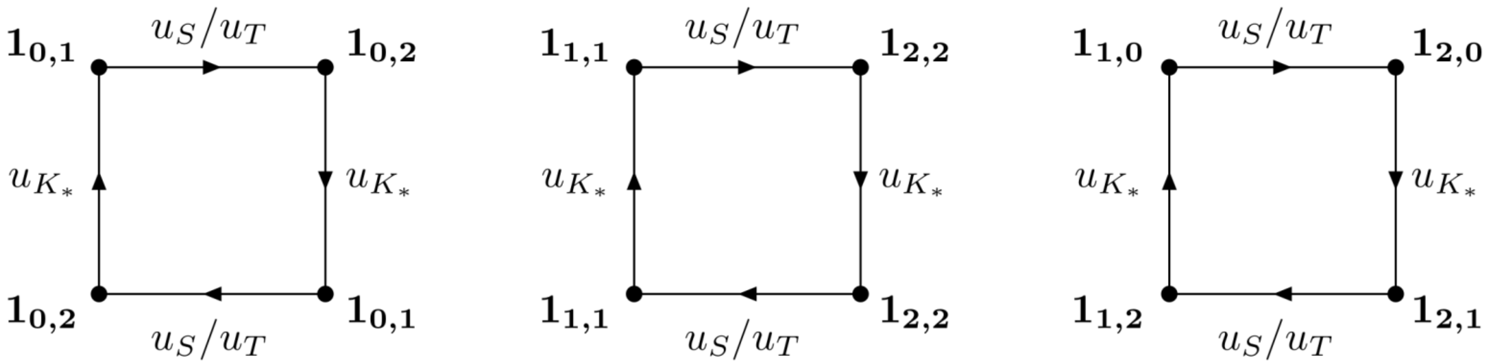
- The superpotential in EFG models is highly constrained because it satisfies both traditional flavor invariance and modular invariance.

- Kahler potential is under control due to the traditional flavor symmetry!

- EFG has a natural UV completion——Heterotic string on orbifold

There is currently no bottom-up EFG model

We choose $\mathbf{G_{ecl}} = \mathbf{\Omega(1) \cong \Delta(27) \rtimes T'}$


$\mathbf{1_{0,1}}$
u_S/u_T
$\mathbf{1_{0,2}}$
u_{K_*}
u_{K_*}
$\mathbf{1_{0,2}}$
$\mathbf{1_{0,1}}$
u_S/u_T
$\mathbf{1_{1,1}}$
u_S/u_T
$\mathbf{1_{2,2}}$
u_{K_*}
u_{K_*}
$\mathbf{1_{1,1}}$
$\mathbf{1_{2,2}}$
u_S/u_T
$\mathbf{1_{1,0}}$
u_S/u_T
$\mathbf{1_{2,0}}$
u_{K_*}
u_{K_*}
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$\mathbf{1_{2,1}}$
u_S/u_T